# ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF TOEPLITZ INTEGRAL OPERATORS ASSOCIATED WITH THE HANKEL TRANSFORM

By

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# Abstract

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ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF TOEPLITZ INTEGRAL OPERATORS ASSOCIATED WITH THE HANKEL TRANSFORM

Thesis under the direction of John V. Baxley, Ph.D., Professor of Mathematics.

Let $\nu$ be a positive constant and let

$$J(x) = x^{1/2-\nu} J_{\nu-1/2}(x), 0 \le x < \infty$$

where

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{\alpha+2k}$$

is the usual Bessel function of order $\alpha$. For a given bounded real function $F$ belonging to $L^1(0,\infty)$, we define for $u, v > 0$

$$\rho(u,v) = \int_0^\infty F(t) J(ut) J(vt) t^{2\nu} dt.$$

For $h \in L^2(0, A; x^{2\nu})$,

$$(T_A h)(x) = \int_0^A h(x)\rho(x,y) y^{2\nu} dy, \quad 0 < x \le A$$

is the Toeplitz integral operator.

We are interested in formulating conditions on $F(t)$ which allow the determination of the asymptotic behavior of the eigenvalues of $T_A$ as $A \to \infty$.

# Chapter 1: Introduction

Let $\nu$ be a positive constant and let

$$J(x) = x^{1/2-\nu} J_{\nu-1/2}(x), \quad 0 \le x < \infty$$

where

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{\alpha+2k}$$

is the usual Bessel function of order $\alpha$. Note that for $\nu < \frac{1}{2}$, $J_{\nu-1/2}(x)$ has a discontinuity at $x = 0$ and that the discontinuity of $J(x)$ at $x = 0$ is removable. We will assume $J(x)$ is its continuous extension in this case. For a given bounded real function $F$ belonging to $L^1(0,\infty)$, we define for $u, v > 0$

$$\rho(u,v) = \int_0^\infty F(t)J(ut)J(vt)t^{2\nu}\, dt.$$

Since $x^{1/2}J_\alpha(x)$ is known to be bounded as $x \to \infty$, then the integrand is bounded by a constant multiple of $|F(t)|$, so this integral exists and is bounded. For $A \ge 1$ and $h \in L^2(0, A; x^{2\nu})$, then

$$(T_A h)(x) = \int_0^A h(x)\rho(x,y)y^{2\nu}\, dy, \quad 0 < x \le A \tag{1.1}$$

is the Toeplitz integral operator. The Hankel transform does not have a standard definition; we will define it on $L^1(0,\infty; x^{2\nu})$ by

$$\hat{f}(x) = \int_0^\infty f(t)J(xt)t^{2\nu}\, dt$$

which differs only slightly from other sources, including [9], [15], [18]. In particular, our definition differs from [9] by a multiplicative constant.

We are interested in formulating conditions on $F(t)$ which allow the determination of the asymptotic behavior of the eigenvalues of $T_A$ as $A \to \infty$.

This problem was previously considered by J.R. Davis in [4] and [5]. Although our definition of $T_A$ differs from that of Davis, so does our definition of $\rho(x,y)$ and $J(x)$, and in fact our $T_A$ is the same as his. Using complicated methods in complex analysis, he found in [4] that for $F(t)$ satisfying the following three conditions:

C1: $F(t)$ is a bounded, continuous, real function defined on $[0,\infty)$ such that $t^2F(t)$ is bounded as $t \to \infty$

C2: $F(0) = M$ and $F(t) < M$ for $t > 0$

C3: $F(t)$ is twice continuously differentiable in a one sided neighborhood of 0, $F'(0) = 0$, and $F''(0) = -2\sigma^2$ for some constant $\sigma$

then the $k^{th}$ largest eigenvalue $\lambda_{k,A}$ of the associated Toeplitz integral operator satisfies

$$\lim_{A\to\infty} A^2(M - \lambda_{k,A}) = \sigma^2 z_k^2, \tag{1.2}$$

where $0 < z_1 < z_2 < \cdots$ are the positive zeros of the Bessel function $J_{\nu-1/2}(z)$. In his later paper [5], he extended his results for functions satisfying these more general conditions:

C4: $F(t)$ is a bounded, real function in $L^1(0, \infty; t^{2\nu})$

C5: $F(0) = M$ is the unique maximum and $\limsup_{t\to\infty} F(t) < M$

C6: $\displaystyle\lim_{t\to 0^+} \frac{F(0) - F(t)}{t^\omega} = \sigma^2$ for some constant $\sigma$.

In this case he was able to find an abstractly defined operator, with positive eigenvalues $0 < \mu_1 \le \mu_2 \le \cdots$ to describe the $k^{th}$ largest eigenvalue of the Toeplitz integral operator:

$$\lim_{A\to\infty} A^\omega(F(0) - \lambda_{k,A}) = \sigma^2 \mu_k. \tag{1.3}$$

In this thesis we reproduce the result in [4] using substantially different methods. We shall prove Equation (1.2) under the following hypotheses. For $F_S(t) = (1+t^2)^{-1}$,

H1: $F(t)$ is bounded and absolutely integrable on $(0, \infty)$

H2: $F(0) - F(t) \ge q^2(1 - F_S(t))$ for $t > 0$, for some constant $q > 0$

H3: $\displaystyle\lim_{t\to 0^+} \frac{F(0) - F(t)}{t^2} = \sigma^2$ for some constant $\sigma$.

It is easy to see that C1 implies H1, C3 implies H3, and if $F(0) = M > 0$, then C1 and C2 imply H2. Note that C4 and C5 imply $F(0) = M > 0$. We suspect that the proof in [4] also requires this hypothesis and that its omission was an oversight. Except for this minor point, our result can be viewed as a generalization of Davis' result in [4]. Comparing our result to the $\omega = 2$ case of [5], we draw attention to two points. First, note that he requires $F(t) \in L^1(0, \infty; t^{2\nu})$; our assumption is less restrictive when $\nu > 0$. Second, the operator with eigenvalues $\mu_k$ is abstractly defined, and it is not clear that the eigenvalues of that operator are related to the zeros of the Bessel function $J_{\nu-1/2}(z)$.

We also obtain partial results related to Equation (1.3) in the case that $\omega$ is an even integer. Specifically we shall show

$$\limsup_{A\to\infty} A^{2n}(F(0)-\lambda_{k,A}) \leq \sigma^2\Lambda_k,$$

where $\Lambda_k$ are positive eigenvalues of a specific differential operator which we will define later.

We follow closely the methods used in [17], which proved similar results for Toeplitz operators associated with the Fourier transform. These methods were previously used on a discrete form of the problem in the case of Fourier series by Parter in [12] and [13] and later by Baxley in [3], in a case involving orthogonal polynomials.

# Chapter 2: Notation and Mathematical Background

We first establish notation we will need throughout this thesis. We will be using three particular Hilbert spaces, namely $L^2(0,1;x^{2\nu})$, $L^2(0,A;x^{2\nu})$ for $A>0$, and $L^2(0,\infty;x^{2\nu})$. We will use $(\cdot\ ,\cdot)$ and $\|\cdot\|$ to denote the inner product and norm of $L^2(0,1;x^{2\nu})$, $(\cdot\ ,\cdot)_A$ and $\|\cdot\|_A$ to denote the inner product and norm of $L^2(0,A;x^{2\nu})$, and $\langle\cdot\ ,\cdot\rangle$ to denote the inner product of $L^2(0,\infty;x^{2\nu})$.

As usual, $C^\infty(a,b)$ will denote the set of infinitely differentiable functions on the interval $(a,b)$ and $C_0^\infty(a,b)$ will denote those functions of $C^\infty(a,b)$ having compact support in $(a,b)$.

We now collect statements of tools we will need later. We will make use of Fubini's Theorem in the following form.

**Theorem 2.1.** *Suppose* $f(x,y)$ *is measurable and* $\int_0^\infty \left[\int_0^\infty |f(x,y)|\,dx\right]\,dy$ *is  nite. Then*

$$\int_0^\infty \left[\int_0^\infty f(x,y)\,dx\right]\,dy = \int_0^\infty \left[\int_0^\infty f(x,y)\,dy\right]\,dx.$$

We will use differentiation under an integral in the particular way. We provide the justification here.

**Lemma 2.2.** *Suppose* $f(x,t)$ *is continuous and bounded and* $f_x(x,t)$ *exists and is bounded on* $(0,1)\times(0,\infty)$ *and* $g$ *is integrable on* $(0,\infty)$*. Then for*

$$H(x) = \int_0^\infty g(t)f(x,t)\,dt,$$

$H(x)$ *is di  erentiable on* $(0,1)$ *and*

$$H'(x) = \int_0^\infty g(t)f_x(x,t)\,dt$$

*for all* $x\in(0,1)$*.*

Proof: By definition,

$$H'(x) = \lim_{h\to 0}\frac{H(x+h)-H(x)}{h},$$

if this limit exists. It suffices to show that the limit exists and has the asserted value for any sequence $\{h_n\}$ with $x+h_n\in(0,1)$ and $h_n\to 0$ as $n\to\infty$. So suppose $\{h_n\}$ is such a sequence. Then

$$\lim_{n\to\infty}\frac{H(x+h_n)-H(x)}{h_n} = \lim_{n\to\infty}\int_0^\infty g(t)\left(\frac{f(x+h_n,t)-f(x,t)}{h_n}\right)\,dt.$$

Since $f_x$ exists and is bounded, by the mean value theorem, there exists $x_n$ between $x$ and $x + h_n$ such that

$$|g(t)| \left| \frac{f(x+h_n,t) - f(x,t)}{h_n} \right| = |g(t) f_x(x_n,t)| \leq K|g(t)|,$$

for some constant $K > 0$. Since $g$ is integrable and $f_x$ is continuous, the Lebesgue Dominated Convergence Theorem gives

$$\lim_{n\to\infty} \frac{1}{h_n} \left( \int_0^\infty g(t) f(x+h_n,t)dx - \int_0^\infty g(t) f(x,t)dt \right) = \int_0^\infty g(t) f_x(x,t)\, dt. \ \square$$

We will need the following approximation theorem from [1].

**Theorem 2.3.** *$C_0^\infty(0,1)$ is a dense subset of $L^2(0,1;x^{2\nu})$.*

We will also need the Ascoli-Arzela Theorem.

**Theorem 2.4.** *Given a sequence of functions uniformly bounded and equicontinuous on a closed interval, there exists a subsequence which converges uniformly on that interval.*

It is easy to show that the hypotheses of the following corollary imply the hypotheses of the theorem just stated.

**Corollary 2.5.** *Given a sequence of functions equicontinuous on a closed interval and bounded at one point in the interval, there exists a subsequence which converges uniformly on that interval.*

Elementary properties of Bessel functions will also be needed. In particular we will use the following identities, which can be found in [10] and [16].

**Lemma 2.6.**

$$\frac{d}{dx}[x^\alpha J_\alpha(x)] = x^\alpha J_{\alpha-1}(x)$$

$$\frac{d}{dx}[x^{-\alpha} J_\alpha(x)] = -x^{-\alpha} J_{\alpha+1}(x)$$

$$\frac{d}{dx}[x^{-\alpha} K_\alpha(x)] = -x^{-\alpha} K_{\alpha+1}(x)$$

The following integral can be found in [11], pg. 336 and is a consequence of a more general result proved in [16] pg. 429.

**Lemma 2.7.**

$$\int_0^\infty \frac{tJ_\nu(at)J_\nu(bt)}{z^2+t^2}\, dt = I_\nu(bz)K_\nu(az), \quad R(z) > 0,\ a \geq b > 0,\ R(\nu) > -1$$

We will need information on the asymptotic behavior of Bessel functions. The following lemma comes from [16], pg. 199-202.

**Lemma 2.8.** *$\sqrt{x}J_\alpha(x)$ is bounded as $x \to \infty$, and $\lim_{x\to\infty} e^x\sqrt{x}K_\alpha(x) = \sqrt{\frac{\pi}{2}}$.*

We will use the following two facts, first proved in [9]. The first lemma is a fairly simple application of the Bessel identities from Lemma 2.6.

**Lemma 2.9.** *Let $\tau f(x) = -\frac{1}{x^2}(x^{2\nu}f'(x))'$. Then $\tau J(xt) = t^2 J(xt)$.*

To avoid confusion, we will sometimes write $\tau_x$ rather than $\tau$ to indicate that derivatives are with respect to $x$. The next lemma is the inversion theorem for the Hankel transform from [9].

**Lemma 2.10.** *If $f, \hat{f} \in L^1(0,\infty; x^{2\nu})$, then $f$ may be rede ned on a set of measure zero so that it is continuous on $(0,\infty)$ and then*

$$f(x) = \int_0^\infty \hat{f}(t)J(xt)t^{2\nu}\,dt.$$

Next is the Parseval Theorem for Hankel transforms. The proof is an application of Fubini's Theorem and the inversion theorem and can be found in [18]. However, some translation is required due to the differences in the definition of the Hankel transform.

**Theorem 2.11.** *If $f, \hat{g} \in L^1(0,\infty; x^{2\nu})$, then*

$$\int_0^\infty f(x)\overline{g(x)}x^{2\nu}\,dx = \int_0^\infty \hat{f}(x)\overline{\hat{g}(x)}x^{2\nu}\,dx.$$

We will also need several results from functional analysis, in particular Hilbert space operator theory, which can be found in many sources, e.g. [2]. A Hilbert-Schmidt kernel of an integral operator is one which is square-integrable.

**Lemma 2.12.** *An integral operator with a Hilbert-Schmidt kernel is a compact operator.*

**Lemma 2.13.** *Let $K$ be a compact linear operator in a Hilbert space $H$. Then*

*(i) only the point zero can be a limit point of the eigenvalues of $K$,*

*(ii) each nonzero eigenvalue of $K$ has nite multiplicity,*

*(iii) there exist only countably many linearly independent eigenvectors belonging to nonzero eigenvalues of $K$.*

Recall that the eigenvalues of a self-adjoint operator are real.

**Theorem 2.14.** *Each nonzero compact self-adjoint operator $T$ has an orthonormal sequence of eigenvectors $e_1, e_2, \ldots$ belonging to its nonzero eigenvalues $\lambda_1, \lambda_2, \ldots$ ($|\lambda_1| \geq |\lambda_2| \geq \cdots$) and these eigenvectors span the range of $T$.*

Since unbounded operators will also appear in our work, we need some basic facts about their adjoints. Suppose $T$ is a linear operator defined on a dense vector subspace of a Hilbert space $H$. If for some $y \in H$, there exists a vector $w \in H$ for which $(Tx, y) = (x, w)$ for all $x$ in the domain of $T$, then we define the adjoint operator $T^*$ by $T^*y = w$ and place $y$ in the domain of $T^*$. The denseness of the domain of $T$ ensures that $w$ is unique. Clearly the zero vector is in the domain of $T^*$ and $T^*(0) = 0$. In an extreme case, this may be the only vector in the domain of $T^*$. We state for reference:

**Lemma 2.15.** *An operator has an adjoint if its domain includes a dense subset, although the adjoint may not have a dense domain.*

A symmetric operator $T$ is one for which the inner product $(Tx, y) = (x, Ty)$ for all $x$, $y$ in the domain of $T$. The definition of the adjoint operator implies that for such a $T$, the adjoint operator $T^*$ is an extension of $T$. An unbounded self-adjoint operator is a symmetric operator whose adjoint's domain is no larger than the domain of $T$.

A critical role will be played by the Friedrichs extension of an unbounded symmetric operator. The following statement concerning the Friedrichs extension is found in [6], pg. 1240-1242.

**Theorem 2.16.** *If an operator $T$, with a dense domain, is symmetric and semi-bounded, then there exists a particular self-adjoint extension which preserves the lower bound, called the Friedrichs extension and denoted $\tilde{T}$. If $g \in D(\tilde{T})$, there exists a sequence $\{g_n\} \subset D(T)$ with $\|g_n - g\| \to 0$ and $(Tg_n, g_n) \to (\tilde{T}g, g)$ as $n \to \infty$.*

The following smoothness result will be necessary. It is proved in [6], pg. 1291-1294. Let $\tau$ be defined as in Lemma 2.9.

**Lemma 2.17.** *If $\lambda$ is a real constant, $f \in L^2(0,1;x^{2\nu})$ and $(f, \tau^k g - \lambda g) = 0$ for every $g \in C_0^\infty(0,1)$, then after modi cation on a set of measure zero, $f \in C^\infty(0,1)$ and $\tau^k f = \lambda f$.*

We will also need a result about Rayleigh quotients which guarantees the existence of positive eigenvalues for compact self-adjoint operators and provides a "maximin" characterization of these eigenvalues. Since we could find a precise statement of this result in the literature only for the finite dimensional case concerning symmetric matrices, we also provide a proof. The result easily found in the literature for compact self-adjoint operators is a complementary "minimax" statement.

**Theorem 2.18.** *Suppose $T$ is a compact self-adjoint operator in a Hilbert space $H$, and let $n$ be a positive integer. If there exists a subspace $M$ of dimension $n$ for which*

$$\inf_{f\in M}\left\{\frac{(Tf,f)}{(f,f)}\right\} > 0,$$

*then $T$ has a least $n$ positive eigenvalues, counting multiplicities. Moreover, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ are the largest $n$ eigenvalues of $T$ arranged in nondecreasing order, with repetitions for multiple eigenvalues, then*

$$\lambda_n = \sup_M \left[\inf_{f\in M}\left\{\frac{Tf,f)}{(f,f)}\right\}\right],$$

*where $M$ ranges over all $n$ dimensional subspaces of $H$.*

Proof: We prove the contrapositive of the first assertion. If $T$ does not have at least $n$ positive eigenvalues, then by the Riesz theorem, we can decompose $H$ into three mutually orthogonal subspaces $H_1$, $H_2$, $H_3$ with $H_1$ being the span of the eigenvectors of $T$ corresponding to positive eigenvalues, $H_3$ being the span of the eigenvectors of $T$ corresponding to the negative eigenvalues, and $H_2$ the null space of $T$. Then by our assumption, the dimension of $H_1$ is less than $n$. Let $u_1, u_2, \cdots, u_n$ be an orthonormal basis for the subspace $M$ of our hypothesis. By the projection theorem, each $u_k$ may be decomposed as a sum $u_k = v_k + w_k$, with $v_k \in H_1$, $w_k$ orthogonal to $H_1$. Then $v_1, v_2, \cdots, v_n$ are $n$ vectors in a subspace $H_1$ of dimension less than $n$. Hence there is a nontrivial linear combination

$$\sum_{k=1}^{n} c_k v_k = 0.$$

Since $u_1, u_2, \cdots, u_n$ are linearly independent, then $u$ defined as

$$u = \sum_{k=1}^{n} c_k u_k = \sum_{k=1}^{n} c_k v_k + \sum_{k=1}^{n} c_k w_k = \sum_{k=1}^{n} c_k w_k$$

is a nonzero vector in $M$ orthogonal to $H_1$. Thus $u \in H_2 + H_3$ and so $\frac{(Tu,u)}{(u,u)} \leq 0$.

To prove the second assertion, let $u_k$ be a normalized eigenvector of $T$ corresponding to $\lambda_k$. Choosing $M_1$ as the $n$ dimensional subspace of $H$ spanned by $u_1, u_2, \cdots, u_n$, a straightforward calculation shows that $\frac{(Tf,f)}{(f,f)} \geq \lambda_n$, for any $f \in M_1$. Also, $\frac{(Tu_n,u_n)}{(u_n,u_n)} = \lambda_n$. Thus, for $M_1$, the relevant infimum equals $\lambda_n$. To complete the proof, we need only show that for any $n$ dimensional $M$, the relevant infimum is less than or equal $\lambda_n$. Let $S$ be the span of $u_1, u_2, \cdots, u_{n-1}$. We can choose an arbitrary orthonormal basis $v_1, v_2, \cdots, v_n$ of $M$ and use the projection theorem to decompose each $v_k = w_k + x_k$ with $w_k \in S$ and $x_k$ orthogonal to $S$. Then $w_1, w_2, \cdots, w_n$ all belong to a subspace of dimension less than $n$ and some nontrivial linear combination

$$\sum_{k=1}^{n} c_k w_k = 0.$$

Thus $v$ defined as

$$v = \sum_{k=1}^{n} c_k v_k = \sum_{k=1}^{n} c_k w_k + \sum_{k=1}^{n} c_k x_k = \sum_{k=1}^{n} c_k x_k$$

is a nonzero vector in $M$ orthogonal to $S$. We can decompose $v = y_1 + y_2 + y_3$ with $y_1 \in H_1$, $y_2 \in H_2$, $y_3 \in H_3$, and $y_1$ orthogonal to $S$. Thus we have $v \in M$ so that

$$\frac{(Tv, v)}{(v, v)} = \frac{(Ty_1, y_1) + (Ty_2, y_2) + (Ty_3, y_3)}{(v, v)} \leq \frac{(Ty_1, y_1)}{(v, v)} \leq \frac{\lambda_n (y_1, y_1)}{(v, v)} \leq \lambda_n. \ \square$$

# Chapter 3: Preliminary Lemmas

**Lemma 3.1.** *$x^\nu J(x)$, $J(x)$, and $J'(x)$ are all bounded for $x \geq 0$.*

Proof: From the definition of $J(x)$ and the series expansion of $J_{\nu-1/2}(x)$, we have

$$x^\nu J(x) = \sqrt{x} J_{\nu-1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{2}}{\Gamma(k+1)\Gamma(k+\nu+1/2)} \left(\frac{x}{2}\right)^{\nu+2k}$$

so $x^\nu J(x)$ is a continuous function for $x \geq 0$. From Lemma 2.8, $J_\nu(x) \leq Mx^{-1/2}$ for large $x$ and some constant $M$, independent of $\nu$. Thus $x^\nu J(x) = \sqrt{x} J_{\nu-1/2}(x)$ is bounded as $x \to \infty$. Therefore by continuity, $x^\nu J(x)$ is bounded for all $x \geq 0$. Since $J(0) = \frac{1}{\Gamma(\nu+1/2)}$, $J(x)$ is also bounded for $x \geq 0$. Using the Bessel identity from Lemma 2.6, we have $J'(x) = -x^{1/2-\nu} J_{\nu+1/2}(x)$. Since the asymptotic behavior of $J_\nu(x)$ is independent of $\nu$, $J'(x)$ behaves similarly to $J(x)$ as $x \to \infty$ (in absolute value). Since $J'(0) = 0$, by continuity $J'(x)$ is bounded for all $x \geq 0$. □

**Lemma 3.2.** *For $F \in L^1(0,\infty)$, the integral de ning $\rho(u,v)$ converges for all $u, v > 0$ and $\rho(u,v)$ is a continuous function of $(u,v)$. Also, $(uv)^\nu |\rho(u,v)|$ is bounded.*

Proof: From Lemma 3.1, for $u, v > 0$,

$$\begin{aligned} \left| \int_0^\infty F(t) J(ut) J(vt) t^{2\nu} dt \right| &\leq \int_0^\infty \left| F(t)(ut)^\nu J(ut)(vt)^\nu J(vt)(uv)^{-\nu} \right| dt \\ &\leq C(uv)^{-\nu} \int_0^\infty |F(t)| dt \end{aligned}$$

for some constant $C$. Since $F(t) \in L^1(0,\infty)$, the integral defining $\rho(u,v)$ converges for all $u, v > 0$. Note that this implies $(uv)^\nu |\rho(u,v)|$ is bounded.

For $u, v > 0$ , let $(u_n, v_n) \to (u, v)$. Then

$$\lim_{n\to\infty} \rho(u_n, v_n) = \int_0^\infty \lim_{n\to\infty} F(t) J(u_n t) J(v_n t) t^{2\nu} dt$$

by the generalized Lebesgue Dominated Convergence Theorem since the absolute value of the integrand is dominated by the integrable function $C|F(t)|(u_n v_n)^{-\nu}$ from above. Then since $J(x)$ is a continuous function, $J(u_n t) \to J(ut)$ and $J(v_n t) \to J(vt)$ and we have $\rho(u_n, v_n) \to \rho(u,v)$, so $\rho(u,v)$ is a continuous function for $u, v > 0$. □

Recall the definition of the differential operator $\tau$ as in Lemma 2.9: $\tau f(x) = -\frac{1}{x^2}\left(x^{2\nu} f'\right)'$. We now define

$$\mathcal{G} = \{f(x) \in C^\infty(0,\infty) :\ x^{2\nu}(\tau^k f)' \to 0 \text{ as } x \to 0^+, \tau^k f \text{ is bounded on } (0,\infty),$$

$$\text{and } (\tau^k f)' \text{ and } \tau^k f \text{ are rapidly decreasing on } (0,\infty) \text{ for all } k \geq 0\}.$$

**Lemma 3.3.** *If* $u \in \mathcal{G} \subset L^2(0,\infty; x^{2\nu})$*, the following conditions hold:*

*(i)* $\tau u \in \mathcal{G}$

*(ii) For* $i + j = k + l$, $\langle \tau^i u, \tau^j u\rangle = \langle \tau^k u, \tau^l u\rangle$

*(iii) For* $0 < x_1 < x_2 < \infty$, $|u(x_2) - u(x_1)|^2 \leq \langle \tau u, u\rangle \int_{x_1}^{x_2} \frac{1}{t^2}\, dt$

*(iv) For* $0 < x_1 < x_2 < \infty$, $\left|{x_2}^{2\nu} u'(x_2) - {x_1}^{2\nu} u'(x_1)\right|^2 \leq \langle \tau^2 u, u\rangle \int_{x_1}^{x_2} t^{2\nu}\, dt$

Proof: Statement *(i)* is true because $u \in \mathcal{G}$ implies that four conditions hold for $k \geq 0$, and then $\tau u \in \mathcal{G}$ only requires that those same conditions hold for $k \geq 1$.

For Statement *(ii)*, it suffices to show that $\langle \tau^i u, \tau^j u\rangle = \langle \tau^{i+j} u, u\rangle$ for all $j \geq 1$.

$$\langle \tau^i u, \tau^j u\rangle = \int_0^\infty \tau^i u\ \overline{\tau^j u} x^{2\nu}\, dx$$

$$= -\int_0^\infty \tau^i u\ \overline{(x^{2\nu}(\tau^{j-1}u)')'}\, dx$$

$$= -(\tau^i u) x^{2\nu} (\tau^{j-1}u)'\Big|_0^\infty + \int_0^\infty x^{2\nu} (\tau^i u)' \overline{(\tau^{j-1}u)'}\, dx$$

Here, the limit term vanishes at infinity because both $\tau^i u$ and $(\tau^{j-1}u)'$ are rapidly decreasing, and it vanishes at 0 because $\tau^i u$ is bounded and $x^{2\nu}(\tau^{j-1}u)' \to 0$. Integrating by parts again,

$$\int_0^\infty x^{2\nu}(\tau^i u)'\overline{(\tau^{j-1}u)'}\, dx = x^{2\nu}(\tau^i u)' \tau^{j-1} u\Big|_0^\infty - \int_0^\infty \frac{1}{x^{2\nu}} (x^{2\nu}(\tau^i u)')' \overline{\tau^{j-1} u} x^{2\nu}\, dx$$

and the limit term vanishes at infinity because both $(\tau^i u)'$ and $\tau^{j-1}u$ are rapidly decreasing, and it vanishes at 0 because $\tau^{j-1}u$ is bounded and $x^{2\nu}(\tau^{j-1}u)' \to 0$. The remaining integral is the inner product of $\tau^{i+1}u$ and $\tau^{j-1}u$, so we have $\langle \tau^i u, \tau^j u\rangle = \langle \tau^{i+1} u, \tau^{j-1} u\rangle$. We repeat this process $j-1$ more times to conclude $\langle \tau^i u, \tau^j u\rangle = \langle \tau^{i+j} u, u\rangle$.

We prove Statement *(iii)* with the fundamental theorem of calculus, Schwarz inequality, and integration by parts. For $0 < x_1 < x_2 < \infty$,

$$|u(x_2) - u(x_1)|^2 = \left|\int_{x_1}^{x_2} \frac{1}{t^\nu} t^\nu u'(t)\, dt\right|^2$$

$$\leq \int_0^\infty t^{2\nu}|u'(t)|^2\,dt\ \int_{x_1}^{x_2}\frac{1}{t^{2\nu}}\,dt$$

$$= \left[t^{2\nu}u'\overline{u}\Big|_0^\infty - \int_0^\infty \frac{1}{t^{2\nu}}(t^{2\nu}u')'\,\overline{u}\,t^{2\nu}\,dt\right]\int_{x_1}^{x_2}\frac{1}{t^{2\nu}}\,dt$$

The limit term vanishes at infinity because $t^{2\nu}u'$ and $u$ are both rapidly decreasing, and it vanishes at 0 because $u$ is bounded and $t^{2\nu}u' \to 0$. Rewriting the remaining integral as an inner product, we have

$$|u(x_2)-u(x_1)|^2 \leq \langle \tau u, u\rangle \int_{x_1}^{x_2}\frac{1}{t^{2\nu}}\,dt.$$

We prove Statement *(iv)* in a similar manner. For $0 < x_1 < x_2 < \infty$,

$$\left|{x_2}^{2\nu}u'(x_2) - {x_1}^{2\nu}u'(x_1)\right|^2 = \left|\int_{x_1}^{x_2}\frac{1}{t^{2\nu}}(t^{2\nu}u'(t))'t^{2\nu}\,dt\right|^2$$

$$\leq \int_0^\infty \left|\frac{1}{t^{2\nu}}(t^{2\nu}u'(t))'\right|^2 t^{2\nu}\,dt\ \int_{x_1}^{x_2} t^{2\nu}\,dt$$

$$= \langle \tau u, \tau u\rangle \int_{x_1}^{x_2} t^{2\nu}\,dt$$

And from Statement *(ii)*, $\langle \tau u, \tau u\rangle = \langle \tau^2 u, u\rangle$, yielding

$$\left|{x_2}^{2\nu}u'(x_2) - {x_1}^{2\nu}u'(x_1)\right|^2 \leq \langle \tau^2 u, u\rangle \int_{x_1}^{x_2} t^{2\nu}\,dt. \ \square$$

**Lemma 3.4.** *If* $f \in \mathcal{G}$*, then* $x^{2k}\hat{f}(x) = \widehat{\tau^k f}(x)$ *for* $k = 1, 2, \ldots$ *and* $\hat{f}$ *is rapidly decreasing on* $(0,\infty)$*.*

Proof: Given the formula for the Hankel transform,

$$\hat{f}(x) = \int_0^\infty f(t)J(xt)t^{2\nu}dt,$$

we multiply the equation by $x^2$, apply Lemma 2.9, and integrate by parts:

$$x^2\hat{f}(x) = \int_0^\infty f(t)x^2J(xt)t^{2\nu}dt = \int_0^\infty f(t)\tau_t J(xt)t^{2\nu}dt = -\int_0^\infty f(t)(t^{2\nu}J'(xt))'dt$$

$$= -f(t)t^{2\nu}J'(xt)\Big|_0^\infty + \int_0^\infty f'(t)J'(xt)t^{2\nu}dt = \int_0^\infty t^{2\nu}f'(t)J'(xt)dt.$$

Since $J'(xt)$ is bounded on $(0,\infty)$ from Lemma 3.1 and $f(t) \in \mathcal{G}$, the boundary term vanishes at infinity because $f$ is rapidly decreasing, and the boundary term vanishes at 0 because $f$ is bounded. Integrating by parts again we have

$$\int_0^\infty t^{2\nu}f'(t)J'(xt)dt = t^{2\nu}f'(t)J(xt)\Big|_0^\infty - \int_0^\infty (t^{2\nu}f'(t))'J(xt)dt = \int_0^\infty \tau_t f(t)J(xt)t^{2\nu}dt$$

Here the boundary term vanishes because $J(xt)$ is bounded from Lemma 3.1 and since $f(t) \in \mathcal{G},\ f'$ is rapidly decreasing and $t^{2\nu} f'(t) \rightarrow$

**Lemma 3.6.** $T_A$ *is a self-adjoint operator.*

Proof: Let $f_1, f_2 \in L^2(0, A; x^{2\nu})$. Note that for $f \in L^2(0, A; x^{2\nu})$, by the Schwarz inequality, $\int_0^A |f(x)x^\nu| dx = (|f|, x^{-\nu})_A \leq \|f\|_A \cdot \|x^{-\nu}\|_A = \|f\|_A \sqrt{A}$. This implies, together with Lemma 3.2,

$$\begin{aligned} \int_0^A \int_0^A |\rho(x,y) f_1(y) f_2(x) y^{2\nu} x^{2\nu}| dy dx &\leq C \int_0^A \int_0^A |f_1(y) f_2(x) y^\nu x^\nu| dy dx \\ &= C \int_0^A |f_1(y) y^\nu| dy \int_0^A |f_2(x) x^\nu| dx \\ &\leq CA \|f_1\|_A \|f_2\|_A \end{aligned}$$

for some constant $C$. Since the integrand is also measurable we may apply Theorem 2.1 in the following way:

$$\begin{aligned} (T_A f_1, f_2)_A &= \int_0^A \left( \int_0^A \rho(x,y) f_1(y) y^{2\nu} dy \right) \overline{f_2(x)} x^{2\nu} dx \\ &= \int_0^A f_1(y) \left( \int_0^A \rho(x,y) \overline{f_2(x)} x^{2\nu} dx \right) y^{2\nu} dy \\ &= \int_0^A f_1(y) \left( \int_0^A \overline{\rho(x,y) f_2(x)} x^{2\nu} dx \right) y^{2\nu} dy \\ &= \int_0^A f_1(y) \overline{T_A f_2(y)} y^{2\nu} dy \\ &= (f_1, T_A f_2)_A. \end{aligned}$$

Now since $D(T_A) = L^2(0, A; x^{2\nu}) = D(T_A{}^*)$, $T_A$ is self-adjoint. $\square$

Since we would prefer to study our operators on a fixed interval of integration we replace $x$ with $Ax$ and $y$ with $Ay$ in Equation (1.1) to obtain

$$(T_A h)(Ax) = A^{2\nu+1} \int_0^1 h(Ay) \rho(Ax, Ay)\, y^{2\nu}\, dy$$

for $0 < x \leq 1$, $A \geq 1$. If we now let $f(y) = h(Ay)$ we may now view the operator $T_A$ defined for $f \in L_2(0, 1; x^{2\nu})$ by the equation

$$(T_A f)(x) = A^{2\nu+1} \int_0^1 f(y) \rho(Ax, Ay) y^{2\nu} dy. \tag{3.1}$$

The restriction of $T_A$ to

$$C_1^\infty(0,1) = \{v \in C^\infty(0,1) : v \text{ is bounded and } v(x) \equiv 0 \text{ for } x \text{ near } 1\}$$

has a useful alternative formula to Equation (3.1), which requires that we extend any $f \in C_1^\infty(0,1)$ to $(0, \infty)$ by defining $f(x) = 0$ for $x \geq 1$ (we will often assume, without comment, that such an extension has been made for functions in $C_1^\infty(0,1)$). For such $f$ we have

$$
\begin{aligned}
(T_A f)(x) &= A^{2\nu+1}\int_0^1 f(y)\rho(Ax,Ay)y^{2\nu}dy \\
&= A^{2\nu+1}\int_0^1\int_0^\infty f(y)F(t)J(Ayt)J(Axt)t^{2\nu}y^{2\nu}dt\,dy
\end{aligned}
$$

Now, $f(y)$ is bounded and $(Axt)^\nu J(Axt)(Ayt)^\nu J(Ayt)$ is bounded from Lemma 3.1. Since $A$ is fixed,

$$|f(y)F(t)J(Ayt)J(Axt)t^{2\nu}y^{2\nu}| \le M|F(t)|x^{-\nu}y^\nu$$

for some constant $M$. Since $y$ varies from 0 to 1 and $F(t)\in L^1(0,\infty)$, the integral converges on $(0,1)\times(0,\infty)$ for $x>0$. Measurability follows from continuity and thus we can apply Theorem 2.1:

$$
\begin{aligned}
(T_A f)(x) &= A^{2\nu+1}\int_0^1 f(y)\;\rho(Ax,Ay)\;y^{2\nu}dy \\
&= A^{2\nu+1}\int_0^\infty f(y)\;\rho(Ax,Ay)\;y^{2\nu}dy \\
&= A^{2\nu+1}\int_0^\infty f(y)\left[\int_0^\infty F(t)J(Axt)J(Ayt)t^{2\nu}dt\right]y^{2\nu}dy \\
&= A^{2\nu+1}\int_0^\infty F(t)\left[\int_0^\infty f(y)J(Ayt)y^{2\nu}dy\right]J(Axt)t^{2\nu}dt \\
&= A^{2\nu+1}\int_0^\infty F(t)\hat{f}(At)J(Axt)t^{2\nu}dt.
\end{aligned}
$$

With a change of variable, replacing $t$ with $t/A$, we have

$$(T_A f)(x) = \int_0^\infty F(t/A)\hat{f}(t)J(xt)t^{2\nu}dt. \tag{3.2}$$

Since $T_A$ is a compact self-adjoint operator, from Lemma 2.13 every nonzero element in the spectrum of $T_A$ is an eigenvalue of finite multiplicity. The next lemma gives further information about these eigenvalues.

**Lemma 3.7.** *Let* $m\le 0$ *and* $M>0$. *Suppose* $m\le F(t)\le M$ *for* $-\infty<t<\infty$. *Then every eigenvalue* $\lambda$ *of the operator* $T_A$ *satis es* $m\le\lambda\le M$.

Proof: Let $\lambda$ be an eigenvalue of $T_A$, and $f$ a corresponding normalized eigenfunction. Let $0<\epsilon<1$. By Lemma 2.3, there exists an $h\in C_0^\infty(0,1)$ such that $\|f-h\|<\epsilon$

and $\|h\| = 1$. Using the boundedness property of $T_A$ and the Schwarz inequality, we have

$$\begin{aligned}(T_A f, f) &= (T_A(f-h) + T_A h, (f-h) + h) \\ &= (T_A(f-h), (f-h)) + (T_A(f-h), h) + (T_A h, (f-h)) + (T_A h, h) \\ &\leq 3\|T_A\|\epsilon + (T_A h, h)\end{aligned}$$

and similarly,

$$\begin{aligned}(T_A h, h) &= (T_A(h-f) + T_A f, (h-f) + f) \\ &\leq 3\|T_A\|\epsilon + (T_A f, f),\end{aligned}$$

so

$$(T_A h, h) - 3\|T_A\|\epsilon \leq (T_A f, f) \leq (T_A h, h) + 3\|T_A\|\epsilon.$$

Further, since $h(t) = 0$ for $t \geq 1$ and from Theorem 2.11,

$$(T_A h, h) = \langle T_A h, h\rangle = \langle F(t/A)\hat{h}, \hat{h}\rangle.$$

Then

$$m\langle\hat{h}, \hat{h}\rangle \leq \langle F(t/A)\hat{h}, \hat{h}\rangle \leq M\langle\hat{h}, \hat{h}\rangle$$

and $\langle\hat{h}, \hat{h}\rangle = \langle h, h\rangle = (h, h) = 1$ together imply

$$m \leq (T_A h, h) \leq M.$$

Thus,

$$m - 3\epsilon\|T_A\| \leq (T_A f, f) \leq M + 3\epsilon\|T_A\|$$

$$m - 3\epsilon\|T_A\| \leq \lambda(f, f) \leq M + 3\epsilon\|T_A\|$$

$$m - 3\epsilon\|T_A\| \leq \lambda \leq M + 3\epsilon\|T_A\|$$

and letting $\epsilon \to 0$, we have $m \leq \lambda \leq M$. □

We now wish to introduce a special case where $F(t)$ is chosen as $F_S(t) = (1+t^2)^{-1}$. In this case we can evaluate the corresponding $\rho_S$ integral:

$$\begin{aligned}\rho_S(u, v) &= \int_0^\infty F_S(t) J(ut) J(vt) t^{2\nu}\, dt \\ &= \int_0^\infty \frac{(ut)^{1/2-\nu} J_{\nu-1/2}(ut)(vt)^{1/2-\nu} J_{\nu-1/2}(vt)}{1+t^2} t^{2\nu}\, dt \\ &= (uv)^{1/2-\nu} \int_0^\infty \frac{t J_{\nu-1/2}(ut) J_{\nu-1/2}(vt)}{1+t^2}\, dt\end{aligned}$$

and from Lemma 2.7, letting $z = 1$, we have

$$\rho_S(u, v) = (uv)^{1/2-\nu} I_{\nu-1/2}(v) K_{\nu-1/2}(u), \quad \text{for } u \geq v > 0. \tag{3.3}$$

Then for $f \in L^2(0,1;x^{2\nu})$,

$$(S_A f)(x) = A^{2\nu+1} \int_0^1 f(y)\rho_S(Ax, Ay)y^{2\nu}dy. \tag{3.4}$$

and for $f \in C_1^\infty(0,1)$,

$$(S_A f)(x) = \int_0^\infty \frac{1}{1+(t/A)^2}\hat{f}(t)J(xt)t^{2\nu}dt \tag{3.5}$$

**Lemma 3.8.** *Let $f \in C_1^\infty(0,1) \cap \mathcal{G}$, and let $g = \left(I + \frac{\tau}{A^2}\right)^n f$. Then*

$$S_A{}^i g = \left(I + \frac{\tau}{A^2}\right)^{n-i} f \quad \textit{for} \quad i = 1, 2, \ldots, n.$$

Proof: With

$$g = \left(I + \frac{\tau}{A^2}\right)^n f = \sum_{j=0}^n \binom{n}{j} \left(\frac{\tau}{A^2}\right)^j f,$$

since $f \in \mathcal{G}$ from Lemma 3.4 we have $\widehat{\tau^j f} = t^{2j}\hat{f}$, so

$$\hat{g} = \sum_{j=0}^n \binom{n}{j} \left(\frac{t^2}{A^2}\right)^j \hat{f} = \left(1 + \frac{t^2}{A^2}\right)^n \hat{f}.$$

Since $f \in C_1^\infty(0,1) \cap \mathcal{G}$, then from Lemma 3.3($i$) we can conclude $g \in C_1^\infty(0,1)$. Thus from Equation (3.5),

$$\begin{aligned} S_A g(x) &= \int_0^\infty F_S(t/A)\hat{g}(t)J(xt)t^{2\nu}\, dt \\ &= \int_0^\infty F_S(t/A)\left(1 + \frac{t^2}{A^2}\right)^n \hat{f}(t)J(xt)t^{2\nu}\, dt \\ &= \int_0^\infty \left(1 + \frac{t^2}{A^2}\right)^{n-1} \hat{f}(t)J(xt)t^{2\nu}\, dt \end{aligned}$$

which is the Hankel transform of $\left(1 + \frac{t^2}{A^2}\right)^{n-1} \hat{f}$. Again from Lemma 3.4 and using Lemma 2.10,

$$S_A g = \left(I + \frac{\tau}{A^2}\right)^{n-1} f.$$

Iterating this process, we have our result. □

**Lemma 3.9.** *Let $f \in C_1^\infty(0,1) \cap \mathcal{G}$. Then $(S_A f)(x)$ and $(S_A f)'(x)$ are both bounded on $x \in [0,1]$ as $A \to \infty$.*

Proof: From Equation (3.5),

$$(S_A f)(x) = \int_0^\infty F_S(t/A)\hat{f}(t)J(xt)t^{2\nu}\,dt$$

and since $0 < F_S(t/A) \le 1$, both $J$ and $J'$ are bounded from Lemma 3.1, and $\hat{f}$ is rapidly decreasing from Lemma 3.4, we can apply Lemma 2.2:

$$(S_A f)'(x) = \int_0^\infty F_S(t/A)\hat{f}(t)J'(xt)t^{2\nu+1}\,dt.$$

Thus $(S_A f)(x)$ and $(S_A f)'(x)$ are integrals which converge for $x \in (0,1)$ independent of $A$. □

**Lemma 3.10.** *Let* $f \in C_1^\infty(0,1)$ *and* $g = S_A f$. *Then there exists* $\epsilon > 0$ *such that*

$$g'(x) = -A\frac{K_{\nu+1/2}(Ax)}{K_{\nu-1/2}(Ax)}g(x) \text{ for } x > 1-\epsilon.$$

Proof: From Equation (3.4),

$$g(x) = A^{2\nu+1}\int_0^1 f(y)\rho_S(Ax,Ay)y^{2\nu}\,dy.$$

Since $f \in C_1^\infty(0,1)$ there exists $\epsilon > 0$ such that $f(x) = 0$ for $x > 1-\epsilon$. Thus

$$g(x) = A^{2\nu+1}\int_0^{1-\epsilon} f(y)\rho_S(Ax,Ay)y^{2\nu}\,dy$$

and for $x > 1-\epsilon$, since $y < x$ in the integrand, we can substitute for $\rho_S(Ax,Ay)$ using Equation (3.3), yielding

$$\begin{aligned} g(x) &= A^{2\nu+1}\int_0^{1-\epsilon} f(y)(Ay)^{1/2-\nu}I_{\nu-1/2}(Ay)(Ax)^{1/2-\nu}K_{\nu-1/2}(Ax)y^{2\nu}\,dy \\ &= \left[A^{2\nu+1}\int_0^{1-\epsilon} f(y)(Ay)^{1/2-\nu}I_{\nu-1/2}(Ay)y^{2\nu}\,dy\right](Ax)^{1/2-\nu}K_{\nu-1/2}(Ax). \end{aligned}$$

Differentiating using Lemma 2.6,

$$g'(x) = \left[A^{2\nu+1}\int_0^{1-\epsilon} f(y)(Ay)^{1/2-\nu}I_{\nu-1/2}(Ay)y^{2\nu}\,dy\right](-A(Ax)^{1/2-\nu}K_{\nu+1/2}(Ax))$$

and so

$$\frac{g'(x)}{g(x)} = -A\frac{K_{\nu+1/2}(Ax)}{K_{\nu-1/2}(Ax)}.$$

Thus, for $x > 1-\epsilon$ we have the identity

$$g'(x) = -A\frac{K_{\nu+1/2}(Ax)}{K_{\nu-1/2}(Ax)}g(x). \;\square$$

**Lemma 3.11.** *Let* $f \in C_1^\infty(0,1) \cap \mathcal{G}$ *and* $g = S_A f$. *Then for* $0 \le x_1 \le x_2 \le 1$,

$$|g(x_2) - g(x_1)|^2 + \left(\int_{x_1}^{x_2} \frac{1}{x^{2\nu}}\, dx\right)\left(A\frac{K_{\nu+1/2}(A)}{K_{\nu-1/2}(A)}|g(1)|^2\right) \le (\tau g, g)\left(\int_{x_1}^{x_2} \frac{1}{x^{2\nu}}\, dx\right).$$

Proof: Let $0 \le x_1 \le x_2 \le 1$, then

$$\begin{aligned}
|g(x_2) - g(x_1)|^2 &= \left|\int_{x_1}^{x_2} \frac{1}{x^{\nu}}\, g'(x)x^{\nu}\, dx\right|^2 \\
&\le \int_{x_1}^{x_2} \frac{1}{x^{2\nu}}\, |g'(x)|^2 x^{2\nu}\, dx \\
&\le \left(\int_{x_1}^{x_2} \frac{1}{x^{2\nu}}\, dx\right)\left(\int_0^1 g'(x)\overline{g'(x)}x^{2\nu}\, dx\right) \\
&\le \left(\int_{x_1}^{x_2} \frac{1}{x^{2\nu}}\, dx\right)\left(x^{2\nu}g'(x)\overline{g(x)}\Big|_0^1 - \int_0^1 \left(x^{2\nu}g'(x)\right)'\,\overline{g(x)}\, dx\right) \\
&\le \left(\int_{x_1}^{x_2} \frac{1}{x^{2\nu}}\, dx\right)\left(g'(1)\overline{g(1)} + (\tau g, g)\right).
\end{aligned}$$

Here the boundary term at 0 vanishes since $g$ and $g'$ are both bounded from Lemma 3.9. Substituting the identity from Lemma 3.10 for $x = 1$, the result follows. □

# Chapter 4: Friedrichs Extension

Let $L_n u = \tau^n u$ for $u \in D(L_n)$ where

$$D(L_n) = \big\{u(x) \in C^\infty(0,1) : u = 0 \text{ near } x = 1,\ (\tau^{n-1}u)' = 0 \text{ near } x = 0,$$
$$\text{and for } n \geq 2,\ x^{2\nu}u',\ x^{2\nu}(\tau u)', \cdots, x^{2\nu}(\tau^{n-2}u)' \to 0 \text{ as } x \to 0^+\big\}.$$

**Lemma 4.1.** *If $u \in D(L_k)$, then $\tau^{k-1}u(x)$, $\tau^{k-2}u(x)$, ... , $u(x)$ are all bounded as $x \to 0^+$.*

Proof: For $u \in D(L_k)$, since $(\tau^{k-1}u(x))' = 0$ for $x$ near 0, $\tau^{k-1}u(x)$ is constant near $x = 0$ and therefore bounded as $x \to 0^+$. Note also that $\tau^{k-1}u(x)$ is a polynomial in $x$ in this neighborhood. We will now show if $\tau^j u(x)$ is a polynomial in $x$ near $x = 0$, then $\tau^{j-1}u(x)$ is also a polynomial in $x$ near $x = 0$ for $j = 1, \ldots, k-1$. Given, for some $n$ and for $x$ near 0,

$$\begin{aligned}
\tau^j u(x) &= \sum_{i=0}^{n} c_i x^i \\
\left(x^{2\nu}(\tau^{j-1}u(x))'\right)' &= -\sum_{i=0}^{n} c_i x^{2\nu+i} \\
x^{2\nu}(\tau^{j-1}u(x))' &= -\sum_{i=0}^{n} \frac{c_i x^{2\nu+i+1}}{2\nu+i+1} + D.
\end{aligned}$$

Since $x^{2\nu}(\tau^{j-1}u(x))' \to 0$ as $x \to 0^+$, $D = 0$. Thus,

$$\begin{aligned}
(\tau^{j-1}u(x))' &= -\sum_{i=0}^{n} \frac{c_i x^{i+1}}{2\nu+i+1} \\
\tau^{j-1}u(x) &= -\sum_{i=0}^{n} \frac{c_i x^{i+2}}{(2\nu+i+1)(i+2)} + E.
\end{aligned}$$

So $\tau^i u(x)$ is a polynomial in $x$ near $x = 0$ for $0 \leq i \leq k-1$ and thus, $\tau^{k-1}u(x)$, $\tau^{k-2}u(x)$, ... , $u(x)$ are all bounded as $x \to 0^+$. □

**Lemma 4.2.** *$C_0^\infty(0,1) \subset D(L_k) \subset D(L_{k+1}) \subset C_1^\infty(0,1) \cap \mathcal{G}$ for all $k \geq 1$.*

Proof: Let $k \geq 1$ be fixed. Let $f \in C_0^\infty(0,1)$, then $f \in C^\infty(0,1)$ and $f$ and all its derivatives vanish in neighborhoods of 0 and 1. Thus, $f \in D(L_k)$.

Now let $g \in D(L_k)$, then $g \in C^\infty(0,1)$ and $g = 0$ near $x = 1$. Since $(\tau^{k-1}g)' = 0$ near $x = 0$, $x^{2\nu}(\tau^{k-1}g)' \to 0$ as $x \to 0^+$. Also, $(\tau^{k-1}g)' = 0$ near $x = 0$ implies all higher derivatives and orders of $\tau$ also vanish near $x = 0$. Thus, $(\tau^k g)' = 0$ near $x = 0$ and $g \in D(L_{k+1})$.

Recall $\mathcal{G} = \{f(x) \in C^\infty(0,\infty) : \; x^{2\nu}(\tau^k f)' \to 0$ as $x \to 0^+, \tau^k f$ is bounded on $(0,\infty)$, and $(\tau^k f)'$ and $\tau^k f$ are rapidly decreasing on $(0,\infty)$ for all $k \geq 0\}$. For $h \in D(L_k)$, we can extend the domain to $(0,\infty)$ continuously by defining $h(x) = 0$ for $x \geq 1$. From Lemma 4.1 and the properties of $D(L_k)$, we have $x^{2\nu}(\tau^k h)' \to 0$ as $x \to 0^+$ and $\tau^k h$ bounded on $(0,\infty)$ for $k \leq n-2$. Then since all higher derivatives and orders of $\tau$ vanish outside a compact subset of $(0,1)$, these two properties hold for all $k > n-2$. Further, since $h$ vanishes on $(1,\infty)$, $(\tau^k h)'$ and $\tau^k h$ are rapidly decreasing, and $h \in \mathcal{G}$. We also note that the conditions of $C_1^\infty(0,1)$ are included in the definition of $D(L_k)$. □

Since $C_0^\infty(0,1)$ is dense in $L^2(0,1;x^{2\nu})$, from Lemma 4.2 and Lemma 2.15, the adjoint operator ${L_k}^*$ exists for all $k \geq 1$.

**Lemma 4.3.** *Let* $u \in D(L_k)$ *and* $0 < x_1 < x_2 \leq 1$. *Then for* $k$ *odd,* $k = 2i+1$*,*

$$\left|\tau^i u(x_2) - \tau^i u(x_1)\right|^2 \leq (L_k u, u) \int_{x_1}^{x_2} \frac{1}{t^{2\nu}}\, dt$$

*and for* $k$ *even,* $k = 2i$*,*

$$\left|{x_2}^{2\nu}(\tau^{i-1}u)'(x_2) - {x_1}^{2\nu}(\tau^{i-1}u)'(x_1)\right|^2 \leq (L_k u, u) \int_{x_1}^{x_2} t^{2\nu}\, dt.$$

Proof: Since $D(L_k) \subset C_1^\infty(0,1) \cap \mathcal{G}$ from Lemma 4.2, these statements follow from Lemma 3.3. Note that we have replaced $\langle \cdot\, , \cdot \rangle$ with $(\cdot\, , \cdot)$ since integrands are zero to the right of 1. □

**Lemma 4.4.** *The operator* $L_k$ *is symmetric and nonnegative for all* $k \geq 1$.

Proof: Symmetry follows directly from Lemma 3.3(*ii*). We note that for $k = 2j$ even,

$$(L_k u, u) = (\tau^k u, u) = (\tau^j u, \tau^j u) = \int_0^1 |\tau^j u(x)|^2 x^{2\nu} dx \geq 0$$

and for $k = 2j+1$ odd,

$$(L_k u, u) = (\tau^k u, u) = (\tau^{j+1} u, \tau^j u) = \int_0^1 |(\tau^j u)'(x)|^2 x^{2\nu} dx \geq 0$$

so $L_k$ is nonnegative for all $k \geq 1$. □

**Lemma 4.5.** *Let* $u \in D(L_k)$*, then* $(\tau^i u, u) \leq (\tau^{i+1} u, u)$ *for* $0 \leq i \leq k-1$.

Proof: Consider the $i = 0$ case. Since $u \in D(L_k) \subset \mathcal{G}$, from Lemma 3.3(*iii*) we have

$$|u(x_2) - u(x_1)|^2 \leq (\tau u, u) \int_{x_1}^{x_2} \frac{1}{t^{2\nu}}\, dt.$$

Letting $x_1 = x$ and $x_2 = 1$ gives

$$|u(x)|^2 \leq (\tau u, u) \int_{x}^{1} \frac{1}{t^{2\nu}}\, dt,$$

and integrating we have

$$\int_0^1 |u(x)|^2\, x^{2\nu}\, dx \leq (\tau u, u) \int_0^1 \int_x^1 \frac{1}{t^{2\nu}}\, dt\; x^{2\nu}\, dx.$$

Since the function $t^{-2\nu}$ achieves its maximum at the left endpoint on $[x, 1]$, we have

$$(u, u) \leq (\tau u, u).$$

If $i$ is even, $i = 2n$, since $u \in D(L_k)$ implies $\tau^n u \in D(L_k)$, from Lemma 3.3(*ii*) and the above argument,

$$(\tau^i u, u) = (\tau^n u, \tau^n u) \leq (\tau^{n+1} u, \tau^n u) = (\tau^{i+1} u, u).$$

If $i$ is odd, $i = 2n + 1$,

$$\|\tau^n u\|^2 = (\tau^n u, \tau^n u) \leq (\tau^{n+1} u, \tau^n u) \leq \|\tau^{n+1} u\|\, \|\tau^n u\|$$

which implies $\|\tau^n u\| \leq \|\tau^{n+1} u\|$, so

$$\begin{aligned} (\tau^i u, u) &= (\tau^{n+1} u, \tau^n u) \leq \|\tau^{n+1} u\|\, \|\tau^n u\| \\ &\leq \|\tau^{n+1} u\|^2 = (\tau^{n+1} u, \tau^{n+1} u) = (\tau^{i+1} u, u). \end{aligned}$$

Thus we have $(\tau^i u, u) \leq (\tau^{i+1} u, u)$ for all $0 \leq i \leq k - 1$. □

For each $k$, from Theorem 2.16 (Lemma 4.4 shows that $L_k$ satisfies the hypotheses), $L_k$ has a particular self-adjoint extension $\tilde{L_k}$ (the Friedrichs extension of $L_k$) which preserves the lower bound.

**Lemma 4.6.** *Let* $u \in D(\tilde{L_k})$*. Then*

*(i)* $u^{(i)}(1) = 0$ *for* $0 \leq i \leq k - 1$

*(ii)* $|u(x_2) - u(x_1)|^2 \leq (\tilde{L_k} u, u) \int_{x_1}^{x_2} \frac{1}{t^2}\, dt$ *for* $0 < x_1 < x_2 \leq 1$

*(iii)* $(u, u) \leq (\tilde{L_k} u, u)$.

Proof: ***(i)*** If $u \in D(\tilde{L_k})$, then from Theorem 2.16 there exists $u_n \in D(L_k)$ such that $\|u_n - u\| \to 0$ and $(L_k u_n, u_n) \to (\tilde{L_k} u, u)$ as $n \to \infty$. Since convergent sequences are bounded, let $M$ be the bound on $(L_k u_n, u_n)$.

From Lemma 4.3, for $k$ even, $k = 2j$,

$$\left| {x_2}^{2\nu} (\tau^{j-1} u_n)'(x_2) - {x_1}^{2\nu} (\tau^{j-1} u_n)'(x_1) \right|^2 \leq (L_k u_n, u_n) \int_{x_1}^{x_2} t^{2\nu} \, dt$$

for $0 < x_1 < x_2 \leq 1$. Then the functions $\{x^{2\nu} (\tau^{j-1} u_n)'(x)\}$ are all zero at $x = 1$ and equicontinuous on $[0, 1]$. Further,

$$\left| \tau^{j-1} u_n(x_2) - \tau^{j-1} u_n(x_1) \right| = \left| \int_{x_1}^{x_2} (\tau^{j-1} u)' \, dt \right| \leq M \int_{x_1}^{x_2} \frac{1}{t^{2\nu}} \, dt,$$

so the functions $\{(\tau^{j-1} u_n)(x)\}$ are zero at $x = 1$ and equicontinuous on any compact subinterval of $(0, 1]$. Continuing in this manner we find all sequences of functions $\{x^{2\nu} (\tau^{\omega} u_n)'(x)\}$ and $\{(\tau^{\omega} u_n)(x)\}$, $0 \leq \omega \leq j - 1$, to be zero at $x = 1$ and equicontinuous on any compact subinterval of $(0, 1]$. Using Corollary 2.5 and a diagonalization argument, we may assume that each of these sequences converges uniformly on any compact subinterval of $(0, 1]$. Since $\|u_n - u\| \to 0$, $u_n$ must be converging uniformly to $u$ on each compact subinterval of $(0, 1]$, and so $u(1) = 0$.

Now, since $x^{2\nu} {u_n}'$ converges, so does ${u_n}'$; let ${u_n}' \to v$. We wish to show that $v = u'$. By the fundamental theorem of calculus, we have $u_n(x) = \int_1^x {u_n}'$. Taking limits of both sides, by the bounded convergence theorem, $u = \int_1^x v$. Since the RHS is differentiable, $u$ must be differentiable, and $u' = v$. Thus, ${u_n}' \to u'$ so $u'(1) = 0$. Since $\tau u_n$ converges, let $\tau u_n \to w$, and so $(x^{2\nu} u_n')' \to -x^{2\nu} w$. By the fundamental theorem of calculus, $x^{2\nu} u_n'(x) = \int_x^1 \tau u_n(t) t^{2\nu} dt$, and taking limits of both sides, by the bounded convergence theorem we get $x^{2\nu} u'(x) = \int_x^1 w(t) t^{2\nu} dt$. Again, since the RHS is differentiable, we may differentiate both sides to conclude that $\tau u = w$, and so $\tau u_n \to \tau u$ and $\tau u(1) = 0$. We repeat this process to find $(\tau^{\omega} u)(1) = 0$ and $(\tau^{\omega} u)'(1) = 0$ for $0 \leq \omega \leq j - 1$.

Then since

$$\tau u = \frac{2\nu}{x} u' + u''$$

and

$$x^{2\nu} (\tau u)' = -2\nu x^{2\nu - 2} u' + x^{2\nu - 1} u'' + x^{2\nu} u''',$$

$\tau u(1) = 0$ and $u'(1) = 0$ imply $u''(1) = 0$; $x^{2\nu} (\tau u)'(1) = 0$, $u'(1) = 0$, and $u''(1) = 0$ imply $u'''(1) = 0$; and so on. Thus, $u^{(i)}(1) = 0$ for $i = 0, 1, \ldots, k - 1$.

A similar argument can be made for $k$ odd, $k = 2j + 1$. Here we would first establish that $\{\tau^j u_n\}$ is zero at $x = 1$ and equicontinuous on any compact subinterval of $(0, 1]$ and continue as before to obtain the same result.

***(ii)*** From Lemma 4.5, we have $(\tau u_n, u_n) \leq (\tau^k u_n, u_n) = (L_k u_n, u_n)$. Then from Lemma 3.3(***iii***),

$$|u_n(x_2) - u_n(x_1)|^2 \leq (L_k u_n, u_n) \int_{x_1}^{x_2} \frac{1}{t^{2\nu}}\, dt$$

for $0 < x_1 < x_2 \leq 1$. Taking limits (through the converging subsequence), we have

$$|u(x_2) - u(x_1)|^2 \leq (\tilde{L_k} u, u) \int_{x_1}^{x_2} \frac{1}{t^{2\nu}}\, dt.$$

***(iii)*** Letting $x_1 = x$ and $x_2 = 1$ and integrating as in the proof of Lemma 4.5, we have $(u, u) \leq (\tilde{L_k} u, u)$. □

**Lemma 4.7.** *$\tilde{L_k}$ has a compact inverse.*

Proof: Let $u$ be an eigenfunction of $\tilde{L_k}$. Then from Lemma 4.6, $(u, u) \leq (\tilde{L_k} u, u) = (\lambda u, u)$ where $\lambda$ is the corresponding eigenvalue. This implies that all eigenvalues of $\tilde{L_k}$ are strictly positive. Hence $\tilde{L_k}^{-1}$ exists. To show $\tilde{L_k}^{-1}$ is compact, suppose there is a sequence $\{\tilde{L_k} u_n\}$ such that $\|\tilde{L_k} u_n\| \leq K$ for $n = 1, 2, \ldots$ . We will show that $\{\tilde{L_k}^{-1}(\tilde{L_k} u_n)\} = \{u_n\}$ has a convergent subsequence. From Lemma 4.6(***iii***), $\|u_n\| \leq \|\tilde{L_k} u_n\|$, and so we have

$$(\tilde{L_k} u_n, u_n) \leq K^2 \text{ for all } n = 1, 2, \ldots$$

Then from Lemma 4.6(***ii***),

$$|u_n(x_2) - u_n(x_1)|^2 \leq K^2 \int_{x_1}^{x_2} \frac{1}{t^{2\nu}}\, dt \text{ for } 0 < x_1 < x_2 \leq 1$$

and also

$$|u_n(x)|^2 \leq K^2 \int_{x}^{1} \frac{1}{t^{2\nu}}\, dt \leq K^2 \frac{1}{x^{2\nu}}.$$

Thus the functions $\{u_n\}$ are equicontinuous and uniformly bounded on compact subintervals of $(0, 1]$. From Theorem 2.4 and a diagonalization argument we may assume the sequence converges uniformly to a limit function $u$ on each compact subinterval of $(0, 1]$. Since the above inequality also holds for the limit function $u$, the bounded convergence theorem guarantees $u_n$ converges to $u$ in $L^2(0, 1; x^{2\nu})$, so $\tilde{L_k}^{-1}$ is compact. □

**Lemma 4.8.** *The range of $\tilde{L_k}$ is all of $L^2(0, 1; x^{2\nu})$.*

Proof: We first show that $R(\tilde{L_k})$ is dense in $L^2(0, 1; x^{2\nu})$. Suppose not; then there is a function $v \neq 0$ in $L^2(0, 1; x^{2\nu})$ which is orthogonal to every element in $R(\tilde{L_k})$, that is $(\tilde{L_k} u, v) = 0 = (u, 0)$, for every $u \in D(\tilde{L_k})$. Then $v \in D(\tilde{L_k}^*) = D(\tilde{L_k})$ and $\tilde{L_k} v = \tilde{L_k}^* v = 0 = \tilde{L_k}(0)$. But this contradicts the fact that $\tilde{L_k}$ is $1-1$ (from Lemma 4.7). Hence $\overline{R(\tilde{L})} = L^2(0, 1; x^{2\nu})$.

It now suffices to show that $\overline{R(\tilde{L_k})} \subset R(\tilde{L_k})$. Let $v \in \overline{R(\tilde{L_k})}$. Then there is a sequence $\{v_n\} \subset R(\tilde{L_k})$ such that $v_n \to v$. Now for each $n$, there exists a $u_n \in D(\tilde{L_k})$ such that $\tilde{L_k} u_n = v_n$. From Lemma 4.6(*iii*),

$$\begin{aligned} \|u_n - u_m\|^2 &= (u_n - u_m, u_n - u_m) \\ &\leq (\tilde{L_k}(u_n - u_m), u_n - u_m) \\ &\leq \|\tilde{L_k}(u_n - u_m)\| \, \|u_n - u_m\| \end{aligned}$$

which implies $\|u_n - u_m\| \leq \|\tilde{L_k}(u_n - u_m)\| = \|v_n - v_m\|$. Now since $\{v_n\}$ is a Cauchy sequence in $L^2(0,1;x^{2\nu})$, so is $\{u_n\}$.

The completeness property of $L^2(0,1;x^{2\nu})$ now guarantees that there exists some $u \in L^2(0,1;x^{2\nu})$ such that $u_n \to u$. We have the following conditions: $u_n \in D(\tilde{L_k})$, $u_n \to u$, and $\tilde{L_k} u_n \to v$. Since $\tilde{L_k}$ is self-adjoint and therefore closed, we conclude that $u \in D(\tilde{L_k})$ and $\tilde{L_k} u = v$. Thus $v \in R(\tilde{L_k})$. □

Because of the previous three lemmas, it follows from Theorem 2.14 that all eigenvalues of the operator $\tilde{L}_n$ are at least 1 and have finite multiplicity, and the eigenfunctions span $L^2(0,1;x^{2\nu})$.

**Lemma 4.9.** *If $w \in N(L_k{}^*)$, then for some constants $a_j$, $w = \sum_{j=0}^{k-1} a_j x^{2j}$.*

For $0 \leq x \leq 1$, we define $Q_0(x) = 1$ and

$$Q_j(x) = [(2\nu+1)\cdot 2 \cdot (2\nu+3) \cdot 4 \cdot (2\nu+5) \cdots 2j]^{-1} x^{2j} \text{ for } i = 1, 2, \ldots k-1$$

and we define $R_0(x) = \int_x^1 t^{-2\nu}\, dt$ and

$$R_{j+1}(x) = \int_x^1 \frac{1}{y^{2\nu}} \left[ \int_y^1 t^{2\nu} R_i(t)\, dt \right] dy \quad \text{for } j = 0, 1, \ldots k-1.$$

The functions $Q_j$ and $R_j$ have the following significance for $j = 0, 1, \ldots k-1$:

$$\tau^j Q_j = (-1)^j \quad \text{and} \quad x^{2\nu}(\tau^j R_j)' = (-1)^{j+1}.$$

Note also that $\tau^i Q_j$ is a polynomial and is therefore bounded near 0 for all $i, j$.

Thus, $\tau^k$ maps all of the functions $Q_0, R_0, \ldots, Q_{k-1}, R_{k-1}$ to 0. Differentiation and the use of the above properties show that the functions $Q_j$ and $R_j$ are linearly independent, and since $\tau^k$ is a $(2k)^{th}$ order differential operator, from Lemma 2.17 the $2k$ functions $\{Q_0, R_0, \ldots, Q_{k-1}, R_{k-1}\}$ must span $N(L_k{}^*)$. Thus for $w \in N(L_k{}^*)$,

$$w = \sum_{j=0}^{k-1} (a_j Q_j + b_j R_j)$$

for some constants $a_j$ and $b_j$. We wish to show that $b_j = 0$ for all $j = 0, 1, \ldots, k-1$.

Suppose $b_\gamma \neq 0$ where $b_j = 0$ for $\gamma + 1 \leq j \leq k-1$. From the definition of the adjoint operator and Lemma 2.17,

$$(\tau^k u, w) = (L_k u, w) = (u, {L_k}^* w) = (u, \tau^k w)$$

for all $u \in D(L_k)$. Choose $u \in C^\infty(0,1)$ such that $u(x) = Q_{k-\gamma-1}(x)$ for $0 \leq x \leq 1/4$ and $u(x) = 0$ for $3/4 \leq x \leq 1$. Repeated integration by parts gives

$$(\tau^k u, w) = \sum_{i=0}^{k-1} (\tau^i u) x^{2\nu} (\tau^{k-i-1} w)' \Big|_0^1 - \sum_{i=0}^{k-1} x^{2\nu} (\tau^i u)' (\tau^{k-i-1} w) \Big|_0^1 + (u, \tau^k w)$$

and the above identity implies

$$\sum_{i=0}^{k-1} (\tau^i u) x^{2\nu} (\tau^{k-i-1} w)' \Big|_0^1 - \sum_{i=0}^{k-1} x^{2\nu} (\tau^i u)' (\tau^{k-i-1} w) \Big|_0^1 = 0.$$

Since $u(x) \equiv 0$ for $3/4 \leq x \leq 1$ (for this particular $u$), all its derivatives vanish on that neighborhood, so our equation simplifies to

$$\lim_{x \to 0} \left[ \sum_{i=0}^{k-1} (\tau^i u) x^{2\nu} (\tau^{k-i-1} w)' - \sum_{i=0}^{k-1} x^{2\nu} (\tau^i u)' (\tau^{k-i-1} w) \right] = 0.$$

Near $x = 0$, $u(x) = Q_{k-\gamma-1}(x)$ so $(\tau^{k-\gamma-1} u)' \equiv 0$ for $0 \leq x \leq 1/4$ (and all higher derivatives vanish as well). Thus we have

$$\lim_{x \to 0} \left[ \sum_{i=0}^{k-\gamma-1} (\tau^i Q_{k-\gamma-1}) x^{2\nu} (\tau^{k-i-1} w)' - \sum_{i=0}^{k-\gamma-2} x^{2\nu} (\tau^i Q_{k-\gamma-1})' (\tau^{k-i-1} w) \right] = 0.$$

Consider the second summation, substituting for $w$,

$$\sum_{i=0}^{k-\gamma-2} x^{2\nu} (\tau^i Q_{k-\gamma-1})' (\tau^{k-i-1} w) = \sum_{i=0}^{k-\gamma-2} x^{2\nu} (\tau^i Q_{k-\gamma-1})' (\tau^{k-i-1} \left[ \sum_{j=0}^{k-1} a_j Q_j + b_j R_j \right])$$

$$= \sum_{i=0}^{k-\gamma-2} \sum_{j=0}^{k-1} \left[ a_j x^{2\nu} (\tau^i Q_{k-\gamma-1})' (\tau^{k-i-1} Q_j) + b_j x^{2\nu} (\tau^i Q_{k-\gamma-1})' (\tau^{k-i-1} R_j) \right].$$

Since all derivatives of $Q_j$ are bounded, the first term in the brackets vanishes as $x \to 0$ (for all $i, j$), leaving

$$\sum_{i=0}^{k-\gamma-2} \sum_{j=0}^{k-1} b_j x^{2\nu} (\tau^i Q_{k-\gamma-1})' (\tau^{k-i-1} R_j).$$

From the properties of $R_j$, $\tau^{k-i-1}R_j = 0$ for $j \leq k-i-2$, and by supposition, $b_j = 0$ for $\gamma+1 \leq j \leq k-1$, so the summation becomes

$$\sum_{i=0}^{k-\gamma-2} \sum_{j=k-i-1}^{\gamma} b_j x^{2\nu}(\tau^i Q_{k-\gamma-1})'(\tau^{k-i-1}R_j)$$

but there are no values for $i$ and $j$ that fit these limits of summation, so the entire quantity vanishes.

Now we consider the first summation and the equation that remains:

$$\lim_{x\to 0}\left[\sum_{i=0}^{k-\gamma-1}(\tau^i u)x^{2\nu}(\tau^{k-i-1}w)'\right] = 0.$$

Substituting for $w$ and rewriting for $x$ near 0 we have

$$\lim_{x\to 0}\sum_{i=0}^{k-\gamma-1}\sum_{j=0}^{k-1}\left[a_j(\tau^i Q_{k-\gamma-1})x^{2\nu}(\tau^{k-i-1}Q_j)' + b_j(\tau^i Q_{k-\gamma-1})x^{2\nu}(\tau^{k-i-1}R_j)'\right] = 0.$$

The first term in the brackets vanishes in the limit as before, leaving

$$\lim_{x\to 0}\left[\sum_{i=0}^{k-\gamma-1}\sum_{j=0}^{k-1} b_j(\tau^i Q_{k-\gamma-1})x^{2\nu}(\tau^{k-i-1}R_j)'\right] = 0.$$

From the properties of $R_j$, $(\tau^{k-i-1}R_j)' = 0$ for $j \leq k-i-2$, and by supposition, $b_j = 0$ for $\gamma+1 \leq j \leq k-1$, so the summation becomes

$$\lim_{x\to 0}\left[\sum_{i=0}^{k-\gamma-1}\sum_{j=k-i-1}^{\gamma} b_j(\tau^i Q_{k-\gamma-1})x^{2\nu}(\tau^{k-i-1}R_j)'\right] = 0.$$

These limits of summation yield only one term, with $i = k-\gamma-1$ and $j = \gamma$, and thus

$$\lim_{x\to 0}\left[b_\gamma(\tau^{k-\gamma-1}Q_{k-\gamma-1})x^{2\nu}(\tau^\gamma R_\gamma)'\right] = \lim_{x\to 0} b_\gamma(-1)^k = 0.$$

Therefore we conclude that $b_\gamma = 0$, a contradiction. Thus for $w \in N({L_k}^*)$,

$$w = \sum_{j=0}^{k-1} a_j Q_j$$

and from the definition of the $Q_j$'s, redefining the $a_j$'s, we have our result. □

**Theorem 4.10.** $D(\tilde{L_k}) = \{u \in D({L_k}^*) : u^{(i)}(1) = 0$ *for* $i = 0, 1, \ldots, k-1\}$

Proof: Since $D(\tilde{L_k}) \subset D({L_k}^*)$ and Lemma 4.6(*i*) implies the boundary conditions are satisfied for $u \in D(\tilde{L_k})$, we need only show the RHS is a subset of the LHS.

Let $u \in D({L_k}^*)$ such that $u^{(i)}(1) = 0$ for $i = 0, 1, \ldots, n-1$. From Lemma 4.8, the range of $\tilde{L_k}$ is all of $L^2(0, 1; x^{2\nu})$, so there exists $v \in D(\tilde{L_k})$ such that $\tilde{L_k} v = {L_k}^* u$. We let $w = u - v$ and wish to show that $w = 0$. Since $D(\tilde{L_k}) \subset D({L_k}^*)$, $w \in N({L_k}^*)$, and from Lemma 4.9, $w$ has the form

$$w = \sum_{j=0}^{k-1} a_j x^{\beta_j}$$

where the $\beta_j$'s are distinct.

Since $u$ satisfies the boundary conditions by hypothesis and $v$ satisfies the boundary conditions by Lemma 4.6(*i*), we have $w^{(i)}(1) = 0$ for $i = 0, 1, \ldots, k-1$. Thus, we have a system of equations

$$\sum_{j=0}^{k-1} a_j p_i(\beta_j) = 0, \quad i = 0, 1, \ldots, k-1$$

where $p_0(\beta) = 1$ and $p_i(\beta) = \beta(\beta - 1) \cdots (\beta - i + 1)$ for $i = 1, 2, \ldots, k-1$.

The determinant of the coefficient matrix in this system is equal, by elementary row operations, to the Vandermonde determinant det $[b_{ij}]$ where $b_{ij} = {\beta_j}^{i-1}$, which is nonzero since the $\beta_j$'s are distinct. Hence $a_0 = a_1 = \cdots = a_{k-1} = 0$, and $w = 0$. □

We now identify the eigenfunctions and eigenvalues of $\tilde{L_1}$.

**Theorem 4.11.** *The eigenvalues $\Lambda_k$ of $\tilde{L_1}$ have multiplicity one and $\Lambda_k = z_k^2$, where $z_k$ is the $k^{th}$ positive zero of the Bessel function $J_{\nu-1/2}$. An eigenfunction corresponding to $\Lambda_k$ is $x^{1/2-\nu} J_{\nu-1/2}(z_k x)$.*

Proof: If $\Lambda$ is an eigenvalue of $\tilde{L_1}$ and $u$ is a corresponding eigenfunction, then $\Lambda > 0$, $u$ is in the null space of $\tilde{L_1} - \Lambda I$ and we have the equation

$$L_1^* u - \Lambda u = \tilde{L_1} u - \Lambda u = 0.$$

By Lemma 2.17, $u$ is infinitely differentiable and $\tau u = \Lambda u$. This equation reduces to

$$u'' + \frac{2\nu}{x} u' + \beta^2 u = 0$$

where $\Lambda = \beta^2$. It is easy to verify that the general solution of this equation is

$$u = c_1 x^{1/2-\nu} u_1(\beta x) + c_2 x^{1/2-\nu} u_2(\beta x),$$

where $u_1$, $u_2$ are any two linearly independent solutions of Bessel's equation. We may choose $u_1(x) = J_{\nu-1/2}(x)$ and $u_2(x) = Y_{\nu-1/2}(x)$. Let $w_2(x) = x^{1/2-\nu} u_2(\beta x)$. Known behavior (see [10]) of $u_2(x)$ as $x \to 0$ shows that

$$\lim_{x \to 0} x^{2\nu} w_2'(x) = d \neq 0$$

for some constant $d$. Choosing $v$ in the domain of $L_1$ so that $v(x) = 1$ in a right neighborhood of $x = 0$, we can integrate by parts to get

$$(L_1 v, w_2) = (\tau v, w_2) = -d + (v, \tau w_2),$$

so that $w_2$ is not in the domain of $L_1^*$. Thus $c_2 = 0$. A similar calculation shows that $x^{1/2-\nu} u_2(\beta x)$ is in the domain of $L_1^*$. Then $u$ is in the domain of $\tilde{L_1}$ if and only if $u(1) = 0$, which occurs if and only if $u_1(\beta) = J_{\nu-1/2}(\beta) = 0$. Thus $\beta$ must be a zero $z_k$ of this Bessel function. □

# Chapter 5: First Main Theorem

Recall the special function $F_S(t) = (1+t^2)^{-1}$, and let $F(t)$ be a function satisfying the following conditions

H4: $F(t)$ is bounded and absolutely integrable on $(0, \infty)$

H5: $F(0) - F(t) \geq q^2(1 - F_S(t))^n$ for $t > 0$, for some constant $q > 0$

H6: $\lim_{t \to 0^+} \dfrac{F(0) - F(t)}{t^{2n}} = \sigma^2$ for some constant $\sigma$.

Note that H1-H3 are a special case of H4-H6, where $n = 1$. Let $M = F(0)$. Then we have

$$\lim_{t \to 0^+} \frac{M - F(t)}{(1 - F_S(t))^n} = \lim_{t \to 0^+} \frac{\frac{M - F(t)}{t^{2n}}}{\frac{(1 - F_S(t))^n}{t^{2n}}} = \sigma^2$$

and so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{M - F(t)}{(1 - F_S(t))^n} - \sigma^2 \right| < \epsilon \ \text{ for } \ 0 \leq t \leq \delta.$$

Define

$$G(t) = \begin{cases} F(t) & 0 \leq t < \delta \\ M - (\sigma^2 + \gamma)(1 - F_S(t))^n & t \geq \delta \end{cases}$$

where $\gamma = \frac{M - F(\delta)}{(1 - F_S(\delta))^n} - \sigma^2$.

**Lemma 5.1.** *Suppose $F(t)$ satis es H4-H6 and let $G(t)$ be de ned as above for an arbitrary $\epsilon > 0$. Then for $t \geq 0$,*

$$M - F(t) < (\sigma^2 + \epsilon)(1 - F_S(t))^n + |G(t) - F(t)|$$

*and the quantity $|G(t) - F(t)|$ is bounded for $t \geq 0$ and vanishes for $0 \leq t < \delta$.*

Proof: We decompose $M - F(t)$ into three terms:

$$M - F(t) = \sigma^2(1 - F_S(t))^n + \left(M - G(t) - \sigma^2(1 - F_S(t))^n\right) + (G(t) - F(t))\,.$$

We consider the second term. For $0 \leq t < \delta$, $G(t) = F(t)$ and from the limiting behavior of $M - F(t)$, for these values of $t$,

$$\left|M - G(t) - \sigma^2(1 - F_S(t))^n\right| < \epsilon(1 - F_S(t))^n.$$

For $t \geq \delta$, $G(t) = M - (\sigma^2 + \gamma)(1 - F_S(t))^n$ and so

$$M - G(t) - \sigma^2(1 - F_S(t))^n = \gamma(1 - F_S(t))^n$$

but since $\gamma$ is defined to be $\frac{M-F(t)}{(1-F_S(t))^n} - \sigma^2$ evaluated at $t = \delta$, $|\gamma| < \epsilon$. Thus for all $t \geq 0$,

$$\left|M - G(t) - \sigma^2(1 - F_S(t))^n\right| < \epsilon(1 - F_S(t))^n,$$

and we have

$$\begin{aligned} M - F(t) &= \left|\sigma^2(1 - F_S(t))^n + \left(M - G(t) - \sigma^2(1 - F_S(t))^n\right) + (G(t) - F(t))\right| \\ &\leq \left|\sigma^2(1 - F_S(t))^n\right| + \left|M - G(t) - \sigma^2(1 - F_S(t))^n\right| + |G(t) - F(t)| \\ &< (\sigma^2 + \epsilon)(1 - F_S(t))^n + |G(t) - F(t)|. \end{aligned}$$

We note that $G(t) = F(t)$ for $0 \leq t < \delta$ and for $t \geq \delta$,

$$\begin{aligned} |G(t) - F(t)| &= |M - (\sigma^2 + \gamma)(1 - F_S(t))^n - F(t)| \\ &\leq M - m + \sigma^2 + \gamma \end{aligned}$$

where $m \leq 0$ is a lower bound for $F(t)$. Thus $|G(t) - F(t)|$ is bounded for all $t \geq 0$. □

**Lemma 5.2.** *Let* $\{f_j\}_{j=1}^k \subset L^2(0, 1; x^{2\nu})$ *be a set of orthonormal functions. For* $0 < \epsilon < 1/k$, *let* $h_j \in L^2(0, 1; x^{2\nu})$ *be de ned such that* $\|f_j - h_j\| < \epsilon$ *for* $j = 1, 2, \ldots, k$. *Then the* $h_j$*'s are linearly independent.*

Proof: Suppose $\displaystyle\sum_{j=1}^k c_j h_j = 0$, and not all of the $|c_j|$ are zero. Let $|c_i|$ be the largest in absolute value of the nonzero constants. We have $\displaystyle\sum_{j=1}^k c_j(h_j - f_j) + \sum_{j=1}^k c_j f_j = 0$. Multiplying both sides by $\overline{f_i} x^{2\nu}$ and integrating gives

$$\left[\sum_{j=1}^k c_j(h_j - f_j, f_i)\right] + c_i = c_i(h_i - f_i, f_i) + c_i + \sum_{j\neq i} c_j(h_j - f_j, f_i) = 0.$$

We then have

$$|c_i| = \left| -\frac{\displaystyle\sum_{j\neq i} c_j(h_j - f_j, f_i)}{1 + (h_i - f_i, f_i)} \right| \leq \frac{\displaystyle\sum_{j\neq i} |c_j| \|h_j - f_j\|}{1 - \|h_i - f_i\|} \leq \frac{\epsilon}{1 - \epsilon}(k - 1)|c_i|.$$

This inequality implies $\epsilon \geq 1/k$, which is a contradiction. □

**Lemma 5.3.** *Let $\mathcal{L} = \text{span}\{g_1, \dots, g_n\}$ be an $n$-dimensional subspace of a Hilbert space $\mathcal{H}$. Then the Rayleigh quotient for a bounded operator $T$*

$$Q(g) = \frac{(Tg, g)}{(g, g)}$$

*achieves a minimum on $\mathcal{L}$, and the minimum is achieved for some $g_0 = \sum_{i=1}^{n} c_i g_i$ such that*

$$\sum_{i=1}^{n} |c_i|^2 = 1.$$

Proof: For $g \in \mathcal{L}$, $g = \sum_{i=1}^{n} c_i g_i$. Since the $g_i$ are given functions, the Rayleigh quotient becomes a function of the coefficient $c_i$'s ($n$-vectors). For a given $g \in \mathcal{L}$, let $d$ be the length of the corresponding $c_i$ vector, and define $\bar{g} = d^{-1} g$. Then for $\bar{g}$, the length of the corresponding coefficient vector is 1. Since

$$Q(\bar{g}) = \frac{(T\bar{g}, \bar{g})}{(\bar{g}, \bar{g})} = \frac{d^{-2}}{d^{-2}} \frac{(Tg, g)}{(g, g)} = Q(g)$$

for $g = \sum_{i=1}^{n} c_i g_i$ we can assume without loss of generality that $\sum_{i=1}^{n} |c_i|^2 = 1$. Then for $\mathbb{S}^n$ the surface of the unit ball in $n$-space,

$$Q : \mathbb{S}^n \to L^2(0, \infty; x^{2\nu}) \to \mathbb{R}$$

is a continuous function on $\mathbb{S}^n$, a compact subset of $\mathbb{R}^n$, and thus $Q$ achieves a minimum. □

**Theorem 5.4.** *Let $k$ be a positive integer. Then for $A$ su ciently large, the integral operator $T_A$, with $F(t)$ satisfying conditions H4-H6, will have at least $k$ positive eigenvalues with $M \geq \lambda_{1,A} \geq \lambda_{2,A} \geq \cdots \geq \lambda_{k,A} > 0$, allowing repetitions for multiple eigenvalues, and we have*

$$\limsup_{A \to \infty} A^{2n}(M - \lambda_{j,A}) \leq \sigma^2 \Lambda_j \quad \textit{for} \quad j = 1, 2, \dots, k,$$

*where $\{\Lambda_j\}$ are the eigenvalues of $\tilde{L}_n$ arranged in nondecreasing order with repetitions for multiple eigenvalues.*

Proof: Let $1/k > \epsilon > 0$ be given. Let $f_j$ be the normalized eigenfunction of $\tilde{L_n}$ corresponding to $\Lambda_j$ for $j = 1, 2, \dots, k$. From Theorem 2.16, there exists $h_j \in D(L_n)$ such that $\|h_j\| = 1$ and

$$\|f_j - h_j\| < \epsilon \ \text{ for } \ j = 1, 2, \dots, k$$

and for $i, j = 1, 2, \ldots, k$,

$$|(h_i, h_j) - (f_i, f_j)| < \epsilon$$
$$|(L_n h_i, h_j) - (\tilde{L_n} f_i, f_j)| < \epsilon.$$

From Lemma 5.2, the $h_j$'s are linearly independent. Now define

$$g_{j,A} = \left(I + \frac{\tau}{A^2}\right)^n h_j.$$

From Lemma 3.8 we have ${S_A}^n g_{j,A} = h_j$, and so the $g_{j,A}$'s are also linearly independent.

Let $\mathcal{M}_k$ denote any $k$-dimensional subspace of $L^2(0,1;x^{2\nu})$. We shall show that for $A$ sufficiently large,

$$\sup_{M_k} \inf_{g \in \mathcal{M}_k} \left[\frac{(T_A g, g)}{(g, g)}\right] > 0.$$

It will then follow from Theorem 2.18 that $T_A$ has at least $k$ positive eigenvalues.

Let $\mathcal{L}_k$ be the span of $g_{j,A}$. Since the $g_{j,A}$'s are linearly independent, $\mathcal{L}_k$ is a particular $k$-dimensional subspace of $L^2(0,1;x^{2\nu})$. Now define $\mu_{k,A}$ as follows:

$$\mu_{k,A} = \sup_{g \in \mathcal{L}_k} \left[M - \frac{(T_A g, g)}{(g, g)}\right] = M - \inf_{g \in \mathcal{L}_k} \left[\frac{(T_A g, g)}{(g, g)}\right].$$

From Lemma 5.3, the Rayleigh quotient $Q(g) = \frac{(T_A g, g)}{(g, g)}$ achieves a minimum on $\mathcal{L}_k$ and there exists $g \in \mathcal{L}_k$ such that $g = \sum_{j=1}^{k} c_j g_{j,A}$ minimizes the Rayleigh quotient and $\sum_{j=1}^{k} |c_j|^2 = 1$. For this particular $g$ (which we now denote $g_A$, for it depends on $A$) we have

$$\mu_{k,A} = M - \frac{(T_A g_A, g_A)}{(g_A, g_A)} = \frac{(M g_A, g_A)}{(g_A, g_A)} - \frac{(T_A g_A, g_A)}{(g_A, g_A)} = \frac{((MI - T_A) g_A, g_A)}{(g_A, g_A)}.$$

We will first consider the numerator $((MI - T_A) g_A, g_A)$. Note that

$$((MI - T_A) g_A)(x) = \int_0^\infty (M - F(t/A)) \hat{g_A}(t) J(xt) t^{2\nu} dt$$

is the Hankel transform of $(M - F(t/A))\hat{g_A}(t)$, and note that $g_A \in C_1^\infty(0,1)$. So

$$((MI - T_A) g_A, g_A) = \langle (MI - T_A) g_A, g_A \rangle$$

and by Theorem 2.11,

$$\langle (MI - T_A) g_A, g_A \rangle = \langle (M - F(t/A)) \hat{g_A}, \hat{g_A} \rangle = \int_0^\infty (M - F(t/A)) |\hat{g_A}(t)|^2 t^{2\nu} dt.$$

From Lemma 5.1,

$$\int_0^\infty (M - F(t/A))|\hat{g_A}(t)|^2 t^{2\nu} dt < (\sigma^2 + \epsilon)\mathcal{I}_1 + \mathcal{I}_2$$

where

$$\mathcal{I}_1 = \int_0^\infty (1 - F_S(t/A))^n |\hat{g_A}(t)|^2 t^{2\nu} dt \quad \text{and}$$
$$\mathcal{I}_2 = \int_0^\infty |G(t/A) - F(t/A)|\,|\hat{g_A}(t)|^2 t^{2\nu} dt.$$

Since $g_A = \sum_{j=1}^k c_j g_{j,A}$ and the Hankel transform is a linear operation, we can write

$$\begin{aligned} \mathcal{I}_1 &= \int_0^\infty (1 - F_S(t/A))^n \left[\sum_{j=1}^k c_j \hat{g}_{j,A}(t)\right] \overline{\left[\sum_{\ell=1}^k c_\ell \hat{g}_{\ell,A}(t)\right]} t^{2\nu} dt \\ &= \sum_{j,\ell=1}^k c_j \overline{c_\ell} \int_0^\infty (1 - F_S(t/A))^n \hat{g}_{j,A}(t) \overline{\hat{g}_{\ell,A}(t)} t^{2\nu} dt. \end{aligned}$$

Because $g_{j,A} = (1 + \frac{\tau}{A^2})^n h_j$ and Lemma 3.4 implies $\widehat{\tau^k h_j} = t^{2k} \hat{h_j}$, we have $\hat{g}_{j,A} = (1 + (t/A)^2)^n \hat{h_j}$. Also, since $(1 - F_S(t/A)) = \frac{(t/A)^2}{1+(t/A)^2}$, we have

$$\begin{aligned} \mathcal{I}_1 &= \sum_{j,\ell=1}^k c_j \overline{c_\ell} \int_0^\infty (1 - F_S(t/A))^n (1 + (t/A)^2)^n \hat{h_j}(t) \overline{(1 + (t/A)^2)^n \hat{h_\ell}(t)} t^{2\nu} dt \\ &= \sum_{j,\ell=1}^k c_j \overline{c_\ell} \int_0^\infty (t/A)^{2n} \hat{h_j}(t) \overline{(1 + (t/A)^2)^n \hat{h_\ell}(t)} t^{2\nu} dt \\ &= \sum_{j,\ell=1}^k c_j \overline{c_\ell} \sum_{m=0}^n \binom{n}{m} \left\langle \left(\frac{t^2}{A^2}\right)^n \hat{h_j}, \left(\frac{t^2}{A^2}\right)^m \hat{h_\ell} \right\rangle. \end{aligned}$$

Then by Theorem 2.11 and Lemma 3.4 we can rewrite these inner products as

$$\mathcal{I}_1 = \sum_{j,\ell=1}^k c_j \overline{c_\ell} \sum_{m=0}^n \binom{n}{m} \left\langle \left(\frac{\tau}{A^2}\right)^n h_j, \left(\frac{\tau}{A^2}\right)^m h_\ell \right\rangle.$$

Since each of these functions is 0 to the right of 1 we can restrict the inner products, giving

$$\mathcal{I}_1 = \sum_{j,\ell=1}^k c_j \overline{c_\ell} \sum_{m=0}^n \binom{n}{m} \left( \left(\frac{\tau}{A^2}\right)^n h_j, \left(\frac{\tau}{A^2}\right)^m h_\ell \right).$$

Multiplying through by $A^{2n}$ and rearranging we have

$$A^{2n}\mathcal{I}_1 = \sum_{j,\ell=1}^{k} c_j\overline{c_\ell}\left[(\tau^n h_j, h_\ell) + \sum_{m=1}^{n}\frac{1}{A^{2m}}\binom{n}{m}(\tau^n h_j, \tau^m h_\ell)\right]$$

and noting that the $h_j$ functions have no dependence on $A$, we conclude that

$$\limsup_{A\to\infty} A^{2n}\mathcal{I}_1 = \sum_{j,\ell=1}^{k} c_j\overline{c_\ell}\ (\tau^n h_j, h_\ell).$$

Then since $\mathcal{I}_1$ is a nonnegative quantity, using the triangle inequality and letting $A \to \infty$ we have

$$\begin{aligned}
\limsup_{A\to\infty} A^{2n}\mathcal{I}_1 &\leq \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\ |(\tau^n h_j, h_\ell) - (\tau^n f_j, f_\ell)| + \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\ (\tau^n f_j, f_\ell) \\
&= \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\ \left|(L_n h_j, h_\ell) - (\tilde{L_n} f_j, f_\ell)\right| + \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\ (\tilde{L_n} f_j, f_\ell) \\
&< \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\ \epsilon + \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\ \Lambda_j(f_j, f_\ell) \\
&\leq \sum_{j,\ell=1}^{k} \epsilon + \sum_{j=1}^{k} |c_j|^2\ \Lambda_j \\
&\leq \epsilon k^2 + \Lambda_k.
\end{aligned}$$

We now consider the integral $\mathcal{I}_2$. From Lemma 5.1, we can write

$$\begin{aligned}
\mathcal{I}_2 &= \int_{\delta A}^{\infty} |G(t/A) - F(t/A)|\, |\hat{g_A}(t)|^2 t^{2\nu} dt \\
&= A\int_{\delta}^{\infty} |G(t) - F(t)|\, |\hat{g_A}(At)|^2 (At)^{2\nu} dt \\
&\leq KA\int_{\delta}^{\infty} |\hat{g_A}(At)|^2 (At)^{2\nu} dt
\end{aligned}$$

for some constant $K$. As shown above (from Lemma 3.4),

$$\hat{g_A}(At) = \sum_{j=1}^{k} c_j(1 + (t/A)^2)^n \hat{h_j}(At) = \sum_{j=1}^{k}\sum_{m=0}^{n}\frac{c_j}{A^{2m}}\binom{n}{m}\widehat{\tau^m h_j}(At).$$

We define $H(t) = \sum_{j=1}^{k}\sum_{m=0}^{n}\binom{n}{m}\widehat{\tau^m h_j}(t)$. Since $\tau^m h_j \in \mathcal{G}$ for all values of $m$ and $j$, from Lemma 3.4, $\widehat{\tau^m h_j}$ is rapidly decreasing for all $m, j$, and so $H(t)$ is a rapidly decreasing function which is independent of $A$. Thus we have, for

Since the inner products $(\tau^m h_j, \tau^p h_\ell)$ are independent of $A$, we have

$$\begin{aligned}\lim_{A\to\infty} (g_A, g_A) &= \sum_{j,\ell=1}^{k} c_j\overline{c_\ell}\,(h_j, h_\ell)\\ &\geq \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\,(f_j, f_\ell) - \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\,|(h_j, h_\ell) - (f_j, f_\ell)|\\ &\geq \sum_{j=1}^{k} |c_j|^2 (f_j, f_j) - \sum_{j,\ell=1}^{k} |c_j\overline{c_\ell}|\,\epsilon\\ &\geq 1 - \epsilon k^2.\end{aligned}$$

Therefore we have

$$\limsup_{A\to\infty} A^{2n}\mu_{k,A} \leq \frac{(\sigma^2+\epsilon)(\epsilon k^2 + \Lambda_k)}{1-\epsilon k^2}.$$

Since the LHS has no dependence on $\epsilon$ and it was chosen arbitrarily, letting $\epsilon \to 0$ we have

$$\limsup_{A\to\infty} A^{2n}\mu_{k,A} \leq \sigma^2\Lambda_k.$$

Thus for sufficiently large $A$, $\mu_{k,A} < M$, and by the Theorem 2.18, we have

$$\lambda_{k,A} = \sup_{\mathcal{M}_k}\left[\inf_{g\in\mathcal{M}_k}\frac{(T_A g, g)}{(g,g)}\right] \geq \min_{g\in\mathcal{L}_k}\frac{(T_A g, g)}{(g,g)} = M - \mu_{k,A} > 0.$$

Thus the operator $T_A$ will have at least $k$ positive eigenvalues with

$$M \geq \lambda_{1,A} \geq \lambda_{2,A} \geq \cdots \geq \lambda_{k,A} > 0.$$

We also see that $M - \lambda_{k,A} \leq \mu_{k,A}$ so that we have

$$\limsup_{A\to\infty} A^{2n}(M - \lambda_{k,A}) \leq \sigma^2\Lambda_k. \ \square$$

# Chapter 6: Second Main Theorem

**Lemma 6.1.** *Let $f \in D(L_1)$ and $g = S_A f$. Then $(I + \frac{\tau}{A^2})g = f$ and $(I - S_A)f = \frac{\tau}{A^2}g$.*

Proof: From Equation (3.5) we have $(S_A f)(x) = \int_0^\infty \frac{1}{1+\frac{t^2}{A^2}} \hat{f}(t) J(xt) t^{2\nu} dt$. By applying $\frac{\tau}{A^2}$, since $J(xt)$ and $J'(xt)$ are bounded from Lemma 3.1 and $\hat{f}(t)$ is rapidly decreasing from Lemma 3.4, we can apply Lemma 2.2 twice to differentiate under the integral:

$$\frac{\tau_x}{A^2}(S_A f)(x) = \int_0^\infty \frac{1}{1 + \frac{t^2}{A^2}} \hat{f}(t) \frac{\tau_x}{A^2} J(xt) t^{2\nu} dt$$

and from Lemma 2.9,

$$\int_0^\infty \frac{\frac{t^2}{A^2}}{1 + \frac{t^2}{A^2}} \hat{f}(t) J(xt) t^{2\nu} dt.$$

Adding this equation to our initial one yields

$$(I + \frac{\tau}{A^2})(S_A f)(x) = \int_0^\infty \hat{f}(t) J(xt) t^{2\nu} dt = f(x).$$

Thus, $(I + \frac{\tau}{A^2})g = f$, and so $\frac{\tau}{A^2}g = f - g = (I - S_A)f$. □

**Theorem 6.2.** *For each integer $k \geq 1$ and $A$ su ciently large, the integral operator $T_A$, with $F(t)$ satisfying conditions H1-H3, will have $k$ positive eigenvalues satisfying $M \geq \lambda_{1,A} \geq \lambda_{2,A} \geq \ldots \geq \lambda_{k,A} > 0$ and*

$$\lim_{A\to\infty} A^2(M - \lambda_{k,A}) = \sigma^2 {z_k}^2$$

*where $z_k$ is the $k^{th}$ positive zero of the Bessel function $J_{\nu-1/2}$.*

Proof: Note that in the case $n = 1$,

$$D(L_1) = \{v \in C^\infty(0, 1) : v = 0 \text{ near } x = 1 \text{ and } v' = 0 \text{ near } x = 0\}.$$

From Theorem 5.4, we know

$$\limsup_{A\to\infty} A^2(M - \lambda_{j,A}) \leq \sigma^2 \Lambda_j \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Define $\alpha_j = \liminf_{A\to\infty} A^2(M - \lambda_{j,A})$. We will show that $\alpha_j = \sigma^2\Lambda_j$ for $j = 1, 2, \ldots, k$, which will complete the proof. Since $\alpha_k = \liminf_{A\to\infty} A^2(M - \lambda_{k,A})$, there exists a sequence $\mathcal{S}$ of real numbers tending monotonically to infinity such that

$$\alpha_k = \lim_{A\to\infty} A^2(M - \lambda_{k,A}), \quad (A \in \mathcal{S})$$

exists. Since the sequences $\{A^2(M-\lambda_{j,A}) : A \in \mathcal{S}\}$, for $1 \le j < k$, are each bounded from Theorem 5.4, by the Bolzano-Weierstrass Theorem we may assume without loss of generality that

$$\alpha_j = \lim_{A\to\infty} A^2(M-\lambda_{j,A}), \ \ (A \in \mathcal{S}) \ \text{ for } \ j = 1, 2, \ldots, k.$$

Let $f_{j,A}$ be a normalized eigenfunction of $T_A$, corresponding to the eigenvalue $\lambda_{j,A}$, for $j = 1, 2, \ldots, k$. Since $D(L_1)$ is dense, we choose $g_{j,A} \in D(L_1)$ such that $\|g_{j,A} - f_{j,A}\| < \min\left\{\frac{1}{A^3\|MI-T_A\|}, \frac{1}{A}\right\}$ and $\|g_{j,A}\| = 1$. Let $u_{j,A} = S_A g_{j,A}$. Then from Lemma 6.1, $(I + \frac{\tau}{A^2})u_{j,A} = g_{j,A}$ and $(I - S_A)g_{j,A} = \frac{\tau}{A^2}u_{j,A}$.

We will now establish a useful inequality. Consider

$$\begin{aligned}
\|g_{j,A} - u_{j,a}\|^2 \ &+ \ \left(\frac{\tau}{A^2}u_{j,A}, u_{j,A}\right) = ((I-S_A)g_{j,A}, (I-S_A)g_{j,A}) + ((I-S_A)g_{j,A}, S_A g_{j,A}) \\
&= \ ((I-S_A)g_{j,A}, g_{j,A}) = \langle (I-S_A)g_{j,A}, g_{j,A}\rangle \\
&= \ \langle (1-F_S(t/A))\hat{g}_{j,A}, \hat{g}_{j,A}\rangle \le \frac{1}{q^2}\langle (M-F(t/A))\hat{g}_{j,A}, \hat{g}_{j,A}\rangle \\
&= \ \frac{1}{q^2}\langle (MI-T_A)g_{j,A}, g_{j,A}\rangle = \frac{1}{q^2}((MI-T_A)g_{j,A}, g_{j,A}) \\
&= \ \frac{1}{q^2}\left[((MI-T_A)f_{j,A}, g_{j,A}) + ((MI-T_A)(g_{j,A}-f_{j,A}), g_{j,A})\right] \\
&\le \ \frac{1}{q^2}\left[(M-\lambda_{j,A}) + \|MI-T_A\|\,\|g_{j,A}-f_{j,A}\|\right] \\
&\le \ \frac{1}{q^2}\left[(M-\lambda_{j,A}) + \frac{1}{A^3}\right].
\end{aligned}$$

From Lemma 3.11, $(\tau u_{j,A}, u_{j,A})$ is a nonnegative quantity, so we can consider the two terms on the LHS of the above inequality separately and then apply Theorem 5.4 to obtain

$$\lim_{A\to\infty}\|g_{j,A} - u_{j,a}\| = 0 \tag{6.1}$$

and $(\tau u_{j,A}, u_{j,A})$ is bounded, by $M_1$ say.

We now wish to show that for $1 \le j \le k$, $\{u_{j,A}(x) : A \in \mathcal{S}\}$ is bounded at $x = 1$ and equicontinuous on compact subsets of $(0, 1]$. Since each term in the inequality of Lemma 3.11 is nonnegative, we can consider the two terms of the LHS separately. We first show equicontinuity on compact subsets of $(0, 1]$. For $0 < \epsilon \le x_1 \le x_2 \le 1$,

$$\begin{aligned}
|u_{j,A}(x_2) - u_{j,A}(x_1)|^2 \ &\le \ (\tau u_{j,A}, u_{j,A})\left(\int_{x_1}^{x_2}\frac{1}{x^{2\nu}}\,dx\right) \\
&\le \ M_1\epsilon^{-2\nu}(x_2 - x_1)
\end{aligned}$$

which implies equicontinuity on $[\epsilon, 1]$. To prove that $\{u_{j,A} : A \in \mathcal{S}\}$ is bounded at $x = 1$, we consider the second term of the inequality of Lemma 29, yielding

$$A\frac{K_{\nu+1/2}(A)}{K_{\nu-1/2}(A)}|u_{j,A}(1)|^2 \le (\tau u_{j,A}, u_{j,A}) \le M_1.$$

Then from Lemma 2.8

$$\limsup_{A\to\infty} |u_{j,A}(1)|^2 \le \lim_{A\to\infty} \frac{M_1}{A}\frac{K_{\nu-1/2}(A)}{K_{\nu+1/2}(A)} = 0$$

and thus $\{u_{j,A} : A \in \mathcal{S}\}$ is bounded at $x = 1$, and in fact $\lim_{A\to\infty} u_{j,A}(1) = 0, (A \in \mathcal{S})$. Thus, from Corollary 2.5, there exists a subsequence $\mathcal{S}'$ of $\mathcal{S}$ such that $\{u_{j,A} : A \in \mathcal{S}'\}$ converges uniformly on $[\epsilon, 1]$. Since $\epsilon > 0$ can be made arbitrarily small, we use a diagonalization argument to find a subsequence $\mathcal{S}_j$ of $\mathcal{S}$ such that $\{u_{j,A} : A \in \mathcal{S}_j\}$ converges uniformly on each compact subset of $(0, 1]$.

Since we can use this argument for each $j$, we first find a subsequence $\mathcal{S}_1$ such that $\{u_{1,A} : A \in \mathcal{S}_1\}$ converges uniformly on each compact subset of $(0, 1]$, then we find a subsequence $\mathcal{S}_2$ of $\mathcal{S}_1$ such that $\{u_{2,A} : A \in \mathcal{S}_2\}$ converges uniformly on each compact subset of $(0, 1]$, and continue this process until we find a subsequence $\mathcal{S}_k$ such that $\{u_{j,A} : A \in \mathcal{S}_k\}$ converges uniformly on each compact subset of $(0, 1]$ for all $j = 1, 2, \ldots, k$ simultaneously. We will assume without loss of generality that $\mathcal{S}$ is this deep subsequence. Also, we define for $0 < x \le 1$,

$$u_j(x) = \lim_{A\to\infty} u_{j,A}(x) \;\; (A \in \mathcal{S}) \;\; \text{for } j = 1, 2, \ldots, k$$

and we note that from the pointwise convergence at $x = 1$ given above,

$$u_j(1) = 0 \;\; \text{for } j = 1, 2, \ldots, k.$$

The goal now is to show that $u_j \in D({L_1}^*)$ and that ${L_1}^* u_j = \frac{\alpha_j}{\sigma^2} u_j$ for each $j$. We begin by investigating the equality

$$\big(A^2(MI - T_A) f_{j,A}, v\big) = \big(A^2(M - \lambda_{j,A}) f_{j,A}, v\big)$$

for a fixed $v \in D(L_1)$ and evaluating the limit of each side as $A \to \infty$ with $A \in \mathcal{S}$. We first show that

$$\lim_{A\to\infty} \text{RHS} = \lim_{A\to\infty} \big(A^2(M - \lambda_{j,A}) f_{j,A}, v\big) = (\alpha_j u_j, v), \;\; (A \in \mathcal{S})$$

and then argue that

$$\lim_{A\to\infty} \text{LHS} = \lim_{A\to\infty} \big(A^2(MI - T_A) f_{j,A}, v\big) = \sigma^2(u_j, \tau v), \;\; (A \in \mathcal{S})$$

for each $j$. This will give the equality, for arbitrary $v \in D(L_1)$,

$$(u_j, \tau v) = \Big(\frac{\alpha_j}{\sigma^2} u_j, v\Big)$$

which implies that $u_j \in D({L_1}^*)$ and that ${L_1}^* u_j = \frac{\alpha_j}{\sigma^2} u_j$. Further, since $u_j(1) = 0$, from Theorem 4.10, $u_j \in D(\tilde{L}_1)$ and so for each $j$, $(\frac{\alpha_j}{\sigma^2}, u_j)$ is an eigenpair of the Friedrichs Extension (although it remains to be shown that $u_j$ is a nontrivial function).

We consider the RHS and write

$$\left(A^2(M-\lambda_{j,A})f_{j,A},v\right) = A^2(M-\lambda_{j,A})(f_{j,A}-u_{j,A},v) + A^2(M-\lambda_{j,A})(u_{j,A},v)$$

and claim that

$$\lim_{A\to\infty} A^2(M-\lambda_{j,A})(f_{j,A}-u_{j,A},v) = 0.$$

From Theorem 5.4, $\limsup_{A\to\infty} A^2(M-\lambda_{j,A}) \leq \sigma^2\Lambda_j$, so it suffices to show that

$$\lim_{A\to\infty} (f_{j,A}-u_{j,A},v) = 0.$$

We decompose this inner product as

$$(f_{j,A}-u_{j,A},v) = (f_{j,A}-g_{j,A},v) + (g_{j,A}-u_{j,A},v)$$

and then

$$(f_{j,A}-g_{j,A},v) \leq \frac{\|v\|}{A}$$

by the choice of $g_{j,A}$. Thus, we have

$$\lim_{A\to\infty} (f_{j,A}-u_{j,A},v) = \lim_{A\to\infty} (g_{j,A}-u_{j,A},v).$$

But from Equation (6.1),

$$\lim_{A\to\infty} \|g_{j,A}-u_{j,A}\| = 0$$

which implies

$$\lim_{A\to\infty} (f_{j,A}-u_{j,A},v) = 0$$

and so

$$\lim_{A\to\infty} \text{RHS} = \lim_{A\to\infty} A^2(M-\lambda_{j,A})(u_{j,A},v) = \alpha_j \lim_{A\to\infty} (u_{j,A},v), \quad (A \in \mathcal{S}).$$

In order to apply the Lebesgue Dominated Convergence Theorem, we need a function that bounds $|u_{j,A}(x)v(x)|x^{2\nu}$. From Lemma 3.11, letting $x_2 = 1$ and $x_1 = x$, and using $(\tau u_{j,A}, u_{j,A}) \leq M_1$, we have

$$\begin{aligned} |u_{j,A}(1) - u_{j,A}(x)|^2 &\leq M_1 \int_x^1 \frac{1}{t^{2\nu}}\, dt \\ |u_{j,A}(1) - u_{j,A}(x)| &\leq \sqrt{M_1}\, \frac{1}{x^\nu} \\ |u_{j,A}(x)| &\leq \sqrt{M_1}\, \frac{1}{x^\nu} + |u_{j,A}(1)| \end{aligned}$$

and since we have the sequence $\{u_{j,A}(1)\}$ bounded, by $M_2$ say, we have

$$|u_{j,A}| \leq \sqrt{M_1}\frac{1}{x^\nu} + M_2 \tag{6.2}$$

and

$$\begin{aligned} |u_{j,A}(x)v(x)|x^{2\nu} &\leq \left(\sqrt{M_1}\,\frac{1}{x^\nu} + M_2\right)|v(x)|x^{2\nu} \\ &\leq (\sqrt{M_1}x^\nu + M_2x^{2\nu})|v(x)| \end{aligned}$$

which is integrable. Thus $\lim_{A\to\infty}(u_{j,A}, v) = (u_j, v)$, and

$$\lim_{A\to\infty} \text{RHS} = (\alpha_j u_j, v).$$

Now we consider the LHS and write

$$\big(A^2(MI - T_A)f_{j,A}, v\big) = \big(A^2(MI - T_A)(f_{j,A} - g_{j,A}), v\big) + \big(A^2(MI - T_A)g_{j,A}, v\big)$$

and

$$\big(A^2(MI - T_A)(f_{j,A} - g_{j,A}), v\big) \leq \frac{\|v\|}{A}$$

by the choice of $g_{j,A}$, so we have

$$\lim_{A\to\infty} \text{LHS} = \lim_{A\to\infty} \big(A^2(MI - T_A)f_{j,A}, v\big) = \lim_{A\to\infty} \big(A^2(MI - T_A)g_{j,A}, v\big)\,.$$

Extending $v$ as before, we can apply Theorem 2.11 to yield

$$\big(A^2(MI - T_A)g_{j,A}, v\big) = \langle A^2(MI - T_A)g_{j,A}, v\rangle = \langle A^2(M - F(t/A))\hat{g}_{j,A}, \hat{v}\rangle.$$

We use the same decomposition of $M - F(t/A)$ as in Lemma 5.1 (with $n = 1$). Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\big\langle A^2(M - F(t/A))\hat{g}_{j,A}, \hat{v}\big\rangle - \big\langle A^2\sigma^2(1 - F_S(t/A))\hat{g}_{j,A}, \hat{v}\big\rangle = \mathcal{N}_1 + \mathcal{N}_2$$

where

$$\mathcal{N}_1 = \big\langle A^2 \left(M - G(t/A) - \sigma^2(1 - F_S(t/A))\right) \hat{g}_{j,A}, \hat{v}\big\rangle$$
$$\text{and } \ \mathcal{N}_2 = \big\langle A^2 \left(G(t/A) - F(t/A)\right) \hat{g}_{j,A}, \hat{v}\big\rangle\,.$$

Consider $\mathcal{N}_2$: since $G(u) = F(u)$ for $u < \delta$,

$$\mathcal{N}_2 = \big\langle A^2 \left(G(t/A) - F(t/A)\right) \hat{g}_{j,A}, \hat{v}\big\rangle = A^2 \int_{\delta A}^{\infty} \left(G(t/A) - F(t/A)\right) \hat{g}_{j,A}(t)\overline{\hat{v}(t)}t^{2\nu}\, dt.$$

Note that from Theorem 2.11 and since $g_{j,A} \in D(L_1)$ we have

$$\int_0^\infty |\hat{g}_{j,A}(t)|^2 t^{2\nu}\, dt = \int_0^\infty |g_{j,A}(t)|^2 t^{2\nu}\, dt = \int_0^1 |g_{j,A}(t)|^2 t^{2\nu}\, dt = \|g_{j,A}\|^2 = 1. \quad (6.3)$$

Using the Schwarz inequality and Equation (6.3), we get

$$\begin{aligned} |\mathcal{N}_2|^2 &\le A^4 \int_{\delta A}^{\infty} |\hat{g}_{j,A}(t)|^2 t^{2\nu}\, dt \int_{\delta A}^{\infty} |(G(t/A) - F(t/A))\hat{v}(t)|^2 t^{2\nu} dt \\ &\le A^4 \int_{\delta A}^{\infty} |(G(t/A) - F(t/A))\hat{v}(t)|^2 t^{2\nu} dt. \end{aligned}$$

With a change of variables we can rewrite the integral as

$$A^5 \int_{\delta}^{\infty} |(G(t) - F(t))\hat{v}(At)|^2 (At)^{2\nu} dt$$

and use the bound $|G(u) - F(u)| < M_3$ from Lemma 5.1 to write

$$|\mathcal{N}_2|^2 \le M_3 A^5 \int_{\delta}^{\infty} |\hat{v}(At)|^2 (At)^{2\nu}\, dt$$

and since $\hat{v}$ is rapidly decreasing from Lemma 3.4, for $A$ sufficiently large,

$$|\hat{v}(At)|(At)^{2\nu} \le (At)^{-6} \text{ on } (\delta, \infty).$$

Then

$$\begin{aligned} |\mathcal{N}_2|^2 &\le M_3 A^5 \int_{\delta}^{\infty} (At)^{-6}\, dt \\ &\le \frac{M_3}{A} \int_{\delta}^{\infty} t^{-6}\, dt \end{aligned}$$

which implies

$$\lim_{A\to\infty} |\mathcal{N}_2| = \lim_{A\to\infty} \left| \left\langle A^2 \left(G(t/A) - F(t/A)\right) \hat{g}_{j,A}, \hat{v} \right\rangle \right| = 0.$$

Consider $\mathcal{N}_1$: from the Schwarz inequality, Lemma 5.1, and Equation (6.3),

$$\begin{aligned} |\mathcal{N}_1|^2 &\le \int_0^{\infty} A^2 \left| M - G(t/A) - \sigma^2(1 - F_S(t/A)) \right| |\hat{g}_{j,A}(t)||\hat{v}(t)| t^{2\nu}\, dt \\ &< \epsilon \int_0^{\infty} A^2 (1 - F_S(t/A)) |\hat{g}_{j,A}(t)||\hat{v}(t)| t^{2\nu}\, dt \\ &\le \epsilon \int_0^{\infty} t^2 |\hat{g}_{j,A}(t)||\hat{v}(t)| t^{2\nu}\, dt \\ &\le \epsilon \int_0^{\infty} |\hat{g}_{j,A}(t)|^2 t^{2\nu}\, dt \int_0^{\infty} t^4 |\hat{v}(t)|^2 t^{2\nu}\, dt \\ &\le \epsilon \int_0^{\infty} t^4 |\hat{v}(t)|^2 t^{2\nu}\, dt \end{aligned}$$

and since $\hat{v}(t)$ is a fixed rapidly decreasing function, $\int_0^\infty t^4|\hat{v}(t)|^2 t^{2\nu}\,dt$ converges to a constant $M_4$ independent of $A$, we have

$$|\mathcal{N}_1| = \left|\left\langle A^2\left(M - G(t/A) - \sigma^2(1 - F_S(t/A))\right)\hat{g}_{j,A}, \hat{v}\right\rangle\right| < \sqrt{\epsilon M_4}.$$

Now we can write

$$\begin{aligned}\limsup_{A\to\infty}\left|\left\langle A^2(M - F(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle\right. &- \left.\left\langle A^2\sigma^2(1 - F_S(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle\right| \\ &\leq \limsup_{A\to\infty}|\mathcal{N}_1 + \mathcal{N}_2| \\ &< \sqrt{\epsilon M_4},\end{aligned}$$

which is true for arbitrary $\epsilon > 0$ and therefore

$$\lim_{A\to\infty}\text{LHS} = \lim_{A\to\infty}\left\langle A^2(M - F(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle = \lim_{A\to\infty}\left\langle A^2\sigma^2(1 - F_S(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle.$$

Using Theorem 2.11 and the fact that $v(t) = 0$ for $t \geq 1$,

$$\begin{aligned}\left\langle A^2\sigma^2(1 - F_S(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle &= \sigma^2\left\langle A^2(I - S_A)g_{j,A}, v\right\rangle \\ &= \sigma^2\left(A^2(I - S_A)g_{j,A}, v\right) \\ &= \sigma^2\left(\tau u_{j,A}, v\right).\end{aligned}$$

Further, integrating by parts,

$$(\tau u_{j,A}, v) = -\,x^{2\nu}u'_{j,A}(x)\overline{v(x)}\Big|_0^1 + x^{2\nu}u_{j,A}(x)\overline{v'(x)}\Big|_0^1 + (u_{j,A}, \tau v).$$

Since $v \in D(L_1)$, both $v$ and $v'$ vanish in a neighborhood of $x = 1$ and are bounded near $x = 0$. Also, from Lemma 3.9, $u_{j,A}$ and $u'_{j,A}$ are both bounded on $[0, 1]$, so all the boundary terms vanish. So we now have

$$\lim_{A\to\infty}\text{LHS} = \sigma^2\lim_{A\to\infty}(u_{j,A}, \tau v).$$

From (6.2), $\left|u_{j,A}(x)\overline{\tau v(x)}\right| x^{2\nu} \leq \left(\sqrt{M_1}x^\nu + M_2x^{2\nu}\right)\left|\overline{\tau v(x)}\right|$ which is integrable, so we can apply the Lebesgue Dominated Convergence Theorem and we have

$$\lim_{A\to\infty}\text{LHS} = \sigma^2(u_j, v), \quad (A \in \mathcal{S}).$$

As argued previously, since $u_j$ satisfies the boundary condition given by Theorem 4.10,

$$(u_j, L_1 v) = (\tilde{L}_1 u_j, v)$$

where $\tilde{L}_1 u_j = \frac{\alpha_j}{\sigma^2} u_j$ for each $j$, and we now wish to show that the $u_j$ are distinct (orthonormal) eigenfunctions of $\tilde{L}_1$. For $1 \leq j, k \leq n$,

$$\begin{aligned} |(u_{j,A}, u_{k,A}) - (f_{j,A}, f_{k,A})| &= |(u_{j,A}, u_{k,A}) - (f_{j,A}, u_{k,A}) + (f_{j,A}, u_{k,A}) - (f_{j,A}, f_{k,A})| \\ &\leq |(u_{j,A} - f_{j,A}, u_{k,A})| + |(f_{j,A}, u_{k,A} - f_{k,A})| \\ &\leq \|u_{j,A} - f_{j,A}\| \, \|u_{k,A}\| + \|f_{j,A}\| \, \|u_{k,A} - f_{k,A}\| \\ &\leq \|u_{j,A} - f_{j,A}\| \left( \|u_{k,A} - f_{k,A}\| + \|f_{k,A}\| \right) + \|u_{k,A} - f_{k,A}\| \\ &\leq \|u_{j,A} - f_{j,A}\| \left( \|u_{k,A} - f_{k,A}\| + 1 \right) + \|u_{k,A} - f_{k,A}\| \end{aligned}$$

We will now argue that for each $j$,

$$\lim_{A \to \infty} \|u_{j,A} - f_{j,A}\| = 0.$$

From our choice of $g_{j,A}$, we have

$$\|u_{j,A} - f_{j,A}\| \leq \|u_{j,A} - g_{j,A}\| + \|g_{j,A} - f_{j,A}\| \leq \|u_{j,A} - g_{j,A}\| + \frac{1}{A}$$

so we know

$$\limsup_{A \to \infty} \|u_{j,A} - f_{j,A}\| \leq \lim_{A \to \infty} \|g_{j,A} - u_{j,A}\|$$

but from Equation (6.1),

$$\limsup_{A \to \infty} \|u_{j,A} - f_{j,A}\| \leq 0$$

and we can conclude

$$\limsup_{A \to \infty} |(u_{j,A}, u_{k,A}) - (f_{j,A}, f_{k,A})| \leq 0$$

and so

$$\lim_{A \to \infty} (u_{j,A}, u_{k,A}) = \lim_{A \to \infty} (f_{j,A}, f_{k,A}) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} .$$

From (6.2),

$$u_{j,A}(x)\overline{u_{k,A}(x)}x^{2\nu} \leq \left( \sqrt{M_1} x^{-\nu} + M_2 \right)^2 x^{2\nu} = M_1 + 2\sqrt{M_1} M_2 x^{\nu} + M_2^2 x^{2\nu}$$

which is integrable, so we can apply the Lebesgue Dominated Convergence Theorem to give

$$\lim_{A \to \infty} (u_{j,A}, u_{k,A}) = (u_j, u_k)$$

and we have that the $u_j$ are orthonormal and therefore are distinct eigenfunctions of $\tilde{L}_1$.

We now show that $\alpha_j = \sigma^2 \Lambda_j$ for each $j = 1, 2, \ldots, k$ to complete the proof. From Theorem 5.4, we have the eigenvalues

$$M \geq \lambda_{1,A} \geq \lambda_{2,A} \geq \cdots \geq \lambda_{k,A}$$

which implies

$$A^2(M-\lambda_{1,A}) \leq A^2(M-\lambda_{2,A}) \leq \cdots \leq A^2(M-\lambda_{k,A})$$

and taking limits as $A \to \infty$ with $A \in \mathcal{S}$ gives

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k.$$

From Theorem 5.4, $\alpha_1 \leq \sigma^2\Lambda_1$ and since $u_1$ is an eigenfunction, $\frac{\alpha_1}{\sigma^2}$ is an eigenvalue no greater than the smallest eigenvalue. Thus, it must be that $\frac{\alpha_1}{\sigma^2}$ is the smallest eigenvalue, so $\alpha_1 = \sigma^2\Lambda_1$. We now assume the inductive hypothesis that $\alpha_j = \sigma^2\Lambda_j$ and show that $\alpha_{j+1} = \sigma^2\Lambda_{j+1}$. From Theorem 5.4,

$$\sigma^2\Lambda_j = \alpha_j \leq \alpha_{j+1} \leq \sigma^2\Lambda_{j+1}$$

and since $\frac{\alpha_{j+1}}{\sigma^2}$ is an eigenvalue, it must be either $\Lambda_j$ or $\Lambda_{j+1}$. Suppose $\frac{\alpha_{j+1}}{\sigma^2} = \Lambda_j$, then $u_j$ and $u_{j+1}$ are both eigenfunctions of $\Lambda_j$, which contradicts Theorem 4.11. Thus, we have $\alpha_{j+1} = \sigma^2\Lambda_{j+1}$ and we conclude that

$$\lim_{A\to\infty} A^2(M-\lambda_{j,A}) = \sigma^2\Lambda_j, \ \ (A \in \mathcal{S}) \ \text{ for } \ j = 1, 2, \ldots, k.$$

Recall from Theorem 4.11, $\Lambda_j = {z_j}^2$ where $z_j$ is the $j^{th}$ positive zero of the Bessel function $J_{\nu-1/2}$. $\Box$

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# Vita

Grey Ballard was born on June 9, 1984 in Winston-Salem, North Carolina. He graduated from Elkin High School in May 2002, then attended Wake Forest University where he majored in Mathematics and Computer Science. He graduated Summa Cum Laude with a Bachelor of Science in 2006 and continued his education the following two years at Wake Forest in the Master of Arts program in Mathematics, obtaining that degree in May 2008. He is a member of Phi Beta Kappa, Pi Mu Epsilon, Upsilon Pi Epsilon, and the American Mathematical Society.