SEMICLASSICAL PREDICTIONS OF THE PREINFLATIONARY ERA, AND LIMITS ON $F(R,T)$ GRAVITY

BY

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Kindred spirits are not so scarce as I used to think. It's splendid to find out there are so many of them in the world.

L.M. Montgomery, *Anne of Green Gables*

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List of Abbreviations

CMB	Cosmic Microwave Background
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	General Relativity
GUT	Grand Unified Theory
ΛCDM	Lambda Cold Dark Matter
QFT	Quantum Field Theory
TOV	Tolman-Oppenheimer-Volkoff

Abstract

This dissertation investigates constraints on the physical applicability of the $f(R,T)$ theory of modified gravity and two predictions of semiclassical gravity on the preinflationary era of our Universe.

The $f(R,T)$ gravity theory modifies the traditional Lagrangian density of general relativity by including a dependence on the trace of the stress-energy tensor T. This modification would have a potentially measurable impact on physical observables, and comparing to observational data would allow for studying the relevance of such theories. We analyze changes to models of Earth's atmosphere under the $f(R,T)$ theory with linear dependence on T and obtain limits to match observational data. Based on the accuracy of the 1976 U.S. Standard Atmosphere model, we find limits on the coefficient χ, which parameterizes the modification, of $|\chi| \lesssim 1.8 \times 10^{-13}$, well below theoretical values used in the literature.

Semiclassical gravity studies quantum fields defined on a classical curved spacetime background. We use this technique to make two predictions about the preinflationary era: whether the anomalous large-scale suppression of the Cosmic Microwave Background (CMB) temperature-anisotropy power spectrum can be explained by a complete solution of the inflaton field, and whether quantum fields that are expected to be significant in the preinflationary era have a radiation-like contribution to the energy density. For the first prediction, we derive complete numerical solutions of the CMB power spectrum using various choices of vacuum states and methods of evolving those states from initial conditions in the preinflationary era. We find, in all cases, a large-scale suppression in the power spectrum, with important state-depended features in the momentum representation. For the second prediction, we analyze whether

the assumption of a radiation-dominated energy density in the preinflationary era can be supported with a semiclassical analysis of relevant quantum fields. We find that non-negligible quantum fields do indeed have radiation-like contributions.

Chapter 1: Introduction

Our best understanding of the Universe comes from two great achievements of the 20th century. First, the theory of general relativity (GR) describes the background structure of the Universe as a spacetime manifold which serves as a stage for the rest of physics. The main idea of GR is that gravitational phenomena are purely geometrical effects, encoded in the metric tensor, and trajectories of particles are nothing more than geodesics on the spacetime manifold. Moreover, the static nature of background space in Newtonian physics is replaced by a dynamical system interacting with the other systems within it. The dynamical nature of spacetime is governed by the Einstein field equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor encoding the curvature of spacetime, $T_{\mu\nu}$ is the stress-energy tensor of the matter content, and G is Newton's gravitational constant. Spacetime as a dynamic entity leads to an understanding of the evolution of our Universe via the study of cosmology.

Second, developed approximately at the same time as GR, quantum theory describes how physical systems operate on the smallest scales. Newtonian notions of a point in phase space are replaced by a state vector in a Hilbert space, and equations of motion are given by the Schrödinger equation in quantum mechanics. Incorporating Lorentz symmetry into quantum theory allows for working with relativistic particles, leading to quantum field theory (QFT).

In the Lagrangian formalism, GR is described by the total action

$$S = \int_{\mathcal{M}} d^4x \mathcal{L} \ , \tag{1.2}$$

which is an integral over the four-dimensional spacetime manifold $\mathcal{M}$ of the total Lagrangian density $\mathcal{L}$. The total Lagrangian density can be expressed as a combination $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_m$ of a gravitational contribution

$$\mathcal{L}_G = \sqrt{-g}\frac{R}{16\pi G} \tag{1.3}$$

and a matter contribution $\mathcal{L}_m$, and the total action can be separated into gravitiational S_G and matter S_m parts, $S = S_G + S_m$. The Einstein equations (1.1) are then obtained as Euler-Lagrange equations by varying the total action with respect to the metric tensor $g_{\mu\nu}$, with the stress-energy tensor resulting from variation of the matter action:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \ , \tag{1.4}$$

where variation is done with respect to the inverse metric for convenience. The Einstein tensor $G_{\mu\nu}$ is similarly obtained from variation of S_G, and the Einstein equations (1.1) are then obtained by variation of the total action.

One fundamental cosmological discovery in the last half century is the observational evidence of accelerated expansion, in which the scale factor a describing the size of the Universe is increasing at an accelerated rate. The Einstein equations (1.1) incorporating a stress-energy tensor $T_{\mu\nu}$ resulting from all known forms of matter are unable to account for this acceleration, so something else must be present in the Universe. The most widely accepted modification producing accelerated expansion is the inclusion of a cosmological constant Λ, an intrinsic energy density of the spacetime vacuum. This is achieved in the Lagrangian description by replacing $R \to R - 2\Lambda$ in the gravitional Lagrangian density (1.3). The ΛCDM cosmological model, in which the Universe is described by a mixture of cosmological constant, cold dark matter, and ordinary matter, is the current best model of our Universe. The cosmological constant appears in modified Einstein equations (1.1) as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{1.5}$$

Through Friedmann dynamics, the inclusion of Λ results in the present day accelerating expansion of the Universe. However, despite its success, many challenges still remain with the ΛCDM model [1].

Modified theories of gravity attempt to do away with the cosmological constant and such unknown quantities while still explaining accelerated expansion. One prevalent class of modified theories is that of $f(R)$ gravity, which in the Lagrangian formulation of gravity replaces the usual linear dependence (1.3) on the scalar curvatuve R in GR with a more general function $f(R)$, leading to a Lagrangian density

$$\mathcal{L}_G = \sqrt{-g}\frac{f(R)}{16\pi} \ , \tag{1.6}$$

and hence the name "$f(R)$ gravity." For example, one simple and surprisingly rich model is that of Starobinsky inflation, in which the Lagrangian includes the linear term of GR plus an R^2 term. Predictions from this theory are compatible with observed data and naturally lead to a period of cosmological inflation. Successes such as this could potentially indicate that the linear theory (1.3) of general relativity is a leading order approximation of a more complete theory.

Another class of modified gravity theories is that of $f(R,T)$ gravity. In [2] an explicit coupling between the scalar curvature and matter Lagrangian $\mathcal{L}_m$ in the gravitiational Lagrangian was proposed, resulting in an extra force in the equations of motion, and this extra force could therefore account for accelerated expansion. One application [3] of such a $f(R,\mathcal{L}_m)$ theory to the cosmological problem reinterpreted the matter term, responsible for accelerated expansion through the extra force, as a matter-dependent cosmological constant $\Lambda(T)$, where T is the trace of the stress-energy tensor, resulting in what is called a $f(R, \Lambda(T))$ theory. Including matter content in the gravitational Lagrangian could instead be considered without a cosmological constant but with T, and hence the so-called $f(R, T)$ gravity,

$$\mathcal{L}_G = \sqrt{-g}\frac{f(R,T)}{16\pi} \ . \tag{1.7}$$

The simplest case of $f(R, T)$ gravity is that of minimal coupling and, therefore, a separable Lagrangian,

$$\mathcal{L}_G = \frac{\sqrt{-g}}{16\pi}\left[f_1(R) + f_2(T)\right] \ . \tag{1.8}$$

This case has garnered much attention since its introduction in [4] because it allows for studying the impact of contributions from T without needing to specify the curvature dependence $f_1(R)$. At low densities, linear T contributions would dominate, and hence linear theories $f_2(T) \propto T$ have been heavily examined [4–7], especially those with

$$f(R,T) = \frac{R}{G} + 2\chi T \ . \tag{1.9}$$

Linear models such as this have produced a wide-ranging set of new predictions, such as cosmological constant-like effects [8], accelerated expansion [9], effects on equilibrium configurations of white dwarfs [5], and wormhole stability [10].

Despite the many predictions of modified gravity theories, little attention has been given to the consequences of these theories given available observational data. For instance, the introduction of new terms may produce additional forces dependent on the coupling constants like χ in the linear theory (1.9). Chapter 2 contains an article I co-authored that analyzed changes to Earth's atmospheric model due to such forces coming from the linear theory (1.9). We found that modifications to the atmospheric profile depend on the coupling constant χ and limits can therefore be placed on χ by comparing to atmospheric data. We found $|\chi| \lesssim 10^{-13}$, which severely restricts new predictions that can be made with this theory.

In addition to accelerated expansion, another fundamental concept in modern cosmology is that of inflation, an era in the early history of the Universe during which vacuum energy dominated and the Universe underwent a brief period of exponential expansion [11]. The basic consequence of such a rapid expansion is that the observable Universe, the region of the Universe we can observe today, would have come from a very small portion of the entire Universe prior to inflation, producing a drastic amplification of otherwise microscopic details and a smoothing out of features, leading to the apparent homogeneity and isotropy observed today. Typically, inflation is achieved by positing a hypothetical scalar *inflaton* field, characterized by a metastable vacuum state with a large vacuum energy, that causes regions of space in the vacuum state to exponentially expand. Inflation would then "freeze-in" vacuum fluctuations of the inflaton field through this rapid expansion, turning them into macroscopic, classical quantities. Energy density perturbations associated with these frozen-in fluctuations would then be possible sources of the large-scale structure we see today.

One particular result of inflation is the prediction of temperature anisotropies in the bath of photons observed to be permeating the Universe, known as the cosmic microwave background (CMB). In the standard Big Bang model, such anisotropies must be encoded in the initial conditions of the Universe in order to produce the CMB temperature anisotropy power spectrum we measure today. This fine tuning of initial conditions is often seen as a weakness of the model, so alternative explanations of the CMB power spectrum are regularly sought. Under inflation, no such fine tuning is required, and anisotropies can be understood as quantum fluctuations that are frozen-in by rapid expansion. There is remarkable agreement between the inflationary theory and measurements of the CMB power spectrum (see Fig. 1 in [12] and Fig. 3 in [11]) on all angular scales below 90°.

However, despite such remarkable agreement, there are several anomalies appearing in the data that have yet to be understood. For instance, there is an apparent suppression of the quadrupole moment that seems to violate statistical isotropy [13]. The most straightforward way to interpret these anomalies would be to accept that they are just low-probability events in the ΛCDM model, occuring in a small fraction of all possible random outcomes for our Universe. Taking this perspective, one could proceed to study how other observations, such as CMB polarization anisotropies, are realized in this low-probability outcome and test for consistency. Such an approach is possible to some extent with existing data, but it is expected that the higher accuracy of future experiments will provide a much richer set of data to test (for example, see Sec. IV in [13]).

It is also worthwhile to understand whether the CMB temperature anomalies may be explained from more fundamental physics. For instance, the precise nature of the inflaton and other quantum fields leading into inflation could lead to perceptible impacts on the CMB. Very little is understood about the preinflationary era, primarily because remnants from this era would be smoothed out and expanded beyond the scale of the observable Universe due to inflation. However, it is still possible to analyze the impact that quantum fields would have on inflation and test what states lead to acceptable models of the CMB temperature anisotropy power spectrum.

Studying the complete preinflationary era in general would require physics above the Planck scale, around 10^{19} GeV, above which quantum gravity effects would need to be taken in account. Currently, we are unable to probe such a high-energy scale directly, so, despite numerous theories, the true nature of quantum gravity is not known. However, if some portion of the preinflationary era had curvature below the Planck scale, around 10^{-4} in Planck units, then gravity would behave classically. In this case, one could make use of the semiclassical approximation, in which gravity behaves classically but matter is still treated as being quantum, using the techniques of quantum field theory in curved spacetime [14]. In this approximation, the stress-energy tensor is replaced by an operator, and the equations of GR are solved classically by approximating $T_{\mu\nu} \to \langle T_{\mu\nu} \rangle$, where $\langle \cdot \rangle$ is the expectation value. For example, the semiclassical version of the Einstein equations (1.1) are

$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle \ . \tag{1.10}$$

Chapter 3 contains an article I co-authored in which we examined the effects of a semiclassical preinflationary era on the energy density of the Universe and the resulting CMB temperature anisotropy power spectrum.

Because of our ignorance of the true nature of the Universe prior to inflation, many inflationary models must make assumptions regarding our small portion of the Universe leading into inflation. If the Universe was indeed semiclassical for some amount of time leading into inflation, then the validity of such assumptions could be tested using the semiclassical approximation. For instance, one assumption often made is that the energy density prior to inflation was radiation dominated [15], corresponding to the leading order contribution to the energy density being proportional to a^{-4},

$$\rho \propto a^{-4} \ , \tag{1.11}$$

where a is the scale factor. Arguing under what conditions this behavior can occur would then lead to a better understanding of the semiclassical preinflationary era and hence support for or a refinement of inflationary models.

Contributions to the energy density from the scalar field are shown to be radiation-like in the semiclassical approximation of the preinflationary era in the Appendix of the article in Chapter 3. The arguments made there for the scalar field are general enough that one might expect a similar set of arguments could be made for other fields that are expected to be relavent in the preinflationary era, namely spin-$\frac{1}{2}$ fields and massless spin-1 fields. However, both of these types of fields prove much more challenging to work with than scalar fields, and the arguments made for the scalar case do not translate without significantly more work. The spin-1 case has gauge symmetry, so one needs to fix a gauge in order to perform calculations. This requires the inclusion of an additional gauge-fixing term as well as an unphysical ghost field contribution necessary for maintaining gauge invariance. The spin-$\frac{1}{2}$ case requires a different mathematical ansatz in order to obtain a properly renormalized energy density in the same manner as the scalar field. Chapter 4 contains an article accepted for publication in Physical Review D I co-authored in which we obtain predictions about the contributions from spin-$\frac{1}{2}$ and spin-1 quantum fields and argue that both indeed produce radiation-like contributions to the energy density in the preinflationary era.

Chapter 2: Limits on $f(R,T)$ Gravity from Earth's Atmosphere

Preface

The following paper was published in Physical Review D [16] and is reproduced here by permission of the American Physical Society. Stylistic variations between this dissertation and the published paper are due to format requirements.

Contributions to this paper were divided between myself and Eric Carlson. Dr. Carlson proposed the original idea of using observational data to obtain limits on the coefficients that parameterize the effects of the modified gravity theory, the set of observable systems to potentially study, and the use of the hydrostatic equation applied to Earth's atmosphere to obtain these limits. I performed the derivation of the hydrostatic equation in our specific modified gravity theory, proposed the use of the U.S. Standard Atmosphere data, and performed the numerical computations in obtaining our results. I wrote the intial manuscript, and we contributed equally to edits leading to the published version.

Abstract

We investigate changes in Earth's atmospheric models coming from the $f(R,T)$ modified theory of gravity, in which the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar and the trace of the stress-energy tensor. We obtain a generic form for the gravitational field equations and derive the hydrostatic equation for Earth's atmosphere for leading order terms $f(R,T) = R + 2\chi T$. Based on the apparent accuracy of the 1976 U.S. Standard Atmosphere model, which varies no more than 10% from observations, we find limits of $-1.6 \times 10^{-13} \lesssim \chi \lesssim 1.8 \times 10^{-13}$.

2.1 Introduction

Modern cosmological observation has revealed the accelerating expansion of the Universe [17–19]. The discovery has become one of the most important developments in modern cosmology due to its apparent inconsistency with the predictions of general relativity, which state that a universe filled with a mixture of ordinary matter and radiation should experience a slowing of the expansion. Novel theories and modifications to general relativity have been proposed to explain the acceleration. Two notions in particular have been heavily investigated: either the Universe contains a great amount of dark energy or the theory of general relativity breaks down on the cosmological scale [20].

One theory that has gained much attention for its ability to explain the expansion is $f(R)$ modified gravity [21]. The gravitational action, from which the Einstein equations are derived, is traditionally a linear function of the scalar curvature R. The $f(R)$ theory of gravity replaces R in the gravitational action with an arbitrary function $f(R)$, with the traditional R being the leading order contribution. Higher order gravity theories, such as the Starobinsky model [22], are thus generally possible.

A generalization of $f(R)$ gravity proposed in Ref. [23] incorporates an explicit coupling between the matter Lagrangian and an arbitrary function of the scalar curvature, which leads to an extra force in the geodesic equation of a perfect fluid. It was later shown that this extra force may account for the accelerated expansion of the Universe [3]. The inclusion of matter terms in the gravitational action was further explored in Ref. [4] in $f(R,T)$ gravity, in which the gravitational Lagrangian density is an arbitrary function of both R and the trace of the stress-energy tensor T. Part of the motivation of $f(R,T)$ gravity is to produce models that can act similarly to a cosmological constant without explicitly including such a constant. The arbitrary dependence on T encapsulates the possible contributions from both nonminimal coupling and explicit T terms.

In general such modified theories can contain higher than first order derivatives in the Lagrangian and will thus suffer Ostrogradsky instability. One can in principle introduce an auxiliary field to the Lagrangian to remove the higher derivatives, resulting in ghosts [24, 25]. A general discussion of such issues is beyond the scope of this paper. For the specific case we will be discussing, this will not be relevant.

Cosmological effects of $f(R,T)$ theories have been explored by choosing several functional forms of f. The separation $f(R,T) = f_1(R) + f_2(T)$ has received much attention because one can explore the contributions from T without specifying $f_1(R)$. For example, in such separable theories, a nonequilibrium picture of thermodynamics at the apparent horizon of the Friedmann-Lemaître-Robertson-Walker universe was studied in Ref. [26].

While higher powers of T have been considered [27], at low densities, the linear contributions will dominate, and linear $f_2(T) \propto T$ is of interest and has been studied [4–7].

Harko *et al.* [4] noted that such a simple model could produce cosmological constant-like effects, but it is easy to see from their formula (29) that an accelerating universe is not possible from this term alone. As we will demonstrate, strong limits on χ, which we use to parametrize linear contributions from T, indicate that this term has no cosmological significance.

Even if $f(R,T)$ is not separable, one would expect that in situations of low curvature and matter density, the linear terms of $f(R,T)$ should dominate. We therefore focus on these terms, and assuming there is no constant term (cosmological constant) we may approximate

$$f(R,T) = R + 2\chi T, \tag{2.1}$$

where the coefficient of R must be 1 to yield conventional gravity in low curvature environments and χ is a single parameter describing the modification of gravity. Note there is no question of Ostrogradsky instability in this theory. Obtaining strong limits on χ could severely affect the possible contributions of this term in astrophysical situations.

One approach to studying the consequences of such gravitational modifications would be comparison with limits from the parametrized post-Newtonian formalism [28]. Strong limits on several parameters from the Solar System and other astrophysical systems can be obtained. Such an approach is nontrivial in this case because the dynamics of this theory are not entirely described in terms of the metric. In particular, as will be shown below, the stress-energy tensor is not conserved, and hence pressureless dust will not generally follow geodesics. We do not pursue this approach here, primarily because we believe we can achieve stronger limits by using the Tolman-Oppenheimer-Volkoff (TOV) equations.

In Ref. [7], modified TOV equations were obtained and used to obtain modifications to models of strange stars. It was found that substantial and potentially measurable changes to such stars occurred if $|\chi| \sim 1$. Observational data for white dwarfs were used in Ref. [5] to obtain a lower limit of $\chi \gtrsim -3 \times 10^{-4}$. This limit was obtained by modeling the interior of the white dwarf as a noninteracting zero temperature electron gas. Indeed, Ref. [5] was unable to obtain self-consistent solutions for $\chi > 0$ because the vanishing of the sound velocity near the surface of the white dwarf did not allow the density to drop to zero at finite radius. However, the surface of the white dwarf is not at zero temperature, and the electron interactions are not negligible, so we believe that positive χ values are also allowed. We have not performed this calculation because we believe we can obtain more stringent limits by considering the Earth's atmosphere.

In this paper, we investigate Eq. (2.1) in the weak-field regime of Earth's atmosphere to further limit the possible range of χ. By modeling the atmosphere as a perfect fluid ideal gas, we obtain modifications to the traditional hydrostatic equation

$$\frac{dp}{dr} = -g\,(p+\rho)\,, \tag{2.2}$$

where p is the isotropic pressure, ρ is the mass density, and g is the acceleration due to gravity. Solutions to the modified hydrostatic equation can then be compared to the atmospheric model given in Ref. [29] to obtain strong limits on χ.

Our paper is structured as follows. The general formalism for $f(R,T)$ gravity is given in Sec. 2.2. In Sec. 2.3, we derive the modified hydrostatic equation in a spherically symmetric, static spacetime. The hydrostatic equation is then obtained for our model of Earth's atmosphere, and computational results and limits on χ are given in Sec. 2.4. Finally, our conclusions are given in Sec. 2.5.

We use the sign conventions of Misner *et al.* [30] with metric signature $(-+++)$ and work in units where $c = G = 1$.

2.2 $f(R,T)$ formalism

The theory of $f(R,T)$ gravity is motivated by the $f(R)$ framework that replaces the standard Hilbert action with an arbitrary function of the Ricci scalar R [25]. Harko *et al.* [4] first proposed $f(R,T)$ gravity by introducing to the gravitational action an arbitrary dependence on the trace of the stress-energy tensor T. The full action is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi}f(R,T) + \mathcal{L}_m\right], \tag{2.3}$$

where the matter Lagrangian $\mathcal{L}_m$ describes any matter contributions. We will follow the derivation given by Ref. [4].[1] Beginning with Eq. (2.3), we define the stress-energy tensor as

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} = -2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m. \tag{2.4}$$

There is an implicit assumption that $\mathcal{L}_m$ does not depend on derivatives of the metric. Variation of Eq. (2.3) with respect to $g^{\mu\nu}$ yields the field equations

$$(R_{\mu\nu}+g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R(R,T)-\frac{1}{2}f(R,T)g_{\mu\nu} = -f_T(R,T)(\Theta_{\mu\nu}+T_{\mu\nu})+8\pi T_{\mu\nu}, \tag{2.5}$$

where $f_R(R,T) \equiv \partial f(R,T)/\partial R$, $f_T(R,T) \equiv \partial f(R,T)/\partial T$, and

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L}_m - 2T_{\mu\nu} - 2g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{2.6}$$

[1] Reference [4] has an equation apparently identical to our Eq. (2.4); however, they should have the opposite sign because they are working with the opposite sign metric. This apparent sign error is canceled by another apparent sign error when they choose $\mathcal{L}_m = -p$ as the perfect fluid matter Lagrangian.

The covariant derivative of Eq. (2.5) can then be written as

$$\nabla_\mu T^{\mu\nu} = \frac{f_T(R,T)}{8\pi - f_T(R,T)} \left\{ \nabla_\mu \left[\ln f_T(R,T)\right] \left(T^{\mu\nu} + \Theta^{\mu\nu}\right) + \nabla_\mu \left(\Theta^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right) \right\}. \tag{2.7}$$

Note that the stress-energy tensor in traditional and $f(R)$ gravity is divergenceless. Applying our explicit form from Eq. (2.1), this simplifies to

$$\nabla_\mu T^{\mu\nu} = \frac{\chi}{4\pi - \chi} \nabla_\mu \left(\Theta^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right). \tag{2.8}$$

2.3 Hydrostatic equation in spherically symmetric $f(R,T)$ gravity

Static, spherically symmetric objects are described by the metric

$$ds^2 = -e^{\nu(r)}\, dt^2 + e^{\lambda(r)}\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{2.9}$$

where $\nu(r)$ and $\lambda(r)$ are metric potentials.

We will consider the stress-energy tensor of a perfect fluid, such that

$$T^{\mu\nu} = (p + \rho)\, u^\mu u^\nu + p g^{\mu\nu}, \tag{2.10}$$

where ρ and p are the energy density and isotropic pressure of the fluid and u^μ is the fluid four-velocity, satisfying $u_\mu u^\mu = -1$ and $u^\mu \nabla_\nu u_\mu = 0$. Equation (2.6) can now be written as

$$\Theta^{\mu\nu} = p g^{\mu\nu} - 2T^{\mu\nu}. \tag{2.11}$$

The field equations in traditional general relativity have no direct dependence on $\mathcal{L}_m$, leading to nonunique choices such as $\mathcal{L}_m = -\rho$ or $\mathcal{L}_m = p$, as discussed in Ref. [31]. In theories with nonminimal coupling of matter to curvature or an action with contributions from T, $\mathcal{L}_m$ explicitly appears in the field equations, and the choices for $\mathcal{L}_m$ become nonequivalent [23]. Following the work of Refs. [4] and [5], we will use $\mathcal{L}_m = p$. Then, the hydrostatic equation from the radial component of Eq. (2.8) is

$$\frac{dp}{dr} = -g\left(\rho + p\right) + \frac{\chi}{8\pi + 2\chi}\frac{d}{dr}\left(\rho - p\right), \tag{2.12}$$

where g is the gravitational acceleration,

$$g \equiv \frac{1}{2}\frac{d\nu}{dr}. \tag{2.13}$$

2.4 Earth's atmosphere and results

In Earth's weak field, the atmosphere can be described using a Schwarzschild metric, for which $e^{\nu(r)} = 1 - 2GM/r$ and thus $g = GM/\left(r^2 - 2GMr\right) = g(r)$. In the atmosphere, $g(r) \approx GM/r^2$. We approximate $p \ll \rho$ and take Earth's atmosphere to be an ideal gas with $\rho = pM/RT$, where M is the atmospheric molar mass, T is the temperature, and R is the ideal gas constant. After reinserting factors of c and assuming constant M, Eq. (2.12) is

$$\frac{dp}{dr} = -\frac{gM}{RT}p + \frac{\chi M c^2}{\left(8\pi + 2\chi\right)R}\frac{d}{dr}\left(\frac{p}{T}\right). \tag{2.14}$$

We now introduce geopotential altitude Z, as defined in the U.S. Standard Atmosphere, 1976 [29] as $g_0\, dZ = g\, dr$, where $g_0 = 9.80665\,\mathrm{m}/s^2$ is the standard surface gravity and $Z = 0$ corresponds to sea level. Hence, Z will not exactly correspond to physical altitude; for example, a geopotential altitude of $Z = 79\,\mathrm{km}$ corresponds to a physical altitude of about $86\,\mathrm{km}$ above sea level. Equation (2.14) in terms of

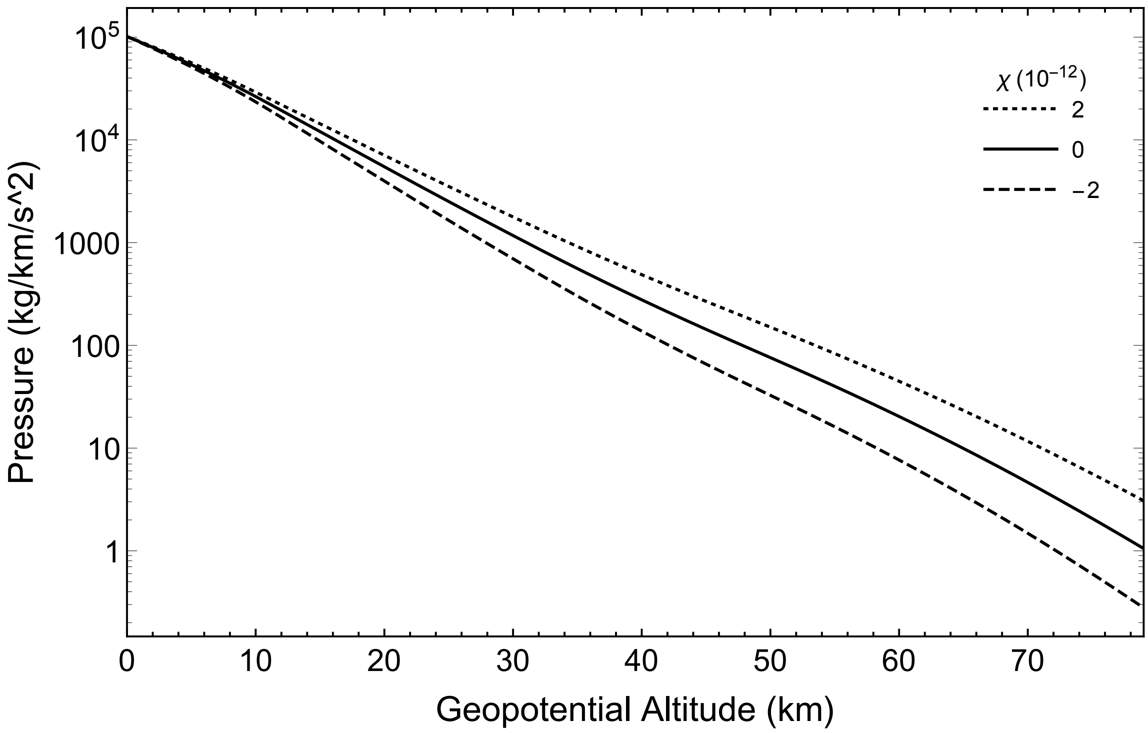

Figure 2.1: Pressure profile numerical solutions as a function of geopotential altitude Z.

geopotential altitude becomes

$$\frac{dp}{dZ} = -\frac{g_0 M}{RT} p + \frac{\chi M c^2}{(8\pi + 2\chi)\, R} \frac{d}{dZ} \left(\frac{p}{T} \right). \tag{2.15}$$

We use the U.S. Standard Atmosphere model for both temperature and atmospheric composition as a function of geopotential height. The U.S. Standard Atmosphere was an update to an existing international model first published in 1958 and updated in 1962, 1966, and 1976 that developed a mathematical model of the atmospheric profile [29]. It proposed a pressure equation similar to Eq. (2.15) without χ terms. The model also developed geopotential profiles for mass density, temperature, and other atmospheric measurements. It separated Earth's atmosphere into upper and lower regions, with the boundary at $Z = 79\,\text{km}$. We focus on the lower region as it is the best measured and understood. In this region, composition is approximately

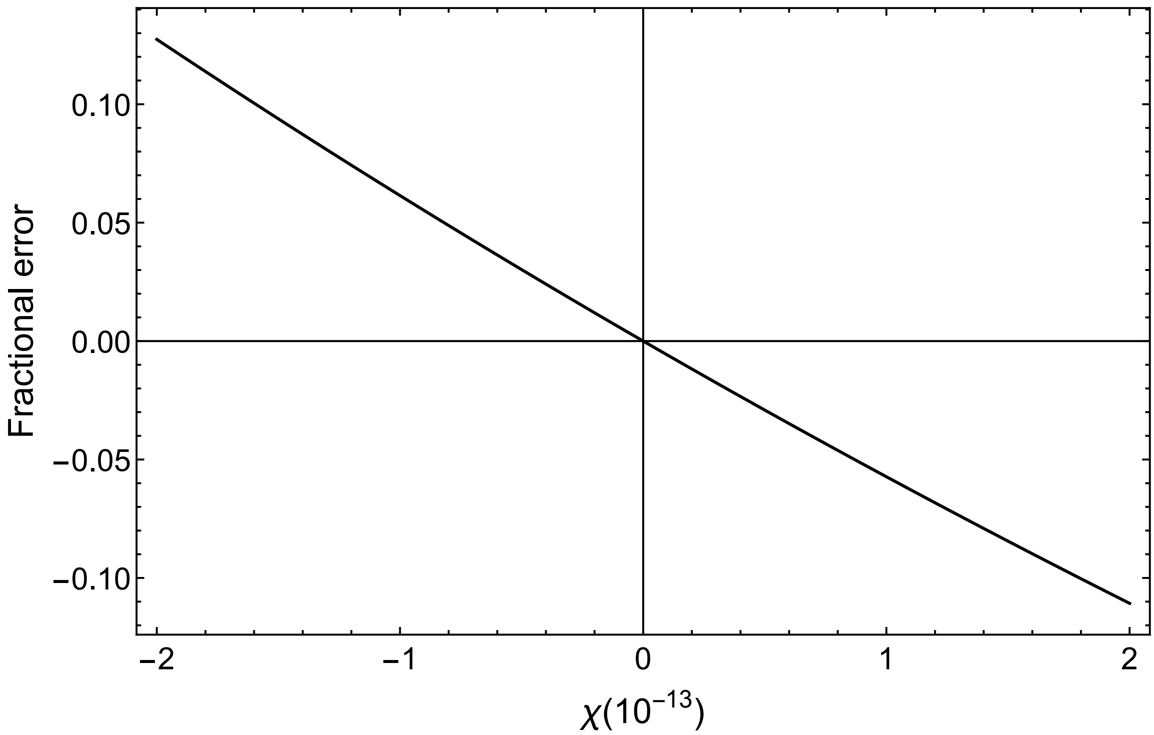

Figure 2.2: Fractional error of numerical solutions from U.S. Standard Atmosphere pressure model at $Z = 79\,\text{km}$, which corresponds to a physical altitude of 86 km.

constant, with $M = 28.9644 \times 10^{-3}\,\text{kg/mol}$, and the temperature is a piecewise linear function of Z.

The final update to the U.S. Standard Atmosphere improved the model such that it differed from atmospheric measurements by at most 10% [29]. We assert that any changes to the predicted pressure profile due to χ terms can be no larger than the largest error of 10% from the model. We numerically solved the differential equation in Eq. (2.15) for pressure given specific values of χ. Figure 2.1 shows pressure profiles for select χ values as a function of Z, and Fig. 2.2 shows the fractional change to the pressure as a function of χ at $Z = 79\,\text{km}$, where the pressure profile difference due to χ is greatest. From the analysis, we obtain approximate limits on χ of

$$-1.6 \times 10^{-13} \lesssim \chi \lesssim 1.8 \times 10^{-13}. \tag{2.16}$$

2.5 Conclusion

We have examined the $f(R,T)$ gravity modified hydrostatic equation in Earth's atmosphere to obtain limits on the leading order contributions from T to the gravitational action. We found that the parameter χ describing the leading order modification to gravity is limited to the range $-1.6\times10^{-13} \lesssim \chi \lesssim 1.8\times10^{-13}$ by examining the atmospheric region below $Z = 79\,\text{km}$ (approximately an altitude of 86 km). These results follow from the assertion that modifications from χ should vary from observational data by at most 10%, which is motivated by the U.S. Standard Atmosphere model having a maximum percent deviation of the same amount.

Within the limits we observe, leading order T terms in the gravitational action would have tiny effects on strange stars (as studied in Ref. [7]) and cosmological models. For example, Ref. [6] found significant cosmological effects for $\chi \sim 1$. Our limits are also 9 orders of magnitude stronger than limits from white dwarfs [5].

Perhaps more promising would be to look at other terms of which the contributions would be small in Earth's atmosphere but could be larger in more extreme situations. The term γRT, for instance, would yield an extra R term in the covariant derivative of the stress-energy tensor, which in environments of large curvature, such as neutron stars, could produce significant changes to traditional models.

Acknowledgments

We would like to thank P. Anderson for his helpful discussion.

Chapter 3: Semiclassical predictions regarding a preinflationary era and its effects on the power spectrum

Preface

The following paper was published in Physical Review D [32] and is reproduced here by permission of the American Physical Society. Stylistic variations between this dissertation and the published paper are due to format requirements. A minor typographical error in the paragraph before Eq. (3.36) is corrected here.

Contributions to this paper were divided between myself, Paul Anderson, Eric Carlson, and Bradley Hicks. I provided the majority of the numerical implementation used in evolving the coefficient functions according to the differential equations given in Eq. (3.34), obtaining a numerical representation of the resulting power spectrum in Eq. (3.48), and obtaining the final multipole moments from the power spectrum using Eq. (3.49). I produced all figures and performed the computational analysis involved in each. I contributed to the analysis of the natural vacuum states, the sudden approximation states, and the adiabatic vacuum states in Secs. 3.4.1, 3.4.2, and 3.6.1 along with Dr. Anderson and Dr. Carlson. I wrote the majority of the content in early drafts of Secs. 3.3, 3.5, and 3.6. I also provided meaningful edits to the entirety of the final published version of the paper.

We work with conformally coupled fields, which are defined as follows. The Lagrangian density for the scalar field of interest is given by

$$\mathcal{L} = \frac{1}{2\sqrt{-g}}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - [m^2 + \xi R]\phi^2\} \; , \tag{3.1}$$

where ξ is the curvature coupling factor. A conformal transformation is a pointwise rescaling of the metric by a function $\Omega(x)$,

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu} \ .$$

A field ϕ is considered conformally invariant if its field equations are invariant under a conformal transformation of the metric and the field

$$\phi(x) \to \bar{\phi}(x) = \Omega(x)^{-1}\phi(x) \ . \tag{3.2}$$

It can be shown that the field is conformally invariant if $m = 0$ and $\xi = 1/6$. For this reason, the coupling $\xi = 1/6$ is called conformal coupling, even when $m \neq 0$. For more details, see pages 43–44 of [14].

Abstract

An investigation is undertaken into the properties and effects of a pre-inflationary era during at least part of which semiclassical gravity was valid. It is argued that if the Universe (or our part of it) was approximately homogeneous and isotropic during that era, then the Universe was likely to have been radiation dominated. A simple model in which the Universe contains classical radiation and a cosmological constant is used to investigate potential effects of such a pre-inflationary era on the cosmic microwave background. The power spectrum is computed using the mode functions of a quantized massless minimally coupled scalar field. Various choices of state for this field are considered, including adiabatic vacuum states of various orders and the vacuum state that would naturally occur if the Universe made a sudden transition from being radiation dominated to de Sitter space. In all cases investigated there is a suppression of the power spectrum at large angles, and when plotted as a function of the momentum parameter, there are always oscillations with state-dependent amplitudes.

3.1 Introduction

In the traditional big bang theory, the Universe began with zero size and an initial curvature singularity. Of course, what this really means is that classical general relativity breaks down, and a description of the very early Universe must come from a quantum theory of gravity. Since it is currently unknown which, if any, of the current quantum gravity candidates is correct, the beginning of the Universe (if there was one) is unknown. However, the paradigm today is that early in its history the Universe underwent a period of inflation. If there was a pre-inflation era, then semiclassical gravity may well have been valid during the latter part of that era. One expects that semiclassical gravity would be valid in the early universe once the curvature is well below the Planck scale. This would be true for spacetime curvatures of order 10^{-4} in Planck units which in the early universe would correspond to energy scales about one hundred times larger than that of the Grand Unified Theory, GUT, energy scale. For this reason it is interesting to explore the predictions that semiclassical gravity makes about the Universe prior to inflation.

There is some ambiguity as to the exact form of the semiclassical Einstein equations due to the unknown sizes of the coefficients of the scalar curvature squared and Ricci squared terms in the gravitational Lagrangian. Renormalization of the stress-energy tensor for quantum fields in curved space requires the existence of such terms. Nevertheless, when gravity is thought of as an effective field theory, one expects that the contributions from such terms to the semiclassical Einstein equations should be relatively small. If that is the case, and if the pre-inflationary Universe, or at least our part of it, was approximately homogeneous and isotropic, then from the point of view of semiclassical gravity, the Universe began with zero size as in the classical big bang model. Of course one of the advantages of inflation is that the part of the Universe we can observe today would have been an extremely small part of the Universe at the onset of inflation. Thus, if there were significant inhomogeneities on larger scales, they would be well outside the current horizon. However, here we make the stronger assumption that any large inhomogeneities were far enough away that the part of the Universe that contained the part we can see today, and was significantly larger than it, was approximately homogeneous and isotropic. Then our argument implies that this portion was radiation dominated at least during the latter part of the pre-inflationary era. Given our ignorance of the pre-inflation era, it is of interest to consider models of this type.

It was argued in [33] that if the Universe began with zero size, then it is possible to define an initial vacuum state for a conformally coupled massive scalar field that, at the initial time, is equivalent to the conformal vacuum state for a conformally coupled massless scalar field. This was done by replacing the mode equation with a Volterra equation that could be solved iteratively. At lowest order it was shown that for any other homogeneous and isotropic state, the stress-energy tensor $\langle T_{\mu\nu} \rangle$ contains terms that have the same form as classical radiation. This is not surprising since it is known [34] that for a conformally invariant field in any other homogeneous and isotropic state than the conformal vacuum the stress-energy tensor has such terms.

In this paper we investigate the properties of this initial vacuum state as well as other homogeneous and isotropic vacuum states in significantly more detail than was done in [33]. We show that for the vast majority of cases where the Universe begins with zero size in any other homogeneous and isotropic state, the stress-energy tensor for a massive conformally coupled scalar field at early times has a term that acts like classical radiation. We use this to make an argument that it is extremely likely that if the Universe had a period before inflation in which the semiclassical approximation was valid, then it expanded in approximately the same way as a radiation-dominated universe during that period. Evidence for a radiation dominated pre-inflationary era has also been found in a model in which the Wheeler-DeWitt equation is solved in the minisuperspace approximation which includes the Hamiltonian for the scale factor when a cosmological constant is present along with the Hamiltonian for a single mode of a massless minimally coupled scalar field [35].

From an observational point of view, the best chance for evidence of a pre-inflationary radiation-dominated phase for the Universe would likely come from the cosmic microwave background, where it has been shown [36] that if inflation did not go on for too long, then there could be significant deviations from the usual prediction if the state of the quantum field differs significantly from the Bunch-Davies state [37–

40].

With this as motivation we consider a simple model in which the Universe contains classical radiation and a cosmological constant. At early times the Universe expands like a radiation-dominated universe and at late times like a de Sitter universe. This model gives a natural onset to inflation described entirely by the cosmological constant. Since we are concerned with the effects on the power spectrum of the pre-inflationary phase, our results are independent of the reheating phase which occurs in most inflationary models and they are also compatible with warm inflation models [41] in which there is a gradual transfer of energy from the inflaton field to the radiation which eliminates the need for a reheating phase.

In many models of inflation the inflaton field is treated as a classical minimally coupled scalar field with a potential while quantum fluctuations of the inflaton field are treated as a quantized massless minimally coupled scalar field. In this paper we are effectively modeling the classical inflaton field with a cosmological constant. We compute the power spectrum using the mode functions for a massless minimally coupled scalar field. The effects of certain types of initial vacuum states for this field on the power spectrum are investigated. One is the natural vacuum state that occurs in a pure radiation-dominated universe that suddenly transforms into de Sitter space. The others are adiabatic vacuum states [42–46] of zeroth-, second-, and fourth-order. There have been several previous calculations of the power spectrum for various models in which the pre-inflationary era was homogeneous, isotropic, and radiation-dominated [15, 47–56]. As is discussed in Sec. 3.6, it appears that in most previous cases a sudden approximation or something similar to one was used. Two exceptions are [54] and [57], where the power spectrum was computed numerically using zeroth-order adiabatic states. A detailed comparison of our results with theirs is given in Sec. 3.6.

In agreement with previous calculations we find that the power spectrum deviates from that of the Bunch-Davies state because the initial vacuum state differs from the Bunch-Davies state. In particular the power spectrum is suppressed at large angles. When plotted in terms of the momentum parameter k there are oscillations for all of the states considered. The largest oscillations come from the sudden approximation and from zeroth-order adiabatic states where the adiabatic matching time (discussed in Sec. 3.4.2) occurs near the onset of inflation. For adiabatic states the oscillations have significantly smaller amplitudes for earlier matching times and for higher order adiabatic states.

In Sec. 3.2 we present our argument that if there was a pre-inflationary phase in which the semiclassical approximation was valid and if the Universe or our part of it was approximately homogeneous and isotropic during that phase, then it is likely that it expanded like a radiation-dominated universe. In Sec. 3.3 we discuss the solution to Einstein's equations for our specific model, which consists of classical radiation and a cosmological constant. The different states that we use for the computations of the power spectrum of the massless minimally coupled scalar field are discussed in Sec. 3.4. A general form for the power spectrum for our model is derived in Sec. 3.5. Some of our computations of the power spectrum are presented, discussed, and compared with previous calculations in Sec. 3.6. A brief summary of our results is given in Sec. 3.7. The Appendix contains details of the calculations related to a possible radiation-dominated pre-inflationary phase. Throughout we use units such that $\hbar = c = G = 1$.

3.2 Prediction regarding a pre-inflationary era

As discussed in the Introduction, we consider the possibility that the Universe, or our part of it, began with zero size in a homogeneous and isotropic state from the point of view of semiclassical gravity. We further assume that there was a pre-inflationary era in which the semiclassical approximation was valid and that during this era interactions between the quantum fields present did not make the dominant contributions to the stress-energy tensors of those fields. In this case we present an argument that it is very likely the Universe was expanding like a radiation-dominated universe during this pre-inflationary era.

As mentioned in the Introduction, in many models of inflation the inflaton field is a massive minimally coupled scalar field that is treated classically. Quantum fluctuations of this field during the period of inflation are generally approximated by a quantized massless minimally coupled scalar field. In such models, the power spectrum is computed from these fluctuations. This is also our approach here. However, most quantum fields that are likely to have had a significant impact on the expansion of the Universe in a semiclassical pre-inflationary era are of spin $\frac{1}{2}$ and spin 1. In the approximation that interactions are neglected, the massless ones are exactly conformally invariant and the massive ones are conformally invariant in the limit that their masses go to zero. Thus the effects of the inflaton field on the expansion during a pre-inflationary phase are expected to be small.

For conformally invariant fields in a spatially flat homogeneous and isotropic spacetime the stress-energy tensor $\langle 0|T_{\mu\nu}|0\rangle$ is composed of two local tensors that contain higher derivative terms [14]. If the semiclassical approximation is valid, then it is usually assumed that these terms are very small[1]. If a conformally invariant field in such a spacetime is in a homogeneous and isotropic state other than the conformal vacuum state, then there is an additional term in its stress-energy tensor that has the same form as that of classical radiation [34]. Therefore, if the early Universe consisted

[1] An important exception is Starobinsky inflation [22], which requires that the coefficient of the R^2 term in the gravitational Lagrangian be of the order 10^9 and that it have a certain sign. We do not consider Starobinsky inflation in this paper.

only of massless conformally invariant quantum fields in homogeneous and isotropic states, and if one or more of the fields was not in the conformal vacuum state, then the Universe would expand like a radiation-dominated universe provided the higher derivative terms made a small contribution to the stress-energy tensor.

Of course many of the quantum fields in the early Universe were massive. We model the spin $\frac{1}{2}$ and spin 1 massive fields with conformally coupled massive scalar fields. The rationale for doing this is that all of these fields are conformally invariant in the massless limit, and at high enough momenta they are effectively massless.

We also restrict our attention to cases when the Universe begins with zero scale factor. We do this, as mentioned in the Introduction, in the same spirit that classical general relativity predicts that the Universe began with an initial singularity.

The metric for a spatially flat homogeneous and isotropic universe is

$$ds^2 = a^2(\eta)\,(-d\eta^2 + d\vec{x}^2)\ , \tag{3.3}$$

with η the conformal time defined by $a\,d\eta = dt$. Scalar fields with arbitrary masses and curvature couplings ξ satisfy the equation

$$\Box\phi - m^2\phi - \xi R\phi = 0\ , \tag{3.4}$$

where the scalar curvature is

$$R = \frac{6a''}{a^3}\ . \tag{3.5}$$

Here primes denote derivatives with respect to η. Expanding the fields in terms of modes in the usual way gives

$$\phi = \int d^3k\ \left[a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\phi_k(\eta) + a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}\phi^*_k(\eta)\right]\ . \tag{3.6}$$

With the definitions

$$\phi_k = \frac{\psi_k}{a} \; , \tag{3.7a}$$

$$\omega_k^2 = k^2 + m^2 a^2 \; , \tag{3.7b}$$

one finds that ψ_k is a solution to the equation [14]

$$\psi_k'' + \left[\omega_k^2 + 6\left(\xi - \frac{1}{6}\right)\frac{a''}{a}\right]\psi_k = 0 \tag{3.8}$$

and satisfies the Wronskian condition

$$\psi_k \psi_k^{*\prime} - \psi_k^* \psi_k' = i \; . \tag{3.9}$$

The classical expression for the stress-energy tensor of an arbitrarily coupled scalar field is [14]

$$T_{\mu\nu} = (1-2\xi)\partial_\mu\phi\partial_\nu\phi + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}\left(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + m^2\phi^2\right) - 2\xi\phi\nabla_\mu\nabla_\nu\phi$$
$$+ 2g_{\mu\nu}\xi^2 R\phi^2 + \xi G_{\mu\nu}\phi^2 \; . \tag{3.10}$$

Substituting (3.6) and (3.7a) into (3.10) then yields an unrenormalized energy density [58]:

$$\rho_u = \frac{1}{4\pi^2 a^4}\int dk\, k^2 \left\{ |\psi_k'|^2 + \omega_k^2 |\psi_k|^2 + 6\left(\xi - \frac{1}{6}\right)\left[\frac{a'}{a}\left(\psi_k'\psi_k^* + \psi_k\psi_k^{*\prime}\right) - \frac{a'^2}{a^2}|\psi_k|^2\right]\right\} . \tag{3.11}$$

Specializing to the case of conformal coupling, $\xi = \frac{1}{6}$, it is helpful to define functions $\alpha_k(\eta)$ and $\beta_k(\eta)$ by the simultaneous equations

$$\psi_k(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}} \left[\alpha_k(\eta) e^{-i\theta_k(\eta)} + \beta_k(\eta) e^{i\theta_k(\eta)}\right] \ , \tag{3.12a}$$

$$\psi_k'(\eta) = \sqrt{\frac{\omega_k(\eta)}{2}} \left[-i\alpha_k(\eta) e^{-i\theta_k(\eta)} + i\beta_k(\eta) e^{i\theta_k(\eta)}\right] \ , \tag{3.12b}$$

where

$$\theta_k(\eta) = \int^{\eta} dx\, \omega_k(x) \ . \tag{3.13}$$

The lower limit for this integral is arbitrary. The Wronskian condition (3.9) becomes

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \ . \tag{3.14}$$

Substituting (3.12a) and (3.12b) into (3.11), setting $\xi = \frac{1}{6}$ and using (3.14) yields

$$\rho_u = \frac{1}{4\pi^2 a^4} \int_0^\infty dk\, k^2 \omega_k \left(1 + 2\,|\beta_k|^2\right) \ . \tag{3.15}$$

Subtracting off the adiabatic counterterms [58, 59] one finds

$$\rho_r = \frac{1}{2\pi^2 a^4} \int_0^\infty dk\, k^2 \omega_k \ |\beta_k|^2 - \frac{m^2}{96\pi^2} \frac{a'^2}{a^4} + \frac{1}{2880\pi^2} \left(-\frac{1}{6} {}^{(1)}H_0{}^0 + {}^{(3)}H_0{}^0 \right) \ . \tag{3.16}$$

The second term on the right is a finite renormalization of $G_0{}^0$, and the last two terms are the higher derivative terms that are assumed to be small.

The first term on the right in (3.16) has the same form as the energy density for classical radiation if $\omega_k|\beta_k|^2$ is independent of time. However, it is actually a function of time, so its behavior at early times needs to be analyzed. This is done in Appendix 3.A, where it is shown that if $\beta_k(\eta)$ is nonzero in the limit $a(\eta) \to 0$, then the initial behavior of the first term in (3.16) is that of classical radiation provided that (i) the integral in (3.16) is finite at η_0; (ii) $|\beta_k(\eta_0)|$ increases slower than k^{-1} at small k; (iii) the derivative $(a^2)'$ has a finite limit as $\eta \to \eta_0$; and (iv) $\int_{\eta_0}^{\eta} |(a^2(x))''| dx$ is finite. If these conditions are satisfied then the resulting solution to the semiclassical Einstein equations will describe a universe that expands like a radiation-dominated universe at early times since the second term in (3.16) is a finite renormalization of ${G_0}^0$ and the last terms are assumed to be negligible.

3.2.1 A "natural" vacuum state

It is well known that in a dynamical spacetime there is usually no state that one can unambiguously label as the vacuum state as there is for free free quantum fields in Minkowski space. However, there can be states which for one reason or another are preferred. One example is the Bunch-Davies state in pure de Sitter space [37–40]. Another is the class of states found in [60] for which at a given moment of time the stress-energy tensor for the quantum field is exactly equal to zero. Here we discuss a different choice for a vacuum state based on the above analysis of states for a massive conformally coupled scalar field.

The state when $\beta_k(\eta_0) = 0$ for the conformally coupled massive scalar field provides a natural definition of a vacuum state if the Universe began with zero size since, as shown above, there is no term in the energy density that acts like classical radiation. One might guess that a similar state would exist, at least in some cases, for nonconformally coupled scalar fields. This is correct, but as we next show, in some important cases the state is problematic for nonconformally coupled scalar fields and potentially problematic for conformally coupled massive scalar fields.

Returning to (3.8), it is useful to define an effective mass

$$M_a^2 = m^2 a^2 + 6\left(\xi - \frac{1}{6}\right)\frac{a''}{a} \,. \tag{3.17}$$

If initially

$$\psi_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \ , \tag{3.18}$$

which is the exact solution for the conformally invariant scalar field in the conformal vacuum state, then one can find a formal solution in terms of a Volterra equation:

$$\psi_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\,\eta} - \frac{1}{k}\int_{\eta_0}^{\eta} dx_1 M_a^2(x_1)\, \sin\left[k(\eta - x_1)\right] \psi_k(x_1) \ . \tag{3.19}$$

This can be solved by iteration to give

$$\psi_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{k^n} I_n(k,\eta) \ , \tag{3.20a}$$

$$I_n(k,\eta) = \int_{\eta_0}^{\eta} dx_1 \int_{\eta_0}^{x_1} dx_2 \cdots \int_{\eta_0}^{x_{n-1}} dx_n M_a^2(x_1) \sin\left[k(\eta - x_1)\right]$$
$$\times \left\{ \prod_{j=2}^{n} M_a^2(x_j) \sin\left[k(x_{j-1} - x_j)\right] \right\} \frac{e^{-ikx_n}}{\sqrt{2k}} \ , \tag{3.20b}$$

where the product is equal to 1 for $n = 1$, $a(\eta_0) = 0$, and either $\eta_0 = -\infty$ or $-\infty < \eta_0 < \infty$. It can be shown that this converges provided that $k^{-1} \int_{\eta_0}^{\eta} dx |M_a^2(x)|$ is finite [33]. We shall restrict our attention to those cases in what follows.

The first term in the sum in (3.20a) can be written as

$$-\frac{1}{k} I_1(k,\eta) = -\frac{1}{2ik}\frac{e^{ik\eta}}{\sqrt{2k}} \int_{\eta_0}^{\eta} dx_1 e^{-2ikx_1} M_a^2(x_1) + \frac{1}{2ik}\frac{e^{-ik\eta}}{\sqrt{2k}} \int_{\eta_0}^{\eta} dx_1 M_a^2(x_1) \ . \tag{3.21}$$

The second term on the right is positive frequency for all times. The first term in some cases has a negative frequency component. This was not noticed in [33]. To see this, assume that there is no divergence in M_a^2 or any of its derivatives at η_0. Then successive integrations by parts can be done. The evaluation of each at the upper limit yields a positive frequency term. The evaluation at the lower limit yields a negative frequency term. Thus if M_a^2 or any of its derivatives is nonzero at η_0 then there is a negative frequency term. Suppose that M_a^2 and its first $(n-1)$ derivatives are zero at η_0 and that the n'th derivative of M_a^2 is nonzero at η_0. Then the vacuum state can only be of adiabatic order $n-1$ if $m \neq 0$ and $\xi = \frac{1}{6}$. The vacuum state can only be of adiabatic order $n+1$ if $m = 0$ and $\xi \neq \frac{1}{6}$. If $m \neq 0$ and $\xi \neq \frac{1}{6}$ then the order of the vacuum state depends on whether it is the n'th derivative of the m^2a^2 term (giving $n-1$) or the other term in (3.17) (giving $n+1$).

From the point of view of pure mathematics, one can think of Eq. (3.8) as a mode equation in flat space with a time dependent potential. If the potential vanishes in the limit $\eta_0 = -\infty$, then it turns on very slowly and this leads to an infinite order initial vacuum state. In the case that η_0 is finite, the potential turns on at the time η_0. How rapidly it turns on depends on how rapidly $a \to 0$. The more rapidly it turns on, the more particle production one would expect to occur due to the 'turn on,' and the lower the order of the adiabatic state that the vacuum state corresponds to. Of course there are cases where the potential is a constant in the limit $\eta \to -\infty$ and cases where it (or one of its derivatives) diverges at $\eta = \eta_0 > -\infty$. We do not consider these cases here.

An important example where the spacetime begins at $\eta_0 = -\infty$ and the vacuum is an infinite-order adiabatic state is de Sitter space in spatially flat coordinates, where $a = \frac{1}{-H\eta}$ with H a constant. The vacuum state in this case is the Bunch-Davies state.

An important example where the vacuum state is a finite-order adiabatic state is when the scale factor can be expanded in the power series $a(\eta) = \sum_{n=1}^{\infty} a_n(\eta - \eta_0)^n$. In general there are two contributions to M_a^2. One comes from the m^2a^2 term, which occurs for any massive field. For it, one finds that if $a_1 \neq 0$, so that the Universe is approximately radiation-dominated at early times, then the vacuum state is at most a first-order adiabatic state. The second contribution to M_a^2 is proportional to $\frac{a''}{a}$. If

$a_1 \neq 0$ then the Volterra solution (3.20) does not work unless $a_2 = a_3 = 0$. In that case, if $m = 0$ and $\xi \neq \frac{1}{6}$ the vacuum state is at most second-order adiabatic if $a_4 \neq 0$, third-order adiabatic if $a_4 = 0$ and $a_5 \neq 0$ and so forth. For the model described in the next section $a_4 = 0$ and $a_5 \neq 0$, so the vacuum state is at most a third-order adiabatic one.

In general it is necessary to have a fourth-order adiabatic state for the stress-energy tensor to be ultraviolet finite. Thus, at least for the model we consider below, the vacuum state discussed here is not acceptable for the massless minimally coupled scalar field. For a conformally coupled massive scalar field it is technically only necessary to have a zeroth-order adiabatic state, so this vacuum state could work. However, if the spacetime is even slightly inhomogeneous or anisotropic, then something akin to a fourth-order adiabatic state would be required to yield a finite stress-energy tensor. Therefore we do not consider this to be an acceptable vacuum state for a massive field for the model considered below or for any model of the Universe in which the expansion approaches that of a radiation-dominated universe at early times but is not exactly equal to that of a radiation-dominated universe.

3.3 Simple model with a radiation-dominated pre-inflationary era

For the rest of this paper we consider a simple model that has a radiation-dominated pre-inflationary era and a late time inflationary era. It consists of classical radiation plus a positive cosmological constant Λ. In this case, one of the Friedmann-Lemaître-Robertson-Walker (FLRW) equations is

$$\frac{a'^2}{a^4} = \frac{\Lambda}{3} + \frac{8\pi c_r}{3a^4} \, , \tag{3.22}$$

where $c_r > 0$ is a constant. The trace of the Einstein equations gives

$$R = \frac{6a''}{a^3} = 4\Lambda \; . \tag{3.23}$$

We use the following scaled variables:

$$\alpha \equiv \left(\frac{\Lambda}{8\pi c_r}\right)^{\frac{1}{4}} a \; , \tag{3.24a}$$

$$\chi \equiv \gamma^{-1}\eta \; , \tag{3.24b}$$

$$\kappa \equiv \gamma k \; , \tag{3.24c}$$

$$\gamma^2 \equiv \frac{3}{\sqrt{8\pi c_r \Lambda}} \; . \tag{3.24d}$$

Equation (3.22) can then be written

$$\frac{d\alpha}{d\chi} = \sqrt{1+\alpha^4} \; . \tag{3.25}$$

Integrating (3.25) and choosing the constant of integration such that $\alpha|_{\chi=0} = 1$ gives

$$\alpha = \frac{\mathrm{cn}\left(2\chi|\frac{1}{2}\right)}{\sqrt{1-\sqrt{2}\,\mathrm{sn}\left(2\chi|\frac{1}{2}\right)\mathrm{dn}\left(2\chi|\frac{1}{2}\right)}} \; , \tag{3.26}$$

where sn, cn, and dn are the Jacobi elliptic functions in the notation of [61]. The limits of χ defined by $\alpha|_{\chi_0} = 0$ and $\alpha|_{\chi_\infty} = \infty$ are given by

$$-\chi_0 = \chi_\infty = \frac{1}{2}K\left(\frac{1}{2}\right) = 0.927037\ldots \; , \tag{3.27}$$

where K is the complete elliptic integral of the first kind. A plot of α during the pre-inflationary era is shown in Figure 3.1.

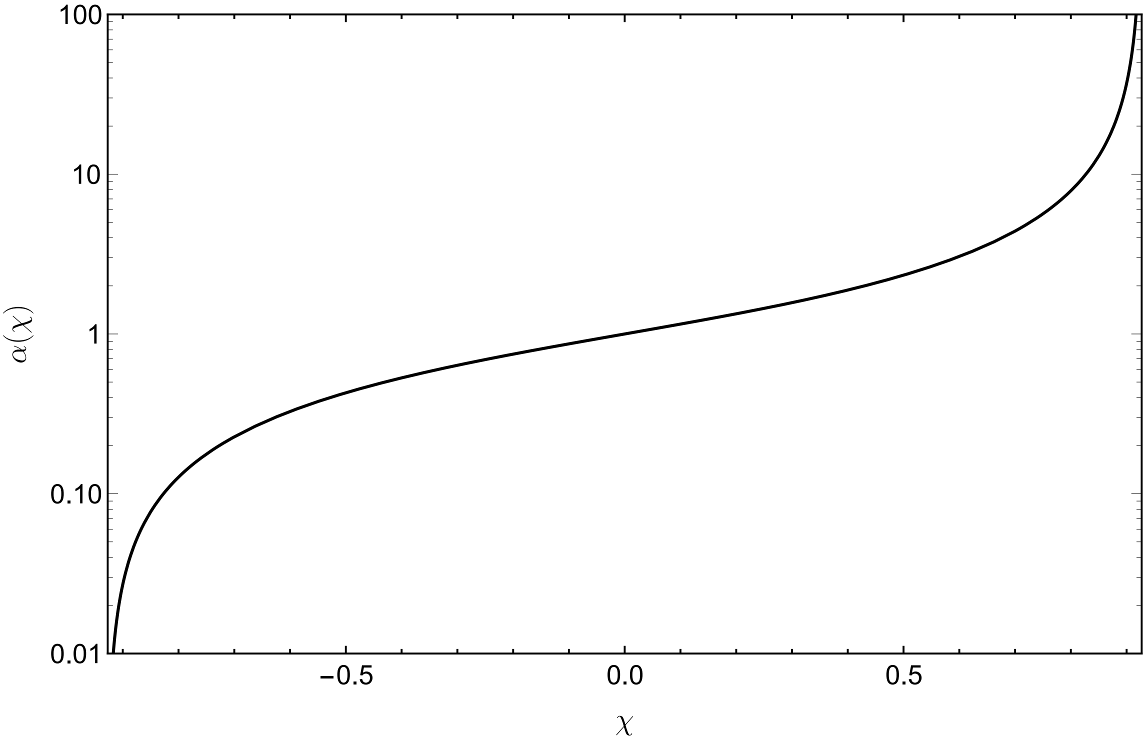

Figure 3.1: Rescaled scale factor $\alpha(\chi)$ over its domain (χ_0, χ_∞).

The mode equation (3.8)written in terms of χ is

$$\frac{d^2\psi_\kappa}{d\chi^2} + \left(\kappa^2 + \gamma^2 M_a^2\right)\psi_\kappa = 0 \ . \tag{3.28}$$

As discussed in the Introduction, in our model we compute the power spectrum using the modes of a massless minimally coupled scalar field, $m = \xi = 0$. For this field substituting (3.17) into (3.28) and using (3.23) and (3.24) gives

$$\frac{d^2\psi_\kappa}{d\chi^2} + \left(\kappa^2 - 2\alpha^2\right)\psi_\kappa = 0 \ . \tag{3.29}$$

The Wronskian condition in terms of scaled conformal time is

$$\psi_\kappa \frac{d\psi_\kappa^*}{d\chi} - \psi_\kappa^* \frac{d\psi_\kappa}{d\chi} = i\gamma \ . \tag{3.30}$$

At late times $\chi \to \chi_\infty$, the spacetime is asymptotically de Sitter, and thus the mode equation (3.29) approaches the mode equation for pure de Sitter space. In pure de Sitter space the solution corresponding to the Bunch-Davies state, which we will call v_κ, is

$$v_\kappa = \sqrt{\frac{\gamma}{2\kappa}} e^{-i\kappa(\chi-\chi_\infty)} \left[1 - \frac{i}{\kappa(\chi-\chi_\infty)}\right] . \tag{3.31}$$

It satisfies the Wronskian condition (3.30). It plus its complex conjugate form a set of linearly independent solutions to (3.29) in the limit $\chi \to \chi_\infty$. For all times it is possible to change variables so that the general solution to the exact mode equation (3.29) is in the form

$$\psi_\kappa = c_1(\kappa;\chi) v_\kappa + c_2(\kappa;\chi) v_\kappa^* , \tag{3.32a}$$

$$\frac{d\psi_\kappa}{d\chi} = c_1(\kappa;\chi)\frac{dv_\kappa}{d\chi} + c_2(\kappa;\chi)\frac{dv_\kappa^*}{d\chi} . \tag{3.32b}$$

The coefficient functions $c_1(\chi)$ and $c_2(\chi)$ are defined by demanding that (3.32) hold for all χ, and are given by

$$c_1(\kappa;\chi) = -\frac{i}{\gamma}\left(\psi_\kappa \frac{dv_\kappa^*}{d\chi} - \frac{d\psi_\kappa}{d\chi} v_\kappa^*\right) , \tag{3.33a}$$

$$c_2(\kappa;\chi) = \frac{i}{\gamma}\left(\psi_\kappa \frac{dv_\kappa}{d\chi} - \frac{d\psi_\kappa}{d\chi} v_\kappa\right) . \tag{3.33b}$$

As de Sitter space is approached in the limit $\chi \to \chi_\infty$, c_1 and c_2 approach constant values. First-order differential equations for c_1 and c_2 can be obtained by using (3.32) in (3.29):

$$\frac{dc_1}{d\chi} = -\frac{2i}{\gamma}\left[\frac{1}{(\chi_\infty-\chi)^2} - \alpha^2(\chi)\right]\left(c_1|v_\kappa|^2 + c_2 v_\kappa^{*2}\right) , \tag{3.34a}$$

$$\frac{dc_2}{d\chi} = \frac{2i}{\gamma}\left[\frac{1}{(\chi_\infty-\chi)^2} - \alpha^2(\chi)\right]\left(c_1 v_\kappa^2 + c_2|v_\kappa|^2\right) . \tag{3.34b}$$

The explicit form for the Bunch-Davies state (3.31) can then be substituted to get expressions where γ cancels out. The state for the massless minimally coupled scalar field can be specified by choosing values of c_1 and c_2 at some specific time χ_m for all values of κ.

3.4 States for the massless minimally coupled scalar field

3.4.1 Sudden approximation

The model described above is designed to give a smooth transition from an initially radiation-dominated universe to de Sitter space, such as would be expected to occur in any realistic model of inflation with an approximately homogeneous and isotropic pre-inflationary phase. An even simpler approximation is to suddenly switch from a pure radiation-dominated universe to a pure de Sitter universe at some particular time. This is called the *sudden approximation.* It has the advantage that the initial vacuum state for the massless minimally coupled scalar field is just the conformal vacuum,

$$\psi_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \,, \tag{3.35}$$

because in a pure radiation-dominated universe the scalar curvature is zero.

The matching can be done by choosing η, a, and a' to be continuous across the matching surface. A complete set of solutions to the mode equation (3.8) for a given value of k in pure de Sitter space consists of the mode function for the Bunch-Davies state and its complex conjugate. Multiplying each of these times a constant and matching the mode functions and their first derivatives at the sudden transition time η_s fixes the values of the two matching coefficients.

Our model is not strictly speaking compatible with a pure radiation-dominated universe because we take R to be a nonzero constant. Similarly, because our model contains radiation, it is not strictly speaking compatible with pure de Sitter space. Nevertheless it is possible to use our model to obtain the same results for the power spectrum as one gets from the sudden approximation. This is useful so that we can directly compare the resulting power spectrum with those of previous calculations as well as with our results for adiabatic vacuum states, which are described below.

Because the matching in the real spacetime is just done at a particular value of the time $\eta = \eta_s$, there is nothing to prevent one from using (3.24b) and (3.24c) for some fixed value of $c_r \Lambda$ to change from η to χ and from k to κ. After doing so, the matching equations are equivalent to (3.33) evaluated at the matching time χ_s. However, for $\chi > \chi_s$ the spacetime is pure de Sitter space, so instead of being starting values for a numerical integration, in the sudden approximation c_1 and c_2 are fixed constants.

In terms of the scaled coordinates, (3.35) becomes

$$\psi_\kappa = \sqrt{\frac{\gamma}{2\kappa}} e^{-i\kappa\chi} \,. \tag{3.36}$$

The Bunch-Davies state in these coordinates in pure de Sitter space is given in (3.31). Substituting into (3.33) at $\chi = \chi_s$ gives

$$c_{1s} = \left[1 + \frac{i}{\kappa(\chi_s - \chi_\infty)} - \frac{1}{2\kappa^2(\chi_s - \chi_\infty)^2}\right] e^{-i\kappa\chi_\infty} \,, \tag{3.37a}$$

$$c_{2s} = -\frac{1}{2\kappa^2(\chi_s - \chi_\infty)^2} e^{i\kappa(\chi_\infty - 2\chi_s)} \,. \tag{3.37b}$$

The sudden approximation is an extreme limit and often results in a state that is not physically acceptable. As shown below, that is the case here. It is for this reason that it is useful to consider a model in which the Universe evolves continuously from a radiation-dominated era to the inflationary era such as the one described in Sec. 3.3.

As shown in Sec. 3.2.1, we have not found a physically acceptable natural initial vacuum state for the massless minimally coupled scalar field in this model. In lieu of a natural initial vacuum state, it is often useful to consider various adiabatic vacuum states, and these are discussed next.

3.4.2 Adiabatic vacuum states

As discussed above, a choice of vacuum state for our model can be made by specifying the values of c_1 and c_2 in (3.33) at some time χ_m for each value of κ. The solutions to the mode equations can then be obtained at any other time by numerically integrating (3.34) forward (or backward) in time. In this section we discuss adiabatic vacuum states. These are exact states for the quantum field that are specified by using a WKB approximation to provide starting values for the modes and their first time derivatives at some particular time that we call the matching time [42–46].

To understand how the WKB expansion works for the scaled coordinates, it is useful to begin with the original coordinates and the original form of the mode equation (3.8). For the massless minimally coupled scalar field, the mode equation can be written in the form

$$\psi_k'' + \left(k^2 - \frac{1}{6}a^2 R\right)\psi_k = 0 \,. \tag{3.38}$$

Note that for the model we are using, $R = 4\Lambda$. Then one makes the change of variable

$$\psi_k = \frac{1}{\sqrt{2W_k}} \exp\left[-i\int_{\eta_1}^{\eta} W_k(\eta')d\eta'\right] \,, \tag{3.39}$$

where η_1 is an arbitrary constant. This automatically ensures the Wronskian condition (3.9), with the result that

$$W^2 = k^2 - \frac{a''}{a} - \left(\frac{W''}{2W} - \frac{3W'^2}{4W^2}\right) \,. \tag{3.40}$$

One starts with zeroth-order in terms of derivatives of the metric and then iterates. At each iteration the new terms contain two more derivatives of the scale factor than the previous ones. Thus

$$W^{(0)} = k \ , \tag{3.41a}$$

$$W^{(2)} = k - \frac{a''}{2ka} \ , \tag{3.41b}$$

$$W^{(4)} = k - \frac{a''}{2ka} + \frac{2a'^2 a'' - 2aa'a''' - 2aa''^2 + a^2 a''''}{8k^3 a^3} \ . \tag{3.41c}$$

An adiabatic state is an exact state for the quantum field that is obtained by using the WKB approximation to some order to fix the starting values for the modes at some particular matching time η_m. One does this by substituting the expression for W at some order into (3.39) and equating with the exact mode function. One does the same for the first time derivative of (3.39). For a given order there are many possible adiabatic states, in part because one obtains different states for different matching times.

For the specific model we are considering and the scaled variables that we are using, one can think of α^2 as being of second adiabatic order and each derivative of α then giving an extra adiabatic order. The reason is that, as mentioned above, the scalar curvature R is a constant for our model and as is seen from Eq. (3.38) is multiplied by a factor of a^2 in the mode equation. Then the WKB approximation in terms of scaled variables is

$$\psi_\kappa = \sqrt{\frac{\gamma}{2W}} \exp\left[-i \int_{\chi_1}^{\chi} W(\chi') d\chi'\right] \tag{3.42}$$

where

$$W^2 = \kappa^2 - 2\alpha^2 - \left[\frac{1}{2W}\frac{d^2 W}{d\chi^2} - \frac{3}{4W^2}\left(\frac{dW}{d\chi}\right)^2\right] \ . \tag{3.43}$$

One easily finds that

$$W^{(0)} = \kappa \,, \tag{3.44a}$$

$$W^{(2)} = \kappa - \frac{\alpha^2}{\kappa} \,, \tag{3.44b}$$

$$W^{(4)} = \kappa - \frac{\alpha^2}{\kappa} + \frac{2\alpha^4 + 1}{2\kappa^3} \,, \tag{3.44c}$$

where we used (3.25) to eliminate all the derivatives of α.

To do the adiabatic matching at a time χ_m, it is easiest to choose the lower limit of the integration variable to be $\chi_1 = \chi_m$. Then

$$\psi_\kappa^W(\chi_m) = \sqrt{\frac{\gamma}{2W}} \,,$$
$$\left(\frac{d}{d\chi}\psi_\kappa^W\right)_{\chi_m} = -i\sqrt{\frac{\gamma W}{2}} - \frac{\sqrt{\gamma}}{(2W)^{3/2}}\frac{dW}{d\chi} \,. \tag{3.45}$$

One can compute these to a particular adiabatic order by substituting for W. Strictly speaking, all that is necessary for the derivative of W is to use the previous adiabatic order, although it is permissible to use the same adiabatic order. The adiabatic state that is generated will be different in the two cases. The result is then substituted into Eq. (3.33) to obtain values for c_1 and c_2 at the time χ_m. This fixes the solutions to those equations.

It is important to note that the WKB approximation breaks down in our model for $\kappa \leq \sqrt{2}\,\alpha$. Thus for any given matching time χ_m there will be values of κ that cannot reasonably be fixed using adiabatic matching. If one wishes to compute the stress-energy tensor for the quantum field, then it will be important to find acceptable starting values for such modes. However, for the purposes of the power spectrum, it is in most cases sufficient to restrict attention to $\kappa > \sqrt{2}\,\alpha(\chi_m)$ for matching times χ_m that occur near the time when the semiclassical approximation becomes valid in the radiation-dominated pre-inflationary phase. Exceptions can occur if the horizon size at the time of the onset of inflation, when scaled to the current time, is significantly smaller than the horizon size today.

3.5 Power Spectrum

The standard power spectrum for the field ϕ given in terms of wave number k and conformal time η is [36]

$$P_\phi(k;\eta) = \frac{k^3}{2\pi^2} \left|\phi_k(\eta)\right|^2 \ . \tag{3.46}$$

Using (3.7a) and the scaled variables (3.24) gives

$$P_\phi(\kappa;\chi) = \frac{\kappa^3 H_\Lambda^2}{2\pi^2\gamma\alpha^2} \left|\psi_\kappa(\chi)\right|^2 \ , \tag{3.47}$$

where $H_\Lambda^2 \equiv \frac{1}{3}\Lambda$. Evaluating (3.47) in the limit $\chi \to \chi_\infty$ using (3.31) and (3.32a), one finds

$$P_\phi(\kappa) = \frac{H_\Lambda^2}{4\pi^2} \left|c_1(\kappa;\chi_\infty) - c_2(\kappa;\chi_\infty)\right|^2 \ . \tag{3.48}$$

The problem of calculating the late time power spectrum therefore reduces to finding $c_1(\kappa;\chi_\infty)$ and $c_2(\kappa;\chi_\infty)$. For the model we are considering this can be accomplished by solving (3.34).

Models of the early Universe are heavily constrained by observations of the CMB as well as measurements of large-scale structure. Variations in the CMB are described in terms of the parameters C_ℓ, which are related to the power spectrum by [62]

$$C_\ell = \frac{4\pi}{9} \int_0^\infty \frac{dk}{k} P_\phi(k) j_\ell^2(k\eta_{\text{eff}}) = \frac{4\pi}{9} \int_0^\infty \frac{d\kappa}{\kappa} P_\phi(\kappa) j_\ell^2(s^{-1}\kappa) \ . \tag{3.49}$$

Here j_ℓ is a spherical Bessel function, $\eta_{\text{eff}} = \frac{r_{\text{eff}}}{a_0}$, where r_{eff} and a_0 are the physical size of the effective horizon and scale factor today, and $s = \frac{\gamma a_0}{r_{\rm eff}}$. We define a_i as the scale factor at the start of inflation, when the radiation and cosmological constant contributions to (3.22) are equal. Then using Eq. (3.24d) and $H_\Lambda^2 = \frac{\Lambda}{3}$, we find $a_i = H_\Lambda^{-1}\gamma^{-1}$, and therefore

$$s = \left(\frac{a_0}{a_i}\right)\left(\frac{H_\Lambda^{-1}}{r_{\rm eff}}\right) . \tag{3.50}$$

This means that s corresponds approximately to the ratio of the size of the horizon at the start of inflation, scaled to the current time, to the effective horizon today.

3.6 Results

Although it is not always obvious exactly how a state for the massless minimally coupled scalar field was chosen, it seems likely that many previous calculations of the power spectrum [15, 47, 49–53, 55] for a radiation-dominated pre-inflationary phase made use of either the sudden approximation or something very similar to it. Therefore it is useful to begin with the power spectrum we obtain for the sudden approximation. Substituting (3.37) into (3.48) one finds

$$\begin{aligned} P_\phi = \frac{H_\Lambda^2}{4\pi^2} + \frac{H_\Lambda^2}{8\pi^2\kappa^4(\chi_\infty - \chi_s)^4}\Big\{1 + \left[2\kappa^2(\chi_\infty - \chi_s)^2 - 1\right]\cos\left[2\kappa(\chi_\infty - \chi_s)\right] \\ - 2\kappa(\chi_\infty - \chi_s)\sin\left[2\kappa(\chi_\infty - \chi_s)\right]\Big\} . \end{aligned} \tag{3.51}$$

This spectrum oscillates and has a peak value about a factor of 1.13 times the Bunch-Davies constant value, independent of the time of the sudden transition χ_s. The resulting spectrum is shown in Fig. 3.2 for the choice $\chi_s = 0$. The large oscillations and enhanced values compared with the power spectrum for the Bunch-Davies state are qualitatively identical with most previous results in the literature.

For the adiabatic vacuum states we considered, starting values for c_1 and c_2 were calculated at the matching time χ_m as discussed above, and then Eqs. (3.34) were

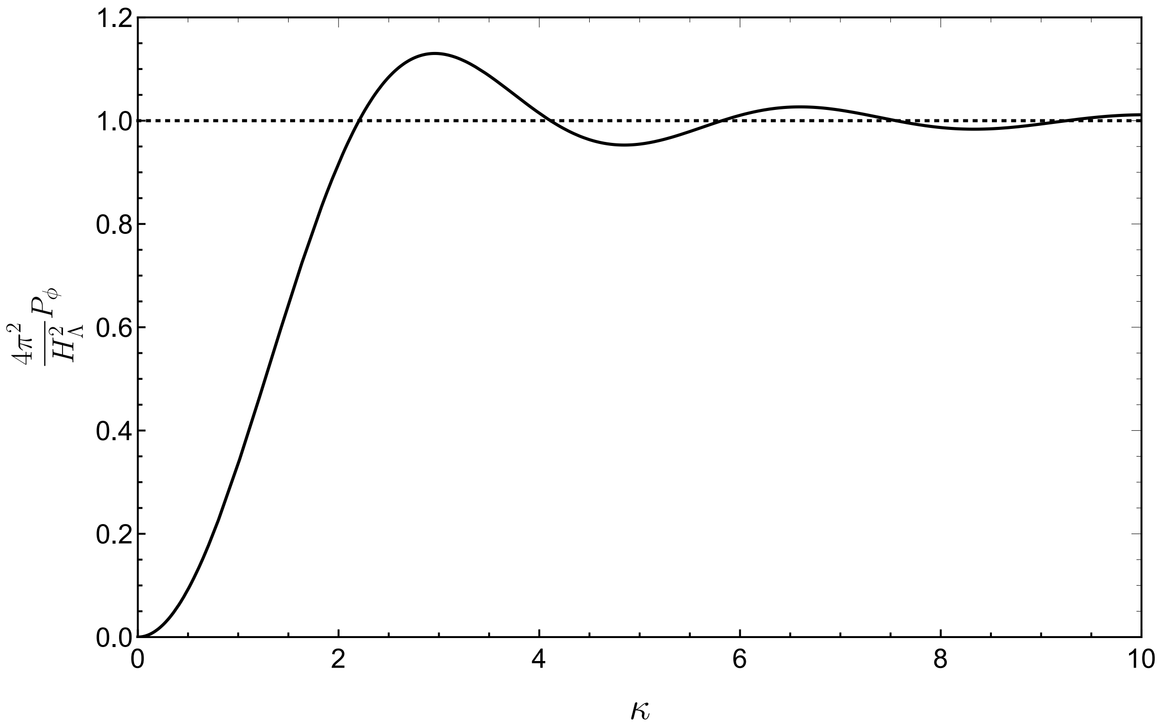

Figure 3.2: Power spectrum from a sudden approximation with the transition occurring at $\chi_s = 0$. Enhancements in the power spectrum are seen compared to the Bunch-Davies value, represented by the dotted line.

solved numerically to find their asymptotic values. These asymptotic values were then substituted into (3.48) to obtain the power spectrum. Our results for the matching time when $\alpha = 0.1$ are shown in Fig. 3.3. Note that if inflation occurs at the GUT scale, then since $\alpha = 1$ is the onset of inflation in our model, the energy at the matching time is ten times larger than the energy at the onset of inflation. However, the energy at the matching time is still one hundred times smaller than the Planck scale, so this is a conservative estimate of when the semiclassical approximation first became valid. For comparison purposes, Fig. 3.4 shows our results for a matching time corresponding to the onset of inflation when $\alpha = 1$. This is too late to be a natural time for the matching but provides an important illustration of how much the adiabatic vacuum states depend on the matching time.

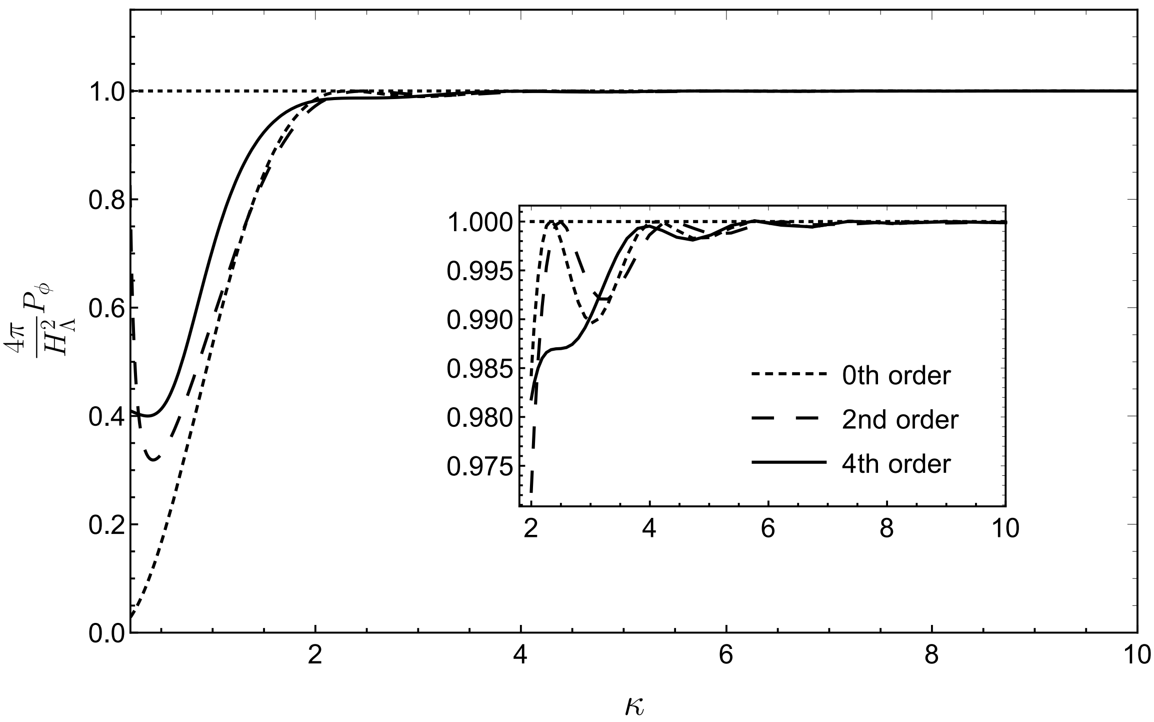

Figure 3.3: Comparison of the power spectra for adiabatic vacuum states of order zero, two, and four, when the matching is done at $\alpha_m = 0.1$ ($\chi_m = -0.827$). Note that all orders show oscillatory behavior, but this behavior is much smaller at higher orders. Note also that the power spectrum never noticeably exceeds 1, the Bunch-Davies value.

As can be seen from the inset of Fig. 3.3, where the matching is done at $\alpha = 0.1$, all three adiabatic orders shown (zeroth, second, and fourth) have a small amount of oscillatory behavior, with the amplitudes of the oscillations decreasing as the adiabatic order increases. Note that in no case is there a noticeable enhancement of the power spectrum above the standard Bunch-Davies state. In contrast, using a matching time when $\alpha = 1$, Fig. 3.4 shows that the oscillations in the power spectra are significantly enhanced in comparison with the earlier matching time for a given adiabatic order.

The one case we are aware of where the power spectrum was computed using adiabatic states for the specific cosmological model we used was in [54]. There, the power spectrum was computed for zeroth-order adiabatic states with various matching

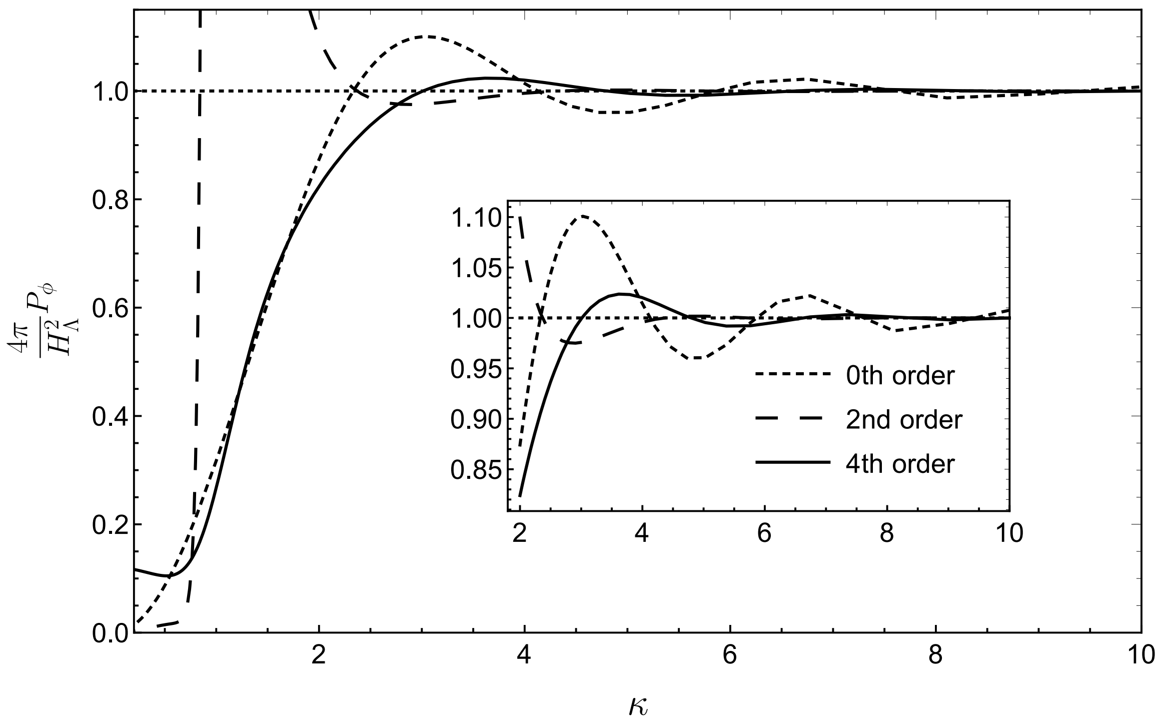

Figure 3.4: Comparison of the power spectra for adiabatic vacuum states of order zero, two, and four, when the matching is done at $\alpha_m = 1$ ($\chi_m = 0$).

times. For relatively late matching times, our computations of the power spectrum for zeroth-order adiabatic vacuum states agree qualitatively with theirs. However, we do find numerical differences when the matching is done at late times that we cannot explain. We also find a qualitative difference when the matching is done at early times in that we always see oscillations in the power spectrum whereas their results show monotonic behavior.

In [57], the power spectrum was computed numerically for a radiation-dominated pre-inflationary phase in the context of slow-roll inflation. The authors assumed a zeroth-order adiabatic state. It is unclear what matching time they used, but if it was near the onset of inflation, then our results agree qualitatively with theirs.

Once the power spectrum has been computed, the angular power spectrum can be calculated using (3.49). Figure 3.5 shows the resulting spectrum for a fourth-order adiabatic state and a matching time of $\alpha = 0.1$. It can be shown that in the limit $s \to \infty$ the resulting angular power spectrum is flat and thus independent of ℓ, but if s is not too large, there is a suppression of the angular power spectrum for small ℓ. Note that for matching at an early time such as $\alpha = 0.1$, any suppression of the $\ell = 2$ component is accompanied by a comparable but smaller suppression of $\ell = 3$ and other small ℓ values. Figure 3.6 compares the results for the zeroth- and fourth-order adiabatic states with matching at $\alpha = 0.1$ with the sudden approximation for the case $s = 0.3$.

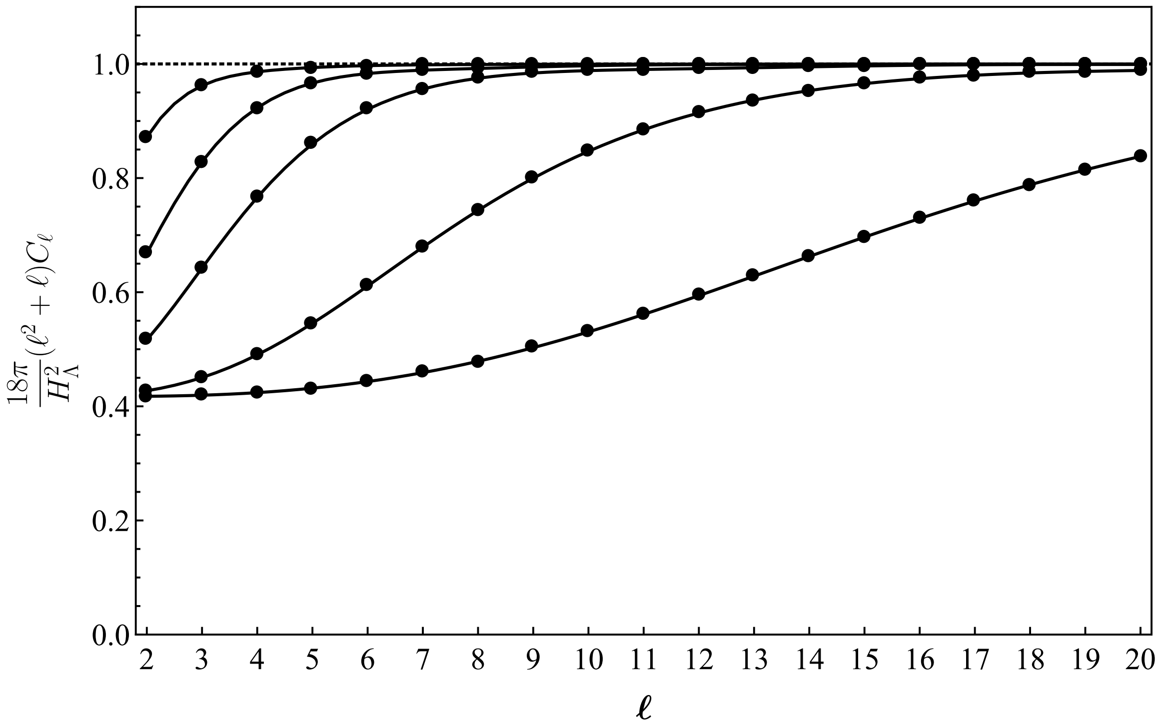

Figure 3.5: Contributions to the angular power spectrum from modes with $\kappa \geq \sqrt{2}\alpha_m$ for a fourth order adiabatic vacuum state with the adiabatic matching done at $\alpha_m = 0.1$. From top to bottom, the curves are for $s = 0.50$, 0.30, 0.20, 0.10, and 0.05.

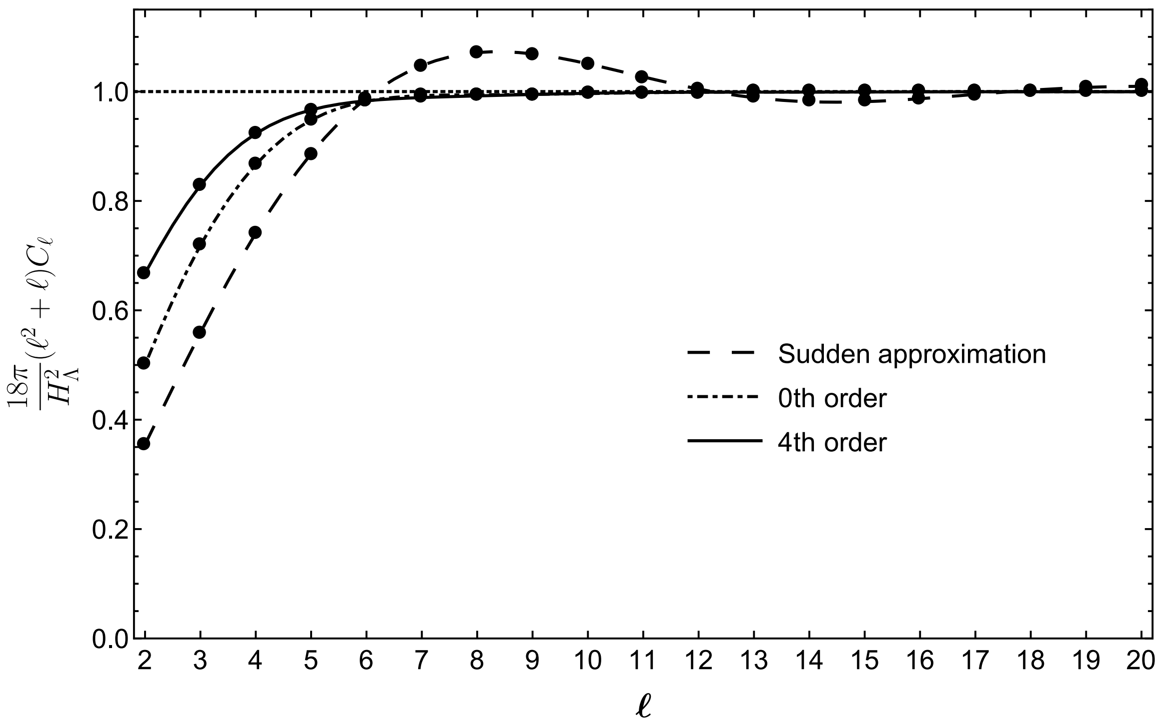

Figure 3.6: Angular power spectrum for the sudden approximation and for the contributions from modes with $\kappa \geq \sqrt{2}\alpha_m$ to the power spectrum for zeroth and fourth order adiabatic vacuum states. For the sudden approximation the matching is done at $\alpha_s = 1$, while for the adiabatic vacuum states the matching was done at $\alpha_m = 0.1$. In each case, $s = 0.3$.

3.6.1 Adiabatic states in de Sitter space

To understand why the oscillations for a given matching time are smaller for larger adiabatic orders, it is useful to switch to pure de Sitter space where analytic solutions to the mode equation are known and the power spectrum for any state can be computed analytically. For simplicity, we will also revert to unscaled variables, so that

$$a_{\rm dS} = -\frac{1}{H\eta} \ , \tag{3.52a}$$

$$v_k = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) \ , \tag{3.52b}$$

$$\psi_k = \frac{1}{\sqrt{2W}} e^{-i\int W d\eta} \ , \tag{3.52c}$$

$$W^2 = k^2 - \frac{a''_{\rm dS}}{a_{\rm dS}} - \left(\frac{W''}{2W} - \frac{3W'^2}{4W^2}\right) \tag{3.52d}$$

We can find c_1 and c_2 directly from Eq. (3.33), which become

$$c_1 = -i\left(\psi_k v_k'^* - \psi_k' v_k^*\right) \ , \tag{3.53a}$$

$$c_2 = i\left(\psi_k v_k' - \psi_k' v_k\right) \ . \tag{3.53b}$$

Recall that primes denote derivatives with respect to η. The power spectrum is given by Eq. (3.48), but c_1 and c_2 are now independent of time

$$P_\phi = \frac{H^2}{4\pi^2}\left|c_1(k) - c_2(k)\right|^2 \ . \tag{3.54}$$

The power spectra for an adiabatic matching time η_m for zeroth-, second-, and fourth-order adiabatic states are

$$P_\phi^{(0)} = \frac{H^2}{4\pi^2} + \frac{H^2}{8\pi^2 k^4 \eta_m^4}\left[1 + \left(2k^2\eta_m^2 - 1\right)\cos(2k\eta_m) - 2k\eta\sin(2k\eta_m)\right] , \tag{3.55a}$$

$$\begin{aligned} P_\phi^{(2)} =& \frac{H^2}{4\pi^2} + \frac{H^2}{8\pi^2 k^4 \eta_m^4 (k^2\eta_m^2 - 1)^3}\big[k^2\eta_m^2 + 1 \\ & - (2k^6\eta_m^6 - 2k^4\eta_m^4 - k^2\eta_m^2 + 1)\cos(2k\eta_m) - 2k\eta_m\sin(2k\eta_m)\big] , \end{aligned} \tag{3.55b}$$

$$\begin{aligned} P_\phi^{(4)} =& \frac{H^2}{4\pi^2} + \frac{H^2}{8\pi^2 k^6 \eta_m^6 (k^4\eta_m^4 - k^2\eta_m^2 + 1)^3}\big[k^6\eta_m^6 + 6k^4\eta_m^4 - 3k^2\eta_m^2 + 1 \\ & + (2k^{12}\eta_m^{12} + 2k^{10}\eta_m^{10} - 8k^8\eta_m^8 + 17k^6\eta_m^6 - 14k^4\eta_m^4 + 5k^2\eta_m^2 - 1)\cos(2k\eta_m) \\ & + (2k^{11}\eta_m^{11} - 8k^9\eta_m^9 + 14k^7\eta_m^7 - 20k^5\eta_m^5 + 8k^3\eta_m^3 - 2k\eta_m)\sin(2k\eta_m)\big] . \end{aligned} \tag{3.55c}$$

For a universe with a pre-inflationary radiation-dominated era, de Sitter space is approached in the late time limit. Therefore, the terms in Eq. (3.55) that provide the leading order behavior in our model are those with the largest power of $k\eta_m$. We find that the oscillatory terms always dominate the non-oscillatory terms. Furthermore, the leading order oscillatory terms have smaller and smaller contributions at higher order, with zeroth-order being $(k\eta_m)^{-2}$, second-order $(k\eta_m)^{-4}$, fourth-order $(k\eta_m)^{-6}$, and so on. That is, oscillatory terms will contribute the most at zeroth order and then contribute less and less as the order is increased.

3.6.2 Discussion

As can be seen in Figs. 3.3 and 3.6, for adiabatic matching at a relatively early time there is a suppression of the power spectrum at small wave numbers κ compared to the spectrum of the Bunch-Davies state. In contrast, for a sudden approximation we find an enhancement at certain values of κ, as seen in Figs. 3.2 and 3.6. The latter behavior agrees qualitatively with the results in [15, 49, 52, 55], where it appears that some type of sudden approximation was used. The sudden approximation was also used and an enhancement of the power spectrum for certain values of the momentum parameter was also observed in [47, 50, 51, 53]. However, our results disagree in detail with theirs. The qualitative differences between the power spectrum for the sudden approximation and for states with adiabatic matching at an early time imply that it is necessary to not use a sudden approximation but instead to work in a spacetime where there is a smooth transition to the inflationary era and compute the power spectrum using solutions to the exact mode equation in that spacetime, which in most cases will be obtained numerically. This point was made in [54], where, as discussed above, the power spectrum was computed for zeroth-order adiabatic vacuum states at various matching times for the same model that we consider here.

For large values of κ the power spectrum approaches the constant value it has for the Bunch-Davies state. That in turn gives an approximately flat spectrum at large ℓ, as shown in Fig. 3.5. A more realistic model would include not just a simple inflation era, but rather an evolving scalar field in an appropriate potential, which would presumably result in a tilted spectrum. This spectrum would then have to be processed through all of the subsequent stages of cosmology to reproduce the CMB that we observe. We think it is likely that our findings of the suppression of small ℓ with no enhancement at any ℓ would persist in a more realistic model of this type.

The suppression of the spectrum at small ℓ could provide one explanation of the anomalously small value of the quadrupole moment of the CMB [63]. In [54], where the same model that we are using was investigated, this suppression at small values of ℓ was observed. By examining Fig. 3.5, we see that any significant suppression of the $\ell = 2$ modes comes at the expense of also suppressing $\ell = 3$, which is not observed. It should also be noted that values which lead to significant suppression of $\ell = 2$ have relatively small s values, such as $s \approx 0.3$. Recall that s corresponds roughly to the ratio of the size of the horizon at the start of inflation, scaled to the current time, to the effective horizon today (see (3.50)). This means that for $s \lesssim 1$ we cannot explain the homogeneity and flatness of the current Universe exclusively in terms of inflation; there must be some mechanism which makes the Universe uniform on scales slightly larger than the horizon size at the start of inflation. However, because of the existence of an inflationary phase, the flatness of the Universe today does not require the ultra-fine tuning that would typically be needed without inflation. It would only take a very modest fine-tuning at the start of inflation to result in an approximately flat Universe today.

It is worth noting that the Trans-Planckian Censorship Conjecture, TCC, [64] which states that trans-Planckian modes should never cross the horizon in an expanding universe and become classical, provides a restriction on the duration of the inflationary phase [65]. In [66] it was argued that it is possible to modify the TCC in ways that allow trans-Plankian modes to cross the horizon. However, with these modifications the TCC still provides a restriction on the duration of inflation.

3.7 Summary and Conclusions

In Sec. 3.2 it was argued that if there was a pre-inflationary era in which the semi-classical approximation in gravity was valid and if the Universe, or our part of it, was approximately homogeneous and isotropic during that time, then it is very likely the Universe expanded like a radiation-dominated universe during that era. The argument is based on several assumptions and a proof that is given in the Appendix. The assumptions are: i. the dominant quantum fields should behave like free fields during that epoch, ii. higher derivative terms necessary for the renormalization of the stress-energy tensor should be small since the semiclassical approximation is assumed to be valid, and iii. massive spin $\frac{1}{2}$ and spin 1 fields can be modeled using massive conformally coupled spin 0 fields. The proof shows that, to leading order in the limit that the scale factor vanishes, the energy density of a massive conformally coupled scalar field is of the same form as that of classical radiation for a very large class of physically acceptable homogeneous and isotropic vacuum states provided that $(a^2)'$ is finite at the initial singularity and $\int_{\eta_0}^{\eta} |[a^2(x)]''|dx$ is also finite.

It was further shown that there is a state for a scalar field with arbitrary curvature coupling that goes like $\psi = (2k)^{-1/2}e^{-ik\eta}$ when the scale factor is small, so long as in the limit that the scale factor vanishes the effective mass term in the mode equation also vanishes. This would seem to be a natural initial vacuum state because the mode equation approaches that of the conformally invariant scalar field in this limit. It was also shown that this state is an infinite-order adiabatic vacuum state if the scale factor vanishes in the limit $\eta \to -\infty$, such as happens in pure de Sitter space, where the vacuum state is the Bunch-Davies state. However, if the scale factor vanishes at a finite value η_0 of the conformal time and one or more of the derivatives of the effective mass term is nonzero at η_0, then this state is only a finite-order adiabatic vacuum state. In particular, if the Universe expanded like a radiation-dominated universe near $\eta = \eta_0$, then for an arbitrarily coupled massive scalar field the state is at most a first-order adiabatic vacuum state. For a massless nonconformally coupled scalar field the state can be of adiabatic order two or higher in the radiation dominated case depending on the detailed behavior of the scale factor near η_0. For the specific model considered here it is at most a third-order adiabatic vacuum state. For the conformally coupled massive scalar field a zeroth-order adiabatic state is enough to give a finite stress-energy tensor in a homogeneous and isotropic spacetime, but only for exact homogeneity and isotropy. For nonconformally coupled scalar fields, both massive and massless, a fourth-order adiabatic state is required for the renormalized stress-energy tensor to be finite. It is important to point out that this doesn't prevent the solution from being used for small and intermediate values of the momentum parameter k, but it does prevent it from being used for arbitrarily large values of k.

In Sec. 3.5 we studied the effects of a radiation-dominated pre-inflationary phase on the power spectrum that is computed using the massless minimally coupled scalar field. To do so we used a simple model in which the classical Einstein equations are solved when that classical radiation and a positive cosmological constant are present. The mode equation was solved for this model for several different states for the quantum field, and the solutions were used to compute the power spectra for these states.

We found that a sudden approximation in which the metric is exactly that of a radiation-dominated universe up to a transition time and is exactly that of de Sitter space afterward gives relatively large oscillations in the power spectrum when it is plotted as a function of the momentum parameter k. For an early matching time, well before the onset of inflation, a zeroth-order adiabatic state such as the one mentioned above gives oscillations with a smaller amplitude, while higher-order adiabatic states at the same matching time give successively smaller amplitude oscillations. Previous investigations [15, 47, 49–53, 55] in which there is a radiation-dominated pre-inflationary phase have found similar results. Those investigations appear to have made use of either a sudden approximation or zeroth-order adiabatic states. In some cases there are disagreements with our results that are discussed in Sec. 3.6.2. All of these calculations, including ours for the better-behaved second- and fourth-order adiabatic states, predict that if inflation did not go on for too long, then there are potentially observable differences in the power spectrum from that of the Bunch-Davies state in de Sitter space. These differences, if observed, would have the potential to give us information about the initial state of the matter fields in the Universe if there was a pre-inflationary radiation-dominated era.

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3.A Proof that the Universe is Radiation Dominated at Early Times

Here we formally prove that in the presence of conformally coupled scalar fields, whether massless or not, to leading order the energy density will scale as $\rho \propto a^{-4}$ at early times for almost any scale factor $a(\eta)$ that vanishes at conformal time η_0 (which may be finite or $-\infty$) and for a large class of physically acceptable homogeneous and isotropic vacuum states. As can be seen from (3.16), this will happen if the integral

$$I(\eta) = \int_0^\infty dk\, k^2 \omega_k(\eta)\, |\beta_k(\eta)|^2 \tag{3.56}$$

has the property $\lim_{\eta \to \eta_0} I(\eta) = I(\eta_0)$ with $0 < I(\eta_0) < \infty$. We begin with the following set of conditions that must be satisfied for the proof to hold:

$$\int_0^\infty dk\, k^3 |\beta_k(\eta)|^2 < \infty \,, \tag{3.57a}$$

$$\int_0^\lambda dk\, k |\beta_k(\eta_0)|^2 < \infty \,, \tag{3.57b}$$

$$\lim_{\eta \to \eta_0} \left[a^2(\eta)\right]' < \infty \,, \tag{3.57c}$$

$$\int_{\eta_0}^\eta dx \left| \left[a^2(x)\right]'' \right| < \infty \,. \tag{3.57d}$$

We expect that the condition (3.57a) will be satisfied by any homogeneous and isotropic vacuum state for which the energy density is finite for times $\eta > \eta_0$. Condition (3.57b) is satisfied so long as there is at most a weak infrared divergence in β_k. Condition (3.57c) means the initial expansion is not too extreme, and (3.57d) will be automatically satisfied for any universe for which (3.57c) applies if $[a^2(\eta)]''$ is always positive or always negative.

To prove that $I(\eta) \to I(\eta_0)$, we will show that for any $\epsilon > 0$, there is a finite time $\bar{\eta}$ such that for $\eta < \bar{\eta}$, $|I(\eta) - I(\eta_0)| < \epsilon$. This can be done by dividing the integral into parts using an infrared cutoff λ and then writing $I(\eta)$ as

$$I(\eta) = I(\eta_0) + \Delta I_1(\eta) + \Delta I_2(\eta) + \Delta I_3(\eta) \; , \tag{3.58}$$

where

$$\Delta I_1(\eta) = \int_0^\lambda dk\, k^2 \left[\omega_k(\eta) \left|\beta_k(\eta)\right|^2 - k \left|\beta_k(\eta_0)\right|^2\right] \; , \tag{3.59a}$$

$$\Delta I_2(\eta) = \int_\lambda^\infty dk\, k^2 \omega_k(\eta) \left[\left|\beta_k(\eta)\right|^2 - \left|\beta_k(\eta_0)\right|^2\right] \; , \tag{3.59b}$$

$$\Delta I_3(\eta) = \int_\lambda^\infty dk\, k^2 \left[\omega_k(\eta) - k\right] \left|\beta_k(\eta_0)\right|^2 \; . \tag{3.59c}$$

To place bounds on these quantities we begin by finding inequalities that α_k, β_k, and their first derivatives satisfy. The inequalities can be obtained using the differential equations satisfied by α_k and β_k. By combining (3.12a) and (3.12b), and substituting the results in (3.8) with $\xi = \frac{1}{6}$, one finds

$$\alpha_k' = \frac{\omega_k'}{2\omega_k} \beta_k e^{2i\theta_k} \; , \tag{3.60a}$$

$$\beta_k' = \frac{\omega_k'}{2\omega_k} \alpha_k e^{-2i\theta_k} \; . \tag{3.60b}$$

Using these equations it is straightforward to show that

$$\frac{\left(|\alpha_k|^2 + |\beta_k|^2\right)'}{|\alpha_k|^2 + |\beta_k|^2} = \frac{\omega_k'}{\omega_k} \frac{2\mathrm{Re}\left(\alpha_k \beta_k^* e^{-2i\theta_k}\right)}{|\alpha_k|^2 + |\beta_k|^2} \leq \frac{\omega_k'}{\omega_k} \frac{2|\alpha_k|\,|\beta_k|}{|\alpha_k|^2 + |\beta_k|^2} < \frac{\omega_k'}{\omega_k} \,, \tag{3.61}$$

where the last inequality can be obtained from $(|\alpha_k| - |\beta_k|)^2 > 0$.

Integrating the inequality (3.61) from η_0 to η yields

$$|\alpha_k(\eta)|^2 + |\beta_k(\eta)|^2 \leq \frac{\omega_k(\eta)}{k} \left[|\alpha_k(\eta_0)|^2 + |\beta_k(\eta_0)|^2\right] \,. \tag{3.62}$$

Multiplying both sides by $\frac{1}{2}\omega_k(\eta)$ and using (3.14), gives

$$\omega_k(\eta) \left(\frac{1}{2} + |\beta_k(\eta)|^2\right) \leq k \left[1 + \frac{m^2 a^2(\eta)}{k^2}\right] \left[\frac{1}{2} + |\beta_k(\eta_0)|^2\right] \,. \tag{3.63}$$

Thus it is also true that

$$\omega_k(\eta)\, |\beta_k(\eta)|^2 < k \left[1 + \frac{m^2 a^2(\eta)}{k^2}\right] \left[\frac{1}{2} + |\beta_k(\eta_0)|^2\right] \,. \tag{3.64}$$

Integrating this over k up to any finite limit λ gives

$$\int_0^\lambda dk\, k^2 \omega_k(\eta)\, |\beta_k(\eta)|^2 < \int_0^\lambda dk\, k \left[k^2 + m^2 a^2(\eta)\right] \left[\frac{1}{2} + |\beta_k(\eta_0)|^2\right] \,. \tag{3.65}$$

Note that because of the assumption (3.57b), the integral on the right is finite and is a strictly increasing function of η. Choosing an arbitrary conformal time η_1, restricting to times $\eta < \eta_1$, and choosing λ to be small enough allows an arbitrarily small upper bound to be placed on the integral. Choosing that bound to be $\frac{1}{3}\epsilon$ gives

$$0 \leq \int_0^\lambda dk\, k^2 \omega_k(\eta)\, |\beta_k(\eta)|^2 < \frac{1}{3}\epsilon \quad \text{for} \quad \eta < \eta_1 \,. \tag{3.66}$$

Equation (3.66) will be true at all early times, including $\eta = \eta_0$, and it is obviously positive, so comparing with (3.59a) we see that the absolute value of the difference between the integral in (3.66) for $\eta > \eta_0$ and the integral evaluated at $\eta = \eta_0$ must also satisfy the same bound, so

$$|\Delta I_1(\eta)| = \left| \int_0^\lambda dk\, k^2 \omega_k(\eta) \, |\beta_k(\eta)|^2 - \int_0^\lambda dk\, k^3 \, |\beta_k(\eta_0)|^2 \right| < \frac{1}{3}\epsilon \quad \text{for} \quad \eta < \eta_1 \; . \tag{3.67}$$

To make progress on ΔI_2, first note that for (3.57a) to be satisfied, $|\beta_k(\eta_0)|$ must fall faster than k^{-2} at large values of k, and it must not diverge as quickly as k^{-2} for small values of k. It follows that $k^2|\beta_k(\eta_0)|$ must have an upper bound for all values of k, which we call B, so that

$$|\beta_k(\eta_0)| < \frac{B}{k^2} \; . \tag{3.68}$$

Next (3.60b) can be integrated to yield

$$\beta_k(\eta) = \beta_k(\eta_0) + \Delta\beta_k(\eta) \; , \tag{3.69a}$$

$$\Delta\beta_k(\eta) = \frac{1}{2} \int_{\eta_0}^{\eta} dx \frac{\omega_k'(x)}{\omega_k(x)} \alpha_k(x) e^{-2i\theta_k(x)} \; . \tag{3.69b}$$

If the condition (3.57a) is satisfied and if $\beta_k(\eta)$ is a continuous function of η for all $k > 0$, then, for any $\lambda > 0$ and all $k \geq \lambda$, there will be an upper bound on the value of $|\beta_k(\eta)|$ for $\eta_0 \leq \eta < \eta_2$, where $\eta_2 > \eta_0$. This bound may depend on η_2, but it will not depend on k. Using (3.14), this means that there exist positive constants $\beta_{\max}$ and $\alpha_{\max}$ such that

$$|\beta_k(\eta)| < \beta_{\max} \; , \tag{3.70a}$$

$$|\alpha_k(\eta)| < \alpha_{\max} = \sqrt{1 + \beta_{\max}^2} \; . \tag{3.70b}$$

Equations (3.69b) and (3.70b) can be used to place a limit on $\Delta\beta_k$:

$$|\Delta\beta_k(\eta)| \leq \frac{1}{2}\int_{\eta_0}^{\eta} dx \left|\frac{\omega_k'(x)}{\omega_k(x)}\right| |\alpha_k(x)| \leq \frac{\alpha_{\max}}{2} \ln\left[\frac{\omega_k(\eta)}{k}\right] < \frac{\alpha_{\max}}{4}\left[\frac{\omega_k(\eta)}{k} - \frac{k}{\omega_k(\eta)}\right] , \tag{3.71}$$

where the fact that ω_k is an increasing function of η has been used along with the identity $\ln(x) < \frac{1}{2}(x - x^{-1})$ when $x > 1$. Thus

$$|\Delta\beta_k(\eta)| < \frac{\alpha_{\max} m^2 a^2(\eta)}{4\, k\, \omega_k(\eta)} . \tag{3.72}$$

Note that $\Delta\beta_k(\eta)$ vanishes in the limit $\eta \to \eta_0$, so $\Delta\beta_k(\eta)$ can be made arbitrarily small by choosing an early enough time η_2. Thus for any $\delta > 0$ it is possible to find a time η_2 such that

$$|\Delta\beta_k(\eta)| < \frac{\delta}{k\omega_k(\eta)} \leq \frac{\delta}{k^2} \quad \text{for} \quad k > \lambda\ ,\ \eta_0 \leq \eta < \eta_2\ . \tag{3.73}$$

To find a bound on $\Delta I_2(\eta)$, it is useful to derive a second bound on $|\Delta\beta_k(\eta)|$. It is easy to show that

$$\Delta\beta_k(\eta) = \frac{i}{4}\int_{\eta_0}^{\eta} dx \frac{\omega_k'(x)}{\omega_k^2(x)} \alpha_k(x) \frac{d}{dx}\, e^{-2i\theta_k(x)} . \tag{3.74}$$

Then, integrating by parts and using (3.60a) gives

$$\begin{aligned}\Delta\beta_k(\eta) =& \frac{i}{4}\left[\frac{\omega_k'(\eta)}{\omega_k^2(\eta)}\alpha_k(\eta)e^{-2i\theta_k(\eta)} - \frac{\omega_k'(\eta_0)}{k^2}\alpha_k(\eta_0)e^{-2i\theta_k(\eta_0)}\right] \\ &- \frac{i}{4}\int_{\eta_0}^{\eta} dx \left\{\left[\frac{\omega_k''(x)}{\omega_k^2(x)} - \frac{2\omega_k'^2(x)}{\omega_k^3(x)}\right]\alpha_k(x)e^{-2i\theta_k(x)} + \frac{\omega_k'^2(x)}{2\omega_k^3(x)}\beta_k(x)\right\} .\end{aligned} \tag{3.75}$$

Using (3.7b) one finds that

$$\Delta\beta_k(\eta) = \frac{im^2}{8}\left[\frac{[a^2(\eta)]'}{\omega_k^3(\eta)}\alpha_k(\eta)e^{-2i\theta_k(\eta)} - \frac{[a^2(\eta_0)]'}{k^3}\alpha_k(\eta_0)e^{-2i\theta_k(\eta_0)}\right]$$
$$- \frac{im^2}{8}\int_{\eta_0}^{\eta} dx \Bigg\{ \frac{[a^2(x)]''}{\omega_k^3(x)}\alpha_k(x)e^{-2i\theta_k(x)}$$
$$- \frac{m^2}{4}\frac{[a^2(x)]'^2}{\omega_k^5(x)}\left[6\alpha_k(x)e^{-2i\theta_k(x)} + \beta_k(x)\right]\Bigg\} . \tag{3.76}$$

There might be some concern that $e^{-2i\theta_k(\eta_0)}$ is ill-defined in the case $\eta_0 = -\infty$, but in this case it is always true that $[a^2(\eta_0)]' = 0$, so this ambiguity is irrelevant. Note that condition (3.57c) must be satisfied for (3.76) to be a well-defined expression.

We now place a limit on $\Delta\beta_k(\eta)$ using (3.76), (3.70a), (3.70b) and the fact that $\omega_k > k > \lambda$:

$$|\Delta\beta_k(\eta)| < \frac{m^2}{8k^3}\alpha_{\max}\left\{\left[a^2(\eta)\right]' + \left[a^2(\eta_0)\right]' + \int_{\eta_0}^{\eta} dx \left|[a^2(x)]''\right|\right\}$$
$$+ \frac{m^4}{32k^3\lambda^2}(6\alpha_{\max} + \beta_{\max})\int_{\eta_0}^{\eta} dx \left[a^2(x)\right]'^2 . \tag{3.77}$$

Note that the conditions (3.57c) and (3.57d) ensure that the first three terms are finite, and therefore there exists a positive constant C_1 such that

$$\frac{m^2}{8}\alpha_{\max}\left\{\left[a^2(\eta)\right]' + \left[a^2(\eta_0)\right]' + \int_{\eta_0}^{\eta} dx \left|[a^2(x)]''\right|\right\} < C_1 \quad \text{for} \quad \eta_0 \le \eta < \eta_2 . \tag{3.78}$$

Since $[a^2(x)]'$ is finite as $\eta \to \eta_0$ its square must be integrable over any finite range. Thus if η_0 is finite the last term in (3.77) is also finite. If $\eta_0 = -\infty$, first note that it must be true that $\int_{\eta_0}^{\eta}[a^2(x)]'dx = a^2(\eta)$ is finite, so $[a^2(\eta)]'$ must fall off faster than $|\eta|^{-1}$ as $\eta \to -\infty$, and hence $[a^2(\eta)]'^2$ falls off faster than $|\eta|^{-2}$ and is also integrable. Therefore in either case the final term in (3.77) is integrable, and there exists some positive constant C_2 such that

$$\frac{m^4}{32\lambda^2}\left(6\alpha_{\max}+\beta_{\max}\right)\int_{\eta_0}^{\eta} dx\,\left[a^2(x)\right]'^2 < C_2 \quad \text{for} \quad \eta<\eta_2\,. \tag{3.79}$$

Substituting (3.78) and (3.79) into (3.77), and noting that for $k>\lambda$ and $\eta<\eta_2$, $\omega_k(\eta)/k<\omega_\lambda(\eta_2)/\lambda$ one finds that

$$|\Delta\beta_k(\eta)| < \frac{C_1+C_2}{k^3} \quad \text{for} \quad \eta<\eta_2\,,\ k>\lambda\,. \tag{3.80}$$

$$|\Delta\beta_k(\eta)| < \frac{D}{k^2\omega_k(\eta)} \quad \text{for} \quad \eta<\eta_2\,,\ k>\lambda\,, \tag{3.81a}$$

$$D = \frac{C_1+C_2}{\lambda}\omega_\lambda(\eta_2)\,. \tag{3.81b}$$

Combining the two limits (3.73) and (3.81a) gives

$$|\Delta\beta_k(\eta)| < \frac{1}{\omega_k(\eta)}\min\left(\frac{\delta}{k},\frac{D}{k^2}\right) \quad \text{for} \quad \eta<\eta_2\,,\ k>\lambda\,. \tag{3.82}$$

It is possible to put a bound on $|\Delta I_2(\eta)|$ in (3.59b) by choosing δ to be small enough so that $\delta\lambda<D$, and using the bounds in (3.68) and (3.82) along with the fact that $\omega_k>k$. The result is:

$$\begin{aligned}|\Delta I_2(\eta)| &= \int_\lambda^\infty dk\,k^2\omega_k(\eta)\left\{2\mathrm{Re}\left[\beta_k(\eta_0)^*\Delta\beta_k(\eta)\right]+|\Delta\beta_k(\eta)|^2\right\}\\ &< \int_\lambda^\infty dk\,k^2\left[\frac{2B}{k^2}\min\left(\frac{\delta}{k},\frac{D}{k^2}\right)+\frac{1}{k}\min\left(\frac{\delta}{k},\frac{D}{k^2}\right)^2\right]\,.\end{aligned} \tag{3.83}$$

The integral in the second line can be computed by dividing it into the integrals

$\int_\lambda^{D/\delta} dk + \int_{D/\delta}^\infty dk$ with the result that

$$|\Delta I_2(\eta)| < (2B\delta + \delta^2) \ln\left(\frac{D}{\delta\lambda}\right) + 2B\delta + \frac{1}{2}\delta^2 \ . \tag{3.84}$$

Then since δ can be made as small as desired by choosing η_2 appropriately, we can use the same bound as in (3.67)

$$|\Delta I_2(\eta)| < \frac{1}{3}\epsilon \quad \text{for} \quad \eta < \eta_2 \ . \tag{3.85}$$

Finally, we can put a limit on ΔI_3, given by (3.59c) by using $\omega_k^2 < (k + \frac{m^2a^2}{2k})^2$ which implies that $\omega_k - k < m^2a^2/2k$, together with (3.68), so that

$$|\Delta I_3(\eta)| < \int_\lambda^\infty dk\, k^2 \frac{m^2a^2(\eta)B^2}{2k^5} = \frac{m^2B^2}{4\lambda^2} a^2(\eta) \ . \tag{3.86}$$

We can make this small by simply making $a(\eta)$ small, so we have

$$|\Delta I_3(\eta)| < \frac{1}{3}\epsilon \quad \text{for} \quad \eta < \eta_3 \ . \tag{3.87}$$

If we then define $\bar\eta = \min(\eta_1, \eta_2, \eta_3)$ and use (3.67), (3.85) and (3.87) in (3.58) we find

$$|I(\eta) - I(\eta_0)| < \epsilon \quad \text{for} \quad \eta < \bar\eta \ . \tag{3.88}$$

Since this can be achieved for any $\epsilon > 0$, we conclude that

$$\lim_{\eta\to\eta_0} I(\eta) = I(\eta_0) = \int_0^\infty dk\, k^3 \left|\beta_k(\eta_0)\right|^2 \ . \tag{3.89}$$

Chapter 4: Argument for the radiation-dominated behavior of matter fields in the preinflationary era

Preface

The following paper was published in Physical Review D [67] and is reproduced here by permission of the American Physical Society. Stylistic variations between this dissertation and the published paper are due to format requirements, and some minor typographical errors have been corrected here.

Contributions to this paper were divided between myself and Eric Carlson. Dr. Carlson proposed the original idea of performing an analysis of Dirac and vector fields similar to that done for scalar fields in the appendix of [32]. I performed the background research into using the mode expansion technique and performing adiabatic regularization to obtain the stress energy density for both types of fields. I performed the majority of the intial analysis of the Dirac fields leading to the result in Eq. (4.52), and we contributed equally to the subsequent analysis and results found in Secs. 4.3.2 and 4.3.3. I performed the majority of the procedure for vector fields to obtain the unrenormalized energy density contributions, the renormalization counterterms, and the final renormalized energy density. I also developed the symbolic manipulation programs involved in obtaining all of our results for both Dirac and vector fields. I wrote the initial manuscript and made the majority of edits leading to the final version submitted for publication, and Dr. Carlson provided guidance and editorial advice to this process.

Abstract

We investigate the leading-order behavior of matter fields in the preinflationary era using the semiclassical approximation. Many inflationary models assume without supporting arguments that the Universe was radiation dominated prior to inflation, leading to modifications of cosmological observables, such as the cosmic microwave background power spectrum. In previous work, we demonstrated that conformally coupled scalar fields do have a radiationlike contribution to the stress-energy tensor at sufficiently early times. In this work, we extend these arguments to apply to massless spin-1 fields and massive or massless spin-$\frac{1}{2}$ fields. We find massless spin-1 fields always have a radiationlike contribution. For spin-$\frac{1}{2}$ fields, we find the contribution at early times is radiationlike assuming this is the dominant contribution to the stress-energy tensor.

4.1 Introduction

The inflationary paradigm explains many characteristics of our observed Universe. Many models of inflation make use of a radiation-dominated behavior of matter fields in the preinflationary era in order to obtain modifications to the standard predictions of inflation and better explain observed phenomena (see [15] and references therein for a review of models using a radiation-dominated preinflationary era). For instance, various models involving the transition from a radiation-dominated era to inflation lead to modifications of the cosmic microwave background anisotropy spectrum, such as a lowering of the quadrupole moment which appears to be anomalously suppressed [63]. The assumption that fields are radiation-dominated prior to inflation is central to such models, though it is nontrivial to demonstrate that such behavior was the case for our Universe, and it would be hopeless to try to detect particles due to these fields today given the effects of inflation. It is therefore interesting to analyze the preinflationary era and the behavior of matter fields in it.

The primary objective of this work is to investigate the behavior of quantum fields in the preinflationary era. A complete analysis of the early universe would require a theory of quantum gravity, which for now is out of reach, but one could anticipate that after the Planck era some span of the preinflationary era would have curvature well below the Planck scale, in which case the semiclassical approximation should hold. In this paper, we will assume the semiclassical approximation to hold in some portion of the preinflationary era, and we will use this framework to analyze quantum fields in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and obtain their corresponding contributions to the energy density.

The case of massive scalar fields was investigated using the semiclassical approximation in [32]. There it was shown that under a set of conditions on the scale factor the fields did result in radiation-dominated contributions to the energy density. The conditions on the scale factor were also argued to likely be valid for our Universe. In the present work, we will investigate the corresponding behavior of other fields, namely massless spin-1 and massive or massless spin-$\frac{1}{2}$ fields.

The analysis for spin-1 and spin-$\frac{1}{2}$ fields is generally more complicated than for scalar fields. For spin-$\frac{1}{2}$ fields, the mass appears explicitly in the expression for the counterterms at higher than zeroth adiabatic order, which leads to additional complications. Performing adiabatic regularization with a zeroth-order parametrization of the states does not explicitly appear to produce finite energy densities until one considers the higher order contributions buried in the parametrization. For an overview of the adiabatic regularization procedure for spin-$\frac{1}{2}$, see [68–70]. For spin-1 fields one must consider the massive and massless cases separately. Massive vector fields do appear in the Standard Model, but they are ultimately due to interactions with the

Higgs field and are out of the scope of this paper. For the massless case one can show that the analysis decomposes into four decoupled copies of a scalar field, as was done in Ref. [71]. There the analysis was performed to obtain the trace anomaly, but obtaining the renormalized energy density from this groundwork is nontrivial.

In the following, we work in Planck units with $c = G = \hbar = 1$ and use the $(-,+,+,+)$ signature for the metric. We assume a spatially flat universe described by the FLRW metric

$$ds^2 = a^2 \left(-d\eta^2 + d\vec{x}^2\right) \ , \tag{4.1}$$

where $a \equiv a(\eta)$ is the scale factor and η is conformal time. We will employ a prime, such as $a' \equiv \partial_\eta a$, to denote differentiation with respect to conformal time. We are interested in taking $a \to 0$ at early times and determining whether one can assume radiation-dominated behavior. However, we do not want to consider times in the Planck era, during which the semiclassical approximation is not assumed to be valid, so we denote by η_0 the earliest conformal time of consideration and implicitly assume it to be past the Planck era. In Planck units, this corresponds to a value of the Hubble parameter $H \equiv \frac{a'}{a^2} \ll 1$. We therefore aim to determine if the renormalized energy density ρ_r for each field is appropriately radiation dominated during a range of conformal time $\eta_0 < \eta < \eta_1$ for a generic scale factor a. We will allow for the possibility of a nonzero mass m for fermions, in which case we will insist that $m \ll 1$ in Planck units. If there are fermion fields with $m \gtrsim 1$, these will have mass comparable to or greater than the Planck mass, and then we will assume the fields have no contribution to the total energy density when $\eta \gtrsim \eta_0$. We will work with quantum fields in any state other than the conformal vacuum state for massless fields.

The body of this article is split into three sections, one for each of the two spins of fields and one for a discussion of the results obtained. Section 4.2 contains the analysis for spin-1 vector fields, beginning with a summary of the adiabatic regularization procedure and concluding with the renormalization of the energy density and our main result for the spin-1 case. Section 4.3 contains the analysis for spin-$\frac{1}{2}$ fields and begins with a summary of the modified adiabatic regularization procedure, as described in [70]. The subsequent renormalization and analysis is more complicated than that for the vector fields and is split into subsections: in Sec. 4.3.1 we give the renormalization counterterms and preview the assumptions built into our analysis; in Sec. 4.3.2, we derive bounds on the renormalized energy density by splitting the contributions into high and low energy terms and analyzing each in turn; and in Sec. 4.3.3 we obtain the leading order behavior of the renormalized energy density and our main result for the spin-$\frac{1}{2}$ case. We close in Sec. 4.4 with a discussion of our main results and final remarks.

4.2 Energy Density for Vector Fields

As discussed in the Introduction, massive and massless vector fields must be treated separately. In the Lagrangian description, the massive field is described by the Proca action [72], which in flat spacetime is a gauge-fixed theory involving the Higgs mechanism. Working with interactions and the curved spacetime form of the action is beyond the scope of this paper. We will focus instead on the massless case.

According to an argument in [34], any conformally invariant theory in a flat FLRW spacetime will have a stress-energy tensor that contains two terms, one of which is radiationlike and the other of which is the anomalous term. Renormalizing the electromagnatic field, however, requires that masses are introduced for the photon and ghost fields, which break the conformal invariance. These masses are then taken to zero to obtain physical results. We therefore feel it is worth working through the details of this procedure to confirm that the argument in [34] works.

The massless vector field in curved spacetime is given by the massless limit of the theory described by Lagrangian [71]

$$\mathcal{L} = \sqrt{-g}\bigg[-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2\xi}\nabla^{\mu}A_{\mu}\nabla^{\nu}A_{\nu} - \frac{1}{2}m^2 g^{\mu\nu}A_{\mu}A_{\nu} + ig^{\mu\nu}\partial_{\mu}\bar{\chi}\partial_{\nu}\chi + im_{\chi}\bar{\chi}\chi\bigg] , \tag{4.2}$$

where A_μ is the four-vector, ξ is the gauge fixing parameter, χ is the (complex) ghost field used to maintain gauge invariance, m_χ is the mass of the ghost field, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Masses are included in the Lagrangian (4.2) in order to properly renormalize the theory. These masses can trivially be taken to zero before computing any unrenormalized observables, but they are crucial in obtaining the appropriate renormalization counterterms. We will make clear when they are to be taken to zero in the following. The energy-momentum tensor from Eq. (4.2) is

$$T_{\mu\nu} = T^{\text{Maxwell}}_{\mu\nu} + T^{\text{G}}_{\mu\nu} + T^{\text{ghost}}_{\mu\nu} , \tag{4.3a}$$

$$T^{\text{Maxwell}}_{\mu\nu} \equiv -\frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\rho\sigma}F_{\alpha\rho}F_{\beta\sigma} + g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \frac{1}{2}g_{\mu\nu}m^2 g^{\alpha\beta}A_\alpha A_\beta + m^2 A_\mu A_\nu , \tag{4.3b}$$

$$T^{\text{G}}_{\mu\nu} \equiv \frac{1}{\xi}\bigg[-\frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}\nabla_\alpha A_\beta\right)^2 + \left(g_{\mu\nu}g^{\rho\sigma}A_\sigma\nabla_\rho - A_\nu\nabla_\mu - A_\mu\nabla_\nu\right)\left(g^{\alpha\beta}\nabla_\alpha A_\beta\right)\bigg] , \tag{4.3c}$$

$$T^{\text{ghost}}_{\mu\nu} \equiv ig_{\mu\nu}g^{\rho\sigma}\partial_\rho\bar{\chi}\partial_\sigma\chi - i\left(\partial_\mu\bar{\chi}\partial_\nu\chi + \partial_\nu\bar{\chi}\partial_\mu\chi\right) + ig_{\mu\nu}m_\chi^2\bar{\chi}\chi . \tag{4.3d}$$

Following the procedure in Ref. [71], one can define the components of the four-vector A_μ as a combination of temporal, transverse, and longitudinal parts,

$$A_\mu \equiv (A_0, B_i + \partial_i A) . \tag{4.4}$$

These components and the ghost field can be expanded in terms of mode functions Y_a, for $a = 0, L, T, \chi$, where L represents longitudinal and T transverse contributions:

$$A_0 = \frac{1}{m^2 a^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \left[a_{\vec{k}}^{(0)} \left(\partial_\eta - \frac{2a'}{a} \right) (maY_0)\, e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^{(3)} kmaY_L e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \right] , \tag{4.5a}$$

$$B_i = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{p=1,2} \left(\epsilon_i^p a_{\vec{k}}^{(p)} Y_T e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \right) , \tag{4.5b}$$

$$A = \frac{1}{m^2 a^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \left(a_{\vec{k}}^{(0)} maY_0 e^{i\vec{k}\cdot\vec{x}} - a_{\vec{k}}^{(3)} \partial_\eta \left(\frac{ma}{k} Y_L \right) e^{i\vec{k}\cdot\vec{x}} + \text{H.c.} \right) , \tag{4.5c}$$

$$\chi = \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\frac{b_{\vec{k}} Y_\chi}{a} e^{i\vec{k}\cdot\vec{x}} + \frac{b_{\vec{k}}^\dagger Y_\chi^*}{a} e^{-i\vec{k}\cdot\vec{x}} \right) , \tag{4.5d}$$

$$\bar{\chi} = \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\frac{\bar{b}_{\vec{k}} Y_\chi}{a} e^{i\vec{k}\cdot\vec{x}} + \frac{\bar{b}_{\vec{k}}^\dagger Y_\chi^*}{a} e^{-i\vec{k}\cdot\vec{x}} \right) , \tag{4.5e}$$

where $a_{\vec{k}}^{(\mu)}$, $b_{\vec{k}}$, and $\bar{b}_{\vec{k}}$ and their Hermitian conjugates are annihilation and creation operators, ϵ_i^p are the two polarization vectors of the transverse modes, and H.c. represents the Hermitian conjugate of all preceding terms. The creation and annihilation operators satisfy the commutation and anticommutation relations

$$\left[a_{\vec{k}}^{(a)}, a_{\vec{k}'}^{(b)\dagger} \right] = \eta^{ab} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') , \tag{4.6a}$$

$$\left\{ b_{\vec{k}}, \bar{b}_{\vec{k}'}^\dagger \right\} = - \left\{ \bar{b}_{\vec{k}}, b_{\vec{k}'}^\dagger \right\} = i (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') , \tag{4.6b}$$

where $\eta^{ab} = \text{diag}(-1, 1, 1, 1)$. The polarization vectors satisfy

$$\sum_i k_i \epsilon_i^p = 0 , \tag{4.7a}$$

$$\sum_i \epsilon_i^p \epsilon_i^{p'} = \delta^{pp'} , \tag{4.7b}$$

$$\sum_{p=1,2} \epsilon_i^p \epsilon_j^p = \delta_{ij} - \frac{k_i k_j}{k^2} \; , \tag{4.7c}$$

with p the polarization index. The modes Y_a satisfy the decoupled set of differential equations

$$\left(\partial_\eta^2 + \Omega_a^2\right) Y_a = 0 \; , \tag{4.8}$$

where

$$\Omega_a^2 \equiv \omega_a^2 + \zeta_a \tag{4.9}$$

is defined for each contribution by

$$\omega_0^2 \equiv k^2 + \xi m^2 a^2 \; , \tag{4.10a}$$

$$\omega_{L,T}^2 \equiv k^2 + m^2 a^2 \; , \tag{4.10b}$$

$$\omega_\chi^2 \equiv k^2 + m_\chi^2 a^2 \; , \tag{4.10c}$$

and

$$\zeta_{0,\chi} \equiv -\frac{a''}{a} \; , \tag{4.11a}$$

$$\zeta_L \equiv \frac{a''}{a} - \frac{2a'^2}{a^2} \; , \tag{4.11b}$$

$$\zeta_T \equiv 0 \; . \tag{4.11c}$$

The modes satisfy the standard normalization conditions for a scalar field,

$$Y_a Y_a^{*\prime} - Y_a' Y_a^* = i \; . \tag{4.12}$$

Note that in the massless limit Y_0 and Y_χ satisfy the same differential equations, so we can choose

$$Y_0 = Y_\chi \quad \text{for} \quad m = m_\chi = 0 \; . \tag{4.13}$$

Using this mode decomposition, one can find contributions to the energy density

$\rho \equiv \langle T_{\hat{0}\hat{0}} \rangle$ from Eqs. (4.3b)–(4.3d) in terms of Y_a:

$$\rho^{\text{Maxwell}} = \lim_{m\to 0} \frac{1}{4\pi^2 a^4} \int_0^\Lambda dk\, k^2 \Bigg[\omega_T^2 \left|Y_T\right|^2 + \left|Y_T'\right|^2 - \left(k^2 + \frac{a'^2}{a^2}\right) \left|Y_0\right|^2 + \frac{a'}{a}\partial_\eta \left(\left|Y_0\right|^2\right)$$
$$- \left|Y_0'\right|^2 + \left(\omega_L^2 + \frac{a'^2}{a^2}\right) \left|Y_L\right|^2 + \frac{a'}{a}\partial_\eta \left(\left|Y_L\right|^2\right) + \left|Y_L'\right|^2 \Bigg] \tag{4.14a}$$

$$\rho^{\text{G}} = \lim_{m\to 0} \frac{1}{2\pi^2 a^4} \int_0^\Lambda dk\, k^2 \left[\left(\omega_0^2 - \frac{1}{2}\xi m^2 a^2 + \frac{a'^2}{a^2}\right) \left|Y_0\right|^2 - \frac{a'}{a}\partial_\eta \left(\left|Y_0\right|^2\right) + \left|Y_0'\right|^2\right] \tag{4.14b}$$

$$\rho^{\text{ghost}} = \lim_{m_\chi\to 0} \frac{1}{2\pi^2 a^4} \int_0^\Lambda dk\, k^2 \left[-\left(\omega_\chi^2 + \frac{a'^2}{a^2}\right) \left|Y_\chi\right|^2 + \frac{a'}{a}\partial_\eta \left(\left|Y_\chi\right|^2\right) - \left|Y_\chi'\right|^2\right] , \tag{4.14c}$$

where Λ is a cutoff regulator which we will later demonstrate can be taken to infinity in the massless limit.

One may adiabatically renormalize the energy density by writing the vacuum states with the standard WKB ansatz

$$Y_a = \frac{1}{\sqrt{2W_a}} e^{-i\int^\eta d\bar{\eta} W_a(\bar{\eta})} , \tag{4.15}$$

where

$$(W_a)^2 = \Omega_a^2 - \left[\frac{W_a''}{2W_a} - \frac{3}{4}\left(\frac{W_a'}{W_a}\right)^2\right] . \tag{4.16}$$

Solutions to Eq. (4.16) can be approximated using $W_a^{(0)} = \Omega_a$ as the lowest order and iterating to higher orders, keeping to the appropriate adiabatic order, given by the number of time derivatives on the scale factor, at each iteration. Substitution of W_a to some adiabatic order A into Eq. (4.15) would then require expanding the square root only to terms of adiabatic order A.

In order to renormalize the energy density, one must take Eq. (4.15) to the appropriate adiabatic order and substitute into Eqs. (4.14a)–(4.14c) to produce the renormalization counterterms. On dimensional grounds, one would need to keep to fourth adiabatic order to renormalize ρ. The fourth order counterterms are

$$\rho_c^{(4)} = \lim_{m\to 0} \frac{1}{4a^2}\int \frac{d^3k}{(2\pi)^3}\bigg\{W_0 + W_L + 2W_T - 2W_\chi + \frac{\omega_0^2}{W_0} + \frac{\omega_L^2}{W_L} + \frac{2\omega_T^2}{W_T} - \frac{2\omega_\chi^2}{W_\chi} + \frac{a'^2}{a^2 W_0} + \frac{a'^2}{a^2 W_L} - \frac{2a'^2}{a^2 W_\chi} + \frac{a'W_0'}{aW_0^2} - \frac{a'W_L'}{aW_L^2} - \frac{2a'W_\chi'}{aW_\chi^2} + \frac{W_0'^2}{4W_0^3} + \frac{W_L'^2}{4W_L^3} + \frac{W_T'^2}{2W_T^3} - \frac{W_\chi'^2}{2W_\chi^3}\bigg\}^{(4)}, \tag{4.17}$$

where $\{\dots\}^{(4)}$ implies that all W_a are taken to fourth order.

In order to analyze the early-time behavior of the energy density, we will parametrize the mode functions in terms of zeroth-order adiabatic states

$$Y_a = \alpha_{k,a} Y_a^{(0)} + \beta_{k,a} Y_a^{(0)*} , \tag{4.18}$$

$$Y_a' = \alpha_{k,a} Y_a^{(0)\prime} + \beta_{k,a} Y_a^{(0)*\prime} . \tag{4.19}$$

From the normalization condition (4.12), one has

$$|\alpha_{k,a}|^2 - |\beta_{k,a}|^2 = 1 . \tag{4.20}$$

One could instead parametrize the mode functions in terms of higher-order adiabatic states, in which case the Bogoliubov coefficients $\alpha_{k,a}$ and $\beta_{k,a}$ would be constant to the given adiabatic order, but zeroth order will be sufficient to properly renormalize the theory. Substituting Eqs. (4.18) and (4.19) into Eq. (4.8), one obtains differential equations for the coefficients,

$$\alpha_{k,a}' = \frac{\Omega_a'}{2\Omega_a}\beta_{k,a} e^{2i\theta_a} \tag{4.21a}$$

$$\beta_{k,a}' = \frac{\Omega_a'}{2\Omega_a}\alpha_{k,a} e^{-2i\theta_a} , \tag{4.21b}$$

where $\theta_a \equiv \int^\eta d\bar{\eta} W_a(\bar{\eta})$.

One can then obtain the renormalized energy density in terms of $\alpha_{k,a}$ and $\beta_{k,a}$ by subtracting these renormalization counterterms from the unrenormalized energy density given in Eqs. (4.14a)–(4.14c). Using a zeroth order parametrization, one finds that the renormalized energy density ρ_r separates into analytic and mode terms

$$\rho_r = \rho_{\text{an}} + \rho_{\alpha\beta} \ , \tag{4.22}$$

where the analytic terms ρ_{an} are finite higher-order terms, independent of the cutoff regulator Λ, coming from the counterterms (4.17),

$$\begin{aligned} \rho_{\text{an}} &= \frac{1}{2880\pi^2}\left[62\,{}^{(3)}H_{00} + \left(3 + \frac{5}{2}\ln\xi\right)\,{}^{(1)}H_{00}\right] \\ &= \frac{1}{2880\pi^2}\left[\frac{186a'^4}{a^6} + \frac{216a'^2a''}{a^5} + \frac{54a''^2}{a^4} - \frac{108a'a'''}{a^4}\right. \\ &\quad \left. + \ln\xi\left(\frac{180a'^2a''}{a^5} + \frac{45a''^2}{a^4} - \frac{90a'a'''}{a^4}\right)\right] , \end{aligned} \tag{4.23}$$

where ${}^{(1)}H_{00}$ and ${}^{(3)}H_{00}$ are higher order corrections to the Einstein field equations [14], and the mode terms $\rho_{\alpha\beta}$ are those coming from the zeroth order parametrization of Eq. (4.18),

$$\begin{aligned} \rho_{\alpha\beta} \equiv \lim_{m,m_\chi\to 0}\frac{1}{a^2}\int_0^\Lambda \frac{d^3k}{(2\pi)^3}\Bigg[&\frac{k^2+\omega_T^2}{a^2\omega_T}\left|\beta_{k,T}\right|^2 + \frac{m^2}{2\omega_L}\left|\beta_{k,L}\right|^2 + \left(\frac{3\omega_0^2+k^2}{2a^2} + \frac{a'^2}{a^4}\right)\frac{\left|\beta_{k,0}\right|^2}{\omega_0} \\ &- \left(\frac{\omega_\chi^2+k^2}{a^2} + \frac{a'^2}{a^4}\right)\frac{\left|\beta_{k,\chi}\right|^2}{\omega_\chi} - \frac{m^2}{\omega_T}\text{Re}\left(\alpha_{k,T}\beta^*_{k,T}e^{-2i\theta_T}\right) + \frac{m^2}{2\omega_L}\text{Re}\left(\alpha_{k,L}\beta^*_{k,L}e^{-2i\theta_L}\right) \\ &+ \frac{1}{\omega_0}\left(\frac{a'^4}{a^2} - \frac{\xi m^2}{2a^2}\right)\text{Re}\left(\alpha_{k,0}\beta^*_{k,0}e^{-2i\theta_0}\right) + \frac{1}{\omega_\chi}\left(m_\chi^2 - \frac{a'^2}{a^4}\right)\text{Re}\left(\alpha_{k,\chi}\beta^*_{k,\chi}e^{-2i\theta_\chi}\right) \\ &- \frac{2a'}{a^3}\text{Im}\left(\alpha_{k,0}\beta^*_{k,0}e^{-2i\theta_0}\right) + \frac{2a'}{a^3}\text{Im}\left(\alpha_{k,\chi}\beta^*_{k,\chi}e^{-2i\theta_\chi}\right)\Bigg] . \end{aligned} \tag{4.24}$$

Assuming $\beta_{k,a}$ falls faster than k^{-2}, integrating the terms in Eq. (4.24) will yield finite results, even if $\Lambda \to \infty$, and hence the massless limit can be freely taken inside the integral. One finds from substitution of Eqs. (4.15), (4.18), and (4.19) into Eq. (4.8) that $\alpha'_{k,0} = \alpha'_{k,\chi}$ and $\beta'_{k,0} = \beta'_{k,\chi}$ after taking the massless limit, which when combined with Eq. (4.13) allows one to choose the coefficients for the 0 and χ contributions to be identical. The mode term contribution to the energy density therefore drastically simplifies to

$$\rho_{\alpha\beta} = \frac{1}{\pi^2 a^4} \int_0^{\Lambda} dk\, k^3 \left|\beta_{k,T}\right|^2 \,. \tag{4.25}$$

The ultraviolet cutoff can now be removed, and we take the limit $\Lambda \to \infty$. The renormalized energy density ultimately only depends on the transverse mode functions, which are the only physical modes of the theory, and higher-order dependencies on the background curvature:

$$\rho_r = \frac{1}{\pi^2 a^4} \int_0^{\infty} dk\, k^3 \left|\beta_{k,T}\right|^2 + \frac{1}{2880\pi^2} \left[62\, {}^{(3)}H_{00} + \left(3 + \frac{5}{2}\ln\xi\right) {}^{(1)}H_{00}\right] \,. \tag{4.26}$$

Assuming the higher-order corrections are subdominant in the semiclassical approximation below the Planck scale, ρ_r for massless vector fields does have the expected radiation-dominated behavior. We note that this result applies to a general scale factor a and agrees with the prediction in [34] that, other than the higher-order terms, all of the contributions in a flat FLRW metric of a conformally invariant field will act like radiation.

The higher-order terms in (4.26) are of the same form as those found for the trace anomaly in [71, 73], in which the ${}^{(1)}H_{00}$ term has a gauge-dependent coefficient. This coefficient corresponds to a $\Box R$ term appearing in the trace anomaly. The exact value of this coefficient is dependent on the regularization scheme used, unlike for scalar and spin-$\frac{1}{2}$ fields which respectively have the same coefficient regardless of regularization scheme.

It is also worth noting that the higher-order terms may not in fact be negligible compared to the rest of Eq. (4.26) when the quantum field contributions themselves are very small or vanish, such as in the case that $\beta_{k,T} = 0$, which corresponds to the conformal vacuum state. The contribution to the total stress-energy tensor from massless spin-1 fields would then not necessarily be radiationlike. However, in such a scenario some other component would likely be dominating the total stress-energy tensor, and the Friedmann equation assures us that the higher-order, geometric contributions will always be much smaller than the other contributions until one approaches the Planck era.

4.3 Energy Density for Dirac Fields

We now turn our attention to Dirac spinor fields in the preinflationary era of an FLRW universe. We follow the work and notation given in Ref. [70]. There, expressions are in terms of cosmic time t, related to conformal time by $a\,d\eta = dt$. We summarize the procedure for obtaining the unrenormalized energy density here.

Consider Dirac spinor fields $\Psi(x)$ that obey the Dirac equation in curved spacetime,

$$(i\gamma^a e_a{}^\mu \nabla_\mu - m)\,\Psi = 0\;, \tag{4.27}$$

where $e_a{}^\mu$ is the vierbein, γ^a are the flat spacetime Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and $\nabla_\mu \equiv \partial_\mu + \Gamma_\mu$ is the covariant derivative associated with the spin connection Γ_μ. For the metric (4.1), the Dirac equation (4.27) becomes

$$\left[\gamma^0\left(\partial_\eta + \frac{3a'}{2a}\right) + \gamma^i\partial_i + ima\right]\Psi = 0\;. \tag{4.28}$$

The field can be written in terms of creation and annihilation operators $D^\dagger_{\vec{k}\lambda}(\eta)$ and

$B_{\vec{k}\lambda}(\eta)$ as

$$\Psi = \sum_{\lambda=\pm\frac{1}{2}} \int d^3k \left(B_{\vec{k}\lambda}\psi_{\vec{k}\lambda} + D^\dagger_{\vec{k}\lambda} C \bar{\psi}^\intercal_{\vec{k}\lambda} \right) \ , \tag{4.29}$$

where C is the charge conjugation matrix, $\lambda = \pm 1/2$ represents the helicity eigenvalue, and

$$\left\{ B_{\vec{k},\lambda}, B^\dagger_{\vec{k},\lambda} \right\} = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k} - \vec{k}') \tag{4.30}$$

and similarly for $D_{\vec{k}\lambda}$ and $D^\dagger_{\vec{k}\lambda}$, with all other anticommutators vanishing. Working in the Dirac-Pauli representation for γ^a, the modes $\psi_{\vec{k}\lambda}$ can be written as

$$\psi_{\vec{k}\lambda}(\eta, \vec{x}) = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{8\pi^3 a^3}} \begin{pmatrix} h^I_k(\eta)\xi_\lambda(\vec{k}) \ , \\ h^{II}_k(\eta)\hat{k}\cdot\vec{\sigma}\xi_\lambda(\vec{k}) \end{pmatrix} \ , \tag{4.31}$$

where $\xi_\lambda(\vec{k})$ are two-component spinors and are eigenvectors of the spin component along the $\vec{k}$ direction, so that $\frac{1}{2}(\hat{k}\cdot\vec{\sigma})\xi_\lambda(\vec{k}) = \lambda\xi_\lambda(\vec{k})$, with normalization $\xi^\dagger_\lambda\xi_\lambda = 1$, and h^I_k and h^{II}_k are scalar functions that satisfy coupled first-order differential equations

$$\partial_\eta h^I_k(\eta) = -ikh^{II}_k(\eta) - ima(\eta)h^I_k(\eta) \ , \tag{4.32a}$$

$$\partial_\eta h^{II}_k(\eta) = -ikh^I_k(\eta) + ima(\eta)h^{II}_k(\eta) \ , \tag{4.32b}$$

and have the normalization

$$\left|h^I_k(\eta)\right|^2 + \left|h^{II}_k(\eta)\right|^2 = 1 \ . \tag{4.33}$$

The energy density for the Dirac field in terms of the mode functions $h^{I,II}_k$ can be written as

$$\rho = \frac{1}{\pi^2 a^4} \int_0^\infty dk k^2 \left[ma \left(\left|h^{II}_k\right|^2 - \left|h^I_k\right|^2 \right) - k \left(h^I_k h^{II*}_k + h^{I*}_k h^{II}_k \right) \right] \ . \tag{4.34}$$

At this point, we will diverge from this procedure coming from [70], who themselves proceeded to obtain counterterms to a generic unrenormalized energy in an FLRW universe. These counterterms were then used to prove conservation of the energy density and were applied to a de Sitter spacetime and a radiation-dominated universe. As demonstrated there, one can make use of adiabatic regularization to renormalize the energy density, but the ansatz used for the WKB approximation of adiabatic states must be a modified form of the standard ansatz. This renormalization procedure was first demonstrated in [68] and has been applied in other cases [69, 74–76]. Here, we will use the energy density (4.34) as well as the modified WKB ansatz to obtain renormalization counterterms, but we will use a different process to obtain an explicit form of the unrenormalized energy density. Namely, we will expand the mode functions $h_k^{I,II}$ in terms of adiabatic modes $g_k^{I,II}$ via a Bogoliubov-like expansion in order to analyze early time behavior using the Bogoliubov coefficients.

First, one expands $h_k^{I,II}$ in terms of adiabatic modes $g_k^{I,II}$,

$$h_k^I = \alpha_k g_k^I - \beta_k g_k^{II*} \ , \tag{4.35a}$$

$$h_k^{II} = \alpha_k g_k^{II} + \beta_k g_k^{I*} \ , \tag{4.35b}$$

where $g_k^{I,II}$ are given to adiabatic order A,

$$g_k^{I,II} = g_k^{I,II(0)} + g_k^{I,II(1)} + \cdots + g_k^{I,II(A)} \ , \tag{4.36}$$

with adiabatic order understood to be the number of conformal time derivatives, and satisfy the differential equations (4.32a) and (4.32b), and α_k and β_k are the time-dependent Bogoliubov coefficients and are constant to order A. Coupled first order differential equations for α_k and β_k can be obtained from Eqs. (4.32a), (4.32b), and

(4.33). The unrenormalized energy density in terms of the adiabatic states is

$$\rho = \frac{1}{\pi^2 a^4}\int_0^\infty dk k^2 \Big\{ 2\left|\beta_k\right|^2 \left[ma\left(\left|g_k^I\right|^2 - \left|g_k^{II}\right|^2 \right) + 2k\mathrm{Re}\left(g_k^I g_k^{II*} \right) \right]$$
$$+ 4ma\mathrm{Re}\left(\alpha_k \beta_k^* g_k^I g_k^{II}\right) + 2k\mathrm{Re}\left[\alpha_k \beta_k^* \left(\left(g_k^{II}\right)^2 - \left(g_k^I\right)^2 \right)\right] + ma\left(\left|g_k^{II}\right|^2 - \left|g_k^I\right|^2 \right)$$
$$- 2k\mathrm{Re}\left(g_k^I g_k^{II*} \right) \Big\} , \tag{4.37}$$

and the adiabatic renormalization counterterms are

$$\rho_c = \frac{1}{\pi^2 a^4}\int_0^\infty dk k^2 \left[ma\left(\left|g_k^{II}\right|^2 - \left|g_k^I\right|^2 \right) - 2k\mathrm{Re}\left(g_k^I g_k^{II*} \right) \right]^{(4)} , \tag{4.38}$$

where $[\ldots]^{(4)}$ indicates that g_k^I, g_k^{II} are fourth order states. In order for the energy density to be renormalized and all divergences eliminated, the adiabatic states in Eq. (4.37) must be at least of the adiabatic order at which the counterterms in Eq. (4.38) are divergent.

In order to obtain forms for the adiabatic states, one can obtain uncoupled second order equations from Eqs. (4.32a) and (4.32b),

$$\left(\partial_\eta^2 + \frac{a'}{a}\partial_\eta - ima' + \omega^2 \right) g_k^I = 0 , \tag{4.39}$$

$$\left(\partial_\eta^2 + \frac{a'}{a}\partial_\eta + ima' + \omega^2 \right) g_k^{II} = 0 , \tag{4.40}$$

where

$$\omega^2 \equiv k^2 + m^2 a^2 , \tag{4.41}$$

and assume formal WKB series solutions, truncating at the desired adiabatic order. However, Ref. [68] and later Refs. [69, 70] pointed out that the usual WKB ansatz does not satisfy the normalization condition (4.33), so one must use a modified WKB ansatz of the form

$$g_k^I = \sqrt{\frac{\omega + ma}{2\omega}} F e^{-i\theta_k} \ , \tag{4.42a}$$

$$g_k^{II} = \sqrt{\frac{\omega - ma}{2\omega}} G e^{-i\theta_k} \ , \tag{4.42b}$$

and the functions

$$F = 1 + F^{(1)} + \cdots + F^{(A)} \ , \tag{4.43a}$$

$$G = 1 + G^{(1)} + \cdots + G^{(A)} \ , \tag{4.43b}$$

$$\theta_k = \int^{\eta} d\tilde{\eta} \left(\omega + \omega^{(1)} + \cdots + \omega^{(A)}\right) \tag{4.43c}$$

are determined by repeated substitution of Eqs. (4.42a) and (4.42b) into Eqs. (4.32a), (4.32b), and (4.33). There is an ambiguity in the exact forms following this method, but all local observables are independent of the ambiguity [68], so one may fix the ambiguity by choosing $F^{(n)}(-m) = G^{(n)}(m)$ for each order $n \geq 1$. However, we are only interested in using zeroth order states, for which one obtains

$$g_k^I = \sqrt{\frac{\omega + ma}{2\omega}} e^{-i\theta_k} \ , \tag{4.44a}$$

$$g_k^{II} = \sqrt{\frac{\omega - ma}{2\omega}} e^{-i\theta_k} \ , \tag{4.44b}$$

which have a normalization from Eq. (4.33) of

$$\left|g_k^I\right|^2 + \left|g_k^{II}\right|^2 = 1 \ . \tag{4.45}$$

Note that we will continue writing θ_k like we have in Eqs. (4.44a) and (4.44b) for simplicity and assume it to be understood that only the zeroth order term is kept. Substituting Eqs. (4.35a) and (4.35b) into the differential equations (4.32a) and (4.32b), one obtains differential equations for α_k and β_k,

$$\alpha_k' = \frac{-kma'}{2\omega^2}\beta_k e^{2i\theta_k} , \tag{4.46}$$

$$\beta_k' = \frac{kma'}{2\omega^2}\alpha_k e^{-2i\theta_k} , \tag{4.47}$$

and substituting them into the normalization condition (4.33), one finds

$$|\alpha_k|^2 + |\beta_k|^2 = 1 . \tag{4.48}$$

4.3.1 Renormalized energy density

In order to obtain finite results so that we may inspect the behavior of the energy density, one must subtract counterterms up to fourth order from Eq. (4.37). Order by order, these counterterms $\rho^{(n)}$ are

$$\rho_c^{(0)} = -\,\omega , \tag{4.49}$$

$$\rho_c^{(2)} = \frac{k^2\omega'^2}{8m^2a^2\omega^3} = \frac{k^2m^2a'^2}{8\omega^5} , \tag{4.50}$$

$$\rho_c^{(4)} = \mathcal{O}\left(k^{-5}\right) . \tag{4.51}$$

The fourth order counterterms produce finite contributions to the energy density. Using the zeroth order adiabatic states (4.44a) and (4.44b) in Eq. (4.34) and subtracting the counterterms (4.49)-(4.51), one obtains the renormalized energy density

$$\rho_r = \frac{1}{\pi^2a^4}\int_0^\infty dk\, k^2\left[2\omega\,|\beta_k|^2 - \frac{k^2m^2a'^2}{8\omega^5}\right] + \frac{2}{2880\pi^2}\left[-\frac{1}{2}{}^{(1)}H_{00} + \frac{11}{2}{}^{(3)}H_{00}\right] , \tag{4.52}$$

where the finite renormalization terms coming from the fourth order counterterms [14],

$$\frac{2}{2880\pi^2}\left[-\frac{1}{2}{}^{(1)}H_{00} + \frac{11}{2}{}^{(3)}H_{00}\right] = \frac{2}{2880\pi^2a^4}\left(\frac{33a'^4}{2a^4} + \frac{18a'a'''}{a^2} - \frac{9a''^2}{a^2} - \frac{36a'^2a''}{a^3}\right) , \tag{4.53}$$

are assumed small beyond the Planck era. As discussed in the spin-1 case, these higher-order terms (4.53) may not be negligible compared to very small or negligible quantum field contributions in Eq. (4.52). However, if this were true, then some other component would likely be dominating the total energy density, and the potentially nonradiationlike behavior of the spin-$\frac{1}{2}$ contribution (4.52) would be subleading. In any case, we may proceed to analyze the quantum field terms in Eq. (4.52) regardless of the size of the higher-order terms to see if they do in fact produce a radiationlike contribution.

We ultimately will attempt to solve the Friedmann equation

$$H^2 = \frac{8\pi}{3}\rho \,, \tag{4.54}$$

where

$$H \equiv \frac{a'}{a^2} \,, \tag{4.55}$$

and ρ contains ρ_r and may also include other terms such as a cosmological constant or other classical contributions. Because we are working with a semiclassical approximation, we do not assume our analysis to be valid during the Planck era. Hence we will work starting at an initial time η_0, which is assumed to be after the Planck era and corresponds to a scale factor a_0 that is above the Planck scale, and demonstrate that $\rho_r \sim a^{-4}$ for some region $\eta_0 < \eta < \eta_1$.

Above the Planck scale, the Hubble parameter $H \lesssim 1$, so given Eqs. (4.55) and (4.53), we will insist that the following set of inequalities of derivatives of the scale factor must hold:

$$a' \lesssim a^2 \,, \tag{4.56}$$

$$a'' \lesssim a^3 \,, \tag{4.57}$$

$$a'a''' \lesssim a^6 \,. \tag{4.58}$$

We will use these inequalities in order to compute the integral in Eq. (4.52).

4.3.2 Bounds on the renormalized energy density

We will begin by splitting the integral into infrared and ultraviolet regions I_1 and I_2 by a cutoff k_c. The infrared contribution is

$$\begin{aligned} I_1 &= \int_0^{k_c} dk\, k^2 \left[2\omega \left|\beta_k\right|^2 - \frac{k^2 m^2 a'^2}{8\omega^5} \right] \\ &= \int_0^{k_c} dk\, k^2 \left[2k \left|\beta_k\right|^2 + 2\left(\omega - k\right) \left|\beta_k\right|^2 - \frac{k^2 m^2 a'^2}{8\omega^5} \right] , \end{aligned} \tag{4.59}$$

and the ultraviolet contribution is

$$I_2 = \int_{k_c}^{\infty} dk\, k^2 \left[2\omega \left|\beta_k\right|^2 - \frac{k^2 m^2 a'^2}{8\omega^5} \right] . \tag{4.60}$$

At time η_0, we define

$$B_0 \equiv \int_0^{k_c} dk\, 2k^3 \left|\beta_k(\eta_0)\right|^2 . \tag{4.61}$$

If B_0 is the dominant contribution to the renormalized energy density at time η, then $\rho_r \propto a^{-4}$ as desired. However, as one may anticipate given the apparent logarithmically divergent term in Eq. (4.52), this may not always be the case. We will investigate this in the following sections. Given that $|\beta_k(\eta_0)| \leq 1$ from Eq. (4.48), one finds from Eq. (4.61) a lower bound on k_c of

$$k_c \gtrsim B_0^{1/4} . \tag{4.62}$$

Ultraviolet region

One is tempted to assume $\beta_k \to 0$ sufficiently quickly at high k, as is often done with scalar fields [32]. However, because the final term in Eq. (4.60) produces a logarithm divergence, it is evident that doing so will introduce a divergent contribution to the energy density. This situation is occurring because until now we have been working with a zeroth order parametrization of the states, but because the logarithmic divergence comes in at higher than zeroth adiabatic order, one would expect to need to work with at least a second order parametrization to eliminate this higher-order divergence. These problems would indeed disappear working with a higher-order parametrization, but it becomes much more difficult to analytically obtain generic bounds on I_2 doing so.

One can instead continue to work with a zeroth-order parametrization using α_k and β_k. We can understand what the appropriate higher order states look like at large k by integrating Eq. (4.47) by parts, which shows that the asymptotic behavior of β_k will take the form

$$\beta_k \xrightarrow[k\to\infty]{} -\frac{ikma'}{4\omega^3}\alpha_k e^{-2i\theta_k} \; . \tag{4.63}$$

This motivates us to use the parametrization $\bar{\beta}_k$, defined as

$$\bar{\beta}_k \equiv \beta_k + \frac{ikma'}{4\omega^3}\alpha_k e^{-2i\theta_k} \; , \tag{4.64}$$

as the appropriate description to use in calculating the energy density at large k. We then anticipate that $\bar{\beta}_k$ will fall faster than the leading order term at large k, so

$$\left|\bar{\beta}_k(\eta_0)\right| < \frac{A_0}{k^2}\left(\frac{k_c}{k}\right)^{b_0} \tag{4.65}$$

for $k > k_c$, some $b_0 > 0$, and A_0 independent of k. The ultraviolet integral (4.60) written in terms of $\bar{\beta}_k$ is

$$\begin{aligned} I_2 = \int_{k_c}^{\infty} dk\, k^2 \Bigg[&\left(1 - \frac{k^2 m^2 a'^2}{16\omega^6}\right)\left(2\omega\left|\bar{\beta}_k\right|^2 + \frac{kma'}{\omega^2}\mathrm{Im}\big(\alpha_k \bar{\beta}_k^* e^{-2i\theta_k}\big)\right) \\ &- \frac{k^4 m^4 a'^4}{128\omega^{11}}\left|\alpha_k\right|^2 \Bigg] \; . \end{aligned} \tag{4.66}$$

Because $\bar{\beta}_k$ encodes the cancellation of the divergent term in the renormalized energy density, one expects that the contributions to the energy density from Eq. (4.66) will converge.

In order for the energy density to be radiation-dominated the ultraviolet contribution (4.66) must either be the dominant contribution and itself radiation-dominated or be subdominant to the radiation-dominated part of the infrared contribution (4.59). As we will demonstrate, every term in Eq. (4.66) is in fact subdominant to B_0 (4.61), which itself produces a radiation-dominated term in the energy density, and hence the latter is true.

To show this, we will first simplify the first factor in Eq. (4.66) by using Eqs. (4.62) and (4.56) and $\omega > k > k_c$ to show that $k^2 m^2 a'^2 \ll 16\omega^6$ and therefore $1 - \frac{k^2m^2a'^2}{16\omega^6} \approx 1$, provided the condition $a^2 \ll B_0^{1/2} m^{-1}$ is satisfied, so we will need

$$a \ll B_0^{1/4} m^{-1} \; . \tag{4.67}$$

This is the first of several conditions we will need. We will consider the complete set of conditions collectively later.

With this simplification, one finds a bound on I_2 of

$$\begin{aligned} |I_2| &\lesssim \int_{k_c}^{\infty} dk \left[2k^2 \omega \left|\bar{\beta}_k\right|^2 + \frac{1}{2} k m a' \left|\bar{\beta}_k\right| \right] + \frac{m^4 a'^4}{512 k_c^4} \\ &< \int_{k_c}^{\infty} dk \left[2k^3 \left|\bar{\beta}_k\right|^2 + k m^2 a^2 \left|\bar{\beta}_k\right|^2 + \frac{1}{2} k m a' \left|\bar{\beta}_k\right| \right] + \frac{m^4 a'^4}{512 k_c^4} \; , \end{aligned} \tag{4.68}$$

where we have bounded the integral on the final term using the normalization (4.48) to obtain $|\alpha_k| < 1$. For B_0 to dominate, the contributions from I_2 must remain less than B_0, and therefore the full contribution from $I_2(\eta) = I_2(\eta_0) + \Delta I_2$ must be less than B_0. We will find the conditions under which $|I_2(\eta_0)|$ and $|\Delta I_2|$, and therefore $|I_2(\eta)|$, are subdominant to B_0.

Given Eq. (4.65), one finds from Eq. (4.68) that the contributions from the integrand of $|I_2(\eta_0)|$ are subdominant to B_0 provided that

$$A_0^2 \ll B_0 \ , \tag{4.69a}$$

$$m^2 a^2 A_0^2 \ll B_0^{3/2} \ , \tag{4.69b}$$

$$m a' A_0 \ll B_0 \ . \tag{4.69c}$$

To satisfy (4.65), one may increase k_c which allows for decreasing A_0 and ensuring (4.69a) can be satisfied. Furthermore, provided Eq. (4.69a) is satisfied and using Eq. (4.56), one can show Eqs. (4.69b) and (4.69c) become

$$a \ll B_0^{1/4} m^{-1} \ , \tag{4.70}$$

$$a \ll B_0^{1/4} m^{-1/2} \ . \tag{4.71}$$

The final term in (4.68) is also less than B_0 provided (4.71) is satisfied.

For the bound on the integral in ΔI_2 to converge, $\Delta\bar{\beta}_k$ must fall faster than k^{-2}. We will assume that $\Delta\bar{\beta}_k$ falls at least as fast as k^{-2-b_Δ}, and then we will need to show

$$\left|\Delta\bar{\beta}_k\right| < \frac{A_\Delta}{k^2}\left(\frac{k_c}{k}\right)^{b_\Delta} \ , \tag{4.72}$$

with A_Δ independent of k and $b_\Delta > 0$ chosen appropriately for each term contributing to $\Delta\bar{\beta}_k$. One finds from Eq. (4.68) that the contributions from ΔI_2 are less than B_0 provided that

$$A_\Delta^2 \ll B_0 \ , \tag{4.73}$$

$$m^2 a^2 A_\Delta^2 \ll B_0^{3/2} \ , \tag{4.74}$$

$$m a' A_\Delta \ll B_0 \ . \tag{4.75}$$

One can obtain bounds on the contributions to $\Delta\bar{\beta}_k$, and hence the conditions under which Eqs. (4.73)–(4.75) are satisfied, using the differential equation for $\bar{\beta}_k$. From Eqs. (4.47) and (4.64), the differential equation is

$$\bar{\beta}_k' = -\frac{ikma''}{4\omega^3}\alpha_k e^{-2i\theta_k} + \frac{ik^2m^2a'^2}{8\omega^5}\beta_k + \frac{3ikm^3aa'^2}{4\omega^5}\alpha_k e^{-2i\theta_k} \,. \tag{4.76}$$

Writing $\bar{\beta}_k(\eta) = \bar{\beta}_k(\eta_0) + \Delta\bar{\beta}_k$, one then finds by integrating by parts on the phase

$$\begin{aligned}\Delta\bar{\beta}_k &= \int_{\eta_0}^{\eta} dx \left[\frac{-ikma''}{4\omega^3}\alpha_k e^{-2i\theta_k} + \frac{ik^2m^2a'^2}{8\omega^5}\beta_k + \frac{3ikm^3aa'^2}{4\omega^5}\alpha_k e^{-2i\theta_k}\right] \\ &= \frac{kma''}{8\omega^4}\alpha_k e^{-2i\theta_k}\bigg|_{\eta_0}^{\eta} + \int_{\eta_0}^{\eta} dx \left[\frac{-kma'''}{8\omega^4}\alpha_k e^{-2i\theta_k} + \frac{3km^3aa'a''}{8\omega^6}\alpha_k e^{-2i\theta_k}\right. \\ &\quad \left. + \frac{k^2m^2a'a''}{16\omega^6}\beta_k + \frac{ik^2m^2a'^2}{8\omega^5}\beta_k + \frac{3ikm^3aa'^2}{4\omega^5}\alpha_k e^{-2i\theta_k}\right] ,\end{aligned} \tag{4.77}$$

and therefore

$$\begin{aligned}\left|\Delta\bar{\beta}_k\right| < &\frac{m\left(|a''(\eta_0)| + |a''(\eta)|\right)}{8k^3} + \frac{1}{8k^3}\int_{\eta_0}^{\eta} dx \left(m\,|a'''| + m^2a'^2\right) \\ &+ \frac{1}{16k^4}\int_{\eta_0}^{\eta} dx\, m^2\,|a'|\,|a''| + \frac{3}{4k^4}\int_{\eta_0}^{\eta} dx\, m^3aa'^2 + \frac{3}{8k^5}\int_{\eta_0}^{\eta} dx\, m^3a\,|a'|\,|a''| \,.\end{aligned} \tag{4.78}$$

Each term in Eq. (4.78) can be written in the form of Eq. (4.72) to obtain conditions under which each will satisfy Eq. (4.73). Term by term, using the inequalities in Eqs. (4.56) and (4.57), we can obtain conditions on the scale factor under which each term is subdominant to the infrared contribution. In order to obtain these conditions, we will assume a', a'', and a''' have definite signs; that is, each of them is either always positive or always negative. The conditions are

$$\frac{ma''}{k^3} = \frac{ma''}{k^2k_c}\left(\frac{k_c}{k}\right) \implies B_0 \gg \left(\frac{ma''}{k_c}\right)^2$$

$$a \ll B_0^{1/4}m^{-1/3} \,, \tag{4.79a}$$

$$\frac{1}{k^3}\int dx\, m^2 a'^2 < \frac{1}{k^3}\int dx\, m^2 a^2 a' = \frac{m^2 a^3}{k^3} = \frac{m^2 a^3}{k^2 k_c}\left(\frac{k_c}{k}\right)$$

$$\Longrightarrow B_0 \gg \left(\frac{m^2 a^3}{k_c}\right)$$

$$a \ll B_0^{1/4} m^{-2/3}\,, \tag{4.79b}$$

$$\frac{1}{k^4}\int dx\, m^2 a' a'' = \frac{m^2 a'^2}{k^4} = \frac{m^2 a'^2}{k^2 k_c^2}\left(\frac{k_c}{k}\right)^2$$

$$\Longrightarrow B_0 \gg \left(\frac{m^2 a'^2}{k_c^2}\right)^2$$

$$a \ll B_0^{1/4} m^{-1/2}\,, \tag{4.79c}$$

$$\frac{1}{k^4}\int dx\, m^3 a a'^2 < \frac{1}{k^4}\int dx\, m^3 a^3 a' = \frac{m^3 a^4}{k^4} = \frac{m^3 a^4}{k^2 k_c^2}\left(\frac{k_c}{k}\right)^2$$

$$\Longrightarrow B_0 \gg \left(\frac{m^3 a^4}{k_c^3}\right)^2$$

$$a \ll B_0^{1/4} m^{-3/4}\,, \tag{4.79d}$$

$$\frac{1}{k^5}\int dx\, m^3 a a' a'' < \frac{1}{k^5}\int dx\, m^3 a^4 a' = \frac{m^3 a^5}{k^5} = \frac{m^3 a^5}{k^2 k_c^3}\left(\frac{k_c}{k}\right)^3$$

$$\Longrightarrow B_0 \gg \left(\frac{m^3 a^5}{k_c^3}\right)$$

$$a \ll B_0^{1/4} m^{-3/5}\,. \tag{4.79e}$$

Even if one of the three quantities a', a'', or a''' is not of definite sign, one can subdivide the integrals appearing in (4.78) into regions in which it *is* of definite sign, and assuming the number of regions is not too large the sum of these regions can be similarly bounded if the inequalities (4.79a)–(4.79e) are satisfied.

We are working at times beyond the Planck era during which the mass is small compared to the Planck mass, and hence $m \ll 1$, so the strongest restriction among Eq. (4.67) and Eqs. (4.79a)–(4.79e) for which the ultraviolet contributions are subdominant to that from B_0 is

$$a \ll B_0^{1/4} m^{-1/3} \, . \tag{4.80}$$

Equation (4.80) is stronger than the conditions in Eqs. (4.70)–(4.71) and those in Eqs. (4.74)–(4.75). One therefore has a complete set of conditions under which the ultraviolet contribution is subdominant to that from B_0: k_c must be large enough such that Eq. (4.69a) is true, and a must be small enough to satisfy Eq. (4.80).

Infrared region

It remains to be shown that B_0 is indeed the dominant contribution to the energy density among all the terms in the infrared contribution (4.59). Given Eq. (4.61), the infrared contribution (4.59) can be written

$$I_1 = B_0 + (B - B_0) + \int_0^{k_c} dk \left[2k^2 (\omega - k) |\beta_k|^2 - \frac{k^4 m^2 a'^2}{4\omega^5} \right] \, , \tag{4.81}$$

where we have defined

$$B \equiv \int_0^{k_c} dk \, 2k^3 \, |\beta_k|^2 \, . \tag{4.82}$$

The $(B - B_0)$ term can be written

$$\begin{aligned} B - B_0 &= \int_0^{k_c} dk \, 2k^3 \left(|\beta_k|^2 - |\beta_k(\eta_0)|^2 \right) \\ &\leq \int_0^{k_c} dk \, 2k^3 \left(|\Delta\beta_k|^2 + 2 \, |\beta_k(\eta_0)| \, |\Delta\beta_k| \right) \, , \end{aligned} \tag{4.83}$$

where $\Delta\beta_k = \beta_k - \beta_k(\eta_0)$. From Eq. (4.47),

$$|\Delta\beta_k| < \int_{\eta_0}^{\eta} dx \, \frac{kma'}{2\omega^2} = \frac{1}{2} \int_{\eta_0}^{\eta} dx \, \partial_x \tan^{-1} \left(\frac{ma}{k} \right) < \frac{ma}{2k} \, , \tag{4.84}$$

which implies

$$\int_0^{k_c} dk\, 2k^3 \left|\Delta\beta_k\right|^2 < \frac{1}{4}k_c^2 m^2 a^2 \tag{4.85}$$

and

$$\int_0^{k_c} dk\; 4k^3 \left|\beta_k(\eta_0)\right| \left|\Delta\beta_k\right| < \frac{2}{3} m a k_c^3 \; , \tag{4.86}$$

so, using Eq. (4.62), one finds $(B - B_0)$ is dominated by B_0 if

$$a \ll B_0^{1/4} m^{-1} \; . \tag{4.87}$$

Similarly, using the normalization condition (4.48) to bound $\left|\beta_k\right|^2 \leq 1$, the next term in Eq. (4.81) is

$$\int_0^{k_c} dk\, 2k^2 \left(\omega - k\right) \left|\beta_k\right|^2 \leq \frac{1}{2} k_c^2 m^2 a^2 \; , \tag{4.88}$$

which is also less than B_0 if $a \ll B_0^{1/4} m^{-1}$. This condition is weaker than Eq. (4.80) and hence will be satisfied if the set of conditions under which the ultraviolet contribution is subdominant to B_0 is satisfied.

The last term in I_1 is

$$\begin{aligned} \int_0^{k_c} dk\, \frac{-k^4 m^2 a'^2}{4\omega^5} &= \frac{m^2 a'^2}{4}\left(\frac{k_c^3}{3\omega_c^3} + \frac{k_c}{\omega_c}\right) + \frac{1}{4} m^2 a'^2 \ln\left(\frac{ma}{\omega_c + k_c}\right) \\ &\approx \frac{1}{3} m^2 a'^2 + \frac{1}{4} m^2 a'^2 \ln\left(\frac{ma}{2k_c}\right) \; , \end{aligned} \tag{4.89}$$

where we have used $a \ll B_0^{1/4} m^{-1} < k_c$ to simplify $\omega_c^2 \equiv k_c^2 + m^2 a^2 \approx k_c^2$.

4.3.3 Friedmann equation

The energy density is given by the sum of the contribution from the infrared and ultraviolet regions. As shown in Secs. 4.3.2–4.3.2, provided one is looking early enough such that Eq. (4.87) and hence Eq. (4.80) are satisfied, the energy density is dominated by the contributions from B_0 and the logarithm in Eq. (4.89):

$$\rho_r \approxeq \frac{1}{\pi^2 a^4}\left[B_0 + \frac{1}{4}m^2 a'^2 \ln\left(\frac{ma}{2k_c}\right)\right] . \tag{4.90}$$

Given the constraint $m^2 a'^2 \ll m^2 a^4 \ll B_0$ from Eq. (4.79c), the logarithm term will be negligible unless the logarithm itself is large. If it is indeed small, then it can be seen that Eq. (4.90) is radiation dominated in general. Even if it is not negligible, we can still demonstrate that it is radiation dominated in some range using the Friedmann equation provided that spin-$\frac{1}{2}$ fields are the dominant contribution to the total energy-density.

Using the Friedmann equation

$$\left(\frac{a'}{a^2}\right)^2 = \frac{8\pi}{3}\rho_r \tag{4.91}$$

with the assumption that the contribution from spin-half fields dominates the energy density, one finds

$$a'^2 \approxeq \frac{8B_0}{3\pi}\left[1 - \frac{2}{3\pi}m^2 \ln\left(\frac{ma}{2k_c}\right)\right]^{-1} . \tag{4.92}$$

The renormalized energy density can be written using (4.92) as

$$\rho_r \approxeq \frac{B_0}{\pi^2 a^4}\left\{1 + \frac{2}{3\pi}m^2\left[\ln\left(\frac{2k_c}{ma_0}\right) - \ln\left(\frac{a}{a_0}\right)\right]\right\}^{-1} . \tag{4.93}$$

At this point, we will choose $a_0 \approx B_0^{1/4}$ to ensure we are past the Planck era, and given the condition (4.80) we have $a_1 \approx B_0^{1/4} m^{-1/3}$, so $m^2 \ln(a/a_0) < m^2 \ln(m^{-1/3}) \ll 1$ is irrelevant compared to the 1 term. Hence, in the range $B_0^{1/4} < a < B_0^{1/4} m^{-1/3}$, the energy density will indeed be radiation dominated.

Note that a' (4.92) is approximately constant in the range of interest, so a'' will be strongly suppressed. This allows one to relax some of the accumulated constraints, for example Eq. (4.79a), implying $\rho \propto a^4$ over a larger range of a. However, limits such as (4.87) will generally not relax, but this is not surprising because at $a = B_0^{1/4} m^{-1}$ one expects fermions to become nonrelativistic. Of course, realistic models would not only have fermion fields but also an inflaton field, which would look like a cosmological constant, but, until the mass term becomes important at $a \sim B_0^{1/4} m^{-1}$ or inflation takes over, things will still be radiation dominated.

4.4 Discussion and Conclusion

In this paper we have analyzed the early-Universe, preinflationary behavior of massless vector fields of spin-1 and massive or massless fermion fields of spin-$\frac{1}{2}$ in the semiclassical approximation. We showed for a range of conformal time after the Planck era that both types of fields have radiation-dominated behavior. Along with the same result obtained for scalar fields in [32], we have demonstrated that all matter fields which one might anticipate to play a role in the preinflationary era do in fact produce a radiation-dominated energy density that is typically assumed in inflationary models.

In Sec. 4.2, we summarized the adiabatic renormalization procedure for a massless vector field in a spatially flat FLRW universe, following the groundwork laid out in [71]. We then used this procedure to renormalize the energy density contribution for such a field following a parametrization of the mode functions in terms of adiabatic states. We found the renormalized energy density to have a radiation-dominated form similar to that for the scalar field in [32].

In Sec. 4.3, we summarized a modified version of the adiabatic renormalization procedure for a massive or massless fermion field, using the modified WKB ansatz given in [68]. We found that parametrizing the mode functions in terms of adiabatic states required higher than zeroth order contributions in order to properly renormalize the energy density at high energies. We then made use of the leading second adiabatic-order contributions to the parametrization coefficients to obtain the leading order behavior of such high-energy contributions and used this to show the high-energy contributions are in fact subdominant to the radiationlike term in the remaining energy density contributions given a set of constraints on the scale factor. These constraints are more stringent than those required for the scalar field [32], for which only a constraint on $(a^2)''$ was necessary. We then found the next-to-leading order behavior of the energy density to be a logarithmic contribution and showed that it is subdominant compared to the radiation-dominated energy density for a range of conformal time beyond the Planck era, where the constraint on the scale factor coming from the analysis of the high-energy contribution served as the upper bound on the range.

Our analysis of the fermion field assumed that fermionic matter was the dominant contribution to the energy density in order to make use of the Friedmann equation to obtain the approximate leading order behavior. This result for fermions is different than that for scalars [32] and vectors, which work in any case. Our argument for fermions also only applies for the range $a_0 < a < a_1$ described in Sec. 4.3, though we expect this range to be in the preinflationary era prior to the fields becoming matter dominated. This result is weaker than those for the other two fields, but in application it is not. One would not be able to observe these matter fields during this era directly but rather through their effects on other phenomena, such as in the cosmic microwave background, so the results are not necessarily weaker in application.

Having demonstrated in the semiclassical approximation that the matter content in the post-Planck, preinflationary era will indeed be radiation-dominated, one can proceed with the procedure for the inflaton field in [32]. There, it was assumed that spin-$\frac{1}{2}$ and spin-1 massive fields could be modeled with conformally coupled massive scalar fields, which were argued to be radiation-dominated themselves in the preinflationary era. Our results that spin-$\frac{1}{2}$ fermion fields and massless spin-1 vector fields are radiation-dominated, themselves, supports the procedure in [32], and the analysis for obtaining a renormalized energy density for a universe with a mixture of cosmological constant and classical radiation, and hence for obtaining the power spectrum of the cosmic microwave background, is identical.

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