CURVES, KNOTS, AND TOTAL CURVATURE

By

CHARLES M. EVANS

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Approved By:

R. Jason Parsley, Ph.D., Co-Advisor ________________

Stephen B. Robinson, Ph.D., Co-Advisor ________________

Examining Committee:

James Kuzmanovich, Ph.D., Chairperson ________________

Hugh N. Howards, Ph.D. ________________

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Abstract

Charles Evans

We present an exposition of various results dealing with the total curvature of curves in Euclidean 3-space. There are two primary results: Fenchel's theorem and the theorem of Fary and Milnor. Fenchel's theorem states that the total curvature of a simple closed curve is greater than or equal to 2π, with equality if and only if the curve is planar convex. The Fary-Milnor theorem states that the total curvature of a simple closed knotted curve is strictly greater than 4π. Several methods of proof are supplied, utilizing both curve-theoretic and surface-theoretic techniques, surveying methods from both differential and integral geometry. Related results are considered: the connection between total curvature and bridge number; an analysis of total curvature plus total torsion; a lower bound on the length of the normal indicatrix.

Chapter 1: Introduction

Differential geometry is like linear algebra in that its umbrella casts a wide mathematical shadow. The fathers of the field are generally considered to be Leonhard Euler and Gaspard Monge, though the study of differential geometry truly began in earnest when Carl Friedrich Gauss, a tremendously influential mathematician himself, took an interest. Reading through a history of this field is like walking through a Mathematics Hall of Fame: Riemann, Bolyai, Lobachevsky, Grassmann, Cayley, Klein, Hilbert, Whitehead, Minkowski, Einstein, Chern, Milnor, and the list of contributors continues. So what exactly is differential geometry? This branch of geometry makes use of differential and integral calculus to address problems of geometry – the name of this field is not poorly chosen. A student studying in this discipline will typically begin with the theory of curves in the plane and in space, then move on to surfaces in three-dimensions; this path follows the historical development of the eighteenth and nineteenth century, and is concerned primarily with questions of extrinsic and intrinsic properties. A later course will expand the field to include a more generalized theory of manifolds and their associated geometric properties. The field itself contains many divisions: Riemannian geometry, contact geometry, Finsler geometry, symplectic geometry, etc., and the power of differential-geometric techniques can hardly be overstated. Applications of these techniques are myriad and far-reaching. These methods appear in physics with relativity, electromagnetism, Lagrangian and Hamiltonian mechanics; in econometrics; in engineering with digital signal processing; in probabilty and statistics; even in a field as disparate as structural geology, where differential-geometric methods are used to analyze geologic structures.

Our primary focus will be curves, both in the plane and in space, and their total curvatures. We will at points appeal to some heavier machinery, and we assume the reader is relatively familiar with some key concepts, so some details are omitted. We do, however, begin with some basic definitions that will be used throughout.

Definition 1.1. A *parameterized differentiable curve* is a differentiable map $\alpha : I \to \mathbb{R}^3$ where I is an open interval in $\mathbb{R}$. A curve is said to *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

Unless otherwise stated, we will assume all curves are regular. We will also assume that α is not constant on any sub-interval of I. Further, given $t \in I$, we define the *arc length* of α from t_0 to t as

$$s(t) = \int_{t_0}^{t} |\alpha'(u)| \, du \, .$$

We know $\alpha'(t) \neq 0$, and so $s(t)$ is differentiable. Thus, $s'(t) = |\alpha'(t)|$. If $s'(t) = |\alpha'(t)| = 1$ for all t, then we say α is *parameterized by arc length*. Given such a curve, we let s denote the arc length parameter and write $\alpha(s)$. We assume all curves are thus parameterized, unless otherwise stated; this restriction is not truly essential, but it is a tremendous help in proving theoretical results. In fact, we can reparameterize α to obtain some curve β, parameterized by arc length, with the same image as α. This is a standard result.

Now let $\alpha(s)$ be a curve parameterized by arc length. The tangent vector $\alpha'(s)$ has unit length, as we have assumed, and so the quantity $|\alpha''(s)|$ measures the rate of change of the angle that neighboring tangents make with the tangent at s. Said differently, $|\alpha''(s)|$ quantifies how quickly the curve pulls away from the tangent at s (within a neighborhood of s). This leads us to a definition.

Definition 1.2. Let $\alpha : I \to \mathbb{R}^3$ be parameterized by arc length. The quantity $|\alpha''(s)| = k(s)$ is the *curvature* of α at s.

Example 1.3. Consider the circle of radius r parameterized by $\alpha(t) = (r\cos t, r\sin t)$ for $t \in [0, 2\pi)$. Then $\alpha(t) = (-r\sin t, r\cos t)$, and $|\alpha(t)| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r$. So certainly this is not an arc length parameterization. However, notice what happens if we let $t = \frac{s}{r}$. Our parameterization is now $\alpha(s) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)\right)$, and $\alpha'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right)$. Thus

$$|\alpha'(s)| = \sqrt{\sin^2\left(\frac{s}{r}\right) + \cos^2\left(\frac{s}{r}\right)} = 1\,.$$

So our new parameterization $\alpha(s)$ is an arc length parameterization. Further, we have

$$\alpha''(s) = \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right)\,.$$

Hence

$$k(s) = |\alpha''(s)| = \sqrt{\frac{1}{r^2}\cos^2\left(\frac{s}{r}\right) + \frac{1}{r^2}\sin^2\left(\frac{s}{r}\right)} = \frac{1}{r}\,.$$

So a circle of radius r has constant curvature equal to $\frac{1}{r}$.

In what follows, we assume the reader is familiar with the inner (or dot) product and cross product of vectors in $\mathbb{R}^3$. The inner product will be denoted $\langle\cdot,\cdot\rangle$.

If $k(s) \neq 0$, there is a well-defined unit vector $\mathbf{n}(s)$ given by $\alpha''(s) = k(s)\mathbf{n}(s)$. This vector is in fact normal to $\alpha'(s)$; to see this, simply differentiate $\langle\alpha'(s), \alpha'(s)\rangle = 1$ to obtain $2\langle\alpha'(s), \alpha''(s)\rangle = 0$. So we call $\mathbf{n}(s)$ the *normal vector* at s. If $\alpha''(s) = 0$, and thus $k(s) = 0$, for some $s \in I$, we call s a *singular point of order* 1. For now, our attentions are focused on curves parameterized by arc length without such singular points. Let $\mathbf{t}(s) = \alpha'(s)$ and notice that $\mathbf{t}'(s) = k(s)\mathbf{n}(s)$. The plane determined by $\mathbf{t}$ and $\mathbf{n}$ is called the *osculating plane* at s.

Remark 1.4. For curves not parameterized by arc length, the tangent vector $\mathbf{t}$ may not be a unit vector, but curvature still makes sense. In this case, $\mathbf{t}$ is given by

$$\mathbf{t}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}\,.$$

Thus

$$k(t) = \left|\frac{d\mathbf{t}}{ds}\right| = \left|\frac{d\mathbf{t}/dt}{ds/dt}\right| = \frac{|\mathbf{t}'(t)|}{|\alpha' t|}\,.$$

(Recall that $s'(t) = |\alpha'(t)|$.)

Now let $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$; so $\mathbf{b}$ is normal to the osculating plane. This vector is the *binormal vector* at s. Since $\mathbf{t}$ and $\mathbf{n}$ are unit length, so is $\mathbf{b}$. And so $\mathbf{b}'(s)$ measures how quickly the curve pulls away from the osculating plane of s (within a neighborhood of s). By differentiating the equations $\langle \mathbf{b}(s), \mathbf{b}(s)\rangle = 1$ and $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$, we obtain $2\langle \mathbf{b}'(s), \mathbf{b}(s)\rangle = 0$ and

$$\begin{aligned}
\mathbf{b}'(s) &= \mathbf{t}'(s) \times \mathbf{n}(s) + \mathbf{t}(s) \times \mathbf{n}'(s) \\
&= [k(s)\mathbf{n}(s)] \times \mathbf{n}(s) + \mathbf{t}(s) \times \mathbf{n}'(s) \\
&= \mathbf{t}(s) \times \mathbf{n}'(s)\,,
\end{aligned}$$

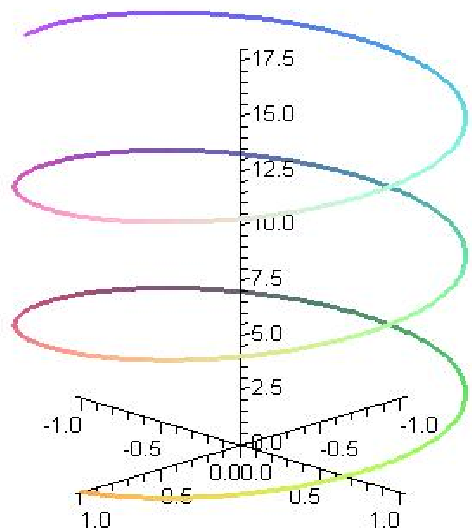
respectively. Thus, we see that $\mathbf{b}'(s)$ is normal to both $\mathbf{b}(s)$ and $\mathbf{t}(s)$. Therefore $\mathbf{b}'(s)$ must be parallel to $\mathbf{n}(s)$. This leads to a definition.

Definition 1.5. Let α be a curve parameterized by arc length without singular points of order 1. The quantity $\tau(s)$ defined by $\mathbf{b}'(s) = \tau(s)\mathbf{n}(s)$ is the *torsion* of α at s.

Example 1.6. An arc length parameterization of a helix is given by

$$\alpha(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)\,,$$

for $s \in [0, 2\pi)$.



Then

$$\mathbf{t}(s) = \alpha'(s) = \left(-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) ,$$

and

$$\mathbf{t}'(s) = \alpha''(s) = \left(-\frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right), 0\right) .$$

Thus

$$k(s) = |\alpha''(s)| = \sqrt{\frac{1}{4}\cos^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{4}\sin^2\left(\frac{s}{\sqrt{2}}\right)} = \frac{1}{2}.$$

From the equation $\mathbf{t}'(s) = k(s)\mathbf{n}(s)$, we have that $\mathbf{n}(s) = \left(-\cos\left(\frac{s}{\sqrt{2}}\right), -\sin\left(\frac{s}{\sqrt{2}}\right), 0\right)$. And thus

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s) = \left(\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) .$$

From this we get

$$\mathbf{b}'(s) = \left(\frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right), 0\right) .$$

The equation $\mathbf{b}' = \tau\mathbf{n}$ tells us that

$$\tau(s) = \mathbf{n}(s) \cdot \mathbf{b}'(s) = -\frac{1}{2}\cos^2\left(\frac{s}{\sqrt{2}}\right) - \frac{1}{2}\sin^2\left(\frac{s}{\sqrt{2}}\right) + 0 = -\frac{1}{2}.$$

So a helix has constant curvature and constant torsion. (Note that $\mathbf{b}' = \tau\mathbf{n}$ imples we could have alternatively computed $\tau = |\mathbf{b}'|$ since $\mathbf{n}$ is unit length.)

Remark 1.7. A more general parameterization for a helix would be $\alpha(t) = (r\cos t, r\sin t, ht)$, for $t \in [0, 2\pi)$. Here, r is the radius and the constant h gives the height between loops. The curvature of this helix is $k(s) = \frac{r}{r^2+h^2}$ and the torsion is $\tau(s) = -\frac{h}{r^2+h^2}$. For the example above we have $r = 1 = h$.

Notice that for each s, we have an orthonormal basis $\{\,\mathbf{t}\,,\,\mathbf{n}\,,\,\mathbf{b}\,\}$, typically referred to as a *frame*. This particular frame is known as the *Frenet frame*. And so far we have seen how to write $\mathbf{t}'$ and $\mathbf{b}'$ in this basis, but not $\mathbf{n}'$. Well, this is not terribly difficult: differentiate the equation $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. This gives us

$$\begin{aligned}
\mathbf{n}' &= \mathbf{b} \times \mathbf{t}' + \mathbf{b}' \times \mathbf{t} \\
&= \mathbf{b} \times (k\,\mathbf{n}) + (\tau\,\mathbf{n}) \times \mathbf{t} \\
&= -k(\mathbf{n} \times \mathbf{b}) - \tau(\mathbf{t} \times \mathbf{n}) = -k\,\mathbf{t} - \tau\,\mathbf{b}\,.
\end{aligned}$$

We thus have the following three equations

$$\begin{aligned}
\mathbf{t}' &= & & k\,\mathbf{n} & & \\
\mathbf{n}' &= & -k\,\mathbf{t} & & & -\tau\,\mathbf{b} \\
\mathbf{b}' &= & & \tau\,\mathbf{n} & &
\end{aligned}$$

known as the *Frenet equations*. These equations are quite important and extremely useful.

A differentiable curve on a closed interval $[a, b]$ is the restriction of a differentiable curve defined on an open interval $I \supset [a, b]$.

Definition 1.8. A curve $\alpha : [a, b] \to \mathbb{R}^3$ is *closed* if

$$\alpha(a) = \alpha(b)$$
$$\alpha'(a) = \alpha'(b)$$
$$\alpha''(a) = \alpha''(b)$$
$$\vdots$$

The curve is *simple* if α is injective on $[a, b)$; that is, if $\alpha(s_1) = \alpha(s_2)$ for $s_1, s_2 \in [a, b)$, then $s_1 = s_2$.

We are primarily concerned with simple closed curves, and we suppose $[a, b] = [0, L]$. So L is the length of α.

Definition 1.9. Let $\alpha : [0, L] \to \mathbb{R}^3$ be a curve parameterized by arc length. The *total curvature* of α is the quantity $k(C) = \int_0^L k(s)\, ds$, where C is the image of α.

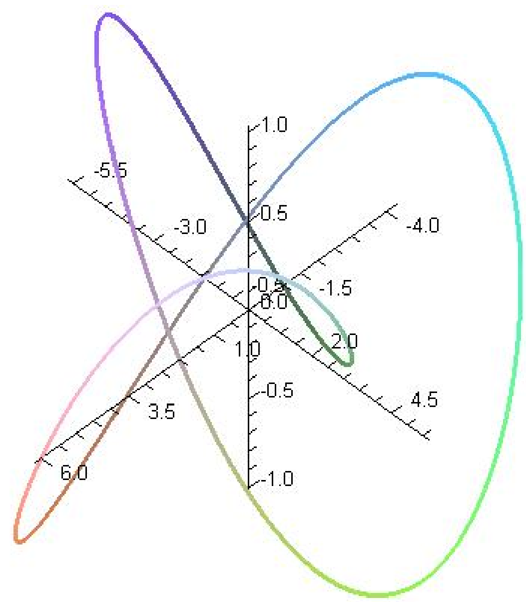
Remark 1.10. For a plane curve $\alpha : [0, L] \to \mathbb{R}^2$, the curvature k is in fact signed. We define the normal vector by requiring that $\{\, \mathbf{t}\, ,\, \mathbf{n}\, \}$ have the same orientation as the natural basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. We then *define* the curvature by the equation $\mathbf{t}' = k\mathbf{n}$. However, it should be clear that $|k|$ agrees with our previous definition. Thus, for a plane curve, the total curvature is defined as $k(C) = \int_0^L |k|\, ds$. For a space curve, we assume $k > 0$.

Example 1.11. Consider again the circle C parameterized by $\alpha(s) = \left(r \cos\left(\frac{s}{r}\right), r \sin\left(\frac{s}{r}\right)\right)$. We saw earlier that $k(s) = \frac{1}{r}$. This circle has length $2\pi r$, and so the total curvature is

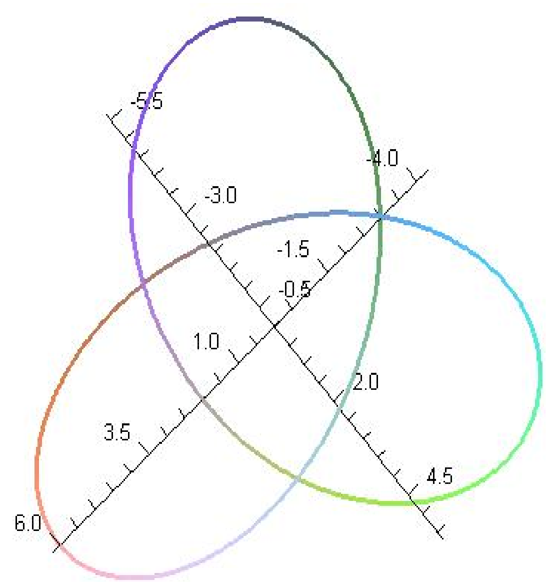
$$k(C) = \int_0^L |k(s)|\, ds = \int_0^{2\pi r} \frac{1}{r}\, ds = \frac{1}{r}(2\pi r - 0) = 2\pi\,.$$

Remark 1.12. If a curve α is not parameterized by arc length, then the total curvature is given by $k(C) = \int\limits_I k(t)|\mathbf{t}(t)|\, dt$.

Example 1.13. Consider the trefoil knot. A parameterization of this is given by $\alpha(t) = (4\cos(2t) + 2\cos(t), 4\sin(2t) - 2\sin(t), \sin(3t))$, and a picture is shown below.



But we can rotate to get a better view.



Calculating the total curvature of this by hand would be a Herculean feat; computing the necessary pieces individually is tedious enough. Maple struggled mightily, but eventually came up with $k(C) \approx 13.035243$. For reference, $4\pi \approx 12.5663706$. And in fact, the trefoil can be continuously deformed so that its total curvature is arbitrarily close to 4π, but will never reach 4π. (This is a *homework exercise* in O'Neill [1]!)

The circle example above illustrates the first major result we will investigate. Fenchel's theorem states that the total curvature of simple closed curved is greater than or equal to 2π, equality holding if and only if the curve is convex planar. This was first proved by Fenchel in 1929 [2].

Chapter 2: Fenchel's Theorem

The nature of mathematics is sometimes unfortunate in the following regard: there is a reasonably straightforward result that we want, but some work and notation is required to reach that end. To prove Fenchel's theorem, which gives us a lower bound on the total curvature on the class of simple closed curves, we appeal to some machinery from surface theory. However, some details will be elided for the sake of brevity. Using this machinery for this proof is akin to a sledgehammer busting a walnut open, but afterward we present a clever little proof with some finesse. In both cases we have some theory to build up.

2.1 Shape Operator and Jacobian

Let p be a point on a surface $M \subseteq \mathbb{R}^3$. Let $\mathcal{N}$ be the unit normal vector field on some neighborhood of p in M. Then $\mathcal{N} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the natural frame field on $\mathbb{R}^3$. The existence of the Euclidean coordinate functions n_i can be proven and is a standard result.

Definition 2.1. For each tangent vector $\mathbf{v}$ to M at p, let

$$S_p(\mathbf{v}) = -\nabla_{\mathbf{v}}(\mathcal{N}) = -\sum_{i=1}^{3} D_{\mathbf{v}}(n_i)\mathbf{e}_i(p)\,,$$

where $D_{\mathbf{v}}(n_i)$ is the directional derivative of n_i (in the direction of $\mathbf{v}$, obviously). This covariant derivative S_p is called the *shape operator* of M at p.

We denote by T_pM the tangent space to our surface M at the point p; this is the set of tangent vectors at p. The shape operator is a map $S_p : T_pM \to T_pM$ and is in fact linear and symmetric. (When convenient or unambiguous, we drop the subscript p on S_p.) At a point p, a normal plane contains the normal vector, and thus contains a vector tangent to the surface. This plane will "cut" the surface in a plane curve. In general, this curve will have different curvatures for different choices of the normal plane at p. The *principal curvatures* at p are the maximum and minimum of this curvature, k_1 and k_2. The directions in the normal plane where the maximum and minimum values occur are the *principal directions*. Gaussian curvature is the product $K = k_1k_2$. We do not prove it, but the eigenvalues of S_p represent the principal curvatures of M at p and the corresponding eigenvectors represent the principal directions of M at p. Thus, $K(p) = k_1k_2 = \det S_p$.

Lemma 2.2. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are linearly independent tangent vectors at* p *of* $M \subseteq \mathbb{R}^3$*, then* $S(\mathbf{v}) \times S(\mathbf{w}) = K(p)\,(\mathbf{v} \times \mathbf{w})$*.*

Proof. Using $\{\,\mathbf{v}\,,\,\mathbf{w}\,\}$ as a basis for T_pM, we have $S(\mathbf{v}) = a\mathbf{v} + b\mathbf{w}$ and $S(\mathbf{w}) = c\mathbf{v} + d\mathbf{w}$. So the matrix of S with respect to our basis is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This means that $K(p) = \det S = ad - bc$. But now

$$\begin{aligned} S(\mathbf{v}) \times S(\mathbf{w}) &= (a\mathbf{v} + b\mathbf{w}) \times (c\mathbf{v} + d\mathbf{w}) \\ &= ad\,(\mathbf{v} \times \mathbf{w}) + bc\,(\mathbf{w} \times \mathbf{v}) \\ &= (ad - bc)\,(\mathbf{v} \times \mathbf{w}) \\ &= K(p)\,(\mathbf{v} \times \mathbf{w}) \ . \end{aligned}$$

□

Remark 2.3. The shape operator is closely related to the second fundamental form: $\mathrm{II}_p(\mathbf{v}, \mathbf{w}) = \langle S_p(\mathbf{v}), \mathbf{w} \rangle$.

This is all we need of the shape operator. Now, however, we need a couple more tools. Let X and Y be surfaces. For any map $f : X \to Y$, there are certain naturally arising associated maps.

Definition 2.4. Given $p \in S$, the induced map $f_{*p} : T_pX \to T_{f(p)}Y$ sending tangent vectors at p in X to tangent vectors at $f(p)$ in Y is called the *differential* of f at p. This map is linear and is sometimes denoted df. We avoid this to alleviate ambiguity.

For our purposes, we also need the differential's dual map

$$f^* : \Lambda(Y) \to \Lambda(X)\,,$$

which sends forms on Y to forms on X.

Definition 2.5. Let X and Y be surfaces oriented by their respective area forms dX and dY. Given $f : X \to Y$, the real-valued function J_f on X such that $J_f\,dX = f^*(dY)$ is called the *Jacobian* of the map f.

The magnitude of the Jacobian at a point measures the rate at which f expands area around p. Beyond this, we omit many details, but the Jacobian is an extremely important tool. We do however need to see how to put all these pieces together.

Let $\mathbf{v}, \mathbf{w} \in T_pX$. Then $J_f(p)\,dX(\mathbf{v}, \mathbf{w}) = f^*(dY)(\mathbf{v}, \mathbf{w}) = dY(f_*\mathbf{v}, f_*\mathbf{w})$. Since $dX(\cdot,\cdot)$ and $dY(\cdot,\cdot)$ represent areas, we see how the Jacobian interacts with area between surfaces. We now have what we need to prove our first theorem.

Theorem 2.6. *The Gaussian curvature K of an oriented surface $M \subseteq \mathbb{R}^3$ is the Jacobian of its Gauss map.*

Proof. Let $N : M \to S^2$ be the Gauss map. Let $\mathcal{N}$ be the unit normal vector field orienting M, and let $\widetilde{\mathcal{N}}$ be the outward normal on S^2. Lastly, denote the area form on S^2 by dS^2. We wish to show that $K\,dM = N^*(dS^2)$. Take two tangent vectors to

M, say $\mathbf{v}$ and $\mathbf{w}$. Using our shape operator lemma we have

$$\begin{aligned}(K\,dM)(\mathbf{v},\mathbf{w}) &= K(p)dM(\mathbf{v},\mathbf{w})\\ &= K(p)\ \langle \mathcal{N}(p), \mathbf{v}\times\mathbf{w}\,\rangle\\ &= \langle \mathcal{N}(p), K(p)(\mathbf{v}\times\mathbf{w})\rangle\\ &= \langle\, \mathcal{N}(p), S(\mathbf{v})\times S(\mathbf{w})\rangle\ .\end{aligned}$$

We also have

$$\begin{aligned}N^*(dS^2)(\mathbf{v},\mathbf{w}) &= dS^2(N_*\mathbf{v}, N_*\mathbf{w})\\ &= \left\langle\, \widetilde{\mathcal{N}}(N(p)), N_*\mathbf{v}\times N_*\mathbf{w}\,\right\rangle\ .\end{aligned}$$

By definition of the Gauss map, and by properties of S^2, we see that $\mathcal{N}(p)$ and $\widetilde{\mathcal{N}}(N(p))$ are parallel. Further, for any tangent vector $\mathbf{u}$ at p, we have $-S(\mathbf{u}) = \nabla_{\mathbf{u}}\mathcal{N} = \sum D_{\mathbf{u}}(n_i)\mathbf{e}_i(p)$ and $N_*\mathbf{u} = \sum D_{\mathbf{u}}(n_i)\mathbf{e}_i(N(p))$. The proof of this last equality is omitted. Thus, $-S(\mathbf{u})$ and $N_*\mathbf{u}$ are parallel for all tangent vectors $\mathbf{u}$. In particular, $-S(\mathbf{v})$ and $-S(\mathbf{w})$ are parallel to $N_*\mathbf{v}$ and $N_*\mathbf{w}$, respectively. Since triple scalar products only rely on Euclidean coordinates of the vectors, we have our result.

□

Now, for $f : X \to Y$, we call $\iint\limits_X f^*(dY) = \iint\limits_X J_f\, dX$ the *algebraic area* of $f(X)$. We have a corollary.

Corollary 2.7. *The total Gaussian curvature of an oriented surface $M \subseteq \mathbb{R}^3$ equals the algebraic area of the image of its Gauss map $N : M \to S^2$.*

Proof. From the theorem above, we have $K\,dM = N^*(dS^2)$. This immediately implies

$$\iint\limits_M K\,dM = \iint\limits_M N^*(dS^2)\,.$$

□

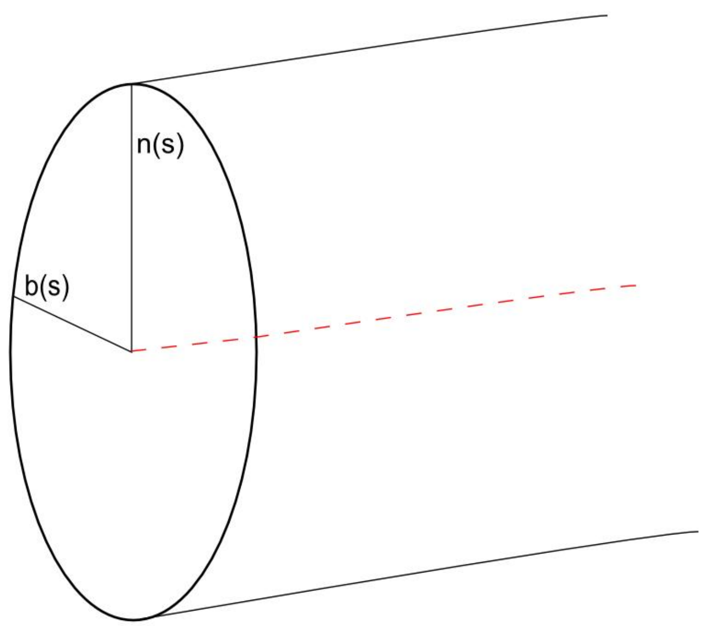
Now, one ought to realize that the above is a very cursory examination of some key ideas in surface theory. There are some very deep mathematical ideas floating around in this section, and as odd as it seems to say, the depth of these concepts is really ancillary to our intended purposes. As long as we have an intuitive understanding, we are okay. Now we must build a tube.

2.2 A Surface-Theoretic Proof

The *tube of radius* r around α is the surface

$$\mathbf{x}(s, v) = \alpha(s) + r(\mathbf{n}(s)\cos v + \mathbf{b}(s)\sin v)$$

with $s \in [0, L]$ and $v \in [0, 2\pi)$.



We assume r is small enough that $rk_0 < 1$, where $k_0 = \max\{|k(s)| : s \in [0, L]\}$. This ensures that the tube T has no self-intersections, so that T is in fact a regular surface. This assumption serves a second purpose demonstrated within the proof.

Theorem 2.8. *(Fenchel) The total curvature of a simple closed curve is greater than or equal to 2π, with equality if and only if the curve is plane convex.*

Proof. Let T be the tube around α. Let $R \subset T$ be the region of T where the Gaussian curvature of T is non-negative. After doing some computations, we see that $\mathbf{x}_s \times \mathbf{x}_v = -r(1 - rk\cos v)[\mathbf{n}\cos v + \mathbf{b}\sin v]$, and so $|\mathbf{x}_s \times \mathbf{x}_v| = r(1 - rk\cos v)$. This gives us that the normal to the tube is $\mathbf{N} = -(\mathbf{n}\cos v + \mathbf{b}\sin v)$. Further, $\mathbf{N}_s \times \mathbf{N}_v = -k\cos v\mathbf{N}$. Notice that we can also write $\mathbf{x}_s \times \mathbf{x}_v = r(1 - rk\cos v)\mathbf{N}$. Thus, we have

$$K(s,v) = \frac{\mathbf{N}_s \times \mathbf{N}_v}{\mathbf{x}_s \times \mathbf{x}_v} = -\frac{k\cos v}{r(1 - rk\cos v)}\,.$$

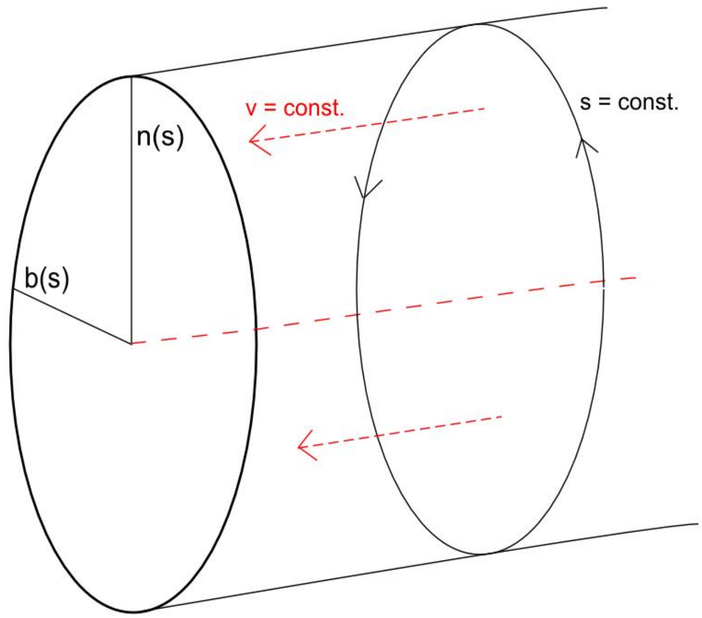
The first equality is a standard result. Therefore, the region R is $\mathbf{x}(D)$, where $D = \left\{(s,v) \in \mathbb{R}^2 \,|\, 0 \le s \le L\,,\ \frac{\pi}{2} \le v \le \frac{3\pi}{2}\right\}$; this requires having $rk_0 < 1$, as we have assumed.

Let $d\sigma$ be the area form on T. Then $d\sigma = |\mathbf{x}_s \times \mathbf{x}_v| dvds$. We now have the following:

$$\iint\limits_R K\,d\sigma = \iint\limits_R K|\mathbf{x}_s \times \mathbf{x}_v|\,dvds = \int_0^L\int_{\pi/2}^{3\pi/2} k\cos v\,dvds = 2\int_0^L k\,ds\,.$$

Now, pick a point $q \in S^2$, and let P be the tangent plane at this point. Take a plane P' parallel to P such that $P' \cap T = \varnothing$, and slide it parallel to itself until it reaches tangency to T at a point p. The tube must lie entirely on one side of P', otherwise it wouldn't hit the point p first, which implies that $K(p) \ge 0$. (This characterization is discussed in do Carmo [3], and we will prove it in the Fary-Milnor theorem.) It follows that the Gauss map N sends p to q, and our choice of q was arbitrary, so $N|_R$ is surjective. By corollary 2.7, this implies that $\iint\limits_R K\,d\sigma \ge \text{area}(S^2) = 4\pi$. Thus, $\int_0^L k\,ds \ge 2\pi$.

We now show that equality holds if and only if α is convex planar.



Consider the coordinate curves of $\mathbf{x}$. If $v = \text{const.}$, then $\mathbf{x}$ yields a curve differing from α only by a translation. If $s = \text{const.}$, then $\mathbf{x}$ yields a circular cross-section of the tube. We note that the Gauss map restricted to a coordinate curve $s = \text{const.}$ is injective and the image is a great circle $\gamma_s \subset S^2$. Let ℓ_s be a coordinate curve for $s = \text{const.}$ For each s, denote by $\gamma_s^+ \subset \gamma_s$ the closed half circle corresponding to points of ℓ_s where $K \geq 0$.

Assume α is convex planar. Then all γ_s^+ have the same endpoints, say p and q, and $\overleftrightarrow{pq}$ is perpendicular to the plane in which α lies. By planar convexity, we have that k never changes sign (we assume $k > 0$); we leave the proof of this fact for later. So N will sweep out exactly the equator on S^2. If N were to "double back" as it went around the equator, then k must have changed sign. Thus, $\gamma_{s_1} \cap \gamma_{s_2} = \{p\} \cup \{q\}$ for $s_1 \neq s_2 \in [0, L)$, and we again apply corollary 2.7 to get that $\iint\limits_R K \, d\sigma = 4\pi$. This immediately implies $\int_0^L k \, ds = 2\pi$.

Assume now that $\int_0^L |k| \, ds = 2\pi$. Then $\iint\limits_R K \, d\sigma = 4\pi$. We claim that all γ_s^+ have the same endpoints, p and q.

Suppose not. Then there exist two distinct great circles γ_{s_1} and γ_{s_2}, with s_1 arbitrarily close to s_2, intersecting in two antipodal points which are not in $N(R \cap Q)$, where $Q = \{x \in T \, : \, K(x) \leq 0\}$. That is, γ_{s_1} and γ_{s_2} intersect in two antipodal points in S^2 that are not mapped from points of zero curvature in T. So there exist at least two points of positive curvature in T that N maps to a single point of S^2; consider, for example, the pre-image under N of $\gamma_{s_1}^+ \cap \gamma_{s_2}^+$. Let $x_1, x_2 \in N^{-1}(\gamma_{s_1}^+ \cap \gamma_{s_2}^+) \subset T$, and let $U_1, U_2 \subset T$ be open neighborhoods about x_1 and x_2, respectively, such that $N(U_i) \subset \gamma_{s_1}^+ \cap \gamma_{s_2}^+$ for $i = 1, 2$. Note that N covers $V = N(U_1) \cap N(U_2)$ twice over, whereas everything else on S^2 is covered only once. By corollary 2.7 again we have

$$\iint\limits_R K \, d\sigma = \text{area}(N(R)) \geq \text{area}(S^2) + \text{area}(V) = 4\pi + \text{area}(V) \, .$$

This means $\iint\limits_R K \, d\sigma > 4\pi$, a contradiction. Thus, all γ_s^+ have the same endpoints, p and q.

Now, the points of zero curvature in T are in the intersections of the binormal $\mathbf{b}$ of α (and its antipode) with T. So $\mathbf{b}(s)$ is parallel to the line $\overleftrightarrow{pq}$ for all s. This implies $\mathbf{t}(s)$ is contained in a plane normal to this line, and so must be the curve α.

Lastly, we show α is convex. We assume α is oriented so that its rotation index is $+1$; we define rotation index in the next section. That we can orient α as desired we show in theorem 2.10.

Certainly it is the case that $2\pi = \int_0^L |k| \, ds \geq \int_0^L k \, ds$; the first equality is by assumption. Let $J = \{s \in [0, L] \, : \, k(s) \geq 0\}$ and let Γ be the image of the tangent map $\mathbf{t}(s)$: this is called the *tangent indicatrix* of α. Then we have $\ell(\Gamma) = \int_0^L |k| \, ds = 2\pi$; that is, the length of the tangent indicatrix is equal to the total curvature. Since α is closed, the vector $\mathbf{t}$ will sweep out at least a full circle along Γ when taken over

J. So $\int_J k\, ds \geq 2\pi$. Thus we have

$$2\pi = \int_0^L |k|\, ds \geq \int_J k\, ds \geq 2\pi\,,$$

which implies that

$$\int_0^L |k|\, ds = \int_J k\, ds = 2\pi\,.$$

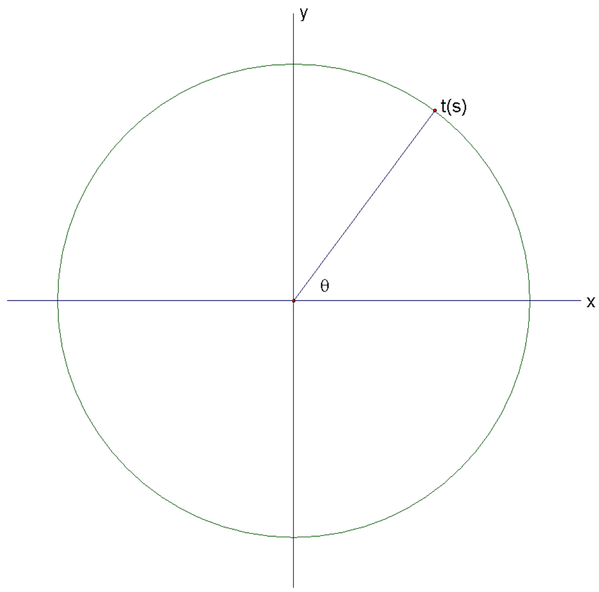
But $2\pi \geq \int_0^L k\, ds$, so $\widetilde{J} = [0, L] \setminus J$ must have measure zero and $\int_{\widetilde{J}} (-k)\, ds = 0$. Then $\int_J k\, ds + \int_{\widetilde{J}} (-k)\, ds = \int_0^L k\, ds = \int_0^L |k|\, ds = 2\pi$. This then implies that $|k(s)| = k(s)$, meaning $k(s) \geq 0$ for all s. Curvature does not change sign if and only if the curve is convex; this is a later theorem. Thus, α is convex and this completes the proof.

□

This proof is due to do Carmo [3] and O'Neill [1]. As is so often the case in mathematics, a more elementary proof can be found. However, we again need several different tools at our disposal. Fortunately, these tools are curve-theoretic.

2.3 Rotation Index and Convexity

Let $\alpha : [0, L] \to \mathbb{R}^2$ be a closed plane curve given by $\alpha(s) = (x(s), y(s))$. The unit tangent vector is $\mathbf{t}(s) = \alpha'(s) = (x'(s), y'(s))$. It is not hard to see that we can consider $\mathbf{t}$ as a map $\mathbf{t} : [0, L] \to S^1$, and we define the tangent indicatrix as $\Gamma = \{\mathbf{t}(s) : s \in [0, L]\} \subseteq S^1$. Let $\theta(s)$ be the angle $\mathbf{t}(s)$ makes with the x-axis.



So $0 \leq \theta(s) < 2\pi$. Notice that we can write $x'(s) = \cos(\theta(s))$ and $y'(s) = \sin(\theta(s))$. Then we observe that $\theta(s) = \tan^{-1}\left(\frac{y'(s)}{x'(s)}\right)$, so $\theta(s)$ is at least locally well-defined and differentiable.

Now,

$$\begin{aligned}
\mathbf{t}'(s) &= (-\sin[\theta(s)]\theta'(s), \cos[\theta(s)]\theta'(s)) \\
&= \theta'(s)(-\sin[\theta(s)], \cos[\theta(s)]) \\
&= \theta'(s)\,\mathbf{n}(s)\,.
\end{aligned}$$

But from the Frenet equations, we know $\mathbf{t}' = k\mathbf{n}$, so we must have $\theta'(s) = k(s)$; this suggests a global definition for $\theta : [0, L] \to \mathbb{R}$ as

$$\theta(s) = \int_0^s k(u)\,du\,.$$

Recall that, since α is planar,

$$k = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}} = \frac{x'y'' - x''y'}{1} = x'y'' - x''y'\,;$$

the first equality is a standard result, and the second equality follows since α is parameterized by arc length. Further, notice that

$$\begin{aligned}\left(\tan^{-1}\left(\frac{y'}{x'}\right)\right)' &= \frac{1}{1+\left(\frac{y'}{x'}\right)^2}\cdot\frac{x'y''-y'x''}{(x')^2} \\ &= \frac{x'y''-x''y'}{(x')^2+(y')^2} \\ &= x'y''-x''y'\,.\end{aligned}$$

Thus, we have $\theta'(s) = k(s) = x'y'' - x''y' = \left(\tan^{-1}\left(\frac{y'(s)}{x'(s)}\right)\right)'$. So our global definition agrees with our local definition up to a constant.

The function θ measures the total rotation of $\mathbf{t}$ as α runs from 0 to s. Our curve being closed means this total angle must be an integral multiple of 2π. Symbolically:

$$\int_0^L k(s)\,ds = \theta(L) - \theta(0) = 2\pi\gamma\,,$$

for some $\gamma \in \mathbb{Z}$.

Definition 2.9. The integer γ thus defined is the *rotation index* of α.

Intuitively, the rotation index tells us how many times $\mathbf{t}$ wraps around S^1. Certainly this value shouldn't depend on the function θ, and in fact it does not: let $\overline{\theta}(s)$ be a function such that $\mathbf{t}(s) = \left(\cos\left(\overline{\theta}(s)\right), \sin\left(\overline{\theta}(s)\right)\right)$. Then we have $\overline{\theta}(s) = \theta(s) + 2\pi j(s)$ for some integer $j(s)$. But θ and $\overline{\theta}$ are both continuous, so $j(s)$ must be continuous. Since $j(s)$ only takes on integer values, it must be constant. Therefore, $\overline{\theta}(L) - \overline{\theta}(0) = \theta(L) + 2\pi j - \theta(0) - 2\pi j = \theta(L) - \theta(0)$. Now we have a theorem.

Theorem 2.10. *(Turning Tangents) The rotation index of a simple closed planar curve is* ± 1*.*

Sketch of Proof. Let $\Delta = \{(s,t) \,:\, s \in [0,L]\,, t \in [0,L]\,, s \le t\} \subset [0,L] \times [0,L]$. Define a map $h : \Delta \to S^1$ by

$$h(s,t) = \begin{cases} \mathbf{t}(s) & \text{if } s = t \\ -\mathbf{t}(s) & \text{if } (s,t) = (0,L) \\ \frac{\alpha(t)-\alpha(s)}{|\alpha(t)-\alpha(s)|} & \text{otherwise}\,. \end{cases}$$

Since α is regular, this map is in fact continuous. It can be shown that there exists a continuous function $\overline{\theta} : \Delta \to \mathbb{R}$ such that $h(s,t) = \left(\cos\left(\overline{\theta}(s,t)\right), \sin\left(\overline{\theta}(s,t)\right)\right)$ for all $(s,t) \in \Delta$.

From our comments just before the theorem, we have

$$\begin{aligned} \int_0^L k\,ds &= \theta(L) - \theta(0) \\ &= \overline{\theta}(L,L) - \overline{\theta}(0,0) \\ &= \underbrace{\overline{\theta}(0,L) - \overline{\theta}(0,0)}_{\varphi_1} + \underbrace{\overline{\theta}(L,L) - \overline{\theta}(0,L)}_{\varphi_2}. \end{aligned}$$

We assume $\alpha(0)$ is at the origin and that C lies in the upper half-plane. Further, we rotate so that $\mathbf{t}(0) = \mathbf{e}_1$, though this may also require reversing orientation.

The quantity φ_1 is the angle the position vector of α turns through, beginning at 0 and ending at π. The quantity φ_2 is the same, except for the negative of the position vector. Thus, $\varphi_1 = \varphi_2 = \pi$; allowing for possible change in orientation, we might have $\varphi_1 = \varphi_2 = -\pi$.

Therefore, $\theta(L) - \theta(0) = \varphi_1 + \varphi_2 = \pm 2\pi$, which implies that the rotation index is ± 1.

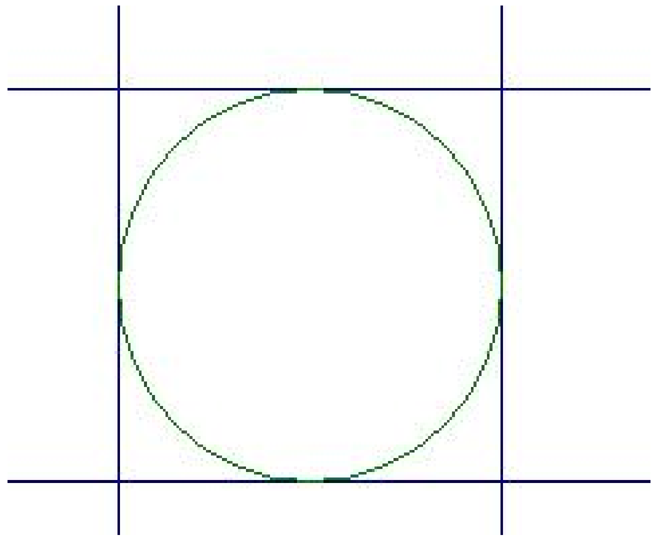
$\square$

This “proof” is due to Shifrin[4].

We now turn to convexity. There is an interesting relationship between the curvature of a curve and convexity, as we shall see.

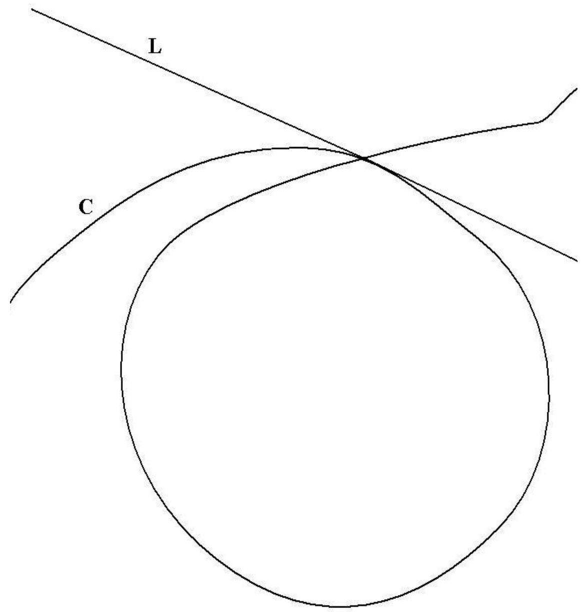
Definition 2.11. A planar curve C is *convex* if C lies entirely in the closed half-plane determined by $\mathbf{t}(s)$ for all $s \in [0, L]$.

A simple example of a convex curve is a circle:



There are many alternative formulations of convexity. One that will be useful is the following: a closed planar curve is convex if every straight line has at most two points in common with the curve. If we recall that C is simply the image in $\mathbb{R}^2$ of some parameterization, it should not be too difficult to see that these definitions agree.

Further, we note that a non-simple curve is non-convex, and so a convex curve must be simple. We do not prove this, but we present a simple illustration.



Here we have a curve C with a tangent line L taken near where C intersects itself.

Theorem 2.12. *A closed planar curve C is convex if and only if it is simple and the signed curvature k does not change sign.*

Proof. Let C be given by $\alpha(s)$. We have $k(s) = \theta'(s)$, so k does not change sign if and only if θ is weakly monotone.

Suppose C is convex; by our remark above, it is simple. Further suppose k does not change sign. Then there are values $s_1 < s_2$ such that $\mathbf{t}(s_1) = \mathbf{t}(s_2)$ and $\theta(s)$ is not constant on $[s_1, s_2]$. Note that $\theta(s_1) = \theta(s_2)$. The theorem of turning tangents tells us that θ takes on every value in $[0, 2\pi)$, so there exists s_3 such that $\mathbf{t}(s_3) = -\mathbf{t}(s_1)$. If all three tangent vectors $\mathbf{t}(s_1)$, $\mathbf{t}(s_2)$, and $\mathbf{t}(s_3)$ are distinct but parallel, then one will be between the other two, separating the curve C. This contradicts our assumption of convexity. Thus, two of the three tangent vectors must be coincident, and so we have a line T which is tangent to C at two distinct points, say p and q. Again by convexity, the whole segment $\overline{pq}$ must be in the region bounded by C. So the common tangent has a well-defined orientation, and the points p and q are actually $\alpha(s_1)$ and

$\alpha(s_2)$. Further, $\mathbf{t}(s)$ is constant on the segment $\overline{pq}$, which implies $\theta(s)$ is constant there as well, a contradiction. Thus, k cannot change sign on.

Conversely, suppose C is simple and that k does not change sign. Orient C so that $k \geq 0$. Then $\theta(s)$ is non-decreasing and, by the theorem of turning tangents, runs from 0 to 2π when $s \in [0, L]$. If $\mathbf{t}(s_1) = \mathbf{t}(s_2)$ for $0 \leq s_1 < s_2 \leq L$, then $\mathbf{t}(s)$ is constant on the interval $[s_1, s_2]$. And so $\alpha([s_1, s_2])$ is a line segment with a fixed tangent.

Suppose C is not convex. Then there exists s_0 such that C has points on both sides of the line T determined by $\mathbf{t}(s_0)$. Consider the "height" function $h_{\mathbf{n}}(s) = \langle \alpha(s) - \alpha(s_0), \mathbf{n}(s_0) \rangle$. Since $[0, L]$ is compact, $h_{\mathbf{n}}$ achieves a maximum and minimum, say at s_1 and s_2, and these clearly occur on opposing sides of T. Notice that $\mathbf{t}(s_0)$, $\mathbf{t}(s_1)$, and $\mathbf{t}(s_2)$ are all parallel. So at least two of the three tangent vectors have the same orientation, say $\mathbf{t}(s_0)$ and $\mathbf{t}(s_1)$. But then $\theta(s_0) = \theta(s_1)$, and since θ is non-decreasing, it must be constant on the interval $[s_0, s_1]$ (assuming $s_1 > s_0$). So $\alpha([s_0, s_1]) \subset T$. But this contradicts the choice of T, and therefore C is convex.

□

The above follows Vaisman [5], with clarifications from do Carmo [3]. And with this we move onto our elementary proof of Fenchel's theorem.

2.4 A Curve-Theoretic Proof

Let C be the curve $\alpha(s) = (x(s), y(s), z(s))$, parameterized by arc length. So we are no longer restricting our attention to planar curves. Recall the definition of the tangent indicatrix: $\Gamma = \{\mathbf{t}(s) : s \in [0, L]\}$, though we note since C is no longer necessarily planar, Γ is not necessarily in S^1. Also, since C may not be planar, we assume $k > 0$. Lastly recall that the length of the tangent indicatrix is equal to the total curvature: $\ell(\Gamma) = \int_0^L k(s)\, ds$.

Lemma 2.13. *The tangent indicatrix Γ of a simple closed curve C does not lie in an open hemisphere of S^2. It lies in a closed hemisphere if and only if C is planar.*

Proof. Without loss of generality, suppose Γ lies in the northern hemisphere. Then $z'(s) \geq 0$ for all $s \in [0, L]$. Since C is closed, we have $z(L) - z(0) = \int_0^L z'(s)\, ds = 0$. This implies $z'(s)$ cannot be strictly positive, and so Γ cannot lie in an open hemisphere.

Suppose Γ lies in a closed hemisphere. Since z' is non-negative and $\int_0^L z'\, ds = 0$, it must vanish identically; $z'(s) \equiv 0$. This implies C lies in some plane $z =$ constant.

Conversely, if C is planar, Γ must lie in a great circle and thus lies in a closed hemisphere.

□

Lemma 2.14. *Let Γ be a closed curve on S^2. If $\ell(\Gamma) < 2\pi$, then Γ lies in an open hemisphere of S^2. If $\ell(\Gamma) = 2\pi$, then Γ lies in a closed hemisphere of S^2.*

Proof. Pick a point $p \in \Gamma$. Let $q \in \Gamma$ be a point such that the curves $\Gamma_1 = pq$ and $\Gamma_2 = qp$ have equal length. Let N be the midpoint of the shorter geodesic (great-circular) arc from p to q. Rotate S^2 so that N is the north pole.

If Γ does not intersect the equator, then it lies in an open hemisphere regardless of its length; so we're done.

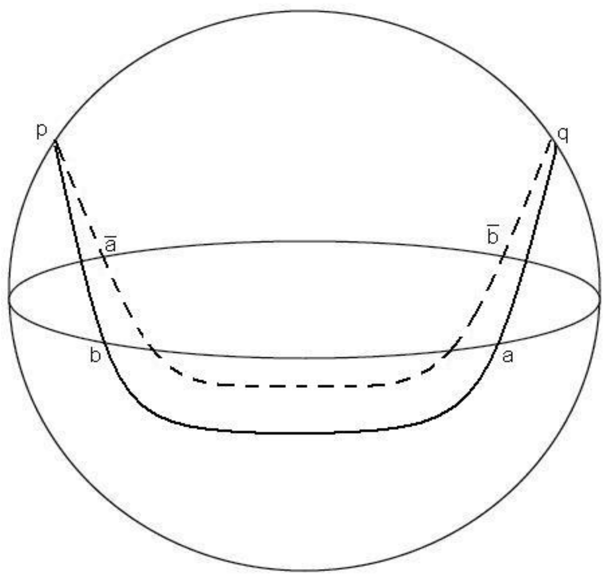
Suppose Γ_1 intersects the equator at some point. Let $\overline{\Gamma}_1$ be the rotation of Γ_1 by π around N, the north pole. So $\ell(\Gamma_1) = \ell(\overline{\Gamma}_1)$. Let $\overline{\Gamma} = \Gamma_1 \cup \overline{\Gamma}_1$. Observe that $\ell(\Gamma) = \ell(\overline{\Gamma})$. Also note that $\overline{\Gamma}$ contains a pair of antipodal equatorial points; let a and b be the equatorial points on Γ_1 and $\overline{a}$ and $\overline{b}$ be the respective antipodes on $\overline{\Gamma}_1$.

Consider joining these points with geodesic arcs (great semi-circles) along the equator.

If Γ_1 intersects the equator, then $\ell(\overline{\Gamma}) \geq 2\pi$ and so $\ell(\Gamma) \geq 2\pi$. Now suppose Γ_1 actually crosses into the southern hemisphere, so that a and b are distinct points. Let $\ell(-)_{\Gamma_1}$ denote a length along Γ_1. Notice that

$$\ell(\overline{\Gamma}) = \ell(\Gamma_1) + \ell(\overline{\Gamma}_1) = \underbrace{\ell(ab)_{\Gamma_1} + \ell(bp)_{\Gamma_1} + \ell(p\overline{a})_{\overline{\Gamma}_1}}_{A} + \underbrace{\ell(\overline{a}\overline{b})_{\overline{\Gamma}_1} + \ell(\overline{b}q)_{\overline{\Gamma}_1} + \ell(qa)_{\Gamma_1}}_{B}.$$

But notice that A and B represent lengths of paths from a to $\overline{a}$, and since each path intersects the equator in exactly one point, b or $\overline{b}$, each of A and B must be greater than the geodesic length from a to $\overline{a}$, which is π. Thus, $\ell(\Gamma) = \ell(\overline{\Gamma}) = A + B > 2\pi$.



So then, if $\ell(\Gamma) < 2\pi$, Γ_1 cannot intersect the equator. This same argument applies to Γ_2, and so Γ must lie in an open hemisphere. If $\ell(\Gamma) = 2\pi$, Γ_1 cannot cross into the southern hemisphere. Again, this reasoning applies to Γ_2, so Γ must lie in a closed hemisphere.

□

Lemma 2.15. *A simple closed planar curve is convex if and only if $\ell(\Gamma) = 2\pi$, where Γ is the tangent indicatrix.*

Proof. Suppose C is convex. Then k does not change sign by theorem 2.12. Orient C so that $k \geq 0$; so $|k| = k$.

Then $\ell(\Gamma) = \int_0^L |k|\, ds = \int_0^L k\, ds = \theta(L) - \theta(0) = 2\pi\gamma$, where γ is the rotation index. Since C is simple, closed, and planar, $\gamma = \pm 1$. But now $\int_0^L k\, ds = 2\pi\gamma$, and $k \geq 0$, so we must have $\gamma = 1$. Thus, $\ell(\Gamma) = 2\pi$.

Conversely, suppose $\ell(\Gamma) = 2\pi$. We may orient C so that $\gamma = 1$. So $\ell(\Gamma) = \int_0^L |k|\, ds = 2\pi = \theta(L) - \theta(0) = \int_0^L k\, ds$. In particular, $\int_0^L |k|\, ds = \int_0^L k\, ds$, implying $|k| = k$. So $k \geq 0$, and thus C is convex.

□

Theorem 2.16. *(Fenchel) The total curvature of a simple closed curve is greater than or equal to* 2π*, with equality if and only if the curve is plane convex.*

Proof. Let Γ be the tangent indicatrix of C. By lemma 2.13, we know Γ cannot lie in an open hemisphere of S^2. Lemma 2.14 then implies $\ell(\Gamma) = \int_0^L |k|\, ds \geq 2\pi$.

If $\ell(\Gamma) = \int_0^L |k|\, ds = 2\pi$, then lemma 2.14 tells us that Γ lies in a closed hemisphere. By lemma 2.13, this implies C is planar. So we have a simple closed planar curve such that $\ell(\Gamma) = 2\pi$; applying lemma 2.15, we see that C must be convex. Lastly, if C is simple closed planar and convex, then lemma 2.15 gives us that $\ell(\Gamma) = \int_0^L |k|\, ds = 2\pi$.

This completes the proof.

□

The proofs of lemmas 2.13 and 2.14 follow Horn [6]. While Fenchel's theorem itself is quite interesting, we note that Borsuk generalized the inequality to curves in arbitrary dimension [7]. Also, sharper results exist; Milnor and Fary gave a lower bound on total curvature for the class of simple closed knotted curves [8, 9]. Example 1.13 demonstrates this result, known today as the Fary-Milnor theorem.

Chapter 3: Fary-Milnor Theorem

We will proceed in a manner similar to that taken with Fenchel's theorem; one proof relies on the tube construction and its surface properties, another more elementary proof makes use of Crofton's formula. And we note now that Crofton's formula also gives a fairly direct proof of Fenchel's theorem, but we omit it. The Fary-Milnor theorem was proved in 1949-1950 as an affirmative response to a conjecture of Borsuk [7]. Fary was 27 years old at the time, while Milnor was an undergraduate of 18 years. One story, perhaps apocryphal, is that Milnor fell asleep in class and when he awoke, there were problems written on the board. Thinking they were homework he copied them down. He later went to his professor saying that he had solved one, but couldn't crack the others. Of course, the one he did get was the Fary-Milnor theorem, and all three problems were actually unsolved at the time. The proofs given below are very different from those found by Milnor and Fary, who used some similar ideas. Both of them attacked the conjecture using inscribed polygons, and their respective techniques will take up chapters four and five. For now, we begin with a definition.

3.1 A Surface-Theoretic Proof

Definition 3.1. A simple closed continuous curve $C \subset \mathbb{R}^3$ is *unknotted* if there exists a homotopy $H_t : S^1 \times I \to \mathbb{R}^3$ such that

$$H_0(S^1) = S^1\,,$$

$$H_1(S^1) = C\,,$$

$$\text{and } \; H_t(S^1) = C_t \approx S^1$$

for all $t \in I$; that is, C_t is homeomorphic to S^1 for all t. When this is not the case, the curve C is said to be *knotted.*

Equivalently, if a curve bounds a surface homeomorphic to a disc, the curve is unknotted; this slightly more intuitive definition will prove more useful for us.

Theorem 3.2. *(Fary-Milnor) The total curvature of a simple closed knotted curve is greater than* 4π*.*

Proof. Let $\alpha(s) = (x(s), y(s), z(s))$ and let $C = \alpha([0, L])$. Let T be the tube around α as we construced in our first proof of Fenchel's theorem, and let $R \subset T$ be the region of T where $K \geq 0$. Take a unit vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v} \neq \mathbf{b}(s)$ for all $s \in [0, L]$.

Define the height function $h_{\mathbf{v}} : [0, L] \to \mathbb{R}$ by $h_{\mathbf{v}} = \langle \alpha(s), \mathbf{v} \rangle$. Notice that

$$h'_{\mathbf{v}}(s) = \langle \alpha'(s), \mathbf{v} \rangle + \langle \alpha(s), \mathbf{0} \rangle = \langle \mathbf{t}(s), \mathbf{v} \rangle .$$

This tells us that s is a critical point of our height function if and only if $\mathbf{v}$ is perpendicular to $\mathbf{t}(s)$. Further,

$$h''_{\mathbf{v}} = \langle \mathbf{t}\,'(s), \mathbf{v} \rangle = k \langle \mathbf{n}, \mathbf{v} \rangle .$$

So at a critical point, $h''_{\mathbf{v}} = k \langle \mathbf{n}, \mathbf{v} \rangle \neq 0$ since $\mathbf{v} \neq \mathbf{b}(s)$ for all s and we are assuming $k > 0$. By the second derivative test, then, our critical points are either maxima or minima.

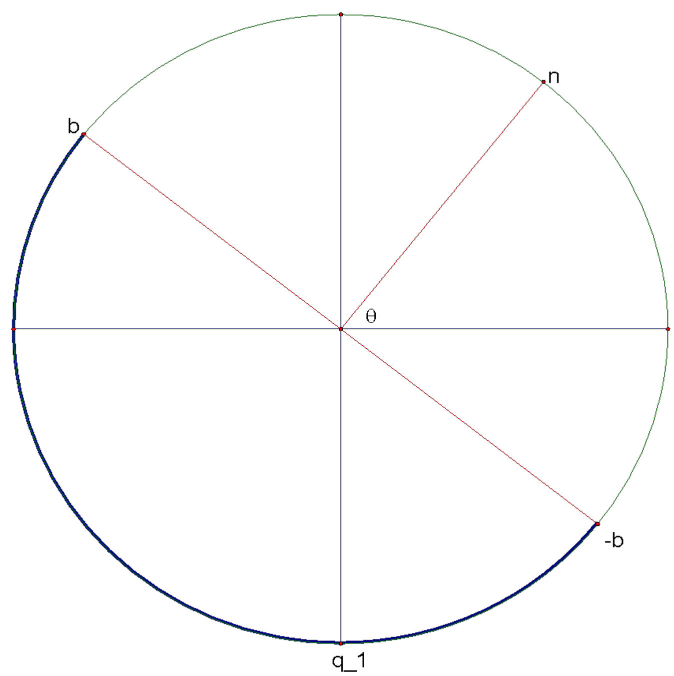
Assume $k(C) = \int_0^L k\, ds \leq 4\pi$. Then $\iint\limits_R K\, d\sigma = 2\int_0^L k\, ds \leq 8\pi$; the first equality was proven in the first proof of Fenchel's theorem.

Now, we claim there exists a vector $\mathbf{v}_0 \notin \mathbf{b}([0, L])$ such that $h_{\mathbf{v}_0}$ has exactly two critical points.

Assume not. Then for all vectors $\mathbf{v} \notin \mathbf{b}([0, L])$, the height function $h_{\mathbf{v}}$ has at least three critical points. Choose one such vector $\mathbf{v}$. We will assume two of the critical points, say s_1 and s_2, are minima; we note that this implies that the third critical point be a maximum, since $[0, L]$ is compact. The case for having two maxima is similar.

Consider a plane P perpendicular to $\mathbf{v}$ such that $P \cap T = \varnothing$. Move P parallel to itself toward the tube T.

Suppose $h_{\mathbf{v}}(s_1) = h_{\mathbf{v}}(s_2)$. In this case, P meets T at two points $q_1 \neq q_2$. Since $\mathbf{v} \notin \mathbf{b}([0, L])$ and P is perpendicular to $\mathbf{v}$, P cannot meet T at points of zero curvature. (Recall from the proof of Fenchel's theorem that the points of zero curvature on T are in the intersection of $\mathbf{b}(s)$ and its antipode with T.) We further claim that P touches T at a point of positive curvature. Consider the circular cross section of T at s_1, and assign a coordinate system so that q_1 is the south pole. (See diagram below.) Then $k\mathbf{n}(s) = \mathbf{t}'(s) = \alpha''(s) = (x''(s), y''(s), z''(s))$. Since $h_{\mathbf{v}}$ is minimized at s_1, we have $h''_{\mathbf{v}}(s_1) = k\langle \mathbf{n}(s_1), \mathbf{v}\rangle = \langle \alpha''(s_1), \mathbf{v}\rangle > 0$. This means that each component of $\alpha''(s_1)$ is non-negative. In particular, $z''(s_1) \geq 0$. But $\mathbf{v} \notin \mathbf{b}([0, L])$ further implies that $z''(s_1) > 0$. Thus, the vertical component of $\mathbf{n}(s_1)$ must be positive; that is, $\frac{z''(s_1)}{k} > 0$. Note that the assumption $k > 0$ is critical here. Thus, $\mathbf{n}(s_1)$ must be in the upper half-plane of our assigned coordinate system. Now, $\mathbf{n}$ makes an angle of θ with the horizontal, and so $\theta \in (0, \pi)$.



The circular cross section taken at $\alpha(s_1)$. Our coordinate system was chosen so that q_1 was the south pole. We showed that $\mathbf{n}$ must lie in the upper half of our coordinate system. The arc subtended by $\mathbf{b}(s_1)$ and $-\mathbf{b}(s_1)$ is indicated by the thicker semi-circle.

Further, we must have that points of positive curvature on this cross section of T lie in the open arc subtended by the diameter formed by $\mathbf{b}(s_1)$ and $-\mathbf{b}(s_1)$; that is, the points of positive curvature lie in $A = \left(\frac{\pi}{2} + \theta, \frac{3\pi}{2} + \theta\right)$. (Recall the definition of the region R.) If $\theta = \varepsilon > 0$, then $q_1 \in A = \left(\frac{\pi}{2} + \varepsilon, \frac{3\pi}{2} + \varepsilon\right)$ since q_1 makes an angle of $\frac{3\pi}{2}$ with the horizontal. Similarly, if $\theta = \pi - \varepsilon$ for $\varepsilon > 0$, we have $q_1 \in A = \left(\frac{3\pi}{2} - \varepsilon, \frac{5\pi}{2} - \varepsilon\right)$. This same argument works for s_2 and q_2. Thus, $K(q_1), K(q_2) > 0$. (This fills a gap mentioned in our proof of Fenchel's theorem.)

Now suppose $h_{\mathbf{v}}(s_1) < h_{\mathbf{v}}(s_2)$. Then P will meet T at a point q_1 with $K(q_1) > 0$ by the same reasoning as above. Take a plane P' parallel to P and at a distance r from P, where r is the radius of the tube. Move P' and P up until P' touches $\alpha(s_2)$. Then P meets T at a point $q_2 \neq q_1$. Again, using the reasoning above, $K(q_2) > 0$.

Notice that in both cases, we must have $N(q_1) = N(q_2)$. Thus, in both cases, we have two points in T with $K > 0$ mapped by N to a single point of S^2. And this occurs for every vector $\mathbf{v} \neq \mathbf{b}([0, L])$. So every vector in S^2 is "covered" by N at least twice, except for the images of the binormals and their antipodes, a set of measure zero. Since q_1 and q_2 are both "hit" by N twice, the single point to which they map in S^2 is covered four times over by N. If we take neighborhoods U_1 and U_2 about q_1 and q_2, then the region $V = N(U_1) \cap N(U_2) \subset S^2$ is covered twice over. By corollary 2.7, we thus have $\iint_R K\, d\sigma = 2\,\text{area}(S^2) + \text{area}(V) > 8\pi$.

This contradicts that $\iint_R K\, d\sigma \leq 8\pi$, and thus proves our claim; that is, there exists a vector $\mathbf{v}_0 \notin \mathbf{b}([0, L])$ such that $h_{\mathbf{v}_0}$ has exactly two critical points. Rotate so that $\mathbf{v}_0$ is parallel to the z-axis. It then follows that $z(s)$ has exactly two critical points. Since $[0, L]$ is compact, these two critical points are a maximum and minimum. Call these critical points s_1 and s_2.

Let P_1 and P_2 be planes perpendicular to $\mathbf{v}_0$ passing through $\alpha(s_1)$ and $\alpha(s_2)$, respectively. Now, α is split into two arcs. Since $h'_{\mathbf{v}_0}(s) \neq 0$ except at s_1 and s_2, one arc is strictly decreasing from $\alpha(s_1)$ to $\alpha(s_2)$ and one strictly increasing from $\alpha(s_2)$ to $\alpha(s_1)$. By the intermediate value theorem, each plane perpendicular to $\mathbf{v}_0$ and between P_1 and P_2 intersects each of these arcs once. Thus, each of these planes intersects C in exactly two points. Join these pairs of points by line segments. This generates a surface bounded by C homeomorphic to a disk. But this implies that C is unknotted, which is a contradiction.

Therefore, we must have $\int_0^L k\, ds > 4\pi$.

□

This proof comes from do Carmo [3], with clarifications from Chern [10] and Spivak [11]. As we have mentioned, Milnor's proof is quite different. In fact, he proved a more general theorem, of which a weak statement of the Fary-Milnor theorem is just a consequence [8]. We will present this result in chapter four.

3.2 Crofton's Formula

Just as with Fenchel's theorem, there is a more elementary, curve-theoretic proof available. We use Crofton's theorem; this theorem deals with the measure of great circles intersecting a curve on the unit sphere, and how that measure relates to the length of the curve. Now, every oriented great circle uniquely determines a *pole*: a point on the sphere normal to the plane containg the great circle; we use the right-hand rule. When we speak of the measure of the great circles, we mean the area of the domain of their poles.

Theorem 3.3. *(Crofton) Let γ be a smooth curve on S^2 with length L. The measure of the oriented great circles meeting γ, each counted a number of times equal to the number of its intersections with γ, is $4L$.*

Proof. Suppose γ is determined by a position vector $e_1(s)$, a unit vector expressed as a function of the arc length parameter s. In a neighborhood of s, let $e_2(s)$ and $e_3(s)$ be unit vectors such that $\langle e_i, e_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq 3$ and $\det(e_1, e_2, e_3) = +1$. That is, $\{e_1, e_2, e_3\}$ determine a positively-oriented frame for γ at each value of s. Differentiating $\langle e_i, e_i \rangle = 1$, we have $\langle e_i', e_i \rangle = 0$. So e_i' is normal to e_i for each $i = 1, 2, 3$. Thus, we have three equations:

$$
\begin{aligned}
e_1'(s) &= a_2 e_2(s) + a_3 e_3(s) \\
e_2'(s) &= -a_2 e_1(s) + a_1 e_3(s) \\
e_3'(s) &= -a_3 e_2(s) - a_1 e_2(s)
\end{aligned}
$$

The skew-symmetry follows from differentiating the equations $\langle e_i, e_j \rangle = 0$ for $i \neq j$. Since e_1 is the position vector, we notice that $e_1'(s) = \mathbf{t}(s)$. And since γ is parameterized by arc length, we have $1 = |\mathbf{t}(s)| = |e_1'(s)| = \sqrt{a_2^2 + a_3^2}$, which implies that $a_2^2 + a_3^2 = 1$. So let $a_2 = \cos(\varphi(s))$ and $a_3 = \sin(\varphi(s))$, where $\varphi(s)$ measures the angle e_1' makes with respect to e_2 in the e_2e_3-plane, which we note is the tangent plane at $e_1(s)$.

An oriented great circle meeting γ at $e_1(s)$ has a pole of the form $Y = \cos\theta\, e_2(s) + \sin\theta\, e_3(s)$, where θ is the angle Y makes with respect to e_2 in the tangent plane. Thus, (s, θ) serve as local coordinates in the domain of poles. Now

$$dY = \underbrace{(-\sin\theta e_2 + \cos\theta e_3)}_{\mathbf{u}}(d\theta + a_1 ds) - \underbrace{(a_2\cos\theta + a_3\sin\theta)e_1}_{\mathbf{v}}\, ds\,.$$

Note that $\mathbf{u}$ and e_1 are orthogonal, and that $|\mathbf{u}| = 1$. Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to Y, the area element of Y in the domain of poles is

$$\begin{aligned}
|dA| = |\mathbf{u}\times\mathbf{v}|d\theta ds &= |\mathbf{u}||\mathbf{v}||\sin\psi|d\theta ds \\
&= |\mathbf{v}|d\theta ds \\
&= |a_2\cos\theta + a_3\sin\theta|d\theta ds \\
&= |\cos\varphi\cos\theta + \sin\varphi\sin\theta| \\
&= |\cos(\varphi - \theta)|d\theta ds\,,
\end{aligned}$$

where we use $|dA|$ because we are calculating area without regard to orientation.

Now, let $Y^\perp$ denote the oriented great circle with Y as its pole. Let $n(Y^\perp)$ be the number of intersection points between γ and $Y^\perp$, counted arithmetically. We do not prove it, but the set of poles Y for which $n(Y^\perp) = \infty$ has zero measure. Then the measure μ described in our theorem is

$$\mu = \iint_{S^2} n(Y^\perp)\,|dA| = \int_0^L\int_0^{2\pi} |\cos(\varphi - \theta)|\, d\theta ds\,.$$

But cosine is even, so $\cos(\varphi - \theta) = \cos(\theta - \varphi)$. Now let $t = \theta - \varphi$. Then $dt = d\theta$, and the measure is translation invariant. Thus we have

$$\begin{aligned}\mu = \int_0^L\int_0^{2\pi} |\cos(\varphi-\theta)|\, d\theta ds &= \int_0^L\int_0^{2\pi} |\cos(\theta-\varphi)|\, d\theta ds \\ &= \int_0^L ds \int_0^{2\pi} |\cos t|\, dt \\ &= \int_0^L ds \left(4\cdot \int_0^{\pi/2} \cos t\, dt\right) \\ &= 4\int_0^L ds \\ &= 4L\,.\end{aligned}$$

□

Chern provided the above proof [10].

Remark 3.4. Crofton's formula can be generalized to any Riemannian surface; the integral is evaluated with respect to the measure on the space of geodesics.

3.3 A Curve-Theoretic Proof

We now give a more elementary proof of the Fary-Milnor theorem making use of Crofton's formula.

Theorem 3.5. *(Fary-Milnor) The total curvature of a knotted simple closed curve is greater than* 4π*.*

Proof. Let C be parameterized by $\alpha(s) = (x(s), y(s), z(s))$; let Γ be the tangent indicatrix. Recall that $\ell(\Gamma) = \int_0^L k\, ds = k(C)$; the total curvature of C is the length of the tangent indicatrix. Let Y be a pole in S^2 and consider the function $h_Y(s) = \langle \alpha(s), Y\rangle$. We know that since $h_Y'(s) = \langle \mathbf{t}(s), Y\rangle$, the value s is a critical point of h_Y if and only if Y is perpendicular to $\mathbf{t}(s)$. Suppose we rotate so that Y is the north pole. Then the critical points of h_Y are exactly the values of s such that $\mathbf{t}(s)$ lies in the equatorial plane. This implies that $n(Y^\perp)$ is exactly the number of minima or maxima of h_Y.

Suppose $k(C) < 4\pi$. By Crofton's Formula we have

$$k(C) = \ell(\Gamma) = \frac{1}{4} \iint_{S^2} n(Y^\perp)\, |dA| < 4\pi\,.$$

The surface area of S^2 is 4π, and so there must be some pole Y such that $n(Y^\perp) < 4$. But $n(Y^\perp)$ must be even, since it is the number of maxima or minima of h_Y. So there exists a pole Y_0 such that $n(Y_0^\perp) = 2$. Again suppose Y_0 is the north pole so that $Y_0^\perp$ lies in the equatorial plane.

Since h_{Y_0} has two extrema, it follows that $z(s)$ has two extrema. But now $[0, L]$ is compact, so $z(s)$ in fact has exactly one maximum and one minimum. From here, our proof proceeds in an identical manner to the previously presented proof: we fill a disk bounded by C, thus reaching a contradiction. So we must have $k(C) \geq 4\pi$.

$\square$

Here we have followed Chern [10] again, with some help from Shifrin [4]. Although this second proof is considered more elementary, the reliance on Crofton's Formula means that there are some non-trivial measure-theoretic considerations in operation. There is always a trade-off.

Chapter 4: Total Curvature and Bridge Number

Earlier we mentioned that Milnor actually proved a more general result, and that the Fary-Milnor theorem was a corollary. This chapter provides this proof. As the title of this chapter suggests, we need a knot-theoretic tool: the bridge number. Milnor's result gives a lower bound on the total curvature in terms of the bridge number of the knot, which is quite amazing.

A knot cannot be embedded in a plane. However, we can situate most of a knot in a plane save for a few "bridges." This idea was first used by Horst Schubert in 1954 [12], and gives us a simple measure of the complexity of a knot. For details on the bridge number, see Adams [13]. This chapter follows Milnor [8].

4.1 Polygons

Before we give a rigorous definition of bridge number, we first talk about polygonal approximations of curves.

Definition 4.1. A *closed polygon* P in $\mathbb{R}^n$, with $n > 2$, is a finite sequence of points a_0, a_1, ..., a_m, with $a_m = a_0$ and $a_i \neq a_{i+1}$, and line segments $\overline{a_i a_{i+1}}$ for $i = 0, 1, \ldots, m-1$.

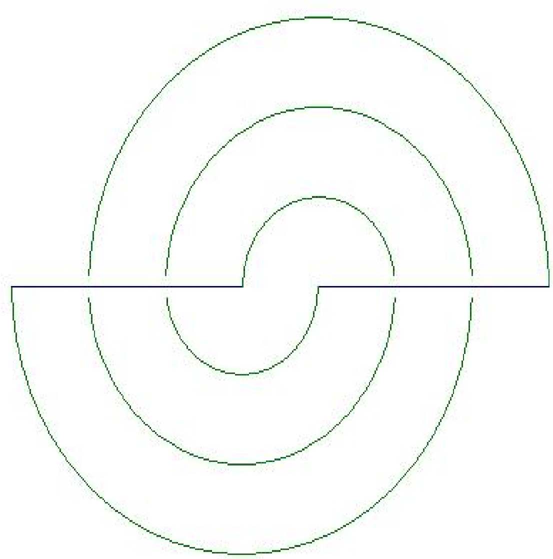
In what follows, we do not distinguish between a vector and a point, as we have done throughout this paper, with all vectors referred to a common origin. So $a_{i+1} - a_i$ is the vector parallel to, and with equal magnitude as, the line segment $\overline{a_i a_{i+1}}$. Let ψ_i be the angle between $a_{i+1} - a_i$ and $a_i - a_{i-1}$ such that $0 \leq \psi_i \leq \pi$. We define the *total curvature of the polygon* P as $k(P) = \sum_{i=1}^{m} \psi_i$.

We say a closed polygon is inscribed in a closed curve $\alpha(s)$ if there exists a collection of values $\{s_i\}$ in $[0, L]$ such that $s_i < s_{i+1}$, $s_{i+m} = s_i + L$, and $a_i = \alpha(s_i)$ for all integers i. (We are taking s_i to signify $s_{(i)}$, where (i) is the least positive residue mod m.) Milnor proves that for any closed polygon P, we have $k(P) = \text{lub}\,\{k(P')\}$ with P' ranging over all polygons inscribed in P. We now define the total curvature of C as $k(C) = \text{lub}\,\{k(P)\}$ with P ranging over all polygons inscribed in C. This definition does agree with the definition that has been used up to this point, though we do not show it here. (We establish a similar result in the next chapter following Fary [9].)

Now, for every closed curve C and unit vector $\mathbf{v}$, let $\mu(C, \mathbf{v})$ be the number of maxima of the function $h_{\mathbf{v}}(s) = \langle \alpha(s), \mathbf{v} \rangle$.

Definition 4.2. The *bridge number* of C is $\text{br}(C) = \min_{\mathbf{v}} \{\mu(C, \mathbf{v})\}$. (Milnor denotes this $\mu(C)$; we defer to the now-standard notation.)

We primarily care about the bridge number of a knot. We note that a knot has bridge number one if and only if it is the unknot, the knot with no crossings. We do not prove it, but intuitively it should be believable. An easy example of a 2-bridge knot is the trefoil. The image below is a 2-bridge presentation of the trefoil.



This is a bird's-eye view. The straight segments are the bridges sitting outside the plane; the spiraling portions are lying in the plane.

Now, for each vector $a_{i+1} - a_i \in \mathbb{R}^n$, we define

$$b_i = \frac{a_{i+1} - a_i}{|a_{i+1} - a_i|}\,.$$

We consider b_i as a point on the sphere S^{n-1}; we say b_i is the *spherical image* of $a_{i+1} - a_i$. Given a polygon P with vertices $a_1, \cdots, a_m$, a spherical polygon Q is formed on S^{n-1} by joining each b_{i-1} to b_i by a geodesic arc of length β_i. This Q is the spherical image of P, which is unique unless we have $b_j = -b_{j+1}$ for some j. We allow for $b_j = b_{j+1}$.

4.2 Milnor's Result

Theorem 4.3. *For any closed curve $C \subset \mathbb{R}^n$, with $n > 1$, the Lebesgue integral $\int_{S^{n-1}} \mu(C, \mathbf{v})\, dS^{n-1}$ exists and is equal to $\frac{M_{n-1}k(C)}{2\pi}$, where $M_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of S^{n-1}, and $\mathbf{v}$ ranges over S^{n-1}.*

We will only consider the case where $n = 3$; the proof for arbitrary dimension is nearly identical. Further, we note that $M_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = 4\pi$, where $\Gamma(z)$ is the Gamma function.

Proof. Suppose $C = P$ for some polygon P. For each $\mathbf{v} \in S^2$, let $S^1_{\mathbf{v}}$ denote the great circle lying in the plane normal to $\mathbf{v}$. So $\mathbf{v}$ is the pole of $S^1_{\mathbf{v}}$. An edge $\overline{b_{i-1}b_i}$ of Q crosses $S^1_{\mathbf{v}}$ if and only if $\langle b_i, \mathbf{v}\rangle$ and $\langle b_{i-1}, \mathbf{v}\rangle$ have opposite signs, so that s_i is an extremum of $h_{\mathbf{v}}$. So if $S^1_{\mathbf{v}}$ contains no vertex of Q, we have that the number of intersections of Q with $S^1_{\mathbf{v}}$ is $2\mu(P, \mathbf{v})$ (since μ is only counting maxima). The set of points $\mathbf{v} \in S^2$ such that $S^1_{\mathbf{v}}$ contains the vertex b_i is exactly $S^1_{b_i}$. Thus, the set of points $\mathbf{v}$ such that $S^1_{\mathbf{v}}$ contains some vertex of Q is $\bigcup_i S^1_{b_i}$. The function $2\mu(P, \mathbf{v})$ is constant on each of the sets $S^2 - S^1_{b_i}$. And we note that even though $\mu(P, \mathbf{v})$ is infinite on each $S^1_{b_i}$, these great circles have measure zero, and so they do not affect the integral of μ. So the integral $\iint_{S^2} 2\mu(P, \mathbf{v})\, dS^2$ exists.

The set of points $\mathbf{v}$ such that $S^1_{\mathbf{v}}$ meets a given edge $\overline{b_{i-1}b_i}$ of length $0 \leq \beta_i \leq \pi$ is a double lune bounded by $S^1_{b_{i-1}}$ and $S^1_{b_i}$. Thus, $\overline{b_{i-1}b_i}$ contributes 1 to $2\mu(P, \mathbf{v})$ if $\mathbf{v}$ is interior to this double lune, and the contribution is 0 is $\mathbf{v}$ is exterior to the double lune. It is not difficult to see that β_i is also the angle between the two great circles bounding our double lune; so the area of the double lune is $\frac{\beta_i M_2}{\pi} = 4\beta_i$. Further, letting ψ_i be the angle between $a_{i+1} - a_i$ and $a_i - a_{i-1}$, we have

$$\psi_i = \cos^{-1}\left(\frac{\langle\, a_{i+1} - a_i \,,\, a_i - a_{i-1} \,\rangle}{|a_{i+1} - a_i||a_i - a_{i-1}|}\right) = \cos^{-1}(\langle\, b_i \,,\, b_{i-1} \,\rangle) = \beta_i \,,$$

with the middle equality following from the definition of b_i. So $\beta_i = \psi_i$, and $\sum_{i=1}^{m} \beta_i = \sum_{i=1}^{m} \psi_i = k(P)$. Thus we have

$$\iint\limits_{S^2} 2\mu(P, \mathbf{v})\, dS^2 = \frac{M_2}{\pi} \sum_{i=1}^{m} \beta_i = \frac{M_2}{\pi} k(P) = 4k(P)\,.$$

Now let C be an arbitrary closed curve parameterized by $\alpha(s)$. Let $\{P_m\}$ be a set of inscribed polygons, given by $\alpha_m(s)$, with vertices $a_1^m = \alpha(s_1^m), \ldots, a_m^m = \alpha(s_m^m)$ such that each P_m contains the vertices of P_{m-1}. We further require that $\lim\limits_{m\to\infty} k(P_m) = k(C)$ and $\lim\limits_{m\to\infty} (s_{j+1}^m - s_j^m) = 0$. Let $h_{\mathbf{v},m}$ be the height function for the polygon $\alpha_m(s)$ in the $\mathbf{v}$ direction.

The values $\mathbf{v}$ for which $h_{\mathbf{v}}$ or $h_{\mathbf{v},m}$, for any m, is constant on some interval form a set of measure zero. They do not affect the integral, so they are ignored in the remainder of the proof.

Certainly it is the case that $\mu(P_{m-1}, \mathbf{v}) \leq \mu(P_m, \mathbf{v}) \leq \mu(C, \mathbf{v})$. Suppose $\mu(C, \mathbf{v})$ is finite. We can select neighborhoods about each of the $\mu(C, \mathbf{v})$ maxima of $h_{\mathbf{v}}$, as well as each of its minima, small enough so that any polygon with a vertex in each neighborhood will have at least $\mu(C, \mathbf{v})$ maxima. For m sufficiently large, this is easy to accomplish. Note that requiring $\lim\limits_{m\to\infty} (s_{j+1}^m - s_j^m) = 0$ is important here.

Suppose $\mu(C, \mathbf{v})$ is infinite. Let $\{s_i\}$ be the set of values of s such that $h_{\mathbf{v}}$ attains a maximum. From analysis, we know that $\{s_i\}$ must contain a monotone subsequence

$\{s_{2i}\}$. So either

$$s_0 < s_2 < \cdots < \lim_{i\to\infty} s_{2i} < s_0 + L\,,$$

or

$$s_0 > s_2 > \cdots > \lim_{i\to\infty} s_{2i} > s_0 - L\,.$$

In either case, we can select intermediate values s_{2i+1} such that $h_{\mathbf{v}}(s_{2i}) > h_{\mathbf{v}}(s_{2i+1})$ and $h_{\mathbf{v}}(s_{2i}) > h_{\mathbf{v}}(s_{2i-1})$. So now, for any $2j$, we select neighborhoods of $\alpha(s_i)$, for $i < 2j$, small enough that any polygon with at least one vertex in each neighborhood has at least $j-1$ maxima. Again, for m sufficiently large, this is clear. Thus, $\mu(P_m, \mathbf{v})$ increases without bound as m goes to infinity.

So in both the finite and infinite cases, we have that $\lim_{m\to\infty} \mu(P_m, \mathbf{v}) = \mu(C, \mathbf{v})$. Further, each of the integrals $\iint_{S^2} \mu(P_m, \mathbf{v})\, dS^2$ exists.

Lastly, we have that $\mu(P_m, \mathbf{v})$ is a non-decreasing sequence of positive functions whose limit exists. By the Lebesgue monotone convergence theorem we have

$$\begin{aligned}
\iint_{S^2} \mu(C, \mathbf{v})\, dS^2 = \iint_{S^2} \lim_{m\to\infty} \mu(P_m, \mathbf{v}) dS^2 &= \lim_{m\to\infty} \iint_{S^2} \mu(P_m, \mathbf{v})\, dS^2 \\
&= \lim_{m\to\infty} \frac{M_2}{2\pi}\, k(P_m) \\
&= \lim_{m\to\infty} 2k(P_m) \\
&= 2k(C)\,.
\end{aligned}$$

□

Corollary 4.4. *For any closed curve C, $k(C) \geq 2\pi\, br(C)$.*

Proof. From our above theorem, we have

$$2k(C) = \frac{M_2}{2\pi}\, k(C) = \iint_{S^2} \mu(C, \mathbf{v})\, dS^2 \geq \iint_{S^2} \mathrm{br}(C)\, dS^2 = 4\pi\, \mathrm{br}(C)\,.$$

□

Notice that we can easily recover Fenchel's theorem. Recovering the Fary-Milnor theorem requires showing that every non-trivial knot has bridge number at least 2. Since we spent some time working through Milnor's method of proof, it seems only fair to devote ourselves to understanding Fary's proof as well: this is the subject of the next chapter.

Chapter 5: Total Curvature and Orthogonal Projections

Fary, whose proof appeared independently of Milnor's, and a little earlier as well, follows a similar path: he uses polygons to approximate a curve. However, Fary is more abstruse in his treatment, as well as more analytical. We will be using the same definition of polygon as Milnor, but we shall require different restrictions on our curves. Further, we will be using a few different, though equivalent, definitions of curvature.

5.1 Curvature and Polygons

Let a closed curve C be given by $\alpha(s) = (x(s), y(s), z(s))$ for $s \in [0, L]$, subject to the following conditions: the tangent $\mathbf{t}$ exists everywhere except a finite number of points corresponding to parameter values $0 \leq d_1 < \cdots < d_n < L$, and $\alpha''(s)$ is continuous in each interval $[d_i, d_{i+1}]$. That is to say, our curve C is piecewise C^2.

Take three points on C and let a, b, and c be the corresponding parameter values. There is a unique circle passing through the these three points; let ρ_{abc} be the radius of this circle.

Definition 5.1. The curvature of C at s is given by

$$k(s) = \lim_{\Delta \to 0} \frac{1}{\rho_{abc}}\,, \tag{5.1}$$

where $\Delta = |a - s| + |b - s| + |c - s|$, provided the limit exists.

Given vectors $\mathbf{u}$ and $\mathbf{v}$, denote the angle between them by $\Phi(\mathbf{u}, \mathbf{v})$, chosen so that $0 \leq \Phi(\mathbf{u}, \mathbf{v}) \leq \pi$.

Definition 5.2. The curvature of C at s is given by

$$k(s) = \lim_{|b-a| \to 0} \frac{\Phi[\mathbf{t}(a), \mathbf{t}(b)]}{|s-a|} = |\alpha''(s)|\,, \tag{5.2}$$

where $a < s < b$.

We note that if $\alpha''(s)$ is continuous, the limits above are equal and attained uniformly. We also remark that we will have have $k(s) \geq 0$, even for planar curves. Recall that $k(C) = \int_0^L k\,ds = \int_0^L \theta'(s)\,ds = \ell(\Gamma)$. There is one last definition of total curvature that will prove useful.

Definition 5.3. Given a closed curve C, the total curvature of C is given by

$$k(C) = \lim_{\max|s_i - s_{i-1}| \to 0} \sum_{i=1}^{n} \Phi[\mathbf{t}(s_i), \mathbf{t}(s_{i-1})] \quad (s_i \neq d_j)\,. \tag{5.3}$$

Now let a polygon P be determined by vertices $a_1, \ldots, a_n$. Then

$$k(P) = \sum_{i=1}^{n} \Phi[\,\overline{a_{i-1}a_i}\,,\, \overline{a_i a_{i+1}}\,]\,,$$

as we had with Milnor's treatment. Lastly, Fary lists as a proposition the following fact:

Fact 5.4. *For a closed planar curve C, $k(C) \geq 2\pi$, with equality if and only if C is convex.*

This is just a weak version of Fenchel's theorem that Fary uses in proving other results. We have already proved it through the course of other proofs, but a short proof is very easily supplied.

Proof. Since C is closed and planar, Γ must be a great circle, and so we immediately have $k(C) \geq 2\pi$. To get the "if and only if," we apply lemma 2.15 and recall that a convex curve must be simple.

□

We now wish to realize the total curvature of a curve by inscribing polygons and taking the limit of their total curvatures. We remarked on such a result earlier, but skipped the proof. Here we present a proof.

Lemma 5.5. *If $p, p', q, q' > 0$ and $\frac{p}{q} \leq \frac{p'}{q'}$, then $\frac{p}{q} \leq \frac{p+p'}{q+q'} \leq \frac{p'}{q'}$.*

Proof. Suppose $\frac{p}{q} \leq \frac{p'}{q'}$. Then $pq' \leq p'q$.

To get get the first inequality, we have the following calculation

$$\begin{aligned} pq' + pq &\leq p'q + pq \\ p(q' + q) &\leq q(p' + p) \\ \frac{p}{q} &\leq \frac{p' + p}{q' + q}. \end{aligned}$$

To get the second inequality, we do a similar calculation

$$\begin{aligned} pq' + p'q' &\leq p'q + p'q' \\ q'(p + p') &\leq p'(q + q') \\ \frac{p + p'}{q + q'} &\leq \frac{p'}{q'}. \end{aligned}$$

Our result follows.

□

Theorem 5.6. *Let P_r be a family of polygons inscribed in C. Suppose that the points of discontinuity of $\mathbf{t}(s)$ are among the vertices of each P_r. Further, suppose $P_r \to C$ as $r \to \infty$, that is, the length of the longest side of P_r tends to zero with $\frac{1}{r}$. Then $\lim_{r\to\infty} k(P_r) = k(C)$.*

Proof. Let $C' \subseteq C$ be the image $\alpha([d, d'])$ for some $d, d' \in [0, L]$ with $d < d'$. Suppose $\alpha''(s)$ is continuous on $[d, d']$. Take $a, b, c \in [d, d']$ with $a < b < c$. Let

$$\begin{aligned} \theta &= \Phi[\alpha(b) - \alpha(a)\, , \, \alpha(c) - \alpha(b)]\, , \\ \theta^* &= \Phi\left[\mathbf{t}\left(\frac{a+b}{2}\right)\, , \, \mathbf{t}\left(\frac{b+c}{2}\right)\right]. \end{aligned}$$

We first wish to show that

$$\lim_{|c-a|\to 0} \frac{\theta}{\theta^*} = 1\,.$$

Let $\varepsilon > 0$. The angle θ goes to zero as $|c-a|$ goes to zero, since $\mathbf{t}(s)$ is continuous on C'. For $|c-a|$ sufficiently small, then, we have

$$\sin\theta \le \theta \le (1+\varepsilon)\sin\theta\,. \tag{5.4}$$

Now, $\frac{b+c}{2} - \frac{a+b}{2} = \frac{c-a}{2}$ and $|\frac{c-a}{2}|$ goes to zero as $|c-a|$ does. So use (5.2) to write

$$k(b) = \lim_{|c-a|\to 0} \frac{\theta^*}{\left(\frac{c-a}{2}\right)}\,.$$

So for $|c-a|$ sufficiently small, we have

$$\left| k(b) - \frac{\theta^*}{\left(\frac{c-a}{2}\right)} \right| < \varepsilon\,.$$

This gives us that

$$(1-\varepsilon)k(b)\,\frac{c-a}{2} \le \theta^* \le (1+\varepsilon)k(b)\,\frac{c-a}{2}\,, \tag{5.5}$$

for $|c-a|$ small enough.

Let $\rho_{abc} = \frac{1}{k^*}$ be the radius of the circle passing through $\alpha(a)$, $\alpha(b)$, and $\alpha(c)$. By (5.1), $k(b) = \lim\limits_{d\to 0} \frac{1}{\rho_{abc}} = \lim\limits_{d\to 0} k^*$, where $d = |a-b| + |c-b|$. But since $|c-a| = |c-b+b-a| \le |c-b| + |b-a|$, we certainly have that $|c-a| \to 0$ as $d \to 0$. So for $|c-a|$ small enough, we have

$$(1-\varepsilon)k(b) \le k^* \le (1+\varepsilon)k(b)\,. \tag{5.6}$$

Lastly, for $|u-v|$ sufficiently small, we have

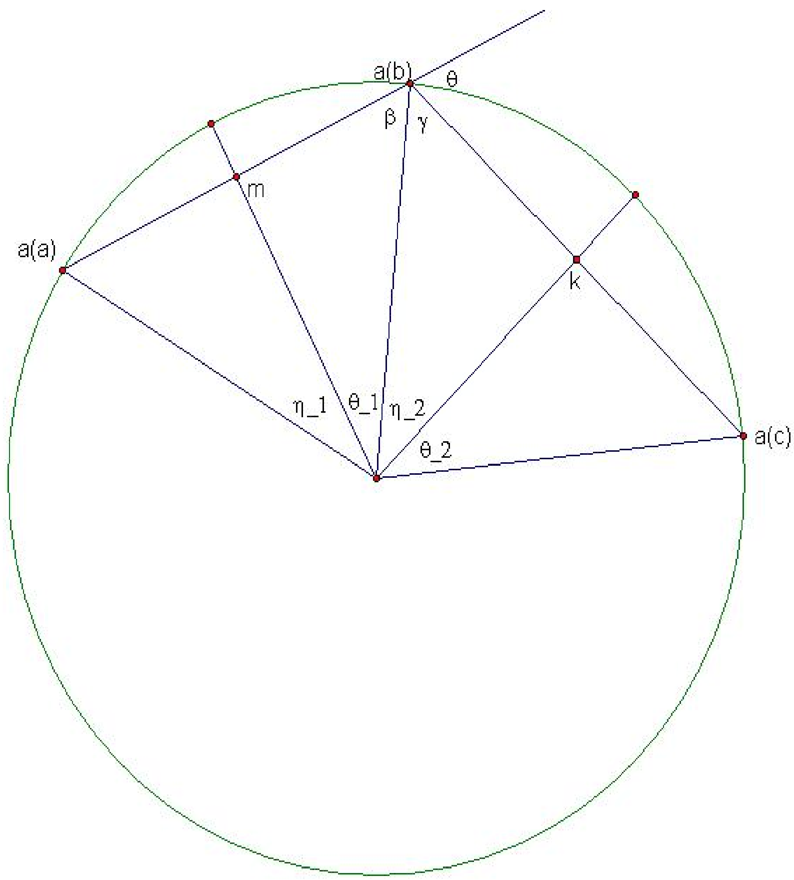
$$1-\varepsilon \le \left|\frac{\alpha(u)-\alpha(v)}{u-v}\right| \le 1+\varepsilon\,. \tag{5.7}$$

So choose $|c-a|$ small enough that inequalities (5.4), (5.5), (5.6), and (5.7) all hold on C'.

Consider the circle passing through $\alpha(a)$, $\alpha(b)$, and $\alpha(c)$ again. Denote the center by r. Let

$$\theta_1 = \Phi\left[\alpha(b) - r\,,\ \frac{\alpha(a)+\alpha(b)}{2} - r\right],$$
$$\theta_2 = \Phi\left[\alpha(c) - r\,,\ \frac{\alpha(b)+\alpha(c)}{2} - r\right].$$

We claim that $\theta_1 + \theta_2 = \theta$. To see this, consider the diagram below.



The triangles $\triangle\alpha(a)\alpha(b)r$ and $\triangle\alpha(b)\alpha(c)r$ are isosceles. So the line segments m and k are the perpendicular bisectors of their respective triangles since they pass through $\frac{\alpha(a)+\alpha(b)}{2}$ and $\frac{\alpha(b)+\alpha(c)}{2}$, respectively. Thus, $\eta_1 = \theta_1$ and $\eta_2 = \theta_2$. This gives us that $\pi = 2\beta + 2\theta_1$ and $\pi = 2\gamma + 2\theta_2$; or, $\beta = \frac{\pi}{2} - \theta_1$ and $\gamma = \frac{\pi}{2} - \theta_2$. And therefore, since $\pi = \theta + \beta + \gamma$, we have $\theta = \pi - \beta - \gamma = \pi - \frac{\pi}{2} + \theta_1 - \frac{\pi}{2} + \theta_2 = \theta_1 + \theta_2$.

Now, from the above diagram, we also see that

$$\sin\theta_1 = \frac{\frac{1}{2}|\alpha(b)-\alpha(a)|}{\rho_{abc}} = \frac{\left|\frac{\alpha(b)-\alpha(a)}{2}\right|}{\left(\frac{1}{k^*}\right)}.$$

A similar calculation with θ_2 gives us

$$\frac{\sin\theta_1}{k^*} = \left|\frac{\alpha(b)-\alpha(a)}{2}\right| \quad \text{and} \quad \frac{\sin\theta_2}{k^*} = \left|\frac{\alpha(c)-\alpha(b)}{2}\right|.$$

So from (5.4) and (5.5), we have the following inequality

$$\frac{\theta}{\theta^*} = \frac{\theta_1+\theta_2}{\theta^*} \leq \frac{(1+\varepsilon)(\sin\theta_1+\sin\theta_2)}{(1-\varepsilon)k(b)\left(\frac{c-a}{2}\right)} = \frac{(1+\varepsilon)k^*\left(\frac{\sin\theta_1}{k^*}+\frac{\sin\theta_2}{k^*}\right)}{(1-\varepsilon)k(b)\left(\frac{b-a}{2}+\frac{c-b}{2}\right)}$$
$$= \frac{(1+\varepsilon)k^*\left(\left|\frac{\alpha(b)-\alpha(a)}{2}\right|+\left|\frac{\alpha(c)-\alpha(b)}{2}\right|\right)}{(1-\varepsilon)k(b)\left(\frac{b-a}{2}+\frac{c-b}{2}\right)} = A.$$

Now applying (5.6), we get

$$A \leq \frac{(1+\varepsilon)^2k(b)\left(\left|\frac{\alpha(b)-\alpha(a)}{2}\right|+\left|\frac{\alpha(c)-\alpha(b)}{2}\right|\right)}{(1-\varepsilon)k(b)\left(\frac{b-a}{2}+\frac{c-b}{2}\right)} = \frac{(1+\varepsilon)^2\left(\left|\frac{\alpha(b)-\alpha(a)}{2}\right|+\left|\frac{\alpha(c)-\alpha(b)}{2}\right|\right)}{(1-\varepsilon)\left(\frac{b-a}{2}+\frac{c-b}{2}\right)} = B.$$

Now, since $a < b < c$, notice that

$$\frac{p}{q} = \frac{\left|\frac{\alpha(b)-\alpha(a)}{2}\right|}{\left(\frac{b-a}{2}\right)} = \left|\frac{\alpha(b)-\alpha(a)}{b-a}\right| \quad \text{and} \quad \frac{p'}{q'} = \frac{\left|\frac{\alpha(c)-\alpha(b)}{2}\right|}{\left(\frac{c-b}{2}\right)} = \left|\frac{\alpha(c)-\alpha(b)}{c-b}\right|.$$

By (5.7), we can choose $|c-a|$ small enough that $\frac{p}{q} \leq 1+\varepsilon$ and $\frac{p'}{q'} \leq 1+\varepsilon$. Choose $|c-a|$ small enough that $\frac{p}{q} \leq \frac{p'}{q'}$, and apply lemma 5.5 to B to get

$$B \leq \frac{(1+\varepsilon)^2(1+\varepsilon)}{1-\varepsilon}.$$

Following our inequalities, we get

$$\frac{\theta}{\theta^*} \leq \frac{(1+\varepsilon)^3}{1-\varepsilon}.$$

But each of (5.4), (5.5), (5.6), and (5.7) is a double inequality, and we have so far used only one inequality from each. An analogous calculation using the other inequalities gives

$$\frac{(1-\varepsilon)^2}{1+\varepsilon} \leq \frac{\theta}{\theta^*}.$$

(Here $1-\varepsilon$ only ends up being squared because (5.4) is not symmetric.) Thus, letting ε go to zero, we get $\frac{\theta}{\theta^*} \to 1$.

Now suppose $\alpha''(s)$ is continuous everywhere and let P_r be a polygon inscribed in C with vertices $\{a_{r_i}\}_{i=1}^{n_r}$. Let $\theta_{r_i} = \Phi[\,\overline{a_{r_{i-1}}a_{r_i}}\,,\,\overline{a_{r_i}a_{r_{i+1}}}\,]$. Note that $k(P_r) = \sum_{i=1}^{n_r} \theta_{r_i}$. Let $\theta^*_{r_i}$ be the angle between tangents on consecutive arcs $a_{r_{i-1}}a_{r_i}$ and $a_{r_i}a_{r_{i+1}}$. By our work above, for sufficiently large r, the ratio $\frac{\theta_{r_i}}{\theta^*_{r_i}}$ is nearly one. We thus have

$$(1-\varepsilon)\sum_{i=1}^{n_r} \theta^*_{r_i} \leq \sum_{i=1}^{n_r} \frac{\theta_{r_i}}{\theta^*_{r_i}} \theta^*_{r_i} \leq (1+\varepsilon)\sum_{i=1}^{n_r} \theta^*_{r_i}\,.$$

This obviously gives us

$$(1-\varepsilon)\sum_{i=1}^{n_r} \theta^*_{r_i} \leq \sum_{i=1}^{n_r} \theta_{r_i} \leq (1+\varepsilon)\sum_{i=1}^{n_r} \theta^*_{r_i}\,.$$

But from definition (5.3), we have that $\sum_{i=1}^{n_r} \theta^*_{r_i}$ is an approximation of $k(C)$. So as $r \to \infty$ we get that $k(P)$ tends to $k(C)$.

If $\alpha''(s)$ is not continuous, we subdivide C into arcs C_k partitioned according to the discontinuous points of $\alpha''(s)$. The endpoints of each C_k are vertices of P_r by supposition, and we use these points to construct partial polygons in each C_k. The total curvatures of these partial polygons tend toward the total curvatures of the C_k's, and the angles near the endpoints of the C_k's tend toward the angles of the left and right tangents of these points.

$\square$

5.2 Orthogonal Projections

With that marathon proof under our belts, we now wish to realize total curvature as the average total curvature of the curve's orthogonal projections. To this end, given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{n}$, let $\mathbf{u}_n$ and $\mathbf{v}_n$ be the orthogonal projections of $\mathbf{u}$ and $\mathbf{v}$ onto a plane whose normal vector is $\mathbf{n}$. Lastly, let $\Phi^*(\mathbf{n};\mathbf{u},\mathbf{v}) = \Phi(\mathbf{u}_n,\mathbf{v}_n)$.

Lemma 5.7. $\omega = \Phi(\mathbf{u}, \mathbf{v}) = \frac{1}{4\pi} \iint\limits_{S^2} \Phi^*(\mathbf{n}; \mathbf{u}, \mathbf{v})\, dS^2$.

Proof. Let $\lambda > 0$. If $\mathbf{v} = \lambda \mathbf{u}$, then both $\Phi(\mathbf{u}, \mathbf{v})$ and $\iint\limits_{S^2} \Phi^*(\mathbf{n}; \mathbf{u}, \mathbf{v})\, dS^2$ are zero. If $\mathbf{v} = -\lambda \mathbf{u}$, then $\Phi(\mathbf{u}, \mathbf{v}) = \pi = \Phi^*(\mathbf{n}; \mathbf{u}, \mathbf{v})$, except for $\mathbf{n} = \pm\mu\mathbf{u}$, for some constant μ, but this forms a set of measure zero and so does not affect the integral. We therefore have

$$\pi = \frac{1}{4\pi} \iint\limits_{S^2} \Phi^*(\mathbf{n}; \mathbf{u}, \mathbf{v})\, dS^2 .$$

So suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly independent; then they determine a plane. Let $\mathbf{n}_0$ be normal to this plane. Let $\mathbf{u}'$ and $\mathbf{v}'$ be vectors such that $\Phi(\mathbf{u}', \mathbf{v}') = \Phi(\mathbf{u}, \mathbf{v})$. Perform a rotation that sends $\frac{\mathbf{u}}{|\mathbf{u}|}$ to $\frac{\mathbf{u}'}{|\mathbf{u}'|}$ and $\frac{\mathbf{v}}{|\mathbf{v}|}$ to $\frac{\mathbf{v}'}{|\mathbf{v}'|}$. Transforming our desired integral by this rotation, we see the integral truly only depends on the angle between $\mathbf{u}$ and $\mathbf{v}$. That is, $\frac{1}{4\pi} \iint\limits_{S^2} \Phi^*(\mathbf{n}; \mathbf{u}, \mathbf{v})\, dS^2 = f(\omega)$, where $\omega = \Phi(\mathbf{u}, \mathbf{v})$. We show $f(\omega)$ is a solution of the functional equation $f(\psi + \eta) = f(\psi) + f(\eta)$ for $0 \leq \psi + \eta \leq \pi$ and $\psi, \eta > 0$. Take three coplanar vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\psi = \Phi(\mathbf{a}, \mathbf{b})$, $\eta = \Phi(\mathbf{b}, \mathbf{c})$, and $\psi + \eta = \Phi(\mathbf{a}, \mathbf{c})$. This gives us $\Phi^*(\mathbf{n}; \mathbf{a}, \mathbf{c}) = \Phi^*(\mathbf{n}; \mathbf{a}, \mathbf{b}) + \Phi^*(\mathbf{n}; \mathbf{b}, \mathbf{c})$. Integrating both sides of this equation over S^2 gives us exactly $f(\psi+\eta) = f(\psi) + f(\eta)$. Moreover, we have $0 \leq f(\omega) \leq \pi$, and recall that $0 \leq \omega \leq \pi$ by definition. Since f is additive, it is at least $\mathbb{Q}$-linear, but it is also the integral of a continuous function, and thus continuous. So f is linear. We have seen already that $f(0) = 0$ and $f(\pi) = \pi$, so we have that $f(\omega) = \omega$. This completes the proof.

□

Given a vector $\mathbf{n}$, let C_n be the projection of C onto the plane with normal vector $\mathbf{n}$.

Theorem 5.8. *If $k(C_n) \leq k$, indepedent of the choice of $\mathbf{n}$, then*

$$k(C) = \frac{1}{4\pi} \iint\limits_{S^2} k(C_n)\, dS^2 .$$

Proof. Choose a family of polygons $\{P_r\}$ inscribed in C so that $\lim_{r\to\infty} k(P_r) = k(C)$. This implies $\lim_{r\to\infty} k(P_{rn}) = k(C_n)$, where P_{rn} is the projection of P_r onto the plane with normal $\mathbf{n}$. Then

$$\begin{aligned}
\frac{1}{4\pi}\iint k(C_n)\,dS^2 &= \frac{1}{4\pi}\iint \lim_{r\to\infty} k(P_{rn})\,dS^2 \\
&= \frac{1}{4\pi}\lim_{r\to\infty}\iint k(P_{rn})\,dS^2 \\
&= \lim_{r\to\infty}\frac{1}{4\pi}\iint \sum_{i=1}^{n_r} \Phi^*[\mathbf{n}; \overline{a_{r_{i-1}}a_{r_i}}, \overline{a_{r_i}a_{r_{i+1}}}]\,dS^2 \\
&= \lim_{r\to\infty}\sum_{i=1}^{n_r} \Phi[\overline{a_{r_{i-1}}a_{r_i}}, \overline{a_{r_i}a_{r_{i+1}}}]\,dS^2 \\
&= \lim_{r\to\infty} k(P_r) \\
&= k(C)\,,
\end{aligned}$$

where the second equality is an application of Lebesgue's monotone convergence theorem, and the fourth equality is from lemma 5.7.

It is possible to have $k(C) < \infty$ and $k(C') = \infty$, for some projection C' of C; hence the hypothesis that $k(C_n) \leq k$.

□

Now a proposition, and then on to Fenchel's theorem

Proposition 5.9. *If $k(C) = 2\pi$, all the projections of C are convex, and thus each have total curvature 2π by proposition 5.4.*

Proof. Suppose not. Then there exists $\mathbf{n}_0$ such that C_{n_0} is not convex. We know that the limit of convex curves is a convex curve, so there is a neighborhood of $\mathbf{n}_0$ such that C_{n_0} is not convex for all $\mathbf{n}$ in this neighborhood. We then have $k(C_n) > 2\pi$ for $\mathbf{n}$ in this neighborhood. From proposition 5.4 and theorem 5.8, we have $k(C) > 2\pi$. This is a contradiction. So we must have that every projection of C is convex.

□

Theorem 5.10. *The total curvature of a simple closed curve is greater than or equal to 2π, with equality if and only if the curve is plane convex.*

Proof. By applying proposition 5.4 and theorem 5.8, we only need consider the case that $k(C) = 2\pi$. Let P, Q, and R be points on C, and let S be a point on the line segment $\overline{PR}$. Consider the projection C' of C onto the plane with normal vector $\overrightarrow{SQ}$. Denote by P', R', Q', and $S' = Q'$ the projections of P, R, Q, and S. From proposition 5.9, C' is convex. Now, the points P', Q', and R' are collinear. That is, C' contains the segment $\overline{P'Q'}$. So then the arc containing P, Q, and R along C is contained in a plane. Similarly, we have this for any three points of C, so C must lie in a plane. Applying fact 5.4, we see that C is convex.

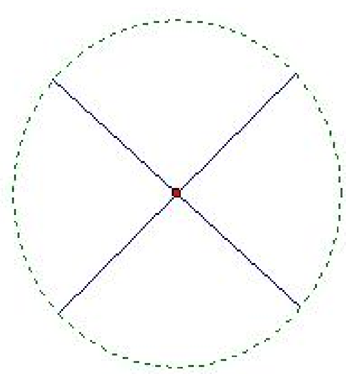
□

We're nearly to the point of the Fary-Milnor theorem; there is one more result we need. A *multiple point* of a curve is a point at which the parameterization is not injective.

Definition 5.11. Let P be a polygon without multiple points. A projection P' of P is *regular* if any line of projection meets at most two sides of P.

Proposition 5.12. *Let P be a knotted polygon and P' a regular projection onto a plane Π. There exists a point $\mathcal{O} \in \Pi$ such that all rays issuing from $\mathcal{O}$ cut P' in $k \geq 2$ different points, or in a multiple point.*

Proof. Since P' is a regular projection, its double points belong to two of its edges by definition. These points cannot be vertices, since that would imply they project down from three edges, and they must be situated on the boundary of four regions. That is, all double points of P' must look as in the picture below:

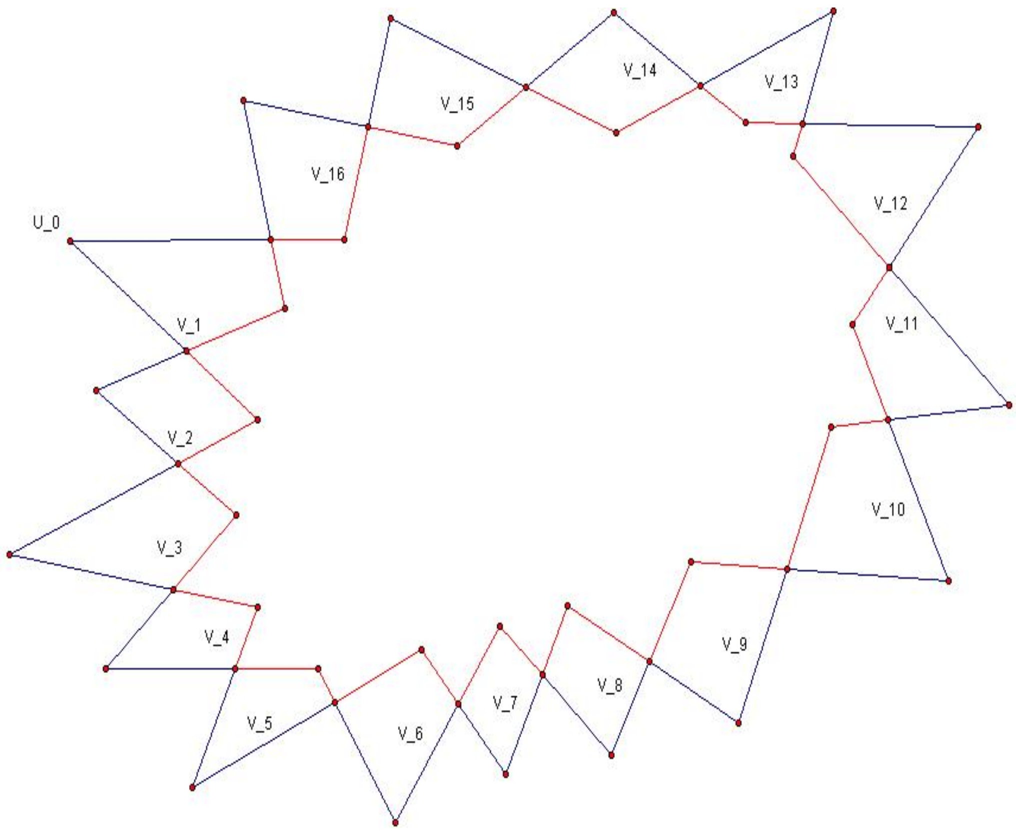


We claim that the polygon P' cuts Π into regions with the following properties: (a) These regions separate into classes $\mathcal{U}$ and $\mathcal{V}$ such that $U \in \mathcal{U}$ touches only elements $V \in \mathcal{V}$; (b) there is a region not adjacent to the region containing to the point at infinity. We first prove (a).

Let U_0 be the region containing the point at infinity, and suppose $U_0 \in \mathcal{U}$. Let $V_1, \ldots, V_i$ be the regions adjacent to U_0. Let $U_1, \ldots, U_j$ be regions adjacent to only the V_i's. We show each of these is in either $\mathcal{U}$ or $\mathcal{V}$, but not both. Suppose not. Then there is a closed polygonal loop L which has an odd number of intersection points with P'.

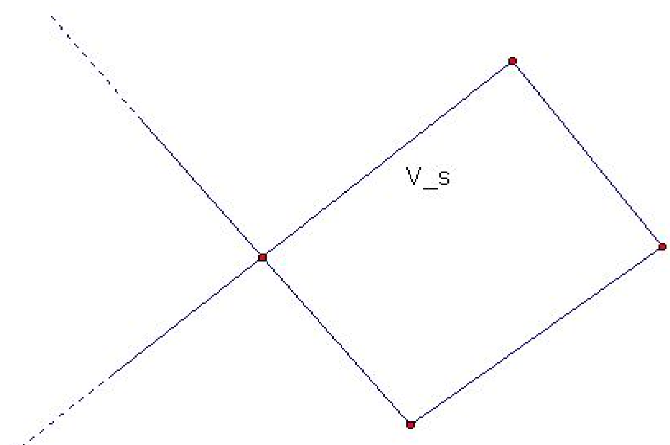
As we generically deform L, the number of intersection points varies but remains odd. However, when L is contracted to a point, the number of intersections must be zero. This is a contradiction. Thus, each of the V_i's must be in $\mathcal{V}$ and each of the U_j's must be in $\mathcal{U}$.

To see (b), suppose the contrary. Then all regions distinct from U_0 have a side in common with it. From V_1 of P', we move to a region V_2 which shares a vertex with V_1. The region V_i will be defined in the same manner as V_{i-1}. If $V_j = V_k$, we can construct a closed polygon L whose interior, by (a), contains a region that the V_i's separate from U_0.



For example, here we have $V_1 = V_{17}$, and the polygon in red is the polygon L. In this case, the interior of L is exactly the region that the V_i's separate from U_0.

Excluding this case, we arrive at a region V_s for which V_{s+1} does not exist. That is, the boundary of V_s is a loop containing only one double point of P'.



By a suitable deformation, we can remove the loop from P'; thus, we can deform

P into a polygon $\widetilde{P}$ such that the projection $\widetilde{P}\,'$ has only one finite region. This implies $\widetilde{P}$ is isotopic to a circle, which is a contradiction. Therefore, there is a region U_1 not adjacent to U_0.

Now, choosing $\mathcal{O} \in U_1$, we see that every ray issuing from $\mathcal{O}$ intersects P' in two or more points.

□

And now we finally come to Fary-Milnor.

Theorem 5.13. *A simple closed knotted curve C has $k(C) \geq 4\pi$.*

Proof. Let P be an inscribed polygon; let P' be a regular projection with vertices $a_1, \ldots, a_n$. We claim

$$k(P') \geq 4\pi\,. \tag{5.8}$$

Choose a point $\mathcal{O}$ according to proposition 5.12 and using the fact that $\mathcal{O}$, a_s, and a_t are not collinear for $1 \leq s, t \leq n$ and $s \neq t$. Let

$$\gamma_k = \Phi\left[\,\overline{\mathcal{O}a_k}, \overline{\mathcal{O}a_{k-1}}\,\right]\,,$$

$$\theta_k = \Phi\left[\,\overline{a_{k-1}a_k}, \overline{a_k a_{k+1}}\,\right]\,,$$

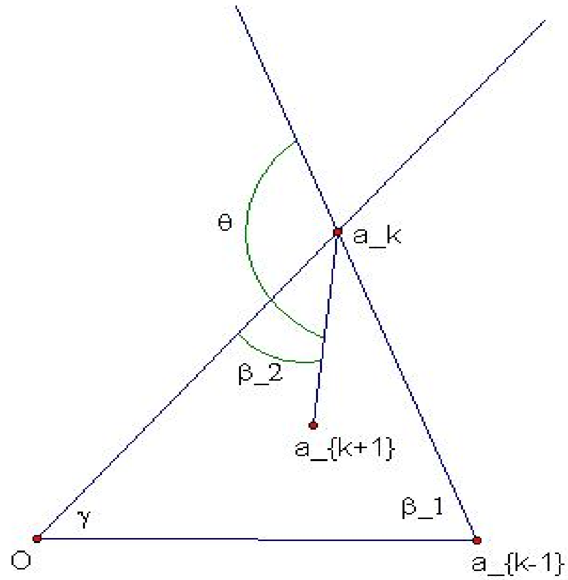
$$\beta_k = \Phi\left[\,\overline{a_k\mathcal{O}}, \overline{a_k a_{k+1}}\,\right]\,.$$

From the definition of $\mathcal{O}$, we have $\sum_{k=1}^{n} \gamma_k \geq 4\pi$. We wish to show that

$$\theta_k \geq \beta_{k-1} + \gamma_{k-1} - \beta_k\,. \tag{5.9}$$

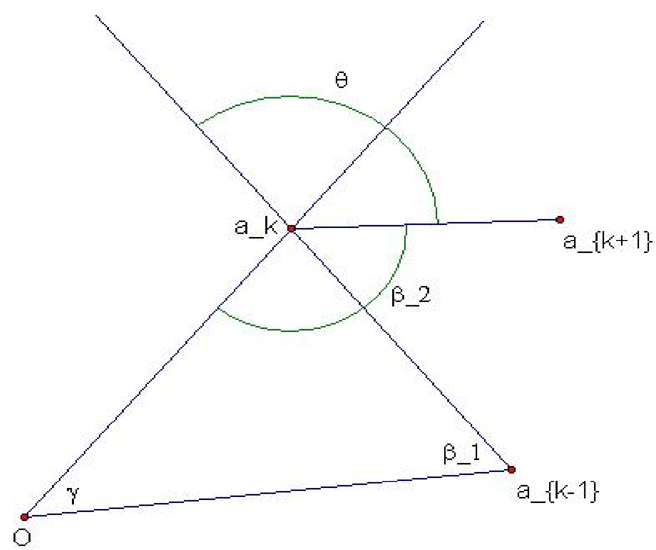
The lines $\overline{\mathcal{O}a_k}$ and $\overline{a_{k-1}a_k}$ cut the plane into four angular "quadrants." Let I denote the region containing triangle $\triangle\mathcal{O}a_{k-1}a_k$. Let II, III, and IV be the other "quadrants" taken counterclockwise from I.

If $a_{k+1} \in I$, we have $\theta_k = \gamma_{k-1} + \beta_{k-1} + \beta_k \geq \beta_{k-1} + \gamma_{k-1} - \beta_k$.

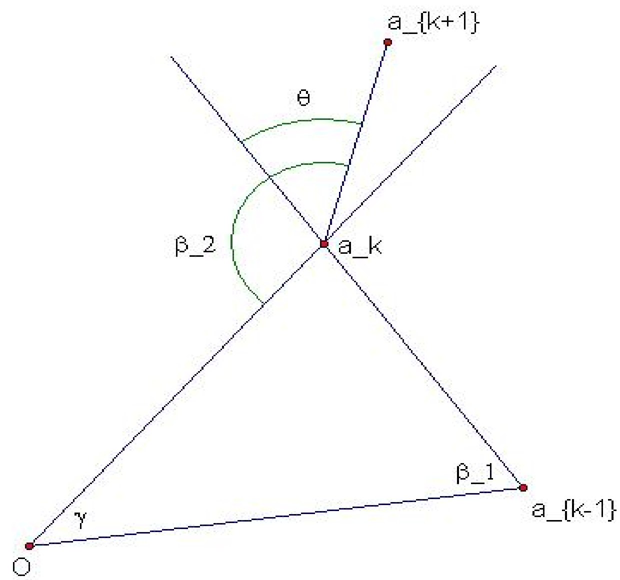


In this picture and the following three, set $\theta = \theta_k$, $\gamma_{k-1} = \gamma$, $\beta_{k-1} = \beta_1$, and $\beta_k = \beta_2$.

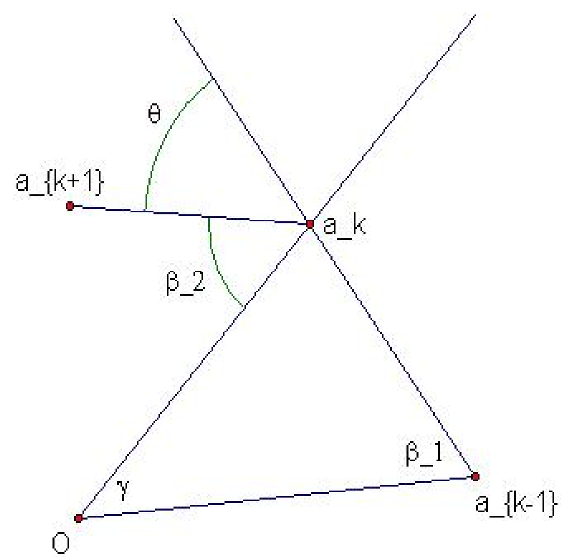
If $a_{k+1} \in II$, we have $\theta_k = 2\pi - (\gamma_{k-1} + \beta_{k-1} + \beta_k) \geq \beta_{k-1} + \gamma_{k-1} - \beta_k$.



If $a_{k+1} \in III$, we have $\theta_k = -\beta_{k-1} - \gamma_{k-1} + \beta_k \geq \beta_{k-1} + \gamma_{k-1} - \beta_k$.



If $a_{k+1} \in IV$, we have $\theta_k = \beta_{k-1} + \gamma_{k-1} - \beta_k$.



This proves inequality (5.9). And thus

$$k(P') = \sum_{k=1}^{n} \theta_k \geq \sum_{k=1}^{n} \gamma_k \geq 4\pi \,.$$

Now let C be a given knot, and let $\varepsilon > 0$. We can thus determine a knotted polygon P inscribed in C such that $k(C) + \varepsilon \geq k(P)$. For all regular projections P_n of P, we have $k(P_n) \geq 4\pi$ by our previous work. The only vectors $\mathbf{n}$ so that this does not hold lie in planes and hyperboloids of finite number, which have zero measure on S^2. Integrating (5.8) over S^2 we get $k(C) + \varepsilon \geq 4\pi$. Letting $\varepsilon \to 0$ we obtain our result.

$\square$

With this we will end our exploration of Fenchel's theorem and the Fary-Milnor theorem. In a later paper, Milnor gave some results focusing on the normal and binormal indicatrices. We will examine these results in the next chapter, and afterward conclude with some ideas for future investigation.

Chapter 6: Further Results: $(k+\tau)(\mathfrak{C})$ and $\omega(C)$

To this point, we have focused our attention solely on the quantity $k(C)$ for some curve C. This quantity, the total curvature, measures the length of the tangent indicatrix. We define similar quantities that measure the lengths of the normal and binormal indicatrices, following in Milnor's footsteps [14]. Let C be a closed C^3 curve.

Definition 6.1. The *total torsion* of a curve C is

$$\tau(C) = \int_C |\tau(s)|\, ds\,.$$

This is a measure of the binormal indicatrix.

Definition 6.2. The *total normalcy* of a curve C is

$$\omega(C) = \int_C \sqrt{k^2(s) + \tau^2(s)}\, ds\,.$$

This is a measure of the normal indicatrix.

For both definitions above, we are assuming $k(s) > 0$ for all $s \in [0, L]$. How are all these quantities related to one another? From the geometric-arithmetic mean inequality, we have $2k|\tau| \leq k^2+\tau^2$. So $(k+|\tau|)^2 = k^2+\tau^2+2k|\tau| \leq 2(k^2+\tau^2)$; taking square roots gives us $k + |\tau| \leq \sqrt{2(k^2+\tau^2)}$. Thus, we have the following string of inequalities:

$$2\pi \leq k(C) \leq \omega(C) \leq k(C) + \tau(C) \leq \sqrt{2}\,\omega(C)\,;$$

the first inequality is Fenchel's theorem, the last is demonstrated by the work above, and the others are obvious.

We also note that if $\tau \equiv 0$, then $\omega(C) = k(C)$.

In the rest of this chapter, we will present an analysis of the quantities $k(C)+\tau(C)$ and $\omega(C)$.

Definition 6.3. An *isotopy* is a continuous map $F_t : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that $F_0 = id_{\mathbb{R}^3}$ and F_t is a homeomorphism for all t.

Two simple closed curves C_1 and C_2 are of the same *isotopy class* (or *type*) if there exists an isotopy F_t such that $F_0(C_1) = C_1$ and $F_1(C_1) = C_2$. That is, F_t sends C_1 to C_2. This isotopy need not be differentiable.

We will show that the greatest lower bound of $k(C) + \tau(C)$ over an isotopy class is of the form $2\pi n$, for some natural number n. Further, we will show that $\omega(C)$ is greater than or equal to 4π provided $\tau(s)$ is not (identically) zero and does not change sign.

6.1 The Quantity $(k + \tau)(\mathfrak{C})$

Let $\mathfrak{C}$ represent an isotopy class of closed curves. For each $\mathfrak{C}$, we let $(k+\tau)(\mathfrak{C})$ denote the greatest lower bound of $k(C) + \tau(C)$ over all representative curves C that are C^3 satisfying $k(s) > 0$. Define $k(\mathfrak{C})$ and $\omega(\mathfrak{C})$ similarly. We need a few lemmas before going on to our main theorems of this section. Given a curve C and a plane P through the origin, let C_P denote the projection of C onto the plane P.

Lemma 6.4. *Given a curve C, the total curvature $k(C)$ is equal to the average of $k(C_P)$ taken over all planes through the origin.*

Proof. This is a restatement of theorem 5.8.

□

Lemma 6.5. *Let $B \subset S^2$ be a curve, let γ be a great circle, and let $n(\gamma \cap B)$ be the number of intersection points of γ and B. Then the length of B is equal to the average of $\pi\, n(\gamma \cap B)$ taken over all great circles γ.*

Proof. This is a restatement of Crofton's theorem. (See 3.3)

□

While lemma 6.4 deals with total curvature, there is a similar assertion for total torsion. Let $\nu(C_P)$ denote that number of points of C such that the osculating plane, the plane determined by $\mathbf{t}(s)$ and $\mathbf{n}(s)$, is perpendicular to P. We can interpret this number as the number of inflection points or cusps of the planar curve C_P.

Lemma 6.6. *Given a curve C, the total torsion $\tau(C)$ is equal to the average of $\pi\nu(C_P)$ taken over all planes P through the origin. The quantity $\nu(C_P)$ is even for almost all such P.*

Proof. Let B denote the binormal indicatrix of C. So the total torsion is equal to the length of B; that is, $\tau(C) = \ell(B)$. Take a great circle γ and let P be the plane containing γ. A binormal vector lies in P if and only if the osculating plane at that point is perpendicular to P. Thus, $\nu(C_P) = n(\gamma \cap B)$. Lemma 6.5 then gives us that $\tau(C) = \ell(B)$ is the average of $\pi\nu(C_P)$ taken over all P passing through the origin.

□

Lastly, we need a two-dimensional analog of lemma 6.4. Let $\mu(C, \mathbf{v})$ denote the number of points of C such that the tangent vector is perpendicular to the unit vector $\mathbf{v}$. This is a slightly different definition than the one given in section 4.1.

Lemma 6.7. *Given a curve C and a plane P, the total curvature $k(C_P)$ is equal to the average of $\pi\mu(C, \mathbf{v})$ taken over all $\mathbf{v} \in S^2$. The quantity $\mu(C, \mathbf{v})$ is even for almost all $\mathbf{v}$.*

Proof. This is a restatement of theorem 4.3.

□

Remark 6.8. That $\mu(C, \mathbf{v})$ is even almost everywhere was proven in theorem 4.3. A similar argument works for the quantity $\nu(C_P)$, but is omitted.

Theorem 6.9. *For each isotopy class* $\mathfrak{C}$ *of closed curves,* $(k + \tau)(\mathfrak{C}) = 2\pi n$ *for some* $n \in \mathbb{N}$*.*

Proof. Take $n \in \mathbb{N}$ and suppose that $2\pi n \leq (k + \tau)(\mathfrak{C}) < 2\pi(n + 1)$. There exists a representative curve C such that $k(C) + \tau(C) < 2\pi(n + 1)$. To prove our result, we will construct a curve isotopic to C with $k + \tau$ arbitrarily close to $2\pi n$.

By lemmas 6.4 and 6.6, there exists a plane projection C_P of C such that $k(C_P) + \pi\nu(C_P) \leq k(C) + \tau(C)$, with $\nu(C_P)$ even. Further, we choose P so that it is not perpendicular to any tangent of C; that is, P is *not* the normal plane, the plane determined by $\mathbf{b}(s)$ and $\mathbf{n}(s)$, for any s. By lemma 6.7, there is a unit vector $\mathbf{v}$ in P such that $\pi\mu(C_P, \mathbf{v}) \leq k(C_P)$, with $\mu(C_P, \mathbf{v})$ even. By our choice of P, we have $\mu(C_P, \mathbf{v}) = \mu(C, \mathbf{v})$. We now have

$$\pi\mu(C, \mathbf{v}) + \pi\nu(C_P) \leq k(C_P) + \pi\nu(C_P) \leq k(C) + \tau(C) < 2\pi(n + 1)\,.$$

That is, $\mu(C, \mathbf{v}) + \nu(C_P) < 2(n + 1)$, and so $\mu(C, \mathbf{v}) + \nu(C_P) \leq 2n$.

Now choose coordinates (x_1, x_2, x_3) so that $\mathbf{v}$ lies in the x_1-axis and P is the x_1x_2-plane. Define an isotopy h_t by $h_t(x_1, x_2, x_3) = (x_1, tx_2, t^2x_3)$. So as $t \to 0$, the tangent vector of $C_t = h_t(C)$ approaches $\pm\mathbf{v}$, except at the $\mu(C, \mathbf{v})$ points where the tangent to C is perpendicular to $\mathbf{v}$. In neighborhoods about each of these points, the tangent to C_t rotates through an angle converging to π as $t \to 0$. So $k(C_t) \to \pi\mu(C, \mathbf{v})$ as $t \to 0$.

Similarly, $\tau(C_t) \to \pi\nu(C_P)$, since P was chosen so it was not perpendicular to any tangent of C. Thus,

$$k(C_t) + \tau(C_t) \to \pi\mu(C, \mathbf{v}) + \pi\nu(C_P) \leq 2\pi n\,.$$

And this completes the proof.

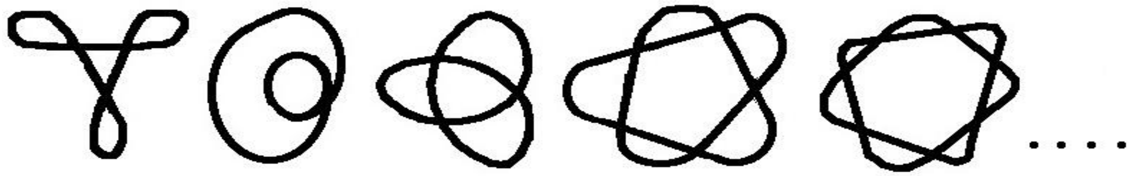
□

From Fary-Milnor, we know $k(C) \geq 4\pi$ for any knot. So $(k+\tau)(\mathfrak{C}) = 2\pi$ only if $\mathfrak{C}$ is the class of unknotted curves. This naturally leads us to ask: what can we say about the case where $(k+\tau)(\mathfrak{C}) = 4\pi$?

Theorem 6.10. *If $(k+\tau)(\mathfrak{C}) = 4\pi$, then $\mathfrak{C}$ is the class of closed 2-strand braids with k crossings, for k odd.*

Proof. Take a representative curve C and plane projection C_P so that $k(C_P) + \pi\nu(C_P) < 6\pi$, with $\nu(C_P)$ even. This is possible by lemmas 6.4 and 6.6. If $\nu(C_P) \geq 2$, then $k(C_P) < 4\pi$, which implies C is unkotted. (This follows from work done in Fary's proof of Fary-Milnor in the previous chapter.) So $\nu(C_P)$ must be zero. Since C_P is planar and has no inflection points, the tangent indicatrix of C_P is a great circle, and so $k(C_P)$ must be a multiple of 2π. If $k(C_P) = 2\pi$, then C_P is homeomorphic to a circle. In this case, C is also homeomorphic to a circle, and thus unknotted. So we must thave $k(C_P) = 4\pi < 6\pi$.

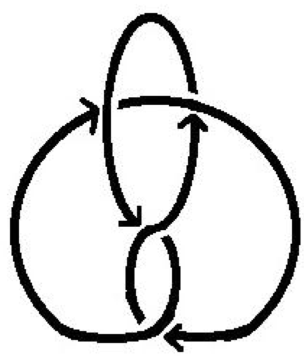
Without loss of generality, assume C_P has only finitely many crossing points. We note that any braid with an even number of crossings is a link, a case we are not considering. Below is a complete list of planar curves with no inflection points, an odd number of crossings, and total curvature equal to 4π.



It should be clear that any knot having one of the above as a plane projection is isotopic to a 2-strand braid.

□

Example 6.11. Consider the class $\mathfrak{C}$ of figure eight knots.



We have that $\nu(C_P) = 2$ for the given projection, and that $\mu(C, \mathbf{v}) = 4$ for a suitably chosen vector $\mathbf{v} \in P$. (Consider $\mathbf{v}$ to be the vector in P pointing "north." Then the tangent to C will be perpendicular to $\mathbf{v}$ four times: at the top of the uppermost arc, at the top of the next-uppermost arc, and twice along the bottom of the knot.) So $(k+\tau)(\mathfrak{C}) \leq (2+4)\pi$, and thus $(k+\tau)(\mathfrak{C}) = 2\pi, 4\pi$ or 6π. But the figure eight is a knot, so $k(C) > 4\pi$ by the Fary-Milnor theorem. So we must have $(k+\tau)(\mathfrak{C}) = 6\pi$. And in fact, $k(\mathfrak{C}) = 4\pi$.

Another way to see this is to note that the class of figure eight knots has Alexander polynomial $t^3 - 3t + 1$, whereas the two-stranded braids have Alexander polynomials $\sum_{k=0}^{2n} (-t)^k$ where n is the number of crossings. This means that $\mathfrak{C}$ is not represented by any two-strand braid. So we again conclude that $(k+\tau)(\mathfrak{C}) = 6\pi$. (For details on the Alexander polynomial see Adams [13] or Reidemeister [15].)

Lemmas 6.4 and 6.6 are clearly proving their worth. Here is another application.

Theorem 6.12. *If C has linking number n with a straight line L, then $k(C)+\tau(C) \geq 2\pi n$.*

Proof. Let P be a plane not perpendicular to L, and let L_P and C_P be the projections of L and C onto P. Since C_P crosses L_P at least $2n$ times, the tangent to C_P must be parallel to L_P at least $2n$ times. Take two consecutive points of C_P such that the tangents are parallel to L_P. If these tangents have opposite directions, the total curvature of the segment joining them along C_P is at least π. If the tangents have the same direction, there must be an inflection point somewhere between the two points. There is a contribution of at least π to $k(C_P) + \pi\nu(C_P)$ in both cases. Therefore, $k(C_P) + \pi\nu(C_P) \geq 2\pi n$.

Applying lemmas 6.4 and 6.6, we have $k(C) + \tau(C) \geq 2\pi n$.

□

6.2 The Quantity $\omega(C)$

For our analysis of the quantity $\omega(C)$, we will need to recall the Frenet equations:

$$\begin{aligned} \mathbf{t}' &= \qquad\quad k\,\mathbf{n} \\ \mathbf{n}' &= -k\,\mathbf{t} \qquad\qquad +\tau\,\mathbf{b} \\ \mathbf{b}' &= \qquad\quad -\tau\,\mathbf{n} \end{aligned}$$

In particular, the equation

$$\mathbf{b}' = -\tau\mathbf{n} \tag{6.1}$$

will be necessary. We should note that in the first chapter, two of the Frenet equations were given as $\mathbf{n}' = -k\mathbf{t} - \tau\mathbf{b}$ and $\mathbf{b}' = \tau\mathbf{n}$; these are more standard. For this chapter, we are choosing the opposite orientation. Now specify two antipodal points as the north and south pole on S^2.

Lemma 6.13. *Let γ and γ' be oriented great circles and suppose γ does not pass through the north and south poles. Let p be the point where γ' crosses south of γ. If the tangent to γ' at p lies north of the equator, then γ' makes a smaller angle with the equator than does γ.*

Proof. Note that p lies on the segment between the southernmost and northernmost points of γ'. So it follows that the southernmost point of γ' lies north of γ, and hence γ' makes a smaller angle with the equator than does γ.

□

From the Frenet equations we have that the tangent vector at any point on the tangent indicatrix, Γ, of C is just the normal vector $\mathbf{n}$. We note that the geodesic curvature of Γ is $\frac{\tau(s)}{k(s)}$; this is a standard result. A positive value for the geodesic curvature means Γ curves toward the binormal $\mathbf{b}$.

Theorem 6.14. *If $\tau(s) \geq 0$, but not identically zero, then $\omega(C) \geq 4\pi$.*

Proof. Recall that $\omega(C)$ measures the length of the normal indicatrix. So by lemma 6.5, it is sufficient to prove that the normal indicatrix intersects every great circle in at least four points. Suppose it intersects some great circle E only twice. Let $\mathbf{n}(s_0)$ and $\mathbf{n}(s_1)$ denote these intersection points. Rotate S^2 so that E is the equator. Let $-\frac{\pi}{2} \leq \varphi(\mathbf{v}) \leq \frac{\pi}{2}$ denote the angle that $\mathbf{v}$ makes with the equator. We may choose s_0, s_1, and the direction north so that

$$|\varphi(\mathbf{b}(s_0))| \geq |\varphi(\mathbf{b}(s_1))| \tag{6.2}$$

and

$$\varphi(\mathbf{n}(s)) > 0 \text{ for } s_0 < s < s_1\,. \tag{6.3}$$

Let γ be the great circle which is tangent to Γ at $\mathbf{t}(s_0)$. We restrict our attention to the interval $[s_0, s_1]$, which is sufficient by symmetry.

Equations (6.1) and (6.3) imply that $\varphi(\mathbf{b}(s))$ is monotone decreasing (and strictly decreasing when $\tau(s) > 0$). From equation (6.2), it then follows that

$$|\varphi(\mathbf{b}(s_0))| \geq |\varphi(\mathbf{b}(s)|\,. \tag{6.4}$$

So if $\varphi(\mathbf{b}(s_0)) = \varphi(\mathbf{b}(s_1))$, then $\tau(s) \equiv 0$ on $[s_0, s_1]$, and, by a similar argument, the same would be true for all values of s. Thus, we must have $\varphi(\mathbf{b}(s_0)) > \varphi(\mathbf{b}(s_1))$; hence, by (6.2) we have $\varphi(\mathbf{b}(s_0)) > 0$.

Now suppose Γ crosses γ at some point $\mathbf{t}(s')$ where $s_0 < s' < s_1$. Further suppose that this is the first such crossing, and let γ' be the great circle that is tangent to Γ at $\mathbf{t}(s')$. Since $k(s) > 0$ and $\tau(s) \geq 0$, the geodesic curvature $\frac{\tau}{k}$ of Γ cannot change sign. So Γ cannot be tangent to γ at a crossing. Thus, γ and γ' are distinct. And since $\varphi(\mathbf{b}(s_0)) > 0$, the point $\mathbf{b}(s_0)$ lies above the equator. By our choice of γ, we see that $\mathbf{b}(s_0)$ is the pole of γ, which implies that γ does not pass through the north and south poles. Because the geodesic curvature of Γ is non-negative, Γ always either curves toward $\mathbf{b}(s)$ or follows a great circle. Therefore Γ either curves north of γ at $\mathbf{t}(s_0)$, or follows the circle γ for some interval and then curves north. In both cases, Γ must cross south of γ at $\mathbf{t}(s')$; thus, γ' crosses south of γ at $\mathbf{t}(s')$. The tangent to γ' at $\mathbf{t}(s')$ is just $\mathbf{n}(s')$, which is north of the equator by (6.3).

Applying lemma 6.12, γ' makes a smaller angle with the equator than γ. But now the poles of γ' and γ are $\mathbf{b}(s')$ and $\mathbf{b}(s_0)$. Therefore

$$|\varphi(\mathbf{b}(s'))| > |\varphi(\mathbf{b}(s_0))|\,.$$

This contradicts (6.4).

Thus, the normal indicatrix must intersect every great circle at least four times. Lemma 6.5 then gives us our result.

□

Therefore, if $\mathfrak{C}$ is the class of knots with $k(s) > 0$ and $\tau(s) \geq 0$ (but not identically zero), then $\omega(\mathfrak{C}) = 4\pi$. And recall that we interpret this as a lower bound on the length of the normal indicatrix over $\mathfrak{C}$. In general, $\omega(\mathfrak{C})$ need not be a multiple of 2π. In fact, for the class $\mathfrak{C}$ of figure eight knots, Milnor proves that $4\pi < \omega(\mathfrak{C}) < 6\pi$ [14].

Chapter 7: Further Investigations

So where do we go from here? There are, perhaps, uncountably many questions one could ask. From Fenchel we have $k(\mathfrak{S}) = 2\pi$, where $\mathfrak{S}$ is the class of simple closed curves. If we now restrict to simple closed knotted curves, Fary-Milnor gives us $k(\mathfrak{C}) = 4\pi$. We also know that $(k+\tau)(\mathfrak{C}) = 2\pi n$ for each isotopy class $\mathfrak{C}$. This immediately gives us a countable number of questions to ask. Adams recently gave a characterization of the class $\mathfrak{C}$ such that $(k+\tau)(\mathfrak{C}) = 6\pi$ using the spiral index and projective superbridge number [16]. But now what about $(k+\tau)(\mathfrak{C}) = 8\pi$? Or 10π? Or $2\pi n$?

There is also the question of links. A link is simply the disjoint union of some number of knots. Let $\mathfrak{L}$ denote some isotopy class of links. As we did in the previous chapter, let $k(\mathfrak{L})$ denote the greatest lower bound of $k(L)$ over all L in $\mathfrak{L}$. Can we show that $k(\mathfrak{L}) = 4\pi$ for a suitable class of links? Similarly, what can we say about $(k+\tau)(\mathfrak{L})$ and $\omega(\mathfrak{L})$?

Let L be a connected sum of knots, $L = K_1 \sharp K_2 \sharp \cdots \sharp K_n$. We certainly have $k(L) \geq 4\pi n$ assuming each K_i is truly knotted. But how does $k(L)$ compare to $k(K_1) + k(K_2) + \cdots + k(K_n)$?

The above is a mere sampling of the questions one could potentially ask, and is fairly heavy on knot-theoretic concerns. These are all tough questions that will keep at least one person thinking for some time.

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Vita

Charles M. Evans

Personal Information

- Born: May 29, 1986, Atlanta, Georgia
- Undergraduate Study: Berry College
Mount Berry, Georgia
B.S. *Cum Laude*, Mathematics, May 2008
- Graduate Study: Wake Forest University
Winston-Salem, North Carolina
M.A., Mathematics, May 2010

Honors, Awards, and Scholarships

- Wake Forest University Mathematics Dept. Teaching Assistantship (2008-2010)
- Berry College Barton Mathematics Award (2007)
- Elizabeth E. Griggs Academic Scholarship, Berry College (2006)
- Berry College Academic Scholarship (2004-2008)
- Second runner up winner, 10th Annual Wake Forest Graduate Research Day poster presentation, "Curves, Knots, and Total Curvature" (2010)

Academic Organizations

- American Mathematical Society
- Mathematical Association of America
- Pi Mu Epsilon, Wake Forest chapter (Inducted 2009, chapter president 2009-2010)