COMMUTATIVITY DEGREE OF FINITE GROUPS

By

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Table of Contents

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Abstract

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COMMUTATIVITY DEGREE OF FINITE GROUPS
Thesis under the direction of Dr. James Kuzmanovich.

The commutativity degree of a group is the probability that two randomly selected (with replacement) elements of the group commute. We find bounds on the commutativity degree of a finite group, equate restricted values of commutativity degree to finite groups with particular structures, compute the commutativity degree for a number of classes of finite groups, and discuss the set of possible values of commutativity degree for finite groups.

Chapter 1: Introduction

The probability that two elements of a group commute is called the commutativity degree of the group. It is well known that no finite group has commutativity degree in the interval $(\frac{5}{8}, 1)$ and that a group G has commutativity degree $P(G) = \frac{5}{8} = \frac{2^2+2-1}{2^3}$ if and only if $G/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We extend this result to any prime p. We expand upon a result from Rusin [43] showing that any group with commutativity degree in the interval $(\frac{1}{2}, \frac{5}{8}]$ has a commutativity degree of $\frac{1}{2}(1 + \frac{1}{2^{2n}})$ for some $n \in \mathbb{N}$ and we construct a class of groups realizing these values for all n. Lescot [32] shows that if a group G has the property $G/Z(G) \cong S_3$, then $P(G) = \frac{1}{2}$. What are the possible values of commutativity degree less than $\frac{1}{2}$? We show that the value $\frac{1}{n}$, for any $n \in \mathbb{N}$, is the commutativity degree of a finite group, and that this group cannot be nilpotent. We will show that, if a group is not nilpotent then its commutativity degree is less than $\frac{1}{2}$, and if a group is not solvable then its commutativity degree is less than $\frac{1}{12}$. We will find additional useful bounds on commutativity degree for different types of groups.

Finding the commutativity degree of a finite group is equivalent to finding the number of conjugacy classes of the group or to finding the number of irreducible characters of the group. This relates commutativity degree to many areas of group theory; there are many questions, and a long history of results, concerning the relationship between irreducible characters of a group and group-theoretic properties of the group.

Our investigation begins with a formal definition and brief introduction to commutativity degree.

1.1 Initial Definitions and Notation

1.1.1 What is the Probability That Two Elements Commute?

Let G be a finite group. Suppose that a random element, x, of G is selected, replaced, and then a second random element y is selected. What is the probability that x and y commute?

Each outcome of this experiment is represented by the ordered pair (x, y) and the sample space is $G \times G$. An outcome (x, y) for which $xy = yx$ is called a commutativity. The set of all commutativities is the event "randomly chosen x and y commute" which we denote by $c(G)$. Explicitly, $c(G) = \{(x, y) : xy = yx\}$. Assuming that all draws are equally likely, the probability that randomly chosen x and y commute is

$$P(c(G)) = \frac{|c(G)|}{|G \times G|}. \tag{1.1}$$

The probability $P(c(G))$ is a property of G called commutativity degree and we shorten the notation to $P(G)$ for convenience. Note that it $xy = yx$ then both (x, y) and (y, x) are elements of $c(G)$. Also, if G is Abelian, then $c(G) = G \times G$ and $P(G) = 1$.

The table of occurrences of commutativities for a group G is called a Commutativity Table. A Commutativity Table contains an entry for each pair in the sample space $G \times G$. Each ordered pair in $c(G)$ is represented by a 1 and all other ordered pairs by a 0. The commutativity degree $P(G)$ is the proportion of nonzero entries to total entries in the table.

For instance, Table 1.1 is the Commutativity Table for the dihedral group D_3.

	e	ρ	ρ^2	r	$r\rho$	$r\rho^2$
e	1	1	1	1	1	1
ρ	1	1	1	0	0	0
ρ^2	1	1	1	0	0	0
r	1	0	0	1	0	0
$r\rho$	1	0	0	0	1	0
$r\rho^2$	1	0	0	0	0	1

Table 1.1: **Commutativity Table for** $D_3 = \langle r, \rho : r^2 = e, \rho^3 = e, \rho r = r\rho^2 \rangle$

There are 30 ordered pairs represented by a 1 on Table 1.1, so the commutativity degree of D_3 is

$$P(D_3) = \frac{|c(G)|}{|G \times G|} = \frac{18}{36} = \frac{1}{2}.$$

As additional examples, Tables 1.2 and 1.3 are Commutativity Tables for the quaternion group Q_8 and the alternating group A_4, respectively.

	e	b	b^2	b^3	a	ab	ab^2	ab^3
e	1	1	1	1	1	1	1	1
b	1	1	1	1	0	0	0	0
b^2	1	1	1	1	1	1	1	1
b^3	1	1	1	1	0	0	0	0
a	1	0	1	0	1	0	1	0
ab	1	0	1	0	0	1	0	1
ab^2	1	0	1	0	1	0	1	0
ab^3	1	0	1	0	1	0	0	1

Table 1.2: **Commutativity Table for** $Q_8 = \langle a, b : a^4 = b^4 = e, a^2 = b^2, ba = ab^3 \rangle$

	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(243)	(142)	(134)	(132)	(143)	(234)	(124)
(1)	1	1	1	1	1	1	1	1	1	1	1	1
(12)(34)	1	1	1	1	0	0	0	0	0	0	0	0
(13)(24)	1	1	1	1	0	0	0	0	0	0	0	0
(14(23)	1	1	1	1	0	0	0	0	0	0	0	0
(123)	1	0	0	0	1	0	0	0	0	0	0	0
(243)	1	0	0	0	0	1	0	0	1	0	1	0
(142)	1	0	0	0	0	0	1	0	0	0	0	1
(134)	1	0	0	0	0	0	0	1	0	1	0	0
(132)	1	0	0	0	1	0	0	0	1	0	0	0
(143)	1	0	0	0	0	0	0	1	0	1	0	0
(243)	1	0	0	0	0	1	0	0	0	0	1	0
(124)	1	0	0	0	0	0	1	0	0	0	0	1

Table 1.3: **Commutativity Table for** A_4

Counting the ratios of entries in Tables 1.2 and Table 1.3 yields that the commutativity degree of Q_8 is

$$P(Q_8) = \frac{40}{64} = \frac{5}{8}$$

and that the commutativity degree of A_4 is

$$P(A_4) = \frac{48}{144} = \frac{1}{3}.$$

1.1.2 Additional Definitions and Notation

Let G be a group. Throughout this paper, we assume that G is finite. Let $g \in G$. The conjugation map $\phi_g : G \to G$ is defined by $\phi_g(x) = gxg^{-1}$. The centralizer of x is the set of elements in G that commute with x, and in terms of conjugation it is the set of $g \in G$ such that $\phi_g(x) = x$. The centralizer of x in G is denoted by $C_G(x)$. The set of all possible images of x under conjugation by elements of G, $\{y \in G : \phi_g(x) = y \text{ for some } g \in G\}$, is the conjugacy class of x. The conjugacy class of $x \in G$ is denoted by $[x]$. The centralizer and conjugacy class of x are related by the equation $|[x]| = [G : C_G(x)]$. Define $k(G)$ to be the number of distinct conjugacy classes of G.

The center of G is the set $\{h \in G : hg = gh \text{ for all } g \in G\}$. The center is denoted by $Z(G)$ or by Z when the referenced group is clear. The commutator subgroup is $< [x, y] = xyx^{-1}y^{-1} : x, y \in G >$ and is denoted by G' or $[G, G]$.

If the normal series of G is

$$G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright ... \triangleright G^{(d)} = \{e\}$$

with $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for $1 \leq i \leq d$, then G is solvable of derived length d.

If the lower central series of G is

$$G = L^{(0)} \triangleright L^{(1)} \triangleright L^{(2)} \triangleright ... \triangleright L^{(n)} = \{e\}$$

with $L^{(i)} = [G, L^{(i-1)}]$ for $1 \leq i \leq n$ or if the upper central series of G is

$$e = Z_0 \subsetneq Z_1 \subsetneq ... \subsetneq Z_n = G$$

with $Z_i = Z(G/Z_{(i-1)})$ for $1 \leq i \leq n$, then G is nilpotent of nilpotence class n.

1.2 Applications of the Definition of Commutativity Degree

In this section we use the definition of commutativity degree to derive the formula $P(G) = \frac{k(G)}{|G|}$, to show that the commutativity degree of a direct product is the product of the commutativity degrees of the factor groups, and to find a general lower bound on the commutativity degree of a non-Abelian group.

1.2.1 A Formula for $P(G)$ Using Basic Definitions

Recall the definition $P(G) = \frac{|c(G)|}{|G \times G|}$. First notice that we can rewrite the event $c(G)$ in terms of the conjugation map.

$$\begin{aligned} c(G) &= \{(x, y) \in G \times G : xy = yx\} \\ &= \{(x, y) \in G \times G : xyx^{-1} = y\} \\ &= \{(x, y) \in G \times G : \phi_x(y) = y\}. \end{aligned} \tag{1.2}$$

Theorem 1.2.1. *Let G be a finite group. Then the commutativity degree of G is $P(G) = \frac{k(G)}{|G|}$.*

Proof. Let $\{[x_1], [x_2], ..., [x_k]\}$ be the set of distinct conjugacy classes of G so that $k = k(G)$. Recall that G is the disjoint union of the distinct conjugacy classes of G. Notice from Equation 1.2 that for $x \in G$, $(x, y) \in c(G)$ if and only if $y \in C_G(x)$. Thus

$$|c(G)| = \sum_{x \in G} |C_G(x)|.$$

It follows that

$$\begin{aligned} |c(G)| &= \sum_{x \in G} |C_G(x)| \\ &= \sum_{i=1}^{k(G)} |[x_i]||C_G(x_i)| \\ &= \sum_{i=1}^{k(G)} [G : C_G(x_i)]|C_G(x_i)| \\ &= \sum_{i=1}^{k(G)} |G| \\ &= k(G)|G|. \end{aligned}$$

Therefore,

$$P(G) = \frac{|c(G)|}{|G \times G|} = \frac{k(G)|G|}{|G|^2} = \frac{k(G)}{|G|}. \tag{1.3}$$

$\square$

Calculating the commutativity degree of a finite group becomes a question of counting the number of conjugacy classes of the group. For some groups of small order, it is possible to explicitly compute the conjugacy classes. As an example, we find the commutativity degree of the symmetric group S_3.

Example 1.2.2. Consider $S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$. The conjugacy classes of S_3 are:

$$\begin{aligned} [e] &= \{e\} \\ [\sigma] &= \{\sigma, \sigma^2\} \\ [\tau] &= \{\tau, \sigma\tau, \sigma^2\tau\}. \end{aligned}$$

The symmetric group S_3 is partitioned into three conjugacy classes so

$$P(S_3) = \frac{3}{|S_3|} = \frac{1}{2}. \blacksquare$$

1.2.2 Direct Products

We can calculate the commutativity degree of a direct product directly from the definition of commutativity degree.

Proposition 1.2.3. *Let H and K be groups and $G = H \times K$. Then $P(G) = P(H)P(K)$.*

Proof. Recall that $P(G) = \frac{|c(G)|}{|G \times G|}$ where

$$c(G) = \{((h_1, k_1), (h_2, k_2)) \in G \times G : (h_1, k_1)(h_2, k_2) = (h_2, k_2)(h_1, k_1)\}.$$

Next we rewrite $c(G)$ in terms of $c(H)$ and $c(K)$ as follows:

$$\begin{aligned} c(G) &= \{((h_1, k_1), (h_2, k_2)) \in G \times G : (h_1h_2, k_1k_2) = (h_2h_1, k_2k_1)\} \\ c(G) &= \{(h_1, h_2) \in H \times H : h_1h_2 = h_2h_1\}\{(k_1, k_2) \in K \times K : k_1k_2 = k_2k_1\} \\ c(G) &= c(H)c(K) \end{aligned}$$

Therefore,

$$P(G) = \frac{|c(G)|}{|G \times G|} = \frac{|c(H)||c(K)|}{|H \times H||K \times K|} = P(H)P(K).$$

$\square$

1.2.3 A Lower Bound from Basic Definitions

We find a general lower bound by applying the definition of commutativity degree.

Proposition 1.2.4. *Suppose* $|G/Z| = l$. *Then* $P(G) \geq \frac{2l-1}{l^2}$.

Proof. Let $|G/Z| = l$. The set of commuting pairs, $c(G)$, contains two copies of $Z(G)$ so that

$$|c(G)| \geq |G \times Z| + |Z \times G| - |Z \times Z|,$$

with the last term accounting for redundancy in counting the pairs of central elements. Then

$$\begin{aligned} P(G) &= \frac{|c(G)|}{|G \times G|} \\ &\geq \frac{|G \times Z| + |Z \times G| - |Z \times Z|}{|G \times G|} \\ &= \frac{|G||Z| + |Z||G| - |Z|^2}{|G|^2} \\ &= \frac{l|Z|^2 + l|Z|^2 - |Z|^2}{l^2|Z|^2} \\ &= \frac{2l-1}{l^2}. \end{aligned}$$

□

1.3 Commutativity Degree and the Class Equation

In this section, we will discuss the class equation as it relates to commutativity degree. To write the class equation, let G be a group and let $\{[x_i] : 1 \leq i \leq k(G)\}$ be the set of distinct conjugacy classes of G. Since the conjugacy classes partition G,

$$|G| = \sum_{i=1}^{k(G)} |[x_i]|. \tag{1.4}$$

Equation 1.4 is called the class equation of G. The number of summands in the class equation equals the number of conjugacy classes in the group, and this makes the class equation one of our primary tools for calculating the number of conjugacy classes of a group. Among our applications of the class equation include derivations of approximately half of the bounds in Chapter 2 and computation of the commutativity degree of many of the groups in Chapter 4.

The class equation is often written in terms of the center. Notice that $x \in Z(G)$ if and only if $[x] = |\{x\}|$. Hence the set $\{|[x]| = 1\} = |Z(G)|$. We can write the class equation in the form

$$|G| = |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} |[x_i]|. \tag{1.5}$$

Since $|[x]| = [G : C_G(x)]$ for all $x \in G$, we can also write the class equation in the form

$$|G| = |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} [G : C_G(x_i)]. \tag{1.6}$$

Next, let $x \in G$. By Lagrange's theorem

$$|G| = |C_G(x)|[G : C_G(x)] = |C_G(x)||[x]|.$$

Thus $|[x]|$ divides $|G|$, and all terms of the class equation divide the order of G.

To calculate the commutativity degree of a group, it is often easier to count the number of terms in Equation 1.4 than to compute each conjugacy class explicitly. We illustrate this with the dihedral group D_4.

Example 1.3.1. Consider $D_4 =< r, \rho : r^2 = e, \rho^4 = e, \rho r = r\rho^3 >$. The class equation of D_4 is

$$|D_4| = |Z| + \sum_{i=|Z|+1}^{k(D_4)} |[x_i]|,$$

with $x_i \notin Z$. Since e and ρ^2 are the only elements of D_4 which commute with all elements in the group, $Z = \{e, \rho^2\}$.

For each remaining conjugacy class $[x_i]$, $|[x_i]| \in \{2, 4\}$ because $|[x_i]|$ divides $|G|$. Suppose there were some $x_i \in D_4$ with $|[x_i]| = 4$. Then $|C_G(x_i)| = 2$ and x_i would commute only with e and itself. Since all elements of D_4 commute with ρ^2, no such element exists. Then the class equation of D_4 written in the form of Equation 1.4 is

$$|D_4| = 1 + 1 + 2 + 2 + 2.$$

Thus the class equation of D_4 has five summands so

$$P(D_4) = \frac{5}{|D_4|} = \frac{5}{8}. \blacksquare$$

1.4 Commutativity Degree and the Degree Equation

The degree equation provides another way to write the order of a group as a sum with $k(G)$ summands. This makes the degree equation our second primary tool for calculating the commutativity degree of a group. Before stating the degree equation, we will briefly discuss the structure of group representations and of group rings, as these pertain to the degree equation.

Let G be a group. The $\mathbb{C}$-vector space with basis $\{g : g \in G\}$ is called the group ring and is denoted by $\mathbb{C}[G]$. Elements of the group ring are written $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{C}$, and multiplication is defined by extending the group multiplication via the distributive law.

A complex representation of G is a homomorphism $\phi : G \to GL_n(\mathbb{C})$; n is called the degree of the representation. Notice that $GL_n(\mathbb{C})$ acts on the vector space of column vectors $V = \mathbb{C}^n$. Also, we can make V into a left $\mathbb{C}[G]$-module by defining $g \cdot \overline{v} = \phi(g)(\overline{v})$ and extending linearly to all of $\mathbb{C}[G]$. Conversely, if M is a left $\mathbb{C}[G]$-module with $\dim_{\mathbb{C}} M = n$, then we can define a representation $\phi_M : G \to GL_n(\mathbb{C})$. To define this representation, first fix a vector space basis for ${}_{\mathbb{C}}M$. If $g \in G$, then

$\lambda_g : M \to M$ defined by $\lambda_g(m) = gm$ is an invertible linear transformation. Define $\phi(g)$ to be the matrix representation of λ_g with respects to the fixed basis. Thus representations of G are equivalent to left modules over $\mathbb{C}[G]$.

The representation ϕ is called irreducible if V has no proper, nontrivial, invariant subspaces; equivalently, V is simple when considered as a left $\mathbb{C}[G]$-module.

The structure of $\mathbb{C}[G]$ is further described using Masche and Wedderburn's theorems. We combine their results as they apply to our situation:

Theorem 1.4.1. *(Masche and Wedderburn) Let G be a finite group. Then the group ring $\mathbb{C}[G]$ can be written as*

$$\mathbb{C}[G] = \mathbb{C} \times M_{n_2}(\mathbb{C}) \times M_{n_3}(\mathbb{C}) \times ...M_{n_l}(\mathbb{C})$$

for some $l \in \mathbb{N}$ and where each integer $n_i \geq 1$.

A finite dimensional algebra of the form described by Theorem 1.4.1 is called semisimple. It is know that over a semisimple finite dimensional algebra, every module is a direct sum of simple modules each of which is isomorphic to a simple left ideal. In terms of Theorem 1.4.1, $\mathbb{C}[G]$ has finitely many nonisomorphic simple modules. Explicitly, these modules are

$$\mathbb{C}, \mathbb{C}^{n_2}, \mathbb{C}^{n_3}, ..., \mathbb{C}^{n_l}.$$

Hence G has l non-equivalent irreducible representations of degree $1, n_2, n_3, \ . \ .. \ n_l$. (The factor $\mathbb{C}$ corresponds to $\phi : G \to \mathbb{C}^\star$ defined by $\phi(g) = 1$ for all $g \in G$.) Hence we have the following equation, which we call the degree equation

$$|G| = 1 + n_2^2 + n_3^2 + ... + n_l^2.$$

It can be shown that l is the number of conjugacy classes of G.

The most common way to write the degree equation requires showing that all representations of G/G' are of degree one. Let $H = G/G'$. Let $\phi : H \to M_{n_i}(\mathbb{C})$ be a representation of H. Note that H is Abelian, so $\mathbb{C}[H]$ is a commutative ring and ϕ must map to a commutative matrix ring. Then $n_i = 1$ and $M_{n_i}(\mathbb{C}) = \mathbb{C}^\star$ is the ring of scalar matrices. It is not difficult to show that all degree one representations of G factor through H. Hence we can write the degree equation as

$$|G| = [G : G'] + \sum_{[G:G']+1}^{k(G)} n_i^2. \tag{1.7}$$

It can also be shown that n_i divides $|G|$ for each i. We omit this proof, but a proof this statement, as well as a more complete discussion of the degree equation, is provided in Chapter 18 of [15].

Since $k(G)$ is the number of conjugacy classes of G, the degree equation is used in a similar manner to the class equation to find the number of conjugacy classes of a group. We include three examples.

Example 1.4.2. Consider the quaternion group
$Q_8 = \langle a, b : a^4 = b^4 = e, a^2 = b^2, ba = ab^3\rangle$. By Equation 1.7,

$$8 = |Q_8| = [Q_8 : Q_8'] + \sum_{i=[Q_8:Q_8']+1}^{k(Q_8)} n_i^2.$$

Since $Q_8' = \{e, b^2\}$, $[Q_8 : Q_8'] = 4$. Then

$$\begin{aligned} |Q_8| &= 1+1+1+1+\sum_{i=5}^{k(Q_8)} n_i^2 \\ &= 1+1+1+1+2^2, \end{aligned}$$

so Q_8 has 5 irreducible representations and thus 5 conjugacy classes. Then $P(Q_8) = \frac{5}{8}$.

Example 1.4.3. We count the number of irreducible representations of A_4. The commutator subgroup of A_4 includes the identity and three pairs of disjoint 2-cycles, so $[A_4 : A_4'] = 3$. Also note that $|A_4| = 12$. Then the degree equation is

$$\begin{aligned} |A_4| &= 1+1+1+\sum_{4}^{k(A_4)} n_i^2 \\ &= 1+1+1+3^2. \end{aligned}$$

Hence A_4 has 4 conjugacy classes and the commutativity degree is $P(A_4) = \frac{1}{3}$. ∎

Example 1.4.4. Consider S_4. Note that $|S_4| = 24$. Since $S_4' = A_4$, $[S_4 : S_4'] = 2$. Then

$$\begin{aligned} |S_4| &= 1 + 1 + \sum_{i=3}^{k(S_4)} n_i^2 \\ &= 1 + 1 + 22. \end{aligned}$$

Then $\sum_{i=3}^{k(S_4)} n_i^2 = 22$ is a sum of squares each greater than 1. If there were an $n_i > 4$ then $n_i^2 > 22$. Hence all $n_i \leq 4$. If there were some $n_i = 4$, then $22 = 16 + \sum_{i=4}^{k(S_4)} n_i^2$ and then $8 = \sum_{i=4}^{k(S_4)} n_i^2$. However, 8 cannot be written as a sum of squares greater than 1. Hence all $n_i \leq 3$. If all the $n_i > 1$ were equal to 2, or all the $n_i > 1$ were equal to 3, then $\sum_{i=3}^{k(S_4)} n_i^2 \neq 22$. Thus there must be some $n_i = 2$ and another $n_j = 3$. It follows that

$$\begin{aligned} |S_4| &= 1 + 1 + 2^2 + 3^2 + \sum_{i=5}^{k(S_4)} n_i^2 \\ &= 1 + 1 + 2^2 + 3^2 + 3^2. \end{aligned}$$

The degree equation for S_4 has 5 terms, so $k(S_4) = 5$ and $P(S_4) = \frac{5}{24}$.∎

Chapter 2: Bounds on Commutativity Degree

In this Chapter, we apply the class and degree equations to find lower and upper bounds on the commutativity degree of finite non-Abelian groups.

In Section 2.1, we use the class equation to derive the upper bound of $\frac{5}{8}$ on the commutativity degree of all finite non-Abelian groups. For a finite group G we find bounds in terms of the smallest prime p dividing $|G/Z|$ called the upper and lower p-bounds. Then we find a second set of bounds called the upper and lower l-bounds in terms of $l = |G/Z|$. An additional set of bounds written in terms of both $|G/Z|$ and p are called the lp-bounds. A final upper bound derived from the class equation is written in terms of a centralizer of maximal order in G.

In Section 2.2, we use the degree equation to find a general pair of lower and upper bounds on the commutativity degree of any finite non-Abelian group in terms of the commutator subgroup. We use a similar method to find another upper bound called the minimal dimension degree equation bound. A corollary of the general upper bound is a second proof of the $\frac{5}{8}$ bound. Two additional upper bounds are written in terms of the derived length of the group. One of these bounds applies to all solvable groups, the second only to p-groups.

In Section 2.3, we find three lower bounds using the structure of the group. The first two bounds are found by counting conjugacy classes. One applies to nilpotent groups and the other to solvable groups. The third bound, which we call the Pyber lower bound, is another bound for solvable groups.

2.1 Bounds from the Class Equation

In this section, we use the class equation to find bounds on the commutativity of non-Abelian finite groups.

2.1.1 The $\frac{5}{8}$ Bound

The upper bound of $\frac{5}{8}$ applies to all non-Abelian finite groups. It is worth emphasizing that there is no group with commutativity degree in the interval $(\frac{5}{8}, 1)$. We derive the $\frac{5}{8}$ bound from the class equation and then include some examples of groups realizing the bound.

Proposition 2.1.1. *Let G be a finite non-Abelian group. Then $P(G) \leq \frac{5}{8}$.*

Proof. Consider the class equation

$$|G| = |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} |[x_i]|.$$

For each i, $|[x_i]| \geq 2$. Then

$$|G| \geq |Z(G)| + 2(k(G) - |Z(G)|), \tag{2.1}$$

and solving for $k(G)$ yields

$$k(G) \leq \frac{|G| + |Z(G)|}{2}.$$

Since G is not Abelian, $G/Z(G)$ is not cyclic. Thus $|G/Z(G)| \geq 4$ and then $|Z(G)| \leq \frac{|G|}{4}$. Therefore

$$k(G) \leq \frac{1}{2}\left(|G| + \frac{|G|}{4}\right) = \frac{5|G|}{8},$$

and

$$P(G) = \frac{k(G)}{|G|} \leq \frac{5}{8}.$$

□

The following example describes some groups realizing the $\frac{5}{8}$ bound.

Example 2.1.2. First, recall from Example 1.3.1 that $P(D_4) = \frac{5}{8}$. Notice that

$$D_4/Z(D_4) = D_4/\{e, \rho^2\} \cong V_4.$$

Secondly, the Quasidihedral Groups,

$$QD_n =< a, b : a^{2n} = b^2 = e, bab = a^{2^n+1} >$$

are a class of groups with commutativity degree $\frac{5}{8}$. In Section 4.1.2, we calculate properties of the Quasidihedral groups and show that it is also true that $QD_n/Z(QD_n) \cong V_4$. ∎

In Proposition 5.2.2, we will generalize this example to show that $P(G) = \frac{5}{8}$ if and only if $|G/Z| \cong V_4$, the Klein 4-group.

2.1.2 p-Bounds

A more specific upper bound on the commutativity degree of a group is written in terms of the smallest prime p dividing $|G/Z(G)|$. This bound is called the upper p-bound.

Proposition 2.1.3. *Let p be the smallest prime dividing $|G/Z(G)|$. Then*

$$P(G) \leq \frac{p^2 + p - 1}{p^3}.$$

Proof. Let p be the smallest prime dividing $|G|$. Let $|G| = pl|Z(G)|$. Then $l \geq p$ since $G/Z(G)$ is not cyclic. Notice that for each $x_i \notin Z(G)$ $|[x_i]| = [G : C_G(x_i)] \geq p$ since $C_G(x_i) \supsetneq Z(G)$.

The class equation yields the bound,

$$|G| \geq |Z(G)| + p(k(G) - |Z(G)|).$$

Solving for $k(G)$ yields

$$k(G) \leq \frac{|G| + (p-1)|Z(G)|}{p}.$$

Then

$$\begin{aligned} P(G) &\leq \frac{|G| + (p-1)|Z(G)|}{p|G|} \\ &= \frac{(p-1)|Z(G)| + pl|Z(G)|}{p^2l|Z(G)|} \\ &= \frac{(p-1) + pl}{p^2l}. \end{aligned}$$

Next consider the ratio

$$\begin{aligned} (\frac{p-1+pl}{p^2l})/(\frac{p-1+p^2}{p^3}) &= \frac{(p-1+pl)p^3}{p^2l(p-1+p^2)} \\ &= \frac{(p-1+pl)p}{l(p-1+p^2)} \\ &= \frac{p^2l + p(p-1)}{p^2l + l(p-1)} \\ &\leq 1. \end{aligned} \tag{2.2}$$

Since this ratio is less than or equal to 1 and the commutativity degree is less than or equal to the numerator $\frac{(p-1)+pl}{p^2l}$, it follows that the commutativity degree is less than or equal to the denominator as well. Hence

$$P(G) \leq \frac{p^2 + p - 1}{p^3}.$$

□

Table 2.1 lists p-upper bounds for G with smallest prime p dividing $|G|$. As p increases, the bound becomes close to $\frac{1}{p}$. For instance, if $p = 641$, the bound is approximately 0.001562 and $\frac{1}{641} \approx 0.001560$. Notice the additional upper bound of $\frac{p+pl-1}{p^2l}$, with $l = |G/Z|$ and p the smallest prime dividing $|G|$, in Equation 2.2. We call this bound the $p^\star$-bound. The $p^\star$-bound provides a slightly improved estimate of $P(G)$. Table 2.2 lists some sample p- and $p^\star$-bounds. Table 2.2 suggests that the

p	2	3	5	7	11	13	17	19
$P-Bound$	0.6250	0.4074	0.2320	0.1603	0.0984	0.0824	0.0621	0.0553
	5/8	11/27	29/125	55/343	25/254	29/352	37/596	41/742

Table 2.1: p**-Upper Bounds**

additional bound reduces the p-bound by at most little more than 0.1, and appears to make a significant difference for small p values. A corresponding lower p-bound can be calculated for a group with $|G/Z(G)| = p^k$.

Proposition 2.1.4. *Let $|G/Z(G)| = p^k$. Then*

$$P(G) \geq \frac{p^k + p^{k-1} - 1}{p^{2k-1}}.$$

Proof. Suppose $|G/Z(G)| = p^k$. Let $x \in G$ such that $x \notin Z(G)$. Then since $x \in C_G(x)$ and $x \notin Z(G)$, $C_G(x) \subsetneq G$. Also $Z(G) \subsetneq C_G(x)$ because $Z(G) \subseteq C_G(x)$ but $x \notin Z(G)$. Thus

$$|Z(G)| < |C_G(x)| < |G|.$$

and then

$$p|Z(G)| \leq |C_G(x)| \leq p^{k-1}|Z(G)|$$

where $|C_G(x)|$ divides $|G|$. Then $|[x]| = [G : C_G(x)]$ and

$$p^{k-1} \geq |[x]| \geq p.$$

From the class equation,

$$|G| \leq |Z(G)| + p^{k-1}(k(G) - |Z(G)|).$$

Solving for $k(G)$ yields

$$k(G) \geq \frac{|G| + (p^{k-1} - 1)|Z(G)|}{p^{k-1}}.$$

p	$l = G/Z$	$\frac{(p-1)+pl}{p^2 l}$		$\frac{p^2+p-1}{p^3}$	
2	3	0.5833	7/12	0.625	5/8
2	5	0.55	11/20	0.625	5/8
2	40	0.5063	81/160	0.6250	5/8
2	100000	0.5000	1/2	0.6250	5/8
3	5	0.3778	17/45	0.4074	11/27
3	7	0.3651	23/63	0.4074	11/27
3	99	0.3356	299/891	0.4074	11/27
5	7	0.2229	39/175	0.232	29/125
5	11	0.2145	59/275	0.232	29/125
7	13	0.1523	97/637	0.1603	55/343
7	49	0.1454	108/743	0.1603	55/343
11	13	0.0971	32/329	0.0984	25/254
11	53	0.0925	27/292	0.0984	25/254
41	43	0.0250	11/441	0.02450	21/841
41	47	0.0249	6/241	0.02450	21/841
41	1009	0.0244	1/41	0.02450	21/841
1009	1013	0.0010	0	0.0010	0
1009	1933	0.0010	0	0.0010	0
1009	3989	0.0010	0	0.0010	0
27107	27109	0.00004	0	0.00004	0

Table 2.2: **Sample Upper p-Bounds and $p^{\star}$-Bounds**

Then

$$
\begin{aligned}
P(G) &\geq \frac{|G| + (p^{k-1} - 1)|Z(G)|}{p^{k-1}|G|} \\
&= \frac{|G/Z(G)| + (p^{k-1} - 1)}{p^{k-1}|G/Z(G)|} \\
&= \frac{p^k + (p^{k-1} - 1)}{p^{k-1}p^k} \\
&= \frac{p^k + p^{k-1} - 1}{p^{2k-1}}.
\end{aligned}
\tag{2.3}
$$

□

In Table 2.3, we compare the lower and upper p-bounds for several p and k values. Observe from Table 2.3 that in the case of $k = 2$, the lower p-bound equals the upper p-bound; hence a group G with $|G/Z(G)| = p^2$ realizes the bound $\frac{p^2+p-1}{p^3}$. This will be illustrated by Proposition 5.2.2, which states that $|G/Z(G)| \cong V_4$ if and only if $P(G) = \frac{5}{8} = \frac{2^2+2-1}{2^3}$. Proposition 5.2.3 is a proof of the general case, and Corollary 2.1.5 is the specific case in which $|G| = p^3$ and G realizes the bound.

Corollary 2.1.5. *Let p be a prime. If G is non-Abelian with $|G| = p^3$ then $P(G) = \frac{p^2+p-1}{p^3}$.*

Proof. Suppose that G is non-Abelian and $|G| = p^3$. First we will show that $|Z(G)| = p$. Since G is a p-group, $|Z(G)| > 1$. Suppose $|Z(G)| = p^2$. Then $[G : Z(G)] = p$, $G/Z(G)$ is cyclic, and G Abelian. Hence $|Z(G)| = p$.

Next suppose $a \in G$ and $a \notin Z(G)$. Since $C_G(a) \supset \{a\}$ and $a \notin Z(G)$, $|C_G(a)| > |Z(G)| = p$. Further, since $a \in C_G(a)$ and $a \notin Z(G)$, $|C_G(a)| < |G| = p^3$. Hence $|C_G(a)| = p^2$ and $|[a]| = [G : C_G(a)] = p$.

k	p	$_{UpperBound}$		$_{LowerBound}$	
2	2	0.6250	5/8	0.6250	5/8
3	2	0.6250	5/8	0.3438	11/32
4	2	0.6250	5/8	0.1797	23/128
10	2	0.6250	5/8	0.0029	2/683
15	2	0.6250	5/8	0.0001	0
2	3	0.4074	11/27	0.4074	11/27
3	3	0.4074	11/27	0.1440	35/243
4	3	0.4074	11/27	0.0489	41/838
10	3	0.4074	11/27	0.0001	0
2	5	0.2320	29/125	0.2320	29/125
3	5	0.2320	29/125	0.0477	37/776
4	5	0.2320	29/125	0.0096	3/313
7	5	0.2320	29/125	0.0001	0
2	41	0.0250	21/841	0.0250	21/841
3	41	0.0250	21/841	0.0006	0
2	641	0.0016	1/640	0.0016	1/640
3	641	0.0016	1/640	0.0000	0

Table 2.3: **Comparison of Upper and Lower p-Bounds**

By the class equation

$$\begin{aligned} |G| &= |Z(G)| + p(k(G) - |Z(G)|) \\ &= p + p(k(G) - p). \end{aligned}$$

Solving for $k(G)$ yields

$$k(G) = p^2 + p - 1.$$

Therefore,

$$P(G) = \frac{p^2 + p - 1}{p^3}.$$

□

To illustrate the generality of Proposition 5.2.3 in comparison to Corollary 2.1.5, we provide an example of an indecomposable group G with commutativity degree $\frac{p^2+p-1}{p^3}$ such that $|G| > p^3$ and $|Z(G)| \neq p$. This example is also an application of GAP. For more information on GAP and more detailed descriptions on the terminology and tasks described in this example, see Appendix 6.1.

Example 2.1.6. We located the group G in the SmallGroup library with SmallGroup ID Tag $[16, 4]$ and found that the presentation of this group is

$$G =< a, b : a^4 = b^4 = e, aba^{-1} = b^3 > .$$

In GAP, we verified that G has $|Z(G)| = 4$, and $k(G) = 10$. Then for the prime $p = 2$,

$$P(G) = \frac{5}{8} = \frac{p^2 + p - 1}{p^3}$$

and $|G/Z(G)| = p^2$, as desired. However, $|G| = p^4$ and $|Z| = p^2$.

Next we will show that G is indecomposable. Notice that $D_8 \times \mathbb{Z}_2$ and $Q_8 \times \mathbb{Z}_2$ are the only non-isomorphic direct products of order 16 with a non-Abelian factor. The SmallGroup ID Tags for these groups are $[16, 12]$ and $[16, 11]$, respectively. Since

G has ID Tag $[16,4]$, it is not a direct product with a non-Abelian factor. Hence it is indecomposable.∎

It is also useful to apply the upper p-bound to a nilpotent group with a non-Abelian factor.

Example 2.1.7. *An Application of the p-Bound to Nilpotent Groups.* Let G be a nilpotent group. Then G is a direct product of p-Sylow subgroups, say $G = \Pi_{i=1}^{r} P_i$ where $|P_i| = p_i^{n_i}$ for some prime p_i and $n_i \in \mathbb{Z}$. The commutativity degree of G is

$$P(G) = \Pi_{i=1}^{r} P(P_i)$$

by Proposition 1.2.3. If, for some i, P_i is Abelian then $P(P_i) = 1$, so the commutativity degree of G is completely determined by the commutativity degree of its non-Abelian p-Sylow factors. That is, for some $s \leq r$, $P(G) = \Pi_{i=1}^{s} P(P_i)$ where each P_i, for $1 \leq i \leq s$, is non-Abelian.

Each non-Abelian p-Sylow factor P_i has commutativity degree bounded by the p-bound

$$P(P_i) \leq \frac{p_i^2 + p_i - 1}{p_i^3},$$

so the commutativity degree of G is bounded by the product of the p-bounds on the non-Abelian factors:

$$P(G) \leq \Pi_{i=1}^{s} \left(\frac{p_i^2 + p_i - 1}{p_i^3} \right).$$

Table 2.4 lists upper bounds for nilpotent groups with given non-Abelian p-Sylow factors: $P_1 = p_1^{n_1}, P_2 = p_2^{n_2}$, and $P_3 = p_3^{n_3}$. All possible products of p-Sylow factors which yield an upper bound greater than 0.05 are listed. ∎

1 Factor		2 Factors		3 Factors	
p_1	Bound	p_1, p_2	Bound	p_1, p_2, p_3	Bound
2	0.625	2, 3	0.2546	2, 3, 5	0.0591
3	0.4074	2, 5	0.1450		
5	0.2320	2, 7	0.1002		
7	0.1603	3, 5	0.0945		
11	0.0984	3, 7	0.0653		
13	0.0824				
17	0.0621				
19	0.0553				

Table 2.4: **Upper p-Bounds for Nilpotent Groups**

2.1.3 l-Bounds and lp-Bounds

Let G be a finite group, let $l = |G/Z(G)|$, and let p be the smallest prime dividing $|G/Z(G)|$. Upper and lower bounds on the commutativity degree of G are calculated from the class equation in terms of l and p. We call these bounds the upper and lower lp-bounds. The l-bounds will be a special case of the lp-bounds.

Proposition 2.1.8. *Let $|G/Z(G)| = l$. If p is the smallest prime such that p divides $|G/Z(G)|$, then*

$$\frac{lp + l - p}{l^2} \leq P(G) \leq \frac{l + p - 1}{pl}.$$

Proof. Let $|G/Z(G)| = l$ and let p be the smallest prime dividing $|G/Z(G)|$. Suppose $a \in G$ and $a \notin Z(G)$. Then $Z \subsetneq C_G(a) \subsetneq G$. Hence

$$p|Z(G)| \leq |C_G(a)| \leq \frac{l}{p}|Z(G)|.$$

and since $[G : C_G(a)] = |[a]|$,

$$p|Z(G)| \leq \frac{|G|}{|[a]|} \leq \frac{l}{p}|Z(G)|.$$

$$\frac{|G|}{p|Z(G)|} \geq |[a]| \geq \frac{p|G|}{l|Z(G)|}.$$

Then

$$\frac{l}{p} \geq |[a]| \geq p. \tag{2.4}$$

We apply the second inequality in Equation 2.4 to the class equation as follows

$$|G| \geq |Z(G)| + p(k(G) - |Z(G)|).$$

Solving for $k(G)$ yields

$$k(G) \leq \frac{|G| + (p-1)|Z(G)|}{p}.$$

Then

$$P(G) \leq \frac{|G| + (p-1)|Z(G)|}{p|G|},$$

and

$$P(G) \leq \frac{l|Z(G)| + (p-1)|Z(G)|}{pl|Z(G)|} = \frac{l+p-1}{pl}.$$

This establishes the upper lp-bound. Next, we apply the first inequality in Equation 2.4 to the class equation as follows

$$|G| \leq |Z(G)| + \frac{l}{p}(k(G) - |Z(G)|).$$

Solving for $k(G)$ yields

$$k(G) \geq \frac{|G| + (\frac{l}{p} - 1)|Z(G)|}{\frac{l}{p}} = \frac{p|G| + (l-p)|Z(G)|}{l}.$$

Then

$$P(G) \geq \frac{p|G| + (l-p)|Z(G)|}{l|G|},$$

and

$$P(G) \geq \frac{lp|Z(G)| + (l-p)|Z(G)|}{l^2|Z(G)|} = \frac{lp+l-p}{l^2}.$$

This establishes the lower lp-bound. $\square$

Since $p = 2$ is the smallest possible prime dividing $|G/Z(G)|$, we can calculate the more general, but less accurate, l-bounds in terms of only $|G/Z(G)|$.

Corollary 2.1.9. *Let G be a non-Abelian group. If $|G/Z(G)| = l$, then*

$$\frac{3l-2}{l^2} \leq P(G) \leq \frac{l+1}{2l}.$$

Proof. Let $|G/Z(G)| = l$ and let p be the smallest prime dividing $|G/Z(G)|$. Since $p \geq 2$,

$$P(G) \leq \frac{l+p-1}{pl} \leq \frac{l+1}{2l}$$

and

$$\frac{3l-2}{l^2} \leq \frac{lp+l-p}{l^2} \leq P(G).$$

□

Notice that the lower bound in Corollary 2.1.9 is a sharper bound than the lower bound $\frac{2l-1}{l^2}$ found using the definition of commutativity degree in Proposition 1.2.4. Table 2.5 contains upper and lower lp- and l-bounds for G with select small composite orders of $G/Z(G)$.

From Table 2.5, notice that the l- and lp-bounds are calculated for groups with $|G/Z| \geq 4$ because if $|G/Z| < 4$ then G is Abelian. Several observations should be noted from the Table. First, for groups with $p = 2$, the l- and lp-bounds are equivalent. For $p > 2$, the lp-bounds are always as sharp or sharper than the l-bounds. For example, when $l = 10$, both types of upper bounds are $\frac{11}{20}$ and both types of lower bounds are $\frac{7}{25}$. Also, when $l = p^2$, the upper and lower lp-bounds are equal. For instance, when $l = 49$, the lower and upper lp-bounds are both $\frac{55}{343}$. This value is the commutativity degree of a group with $|G/Z| = 49$. In this case, $l = p$, and the lp-bounds equals the upper p-bound.

2.1.4 Centralizer Upper Bound

A fourth type of upper bound on the commutativity degree of a non-Abelian group derived from the class equation is written in terms of a centralizer of maximal order. The proof is adopted from Guralnick [21].

l	$L\!-\!Upper$		$L\!-\!Lower$		$LP\!-\!Upper$		$LP\!-\!Lower$	
4	0.6250	5/8	0.6250	5/8	0.6250	5/8	0.6250	5/8
6	0.5833	7/12	0.4444	4/9	0.5833	7/12	0.4444	4/9
8	0.5625	9/16	0.3438	11/32	0.5625	9/16	0.3438	11/32
9	0.5556	5/9	0.3086	25/81	0.4074	11/27	0.4074	11/27
10	0.5500	11/20	0.2800	7/25	0.5500	11/20	0.2800	7/25
12	0.5417	13/24	0.2361	17/72	0.5417	13/24	0.2361	17/72
14	0.5357	15/28	0.2041	10/49	0.3810	8/21	0.2704	53/196
15	0.5333	8/15	0.1911	43/225	0.5333	8/15	0.1911	43/225
16	0.5313	17/32	0.1797	23/128	0.5313	17/32	0.1797	23/128
21	0.5238	11/21	0.1383	61/441	0.3651	23/63	0.1837	9/49
25	0.5200	13/25	0.1168	73/625	0.2320	29/125	0.2320	29/125
27	0.5185	14/27	0.1084	79/729	0.3580	29/81	0.1440	35/243
33	0.5152	17/33	0.0891	22/247	0.3535	35/99	0.1185	43/363
35	0.5143	18/35	0.0841	75/892	0.2229	39/175	0.1673	41/245
39	0.5128	20/39	0.0756	31/410	0.3504	41/117	0.1006	17/169
45	0.5111	23/45	0.0657	31/472	0.2178	49/225	0.1309	53/405
49	0.5102	25/49	0.0604	34/563	0.1603	55/343	0.1603	55/343
51	0.5098	26/51	0.0581	40/689	0.3464	53/153	0.0773	67/867
55	0.5091	28/55	0.0539	43/798	0.2145	59/275	0.1074	13/121

Table 2.5: l-**Upper,** l-**Lower,** lp-**Upper, and** lp-**Lower Bounds**

Proposition 2.1.10. *For some $x \in G$ so that $x \notin Z(G)$, $P(G) \leq \frac{3}{2[G:C_G(x)]}$. If $Z(G) = \{e\}$, then $P(G) \leq \frac{1}{[G:C_G(x)]}$ for some $x \in G$ such that $x \neq \{e\}$.*

Proof. Choose $x = xZ \in G/Z(G)$ such that $c = |C_G(x)|$ is of maximal order. Let $y \in G$ so that $y \notin Z(G)$. Then $c \geq |C_G(y)|$, and

$$\frac{|G|}{c} \leq \frac{|G|}{|C_G(y)|}. \tag{2.5}$$

Applying Equation 2.5 to the class equation as follows

$$\begin{aligned} |G| &= |Z(G)| + \sum_{Z(G)}^{k(G)} [G : C_G(x_i)] \\ &\geq |Z(G)| + \sum_{Z(G)}^{k(G)} \frac{|G|}{c} \\ &= |Z(G)| + \frac{|G|}{c}(k(G) - |Z(G)|). \end{aligned}$$

Then solving for $k(G)$ yields

$$k(G) \leq c - \frac{c|Z(G)|}{|G|} + |Z(G)|.$$

Since $k(G), c$, and $|Z(G)| \in \mathbb{N}$, it follows that $\frac{c|Z(G)|}{|G|} \in \mathbb{Z}$ and

$$k(G) \leq c - 1 + |Z(G)|.$$

Next,

$$P(G) \leq \frac{c}{|G|} + \frac{|Z(G)| - 1}{|G|}. \tag{2.6}$$

Also,

$$Z(G) - 1 \leq Z(G) \leq \frac{|C_G(x)|}{2}.$$

Hence

$$P(G) \leq \frac{3}{2[G : C_G(x)]}. \tag{2.7}$$

Next suppose $Z(G) = \{e\}$. Then Equation 2.6 may be rewritten as follows:

$$\begin{aligned} P(G) &\leq \frac{c}{|G|} + \frac{1-1}{|G|} \\ &= \frac{1}{[G : C_G(x)]}. \end{aligned} \tag{2.8}$$

$\square$

Notice from the proof of Proposition 2.1.10 that to apply the upper bound in Equation 2.7 or Equation 2.8, it is necessary to find a element in G with centralizer of maximal order. We will illustrate the case when $Z(G) = \{e\}$ and $P(G) \leq \frac{1}{[G:C_G(x)]}$ with two examples.

Example 2.1.11. *A Bound on S_n.* Let $n \geq 4$ and consider S_n. It can be shown that $Z(S_n) = \{e\}$. First we will find an $x \in S_n$ with $|C_{S_n}(x)|$ of maximal order. Let $\rho \in S_n$. Recall the well known result that $[\rho]$ is the set of elements of S_n with the same cycle structure as ρ. Dummit and Foote, [15] (4.3, Proposition 11), prove this result, and we discuss it in more detail when computing the commutativity degree of S_n in Section 4.2. Since $|C_{S_n}(\rho)| = \frac{|G|}{|[\rho]|}$, a minimal order of $[\rho]$ corresponds to the desired maximal order $C_G(\rho)$. Hence we will find cycle structure ρ having minimal order. Suppose $\gamma = \rho_1\rho_2$, where ρ_1 ia a cycle and ρ_2 is a product of disjoint cycles each disjoint from ρ_1. Then $|[\gamma]| = |[\rho_1]||[\rho_2]|$ since the number of conjugates of ρ_i equals the number of elements of the same cycle structure as ρ_i. It follows that a conjugacy class of minimal order must be the class of a cycle. Next let $\alpha \in S_n$ be a cycle of length m. We can count the number of m cycles as follows

$$|[\alpha]| = \frac{(n)(n-1)...(n-m+1)}{m} = \frac{P(n,m)}{m}.$$

We will show by induction on the length k of a k-cycle in S_n that

$$|[(12)]| = \frac{n(n-1)}{2} < \frac{(n)(n-1)...(n-k+1)}{k}$$

for all k-cycles such that $2 < k \leq n$ and $n > 4$. Let $k = 3$. Then

$$\begin{aligned}
\frac{(n)(n-1)(n-2)}{3} &\geq \frac{(n)(n-1)}{2}\frac{2(n-2)}{3} \\
&\geq \frac{(n)(n-1)}{2} \cdot 1 \\
&= |[(12)]|.
\end{aligned}$$

Suppose the hypothesis is true for $k = n - 1$ and consider $k = n$. Then

$$\begin{aligned}
\frac{(n)(n-1)...(n-k+1)}{k} &= \frac{(n)(n-1)...(1)(n-2)}{n(n-2)} \\
&\geq \frac{(n)(n-1)}{2}\frac{(2)(n-2)}{n} \\
&\geq \frac{(n)(n-1)}{2} \cdot 1 \\
&\geq [(12)].
\end{aligned}$$

Therefore, $[(12)]$ is of minimal order, and $C_{S_n}((12))$ is of maximal order. Also,

$$|C_{S_n}((12))| = \frac{2n!}{n(n-1)}.$$

Therefore

$$P(S_n) \leq \frac{2(n-2)!}{n!} = \frac{2}{n(n-1)}.$$

Table 2.6 lists the bound of $P(S_n)$ and the value of $P(S_n)$ for small n. Observe that as n increases, the bound becomes large relative to the actual commutativity degree. However both the bound and actual commutativity degree approach the same value, 0, as n increases. ∎

n	$P(S_n)$	Bound S_n
2	1	1
3	0.5	0.3333
4	0.2083	0.1667
5	0.0583	0.1
6	0.0153	0.0667
7	0.0030	0.0476
8	0.0005	0.0357

Table 2.6: **A Comparison of Bounds on $P(S_n)$ and the Value of $P(S_n)$**

Our second application of the bound $\frac{1}{[G:C_G(x)]}$ shows that this bound not useful for all groups.

Example 2.1.12. Consider the dihedral groups, D_n with odd n. Since $|Z(D_n)| = 1$, we apply the bound $P(D_n) \leq \frac{1}{[G:C_G(x)]}$ for an element x with centralizer $C_G(x)$ of maximal order. Observe from the tabulated conjugacy classes in Table 4.16 that the centralizer of maximal order has order n. Hence the bound $P(G) \leq \frac{1}{[G:C_G(x)]}$ yields a bound of only

$$P(D_{2n}) \leq \frac{1}{2}.$$

However, in Section 4.1.2 we also show that as , $P_{n\to\infty}(D_n) = \frac{1}{4}$ and for $n \geq 5$, $P(D_n)$ is close to $\frac{1}{4}$. ∎

2.2 Bounds From the Degree Equation

In this section, we discuss various bounds derived from the degree equation.

2.2.1 Generic Bounds from the Degree Equation

A pair of general bounds on all non-Abelian groups written in terms of the commutator subgroup is obtained from the degree equation. We will refer to these bounds as the upper and lower degree equation bounds.

Proposition 2.2.1. *Given a finite group G,*

(1) $P(G) \leq \frac{1}{4}(1 + \frac{3}{|G'|})$, *and*

(2) $\frac{1}{|G'|} \leq P(G)$.

Proof. 1. Recall the degree equation:

$$|G| = [G : G'] + \sum_{i=[G:G']+1}^{k(G)} (n_i)^2$$

with each $n_i \geq 2$. It follows that

$$|G| \geq [G : G'] + 4(k(G) - [G : G']).$$

Solving for $k(G)$,

$$k(G) \leq \frac{1}{4}(|G| + 3[G : G']).$$

Finally,

$$P(G) \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right). \tag{2.9}$$

2. Since $[G : G']$ counts irreducible characters of degree one, $[G : G'] < k(G)$. Then

$$\frac{|G|}{|G'||G|} \leq \frac{k(G)}{|G|} = P(G)$$

and so

$$\frac{1}{|G'|} \leq P(G).$$

□

An application of the upper degree equation bound provides a simple alternate proof of the upper bound of $\frac{5}{8}$ on the commutativity degree of all non-Abelian groups.

Corollary 2.2.2. *There are no finite groups with commutativity degree in the interval* $(\frac{5}{8}, 1)$.

Proof. Suppose G is a group with $P(G) > \frac{5}{8}$. Solving the upper degree equation bound,

$$\frac{5}{8} < P(G) < \frac{1}{4}\left(1 + \frac{3}{|G'|}\right),$$

for G' yields $|G'| < 2$. Hence $|G'| = 1$ and G is Abelian. Therefore, $P(G) = 1$. □

2.2.2 Minimal Dimension Degree Equation Bound

Guralnick [21] improves the upper degree equation bound by finding an upper bound in terms of the smallest degree, d, of a nonlinear representation of G. A nonlinear representation of G is any representation of degree $d > 1$ and a linear representation of G is a degree one representation. Proposition 2.2.3 is a version of this proof.

Proposition 2.2.3. *If G is non-Abelian and d is the smallest degree of a nonlinear representation of G, then*

$$P(G) \leq \frac{1}{d^2} + \left(1 - \frac{1}{d^2}\right)\frac{1}{|G'|}.$$

Equality follows if all nonlinear representations of G are of degree d.

Proof. Let d the smallest degree of a nonlinear representation of G. Recall the degree equation:

$$|G| = [G : G'] + \sum_{[G:G']}^{k(G)} n_i^2.$$

The term $[G : G']$ counts the number of linear irreducible representations. Since $d \leq n_i$ for $[G : G'] < i \leq k(G)$ there can be at most $\frac{1}{d^2}(|G| - [G : G'])$ nonlinear irreducible representations. Hence by the degree equation,

$$k(G) \leq [G : G'] + \frac{1}{d^2}\left(|G| - [G : G']\right). \tag{2.10}$$

Then,

$$\begin{aligned} P(G) &\leq \frac{1}{|G'|} + \frac{1}{d^2}\left(1 - \frac{1}{|G'|}\right) \\ &= \frac{1}{d^2} + \left(1 - \frac{1}{d^2}\right)\frac{1}{|G'|}. \end{aligned} \tag{2.11}$$

If each n_i corresponding to a nonlinear representation is $n_i = d$, equality follows in Equation 2.10 and 2.11. □

Next we provide an example of the case of equality in Equation 2.11 by describing a class of groups that realize the minimal dimension degree equation bound. Equality holds in the bound in Equation 2.11 for other types of groups as well, including the dihedral groups.

Example 2.2.4. *A Class of Groups realizing the Minimal Dimension Degree Equation Bound.* The Mersenne Group G_p of order $|G_p| = p(2^k)$, where $p = 2^k - 1$ is a Mersenne Prime, is constructed in Example 5.3.4. Notice that

$$|G_p| = (2^k - 1)(2^k) = 2^2k - (2)2^k + 1 + 2^k - 1 = p + p^2.$$

In Example 5.3.4, we also show that $P(G_p) = \frac{1}{p}$. It follows that $k(G_p) = 2^k = p + 1$. From Example 5.3.4, it is easy to see that $|G_p'| = 2^k$ and then that $[G_p : G_p'] = p$. The degree equation is

$$|G_p| = p + \sum_{i=1}^{j} n_i^2.$$

Then $j = 1$ since $k(G_p) = p + 1$, and it follows that

$$|G_p| = p + p^2 = p + n_i^2.$$

Hence $n_i = p$ and the degree equation may be rewritten as

$$|G_p| = p + p^2.$$

The only irreducible character of degree greater than 1 has degree p. Then the degree equation bound yields

$$P(G_p) \leq \frac{1}{p^2} + (1 - \frac{1}{p})\frac{1}{2^k} = \frac{1}{p}$$

and $\frac{1}{p}$ is the commutativity degree of the group. ∎

Next we compare the minimal dimension bound from Proposition 2.2.3 to the upper and lower degree equation bounds determined in Proposition 2.2.1. Let G be a group with a fixed order of G'. First suppose $d = 2$. Then the minimal dimension upper bound equals the upper degree bound:

$$\frac{1}{d^2} + \left(1 - \frac{1}{d^2}\right)\frac{1}{|G'|} = \frac{1}{4}\left(1 + \frac{3}{|G'|}\right).$$

However, if we suppose that the minimal dimension d increases

$$\lim_{d\to\infty}\left[\frac{1}{d^2} + \left(1 - \frac{1}{d^2}\right)\frac{1}{|G'|}\right] = \frac{1}{|G'|},$$

As $d \to \infty$, the minimum dimension bound converges to the lower bound of $\frac{1}{|G'|}$ in Proposition 2.2.1. This suggests that for groups with a large minimum nonlinear irreducible representation degree, $\frac{1}{|G'|}$ is a good approximation of the commutativity degree of the group.

Table 2.7 includes degree equation bounds for G with $|G'| \leq 16$. The first row lists $|G'|$, the next row lists the upper bound from Proposition 2.2.1 (i.e. the minimal dimension bound with $d = 2$), and the last row lists the lower bound from 2.2.1. All intermediate rows list minimum dimension bounds for select small d.

G'	2	3	4	5	6	7	8	9	10	11	12	13
Bound												
Upper	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{7}{16}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{5}{14}$	$\frac{11}{32}$	$\frac{1}{3}$	$\frac{13}{40}$	$\frac{7}{22}$	$\frac{5}{16}$	$\frac{4}{13}$
d=3	$\frac{5}{9}$	$\frac{11}{27}$	$\frac{1}{3}$	$\frac{13}{45}$	$\frac{7}{27}$	$\frac{5}{21}$	$\frac{2}{9}$	$\frac{17}{81}$	$\frac{1}{5}$	$\frac{19}{99}$	$\frac{5}{27}$	$\frac{7}{39}$
d=4	$\frac{17}{32}$	$\frac{3}{8}$	$\frac{19}{64}$	$\frac{1}{4}$	$\frac{7}{32}$	$\frac{11}{56}$	$\frac{23}{128}$	$\frac{1}{6}$	$\frac{5}{32}$	$\frac{13}{88}$	$\frac{9}{64}$	$\frac{7}{52}$
d=5	$\frac{13}{25}$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{29}{125}$	$\frac{1}{5}$	$\frac{31}{175}$	$\frac{4}{25}$	$\frac{11}{75}$	$\frac{17}{125}$	$\frac{7}{55}$	$\frac{3}{25}$	$\frac{37}{325}$
d=10	$\frac{101}{200}$	$\frac{17}{50}$	$\frac{103}{400}$	$\frac{26}{125}$	$\frac{7}{40}$	$\frac{53}{350}$	$\frac{107}{800}$	$\frac{3}{25}$	$\frac{86}{789}$	$\frac{1}{10}$	$\frac{37}{400}$	$\frac{28}{325}$
Lower	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$	$\frac{1}{12}$	$\frac{1}{13}$

Table 2.7: **Degree Equation Bounds**

2.2.3 Derived Length Upper Bounds

The following bounds from the degree equation are written in terms of the derived length of the group. The first is an upper bound on non-Abelian solvable groups. The second improves upon the upper p-bound of $\frac{p^2+p-1}{p^3}$ for p-groups. The derived length bounds appear in Guralnick [21].

Lemma 2.2.5. *No group of order less than* 24 *has derived length greater than* 2.

Proof. Let $|G| < 24$. First, if $|G|$ is prime, then G is of derived length one. Next, using the Sylow Theorems, it is easily shown that G has a proper normal p-Sylow subgroup N for all composite orders of G except 8 and 16. Further, in each of these cases, G/N and N are Abelian so G has derived length of 2.

If $|G| = 8$, either G is Abelian and derived length one, $G \cong D_8$ and derived length two, or $G \cong Q_8$ and derived length two. Similarly, by classification of groups of order 12, the derived length is less than or equal to two if $|G| = 12$. If $|G| = 16$, then G has a normal subgroup of order $4 = 2^2$ and hence has derived length 2. $\square$

Proposition 2.2.6. *1. Let G be solvable and let the derived length of G be $d \geq 4$. Then*

$$P(G) \leq \frac{4d-7}{2^{d+1}}.$$

2. Let p be prime and let G be a finite p-group of derived length $d \geq 2$. Then

$$P(G) \leq \frac{p^d + p^{d-1} - 1}{p^{2d-1}}.$$

Proof. 1. We proceed by induction on d. Let H be a finite solvable group of derived length 4. If $|H'| < 24$, then by Lemma 2.2.5 the derived length of $H' \leq 2$, and then the derived length of H is $d \leq 3$. Hence $|H'| \geq 24$. Then by the upper degree equation bound (Proposition 2.2.1),

$$P(H) \leq \frac{1}{4} + \frac{3}{4|H'|} \leq \frac{9}{32}.$$

Notice that for $d = 4$,

$$\frac{9}{32} = \frac{4d-7}{2^{d+1}}.$$

Suppose that the derived length of G is $d > 4$ and assume that the result is true for groups of derived length $d - 1$.

By Proposition 3.1.5 and the comment thereafter, $P(G) \leq P(G/N)$ for any normal subgroup N of G. Let N_0 be a normal subgroup of G maximal with respect to the property that the derived length of G equals the derived length of G/N_0.

First we will show that G has a unique minimal normal subgroup, M. Since G/N_0 has the property that every normal subgroup has derived length less than d, we can assume that every proper subgroup of a homomorphic image of G has derived length less than d. Next suppose G has two distinct minimal normal subgroups, M_1 and M_2. The derived lengths of G/M_1 and G/M_2 are both less than d, and so the derived length of $G/M_1 \times G/M_2$ is less than or equal to the derived length of both G/M_1 and G/M_2. Hence the derived length of $G/M_1 \times G/M_2$ is less than d. Since the intersection of normal subgroups is normal, $M_1 \cap M_2 \lhd G$. Further, since M_1 and M_2 are minimal normal, $M_1 \cap M_2 = \{e\}$. Thus G embeds in the direct product, $G/M_1 \times G/M_2$ under the mapping $g \mapsto (gM_1, gM_2)$. Then the derived length of G is less than d, a contradiction. Therefore, there is a unique minimal normal subgroup $M \lhd G$. Also notice that the derived length of G/M is $d-1$.

Notice that each irreducible character either has the property that $M \leq \ker\chi$ or has the property that $M \nleq \ker\chi$. (We write $\ker\chi$ to denote the kernel of the representation associated to χ). First we will address those characters χ satisfying $M \nleq \ker\chi$.

Let χ be an irreducible character of G such that $M \nleq \ker\chi$. Since $\ker\chi \lhd G$ and M is the minimal normal subgroup of G, if $M \ntriangleleft \ker\chi$ then $\ker\chi = \{e\}$ and χ is faithful. Hence in this case the degree of the representation of χ is $n = \chi(1)$. By Dixon, [12] (Theorem (2) and comments thereafter), it follows that

$$d \leq \frac{5(\log_3 n + 1)}{2} \approx 1.58 \log_2 n.$$

Hence

$$d \leq 2\log_2(n) \leq 2 + 2\log_2(\chi(1)).$$

Solving for $\chi(1)^2$ yields

$$\chi(1)^2 \geq 2^{d-2}.$$

Finally, we use the degree equation to count the conjugacy classes in G as follows:

$$k(G) = k(G/M) + \frac{|G| - [G:M]}{2^{d-2}}$$

where the first term counts the number of irreducible characters satisfying $M \leq \ker\chi$. These are the irreducible characters that factor through G/M. The second term counts those characters with $M \nleq \ker\chi$.

Then

$$P(G) \leq \frac{k(G/M)}{|G|} + \frac{|G| - [G:M]}{|G|2^{d-2}} = \frac{P(G/M) - 2^{d-2}}{|M|} + \frac{1}{2^{d-2}}.$$

By induction, $P(G/M) \leq \frac{4d-11}{2^d}$. Then

$$P(G/M) - \frac{1}{2^{d-2}} \leq \frac{4d-15}{2^d}$$

and since $|M| \geq 2$,

$$\frac{P(G/M) - 2^{d-2}}{|M|} \leq \frac{4d-15}{2^{d+1}}.$$

It follows that

$$\begin{aligned} P(G) &\leq \frac{P(G/M) - 2^{d-2}}{|M|} + \frac{1}{2^{d-2}} \\ &\leq \frac{4d-15}{2^{d+1}} + \frac{1}{2^{d-2}} \\ &= \frac{4d-7}{2^{d+1}}. \end{aligned}$$

2. Again, we proceed by induction on d. Let G have derived length $d = 2$. By Proposition 2.1.3, $P(G) \leq \frac{p^2+p-1}{p^3}$ because p is the smallest prime dividing $|G/Z(G)|$.

Suppose that $d > 2$ and assume that the result is true for $d-1$. As in part (1) we can assume that every proper subgroup of a homomorphic image of G has derived length less than d and that G has a unique proper minimal normal subgroup, M, with derived length $d-1$. In this case, $M \leq Z(G)$ because both the center and M are normal p groups and M is minimal. Also, $|M| = p$.

Also similarly to part (1), every irreducible character χ such that $M \nleq \ker \chi$ is faithful. In this case, the degree of χ is $n \geq p^{d-1}$, [21], Theorem 12.

Again we count those characters with $M \leq \ker \chi$ and those with $\ker = \{e\}$ separately to find the number of conjugacy classes of G:

$$k(G) \leq k(G/M) + \frac{|G| - [G:M]}{p^{d-1}}.$$

Then

$$P(G) \leq \frac{k(G/M)}{|M|} + \frac{1 - \frac{1}{|M|}}{p^{d-1}} = \frac{P(G/M)}{p} + \frac{p-1}{p^{2d-1}}.$$

Also, by induction,

$$P(G/M) \leq \frac{p^d + p^{d-1} - 1}{p^{2d-2}}$$

and so

$$\begin{aligned} P(G) &\leq \frac{p^d + p^{d-1} - p}{p^{2d-2}p} + \frac{p-1}{p^{2d-1}} \\ &= \frac{p^d + p^{d-1} - 1}{p^{2d-1}}. \end{aligned}$$

□

Example 2.2.7. Let G be a p-group with order $|G| = p^n$ with $n > 2$. Then G has derived length of at most $d = \lceil \frac{n}{2} \rceil$. So suppose $d = \lceil \frac{n}{2} \rceil$. The upper p-bound is

$$P(G) \leq \frac{p^2 + p - 1}{p^3} = \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}$$

and the upper bound from Proposition 2.2.6 (2) is

$$P(G) \leq \frac{p^{\lceil \frac{n}{2} \rceil} + p^{\lceil \frac{n}{2} \rceil - 1}}{p^{n-1}} = \frac{1}{p^{\lceil \frac{n-2}{2} \rceil}} + \frac{1}{p^{\lceil \frac{n}{2} \rceil}} - \frac{1}{p^{n-1}}.$$

For large p, the p-bound is close to $\frac{1}{p}$ and the new bound close to $\frac{1}{p^{\lceil \frac{n-2}{2} \rceil}}$. For $n > 2$, the new bound is a smaller and closer bound. ∎

2.3 Additional Lower Bounds

The following bounds are derived using the structure of the group rather than the class or degree equation. We count conjugacy classes to find a lower bound for the commutativity degree of nilpotent groups and a lower bound for the commutativity degree of solvable groups. However, this bound for solvable groups is very general. We find a second sharper lower bound on the commutativity degree of solvable groups, which we call the Pyber Lower Bound [41].

2.3.1 A Lower Bound for Nilpotent Groups and Solvable Groups

In 1968 Erdös and Turan [16] proved that $k(G) \geq \log_2(\log_2 |G|)$. Sherman [45] provided a significantly greater lower bound on the commutativity degree of nilpotent groups by proving that $k(G) \geq \log_2 |G|$ for nilpotent groups. We provide a proof of Sherman's result.

Proposition 2.3.1. *If G is a finite nilpotent group of nilpotence class n, then*

$$P(G) \geq \frac{n|G|^{\frac{1}{n}} - n + 1}{|G|}.$$

Proof. Let

$$e = Z_0 \subsetneq Z_1 \subsetneq ... \subsetneq Z_n = G$$

be the upper central series of G. Then G is the disjoint union

$$G = Z_0 \cup (Z_1 - Z_0) \cup (Z_2 - Z_1) \cup ... \cup (Z_n - Z_{n-1}).$$

Since each $Z_i \lhd G$ each set $Z_i - Z_{i-1}$ is a disjoint union of conjugacy classes of G.

Let $g \in G$ and suppose $x \in Z_i - Z_{i-1}$ for some i, $1 \leq i \leq n$. Since $Z_i/Z_{i-1} = Z(G/Z_{i-1})$, $xZ_{i-1} \in Z(G/Z_{i-1})$. Then the commutator $x^{-1}g^{-1}xg \in Z_{i-1}$ because Z_{i-1} is the identity of the group G/Z_{i-1}. Hence $g^{-1}xg \in xZ_{i-1}$ and so $[x] \subseteq xZ_{i-1}$. Then

$$|[x]| \leq |xZ_{i-1}| = |Z_{i-1}|.$$

Since $Z_i - Z_{i-1}$ is the disjoint union of conjugacy classes, there are at least

$$\frac{|Z_i| - |Z_{i-1}|}{|Z_{i-1}|} = \frac{|Z_i|}{|Z_{i-1}|} - 1$$

conjugacy classes in $Z_i - Z_{i-1}$.

Counting conjugacy classes (beginning with the center), then applying the arithmetic-geometric mean inequality yields

$$
\begin{aligned}
k(G) &\geq 1 + \sum_{i-1}^{n} \left(\frac{|Z_i|}{|Z_{i-1}|} - 1 \right) \\
&= 1 + \sum_{i-1}^{n} \left(\frac{|Z_i|}{|Z_{i-1}|} \right) - n \\
&= \left(\frac{1}{n} \right) \sum_{i-1}^{n} \left(\frac{|Z_i|}{|Z_{i-1}|} \right) - n + 1 \\
&\geq \left(\prod_{i-1}^{n} \left(\frac{|Z_i|}{|Z_{i-1}|} \right) \right)^{\frac{1}{n}} - n + 1 \\
&= n|G|^{\frac{1}{n}} - n + 1.
\end{aligned}
$$

Then

$$
P(G) \geq \frac{n|G|^{\frac{1}{n}} - n + 1}{|G|}. \tag{2.12}
$$

□

Corollary 2.3.2. *If G is a finite nilpotent group of nilpotence class n, then $k(G) > \log_2 |G|$ and*

$$
P(G) > \frac{\log_2 |G|}{|G|}.
$$

Proof. Consider the function $h(x) = (nx^{\frac{1}{n}} - n + 1) - (\log_2 x)$. Then

$$
\begin{aligned}
h'(x) &= x^{\frac{1}{n}-1} - \frac{1}{x \ln 2} \\
&= \frac{1}{x} \left(x^{\frac{1}{n}} - \frac{1}{\ln 2} \right),
\end{aligned}
\tag{2.13}
$$

and critical points occur at $x_1 = 0$ and $x_2 = \frac{1}{(\ln 2)^n}$. We restrict x to $x > 0$ (we are only concerned with values of x that may equal $|G|$ for some G), so we only need to check for a minimum may only occur at x_2. Since

$$
h'(1) = 1 - \frac{1}{\ln 2} < 0,
$$

$$h'(2^n) = \frac{1}{2^n}\left(2 - \frac{1}{\ln 2}\right) > 0,$$

and $1 < x_2 < 2^n$, it follows that the function has a minimum when $x = x_2$. Then the minimum value of the function is

$$\begin{aligned} h\left(\frac{1}{\ln 2^n}\right) &= n\left(\frac{1}{\ln 2}\right) - n + 1 - n\log_2\left(\frac{1}{\ln 2}\right) \\ &= n\left(\frac{1}{\ln 2} - 1 + \log_2(\ln 2)\right) + 1 \\ &= \frac{n}{\ln 2}\left(1 - \ln 2 + \ln(\ln 2)\right) + 1 \\ &> 0. \end{aligned}$$

Substitute $|G|$ for x and also notice from Equation 2.12 that $k(G) \geq n|G|^{\frac{1}{n}} - n + 1$. Then $0 < h(x) \leq k(G) - \log_2|G|$ for all orders of G. Hence $k(G) > \log_2|G|$ and

$$P(G) > \frac{\log_2|G|}{|G|}.$$

$\square$

A similar proof provides a general lower bound for solvable groups.

Proposition 2.3.3. *If G is solvable of derived length d, the*

$$P(G) \geq \frac{d+1}{|G|}.$$

Proof. The normal series of G is

$$G = G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \rhd \ldots \rhd G^{(d)} = \{e\}$$

with $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for $1 \leq i \leq d$. Then G is the disjoint union

$$G = \{e\} \cup (G^{(d-1)} - G^{(d)}) \cup (G^{(d-2)} - G^{(d-1)}) \cup \ldots \cup (G(0) - (1)).$$

Since each $G^{(i)}$ char $G^{(i-1)}$, $G^{(i)} \lhd G$. Hence each set $G^{(i)} - G^{(i)}$ is a disjoint union of conjugacy classes of G. Then $k(G) \geq d+1$ and

$$P(G) \geq \frac{d+1}{|G|}.$$

$\square$

Although this bound is a rather rough estimate, it does prove to be a sharp bound as illustrated by the following example.

Example 2.3.4. The normal series for D_3 is $D_3 \geq < \rho > \geq \{e\}$, so the derived length of D_3 is 2. Then the bound from Proposition 2.3.3 is $P(D_3) \geq \frac{2+1}{6} = \frac{1}{2}$. By calculations in Section 4.1.2, $P(D_3) = \frac{1}{2}$. ∎

Further, the bound $\frac{d+1}{|G|}$ is a closer bound than the historical bound $\log_2 \log_2 |G|$ in some cases. Table 2.8 tabulates the bounds $\log_2 \log_2 |G|$, $\log_2 |G|$, and $\frac{d+1}{|G|}$ for select groups of small order. Note that the second bound applies only to nilpotent groups and third only to solvable groups. Also, since the derived length of all non-Abelian solvable groups less than order 60 is $d = 2$ (See Proposition 3.3.1), this third bound is $\frac{d+1}{|G|} = \frac{3}{|G|}$ for each of the groups in this table. The bound of $\frac{3}{|G|}$ on the commutativity degree of solvable groups is a closer bound than $\log_2 \log_2 |G|$ for groups of order less than 26.

$\lvert G\rvert$	$\log_2 \log_2 \lvert G\rvert$	$\log_2 \lvert G\rvert$	$\frac{3}{\lvert G\rvert}$
4	0.2500	0.5000	*na*
6	0.2284	0.4308	0.5000
8	0.1981	0.3750	0.3750
9	0.1849	0.3522	0.3333
10	0.1732	0.3322	0.3000
12	0.1535	0.2987	0.2500
14	0.1378	0.2720	0.2143
24	0.0915	0.1910	0.1250
25	0.0886	0.1858	0.1200
26	0.0859	0.1808	0.0385

Table 2.8: **Lower Bounds**

2.3.2 Pyber's Solvable Group Lower Bound

An improved lower bound on the commutativity degree of solvable groups is written in terms of the derived length of a solvable group as discussed in Pyber [41].

Lemma 2.3.5. *For any finite group G, $k(G) \geq |G/G'|$.*

Proof. Since all non-Abelian groups have a representation of degree greater than one, by the degree equation $k(G) > [G : G'] = |G/G'|$. $\square$

Proposition 2.3.6. *If G is a finite solvable group of derived length d, then*

$$k(G) \geq |G|^{\frac{1}{2^d-1}}$$

and

$$P(G) \geq \frac{1}{|G|^{\frac{2^d-2}{2^d-1}}} > \frac{1}{|G|}.$$

Proof. Let the derived series of G be

$$G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \triangleright G^{(d)} = \{e\}$$

with $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for $1 \leq i \leq d$. Since G is solvable, each factor $G^{(i-1)}/G^{(i)}$ is Abelian. By Proposition 3.1.4,

$$\begin{aligned} k(G) &\geq \frac{k(G^{(1)})}{|G/G^{(1)}|} \\ &\geq \frac{k(G^{(i-1)})}{|G/G^{(1)}||G(1)/G^{(2)}|...|G^{(i-2)}/G^{(i-1)}|} \\ &\geq \frac{k(G^{(i-1)})}{|G/G^{(i-1)}|}. \end{aligned} \tag{2.14}$$

By Lemma 2.3.5, $k(G^{(i-1)}) \geq |G^{(i-1)}/G^i|$. Then

$$k(G) \geq \frac{|G^{(i-1)}/G^{(i)}|}{|G/G^{(i-1)}|}.$$

Next we induct on the length of the derived series. For each i,$i = 1, ..d$, we can rewrite $G/G^{(i)}$ as follows

$$\begin{aligned}
|G/G^{(i)}| &= |G/G^{(i-1)}||G^{(i-1)}/G^{(i)}| \\
&\leq |G/G^{(i-1)}|^2 k(G)... \\
&\leq (|G/G^{(i-i)}|)^{2i-1}(k(G)^{2i-1}) \\
&= k(G)^{2^i-1}
\end{aligned}$$

Therefore $|G| = |G/G^{(d)}| \leq k(G)^{2^d-1}$, so

$$k(G) \geq |G|^{\frac{1}{2^d-1}}$$

and

$$P(G) \geq \frac{1}{|G|^{\frac{2^d-2}{2^d-1}}}.$$

□

2.4 Summary of Bounds

In Table 2.9, we summarize the bounds discussed in this chapter. In the table, p is the smallest prime dividing $|G/Z|$, $l = [G : Z]$, $C_G(x)$ is a maximal noncentral centralizer in G, d is the smallest degree of a non-linear representation of G, d_1 is the derived length of G, and $k \in \mathbb{Z}$ such that $|G/Z(G)| = p^k$.

UPPER BOUNDS			
Bound Name	Bound	Restrictions	Method
$\frac{5}{8}-$bound	$\frac{5}{8}$		Class Eqn.
p-bound	$\frac{p^2+p-1}{p^3}$		Class Eqn.
Alt. p-bound	$\frac{(p-1)+pl}{p^2 l}$		Class Eqn.
lp-bound	$\frac{l+p-1}{pl}$		Class Eqn.
l-bound	$\frac{l+1}{2l}$		Class Eqn.
centralizer bound (1)	$\frac{3}{2[G:C_G(x)]}$	$Z(G) \neq \{e\}$	Class Eqn.
centralizer bound (2)	$\frac{1}{[G:C_G(x)]}$	$Z(G) = \{e\}$	Class Eqn.
deg. eqn. bound	$\frac{1}{4}(1+\frac{3}{\vert G'\vert})$		Deg. Eqn.
min. dim. deg. eqn. bound	$\frac{1}{d^2}+(1-\frac{1}{d^2})\frac{1}{\vert G'\vert}$		Deg. Eqn.
solvable group bound	$\frac{4d_1-7}{2^{d_1+1}}$	$d_1 \geq 2$ G solvable	Deg. Eqn.
$p-$group bound	$\frac{p^{d_1}+p^{d_1-1}-1}{p^{2d_1-1}}$	$d_1 \geq 4$ G a p-group	Deg. Eqn.

LOWER BOUNDS			
Bound Name	Bound	Restrictions	Source
generic bound	$\frac{2l-1}{l^2}$		Definitions
p-bound	$\frac{1}{p^{k-1}}+\frac{1}{p^k}-\frac{1}{p^{2k-1}}$		Class Eqn.
lp-bound	$\frac{lp+l-p}{l^2}$		Class Eqn.
l-bound	$\frac{3l-2}{l^2}$		Class Eqn.
deg. eqn. bound	$\frac{1}{\vert G'\vert}$		Deg. Eqn.
solvable group bound	$\frac{d_1+1}{\vert G\vert}$	G solvable	Counting Classes
nilpotent group bound	$\log_2 \vert G\vert$	G nilpotent	Counting Classes
Pyber bound	$\frac{1}{\vert G\vert^{\frac{2d_1-2}{2d_1-1}}}$	G solvable	Counting Classes

Table 2.9: **Bounds**

Chapter 3: Structural Results

In Section 2.3 we derived several bounds on the commutativity degree of nilpotent and solvable groups. Can we determine more information about the commutativity degree of a group if the group is nilpotent or solvable? What about if the group has a normal subgroup or if we can write the composition series of the group? In this chapter, we develop restrictions on the commutativity degree of a group resulting from the structure of the group.

First we discuss bounds on the commutativity degree of a group in terms of the commutativity degree of a subgroup or normal subgroup of the group. Let $H \leq G$ and $N \lhd G$. We show that

$$\frac{1}{[G:H]^2} P(H) \leq P(G) \leq P(H),$$

and

$$P(G) \leq P(N)P(G/N).$$

Then we describe conditions required for the bounds to be realized. These bounds, first given by Gallagher [20], are used by a number of authors to prove multiple results. To demonstrate the usefulness of these bounds, we apply these bounds to find additional bounds on the commutativity degree in terms of a type of subgroup called a section, in terms of the factors of the composition series of a group, and in terms of the factors of a semidirect product.

In the remaining two sections, we discuss commutativity degree in terms of nilpotence and solvability. In Section 3.2 we show that if the commutativity degree of a group is greater than $\frac{1}{2}$, then the group is nilpotent with a specific structure. If the commutativity degree of the group is greater than $\frac{1}{12}$, then the group is solvable. In

Section 3.3, we describe groups of commutativity degree greater than or equal to $\frac{1}{12}$ in terms of the solvability of the group.

3.1 Subgroups and Normal subgroups

Let G be a group, $H \leq G$, and $N \lhd G$. In this section, we find bounds on $P(G)$ in terms of $P(H)$ and $P(N)$ and then show the cases when $P(G) = P(H)$. We will also use these bounds to find additional bounds on specific types of groups. Most of the propositions in this section appear or are referenced in multiple sources, each crediting Gallagher [20] for the original result.

Proposition 3.1.1. *Let H be a subgroup of G. Then*

$$\frac{P(H)}{[G:H]^2} \leq P(G) \leq P(H).$$

Proof. Let $x \in G$. Then $H \cap C_G(x) = C_H(x)$ and $C_H(x) \leq C_G(x)$. Suppose $|C_G(x)/C_H(x)| = m$ and let the distinct coset representatives of $C_H(x)$ in $C_G(x)$ be $\{g_i : 1 \leq i \leq m\}$. Consider the cosets g_iH and g_jH in G. We claim that g_iH and g_jH are distinct cosets of H in G for $i \neq j$. Suppose not. Then $g_i = g_jh$ for some $h \in H$. Then

$$h = g_j^{-1}g_i \in C_G(x) \cap H = C_H(x),$$

and this implies $g_iC_H(x) = g_jC_H(x)$, a contradiction. Hence each distinct coset of $C_H(x)$ is also a distinct coset of H in G. Therefore $[C_G(x) : C_H(x)] \leq [G : H]$.

Next, by Lagrange's Theorem,

$$|C_G(x)| = [C_G(x) : C_H(x)]|C_H(x)|.$$

Then

$$|C_G(x)| \leq [G : H]|C_H(x)|. \tag{3.1}$$

Summing over G yields

$$\sum_{x\in G}|C_G(x)| \leq [G:H]\sum_{x\in G}|C_H(x)|. \tag{3.2}$$

If $x \in G$ and $y \in C_H(x)$ then $yxy^{-1} = x$ and $y \in H$. In this case, $xyx^{-1} = y$ so $x \in C_G(y)$ and we may rewrite the righthand side of Equation 3.2 as follows:

$$\sum_{x\in G}|C_G(x)| \leq [G:H]\sum_{y\in H}|C_G(y)|. \tag{3.3}$$

Applying Equation 3.1 to Equation 3.3 yields

$$\sum_{x\in G}|C_G(x)| \leq [G:H]^2\sum_{y\in H}|C_H(y)|. \tag{3.4}$$

Finally,

$$\begin{aligned} P(G) &= \frac{\sum_{x\in G}|C_G(x)|}{|G|^2} \\ &\leq \frac{[G:H]^2\sum_{y\in H}|C_H(y)|}{|G|^2} \\ &= \frac{[G:H]^2\sum_{y\in H}|C_H(y)|}{[G:H]^2|H|^2} \\ &= P(H). \end{aligned} \tag{3.5}$$

Therefore $P(G) \leq P(H)$.

For each $x \in G$, $|C_H(x)| = |C_G(x)|$ or $C_G(x)$ contains no elements of H. Thus $|C_H(x)| \leq |C_G(x)|$ and

$$\begin{aligned} P(G) &= \frac{\sum_{x\in G}|C_G(x)|}{|G|^2} \\ &\geq \frac{\sum_{x\in G}|C_H(x)|}{|G|^2} \\ &\geq \frac{\sum_{x\in H}|C_H(x)|}{|G|^2} \\ &= \frac{|H|^2\sum_{x\in H}|C_H(x)|}{|H|^2|G|^2} \\ &= \frac{1}{[G:H]^2}P(H). \end{aligned}$$

Therefore, $\frac{P(H)}{[G:H]^2} \leq P(G)$. $\square$

Corollary 3.1.2. *If H is Abelian then $P(G) \geq \frac{1}{[G:H]^2}$.*

Proof. By Proposition 3.1.1,

$$P(G) \geq \frac{P(H)}{[G:H]^2} = \frac{1}{[G:H]^2}.$$

□

We will compare the lower bound of $\frac{1}{|G'|}$ from Proposition 2.2.1 to the bound $\frac{1}{[G:H]^2}$ from Corollary 3.1.2. If H is the largest Abelian subgroup of G, $[G:H]$ is the smallest index of an Abelian subgroup of G, and $\frac{1}{[G:H]^2}$ the largest possible lower bound of the form $\frac{1}{[G:H]^2}$. Recall G' is the smallest subgroup such that G/G' is Abelian. Hence if $|G'|$ is small relative to $|G|$, then the bound $\frac{1}{|G'|}$ is a larger bound, and if $|G'|$ is large relative to $|G|$, then the bound $\frac{1}{[G:H]}$ becomes a better bound. To illustrate this, we will use the dihedral groups.

Example 3.1.3. Let $n > 4$ and consider the dihedral group

$$D_n = < r, \rho : r^2 = \rho^n = e, \rho r = r\rho^{n-1} > .$$

Let $H = < \rho >$. Then $H \leq D_n$, H is cyclic of order n, and $[D_n : H] = 2$. By Corollary 3.1.2

$$P(G) \geq \frac{1}{[D_n:H]^2} = \frac{1}{4}.$$

This is an appropriate lower bound for the dihedral groups because the asymptotic commutativity degree of the dihedral groups is

$$\lim_{n\to\infty} P(D_n) = \frac{1}{4},$$

which we will discuss and calculate in Section 4.1.2.

Since $D_n' = < \rho >$ as well, the bound $\frac{1}{|G'|}$ is

$$P(D_n) \geq \frac{1}{D_n'} = \frac{1}{n}.$$

For $n > 4$, $\frac{1}{n} < \frac{1}{4}$. Further, as $n \to \infty$ this bound approaches zero. notice that the bound $\frac{1}{[G:H]^2}$ is a closer bound and $|G'| = \frac{|D_n|}{2}$ is large relative to $|G|$.∎

Next we address a normal subgroup $N \lhd G$. We will find a lower and upper bound on $P(G)$ in terms of $P(N)$. Using the upper bound, we will also describe the case when $P(G) = P(N)$ and then prove several corollaries.

Proposition 3.1.4. *If $N \lhd G$ then $k(G) \geq k(N)/|G/N|$ and $P(G) \geq P(N)/|G/N|^2$.*

Proof. Let $N \lhd G$, $g \in G$ and $n \in N$. Then $gng^{-1} \in N$, and the conjugacy classes of G are partitioned into those contained in N and those disjoint from N. However, two elements contained in N may be conjugates in G but not in N.

Suppose n_1, n_2 are conjugate in G and $n_1N = n_2N$. In G, $[gn_1g^{-1}] = [gn_2g^{-1}]$. Hence the number of G-conjugacy classes in N is at least $\frac{k(N)}{|G/N|}$, and then

$$k(G) \geq \frac{k(N)}{|G/N|}.$$

Thus

$$\begin{aligned} P(G) &\geq \frac{k(N)}{|G||G/N|} \\ &= \frac{k(N)}{|N|}\frac{|N|^2}{|G|^2} \\ &= \frac{P(N)}{[G:N]^2}. \end{aligned}$$

□

Proposition 3.1.5. *Let $N \lhd G$. Then $P(G) \leq P(G/N)P(N)$. Equality holds if and only if $C_G(x)N = C_G(xN)$ for each $x \in G$ and in the case of equality G/N is Abelian.*

Proof. Let $x \in G$. By the diamond isomorphism theorem,

$$\frac{C_G(x)}{C_N(x)} \cong \frac{C_G(x)N}{N} \subseteq C_{G/N}(xN)$$

Then

$$|C_G(x)| \leq |C_{G/N}(xN)| \cdot |C_N(x)| \tag{3.6}$$

In the case of $C_G(x)N = C_G(xN)$,

$$\frac{C_G(x)}{C_N(x)} \cong \frac{C_G(x)N}{N} = \frac{C_G(xN)}{N} = C_{G/N}(xN)$$

and equality holds in Equation 3.6.

Summing over G yields

$$\sum_{x\in G} |C_G(x)| \leq \sum_{x\in G} |C_{G/N}(xN)| \cdot |C_N(x)|. \tag{3.7}$$

Let $\{x_iN : 1 \leq i \leq r\}$ be the set of distinct cosets of N. Since the disjoint union $\bigcup_{i=1}^{r} x_iN = G$, summing over G is equivalent to summing over each coset, then summing over the set of cosets. That is,

$$\sum_{x\in G} |C_G(x)| \leq \sum_{xN\in G/N} |C_{G/N}(xN)| \sum_{y\in xN} |C_N(y)|. \tag{3.8}$$

If $y \in xN$ and $z \in C_N(y)$, then $zyz^{-1} = y$ and $z \in N$. Equivalently, $yzy^{-1} = z$, so $y \in C_N(z) \cap xN = C_{xN}(z)$. Hence we may rewrite the last term of Equation 3.8 as follows:

$$\sum_{x\in G} |C_G(x)| \leq \sum_{xN\in G/N} |C_{G/N}(xN)| \sum_{z\in N} |C_{xN}(z)| \tag{3.9}$$

with equality holding the case of $C_G(x)N = C_G(xN)$ in Equations 3.7, 3.8, and 3.9.

Next let $z \in N$ and let xN be a coset of N. Define $C_{xN}(y)$ to be the set of elements in the set xN that commute with y. Suppose that $C_{xN}(z) \neq \emptyset$ and let $a \in C_{xN}(z)$. Then $a = xn$ for some $n \in N$, so $xnz(xn)^{-1} = z$. Then $nzn^{-1} = x^{-1}zx$ and $x^{-1}zx$ is conjugate to z in N. Then $nzn^{-1} \in [z]$ so n is in a coset of z in N. Hence $C_{xN}(z)$ is a coset of $C_N(z)$ so that

$$|C_{xN}(z)| = \begin{cases} |C_N(z)| & \text{if } C_{xN} \neq \emptyset \\ 0 & \text{if } C_{xN}(z) = \emptyset. \end{cases}$$

Thus

$$|C_{xN}(z)| \leq |C_N(z)|. \tag{3.10}$$

Notice that $C_{xN} \neq \emptyset$ for all $x \in G$ when $C_G(x)N = C_G(xN)$ and in this case equality holds in Equation 3.10.

Rewriting the righthand side of Equation 3.9, we conclude that

$$\sum_{x \in G} |C_G(x)| \leq \sum_{xN \in G/N} |C_{G/N}(xN)| \sum_{z \in N} |C_N(z)| \tag{3.11}$$

with equality holding the case of $C_G(x)N = C_G(xN)$.

Finally, we apply the definition of commutativity degree:

$$\begin{aligned} P(G) &= \frac{\sum_{x \in G} |C_G(x)|}{|G|^2} \\ &\leq \frac{\sum_{xN \in G/N} |C_{G/N}(xN)| \sum_{z \in N} |C_N(z)|}{[G:N]^2 |N|^2} \\ &= P(G/N)P(N) \end{aligned}$$

with equality holding the case of $C_G(x)N = C_G(xN)$. Also, if $P(G) = P(G/N)P(N)$, then $P(G/N) = 1$ because $P(G) \leq P(N)$ by Proposition 3.1.1. Hence G/N is Abelian in the case when $C_G(x)N = C_G(xN)$. □

We know that the commutativity degree of a group is less than or equal to the commutativity degree of any of its subgroups. Next we describe the case of equality in the commutativity degree of a group and subgroup. Then we address the commutativity degree of a homomorphic image of a subgroup.

Corollary 3.1.6. *If $H \leq G$ and $P(G) = P(H)$ then $H \lhd G$ and G/H is Abelian.*

Proof. Suppose that $P(G) = P(H)$. Then equality holds through Equation 3.5 in Proposition 3.1.1, and in particular

$$\frac{\sum_{x \in G} |C_G(x)|}{|G|^2} = \frac{[G:H]^2 \sum_{y \in H} |C_H(y)|}{|G|^2}.$$

Hence

$$\sum_{x\in G}|C_G(x)| = [G:H]^2 \sum_{y\in H}|C_H(y)|, \tag{3.12}$$

$$\sum_{x\in G}|C_G(x)| = [G:H] \sum_{y\in H}|C_G(y)|, \tag{3.13}$$

and

$$\sum_{x\in G}|C_G(x)| = [G:H] \sum_{x\in G}|C_H(x)|. \tag{3.14}$$

Note that Equations 3.12, 3.13, and 3.14 are the analogous cases of equality to Equations 3.4, 3.3, and 3.2 respectively. Equality must hold in each term of 3.14 (because if one term was a strict inequality, the entire sum would be a strict inequality). Hence for all $x \in G$

$$|C_G(x)| = [G:H]|C_H(x)|.$$

Then for each $x \in H$,

$$\begin{aligned} |C_H(x)H| &= \frac{|C_G(x)||H|}{|C_H(x)|} \\ &= \frac{[G:H]|C_H(x)||H|}{|C_H(x)|} \\ &= |G|. \end{aligned}$$

Next let $g \in G$ and $x \in H$. Then $g = ch$ for some $c \in C_G(x)$ and $h \in H$. Then

$$g^{-1}xg = h^{-1}c^{-1}xch = h^{-1}xh \in H.$$

Hence $H \lhd G$.

Next, since $H \lhd G$, $P(G) \leq P(H)P(G/H)$ by Proposition 3.1.5. Then $P(G/H) = 1$ because $P(G) = P(H)$. □

Notice that if $G = H \times C$ for an Abelian subgroup C, then $P(G) = P(H)$ trivially. Next we provide a nontrivial example of the equality $P(G) = P(H)$ for some subgroup H of G.

Example 3.1.7. Consider the class of groups

$$G_m =< a, b : a^3 = b^{2^m} = 1, bab^{-1} = a^{-1} >$$

for $m \in \mathbb{N}$. Notice that

$$G_1 =< a, b : a^3 = b^2 = 1, , bab^{-1} = a^{-1} >\cong S_3.$$

We will discuss properties of G_m groups in Section 4.1.2. We show that the commutativity degree is $P(G_m) = \frac{1}{2}$ for each G_m group. We also find that, for $m > 1$, $G_1 \leq G_m$. By comparing the relations of G_1 to S_3, it is easy to see that $G_1 \cong S_3$. Hence $P(S_3) = P(G_m) = \frac{1}{2}$ and $S_3 \cong G_1 \leq G_m$ for $m > 1$.

This example generalizes to $G_m \leq G_r$ with $P(G_m) = P(G_r)$ if there is some $n \in \mathbb{N}$ such that $r = n^m$. ∎

Several additional bounds result from the bound $P(G) \leq P(N)P(G/N)$ for $N \lhd G$. Let $Y \leq G$, let $Z \lhd Y$, and let $X = Y/Z$. Then X is called a section of G. The next bound is written in terms of a section of G.

Corollary 3.1.8. *Let X be a section of G. Then $P(G) \leq P(X)$.*

Proof. Since $Y \leq G$, $P(G) \leq P(Y)$. Also, since $Z \lhd Y$, $P(G) \leq P(Y/Z)$. Therefore, $P(G) \leq P(Y/Z) = P(X)$. □

Now we apply Proposition 3.1.5 to the composition series of a group to find another upper bound.

Corollary 3.1.9. *Let G be a group and let the set $\{S_i : 1 \leq i \leq t\}$ denote the non-Abelian composition factors of G. Then $P(G) \leq \Pi_{i=1}^t P(S_i)$.*

Proof. Since G is finite, there exists a normal series

$$G \rhd N_1 \rhd N_2 \rhd \ldots \rhd N_r = \{e\}$$

with $r \geq t$. By Proposition 3.1.5, $P(G) \leq P(N_1)P(G/N_1)$. Likewise, for $i \leq r$, $P(N_{i-1}) = P(N_i)P(N_{i-1}/N_i)$. Hence

$$P(G) \leq P(G/N_1)P(N_1/N_2)...P(N_{r-1}/N_r).$$

Let $S_1, S_2, ..S_t$ be the non-Abelian factors. Since $P(N_{i-1}/N_i) = 1$ when N_{i-1}/N_i is Abelian,

$$P(G) \leq \Pi_{i=1}^t P(S_i).$$

□

Next we apply Proposition 3.1.5 to semidirect products. To construct an external direct product, let N and H be groups such that there is a homomorphism $\phi : H \to Aut(N)$. Then the external semidirect product is the set $N \times H$ with multiplication defined by

$$(n_1, h_1)(n_2, h_2) = (n_1\phi(h_1)(n_2), h_1h_2)$$

and is denoted by $G = N \rtimes H$.

A group G is called an internal semidirect product if $N, H \leq G$ and the following conditions hold: $N \lhd G$, $N \cap H = \{e\}$, and $NH = HN = G$. Then there is a homomorphism $\phi : G \to Aut(N)$ satisfying $\phi(h)(n) = hnh^{-1}$ for all $h \in H$, and $G \cong N \rtimes H$.

Corollary 3.1.10. *Let $G = H \rtimes K$. Then $P(G) \leq P(H)P(K)$.*

Proof. Since $H \lhd G$, by Proposition 3.1.5 $P(G) \leq P(G/H)P(H)$. Then since $K \subset G/H$,

$$P(G) \leq P(K)P(H).$$

□

Next we extend this result to a specific type of semidirect product, the wreath product. Let H be a group and let $N = H \times H \times ... \times H$ be the direct product of n copies of H. Let $A \leq S_n$. Let $\sigma \in A$ and suppose there is homomorphism $\phi : G \longrightarrow Aut(N)$ so that

$$\phi(\sigma) \cdot (h_1, h_2, ...h_n) = (h_{\sigma(1)}, h_{\sigma(2)}, ...h_{\sigma(n)}).$$

Then G is the semidirect product of N by A with multiplication as follows: Let $(n, \sigma) \in G$ and $(n', \sigma') \in G$. Then

$$(n, \sigma)(n', \sigma') = (n\sigma \cdot n', \sigma\sigma').$$

Then G is called the wreath product of N by A with respects to n and is denoted by $G = N$ Wr A. Notice that $|G| = |N||A| = |H|^n|A|$.

Corollary 3.1.11. *Let $N = \prod_{i=1}^{n} H$ and let G be the wreath product of N by K with respects to n. Then $P(G) \leq P(H)^n P(K)$.*

Proof. Since $G = N \rtimes K$, $N \lhd G$. Then $P(G) \leq P(N)P(K) = P(H)^n P(K)$. $\square$

It is not surprising that we only find a very general bound on the commutativity degree of a semidirect product using these direct methods. Finding the number of conjugacy classes, that is; the number of irreducible characters, of a semidirect product is not an elementary question. In fact, it was not until 2004 that this question was partially answered as the solution to the "$k(GV)$ problem".

The "$k(GV)$-problem" was a conjecture that the number of distinct conjugacy classes of a specific type of semidirect product, denoted GV , is bounded by the order of the subgroup V. This means the commutativity degree is bounded by

$$P(GV) \leq \frac{|V|}{|GV|} = \frac{1}{|G|}.$$

The semidirect product in the conjecture is restricted to the case when an elementary Abelian p-group V acts faithfully and irreducibly on a group $G \leq GL(V)$. (G is called a p'-group.) Knörr [29] proposed the "$k(GV)$-problem" in 1984 and used a type of character called a generalized character to verify the existence of a bound on a subclass of the semidirect products in question. Knörr's "$k(GV)$-problem" was a specific case of a series of more general questions stated by Brauer in the 1960s.

In 2004 Gluck, Magaard, Reise, and Schmid published the solution to the "$k(GV)$-problem", answering Knörr's question affirmatively. Although the "$k(GV)$-problem" was only a specific case of Brauer's questions, the solution to Knörr's problem is significant in itself. As Knörr noted when he stated the problem, "an affirmative answer would give information on all faithful (and irreducible) representations of finite groups over nearly all finite fields ...". ([29], 181). Notice that this solution, which took twenty years to complete, only addresses bounds on the commutativity degree of certain types of semidirect products.

3.2 Nilpotent Groups

In this section we show that $\frac{1}{2}$ is an upper bound on the commutativity degree of non-nilpotent groups. Then we show that this is the least upper bound by describing the class of non-nilpotent groups having commutativity degree $\frac{1}{2}$. We also describe the structure of groups with commutativity degree greater than $\frac{1}{2}$. Much of this section appears in Lescot [31], [32], [33], and [34].

Lemma 3.2.1. *Let G be a finite group. If $|G'| = 2$, then $G' \subseteq Z(G)$.*

Proof. Let $G' = \{e, a\}$. Suppose $a \notin Z(G)$. Let $b \in G$ such that $aba^{-1} = x$ with $x \neq b$. Then

$$aba^{-1}b^{-1} = xb^{-1} \in G',$$

so $xb^{-1} \in \{e, a\}$. If $xb^{-1} = e$, then a commutes with b, a contradiction. If $xb^{-1} = a$, then $aba^{-1}b^{-1} = a$. This implies $bab^{-1} = e$ and it follows that $a = e$, another contradiction. Therefore $a \in Z(G)$ and $G' \subseteq Z(G)$. □

Proposition 3.2.2. *(Lescot [32]). (1.) If G is a group such that $P(G) > \frac{1}{2}$, then $|G'| < 3$.*

(2.) If G is a group such that $P(G) > \frac{1}{2}$, then G is nilpotent.

Proof. (1.) Suppose $P(G) > \frac{1}{2}$. By the upper degree equation bound on $P(G)$,

$$\frac{1}{2} < P(G) \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right)$$

Solving this equation for $|G'|$ yields $|G'| < 3$. This establishes (1).

(2.) If $|G'| = 1$ then $G' = \{e\}$. Hence G is Abelian and nilpotent.

Assume $|G'| = 2$. By Lemma 3.2.1 $G' \subseteq Z(G)$. Hence $G/Z(G)$ is Abelian, so the upper central series terminates after two terms and G is nilpotent of nilpotence class 2. This establishes (2). □

Proposition 3.2.3. *If G is not nilpotent and $P(G) = \frac{1}{2}$, then*

1. *$G/Z(G) \cong S_3$.*
2. *If the order of $Z(G)$ is odd, then G has a subgroup $H \cong S_3$ such that $G = H \times Z(G)$.*

Proof. 1. First let $H = G/Z(G)$. Then by Proposition 3.1.5,

$$P(G) \leq P(Z(G))P(H) = P(H).$$

Thus $P(H) \geq \frac{1}{2}$. If $P(H) > \frac{1}{2}$ then G is nilpotent.

Suppose that $P(H) = \frac{1}{2}$. Then equality holds in the result of Proposition 3.1.5. Let $g \in G$. Then since $Z(G) \subseteq C_G(x)$,

$$C_H(gZ(G)) = C_G(g)Z(G)/Z(G) = C_G(g)/Z(G).$$

If $gZ(G) \in Z(H)$, then

$$C_H(gZ(G)) = H = C_G(g)/Z(G).$$

It follows that $G = C_G(g)$ and $g \in Z(G)$. Since $Z(G)$ is the identity element in H, $Z(H) = e$.

Next we will consider two cases. In the first case, suppose that $|H| < 10$. Observe from Table 3.1 that $H \cong S_3$ since $P(H) = \frac{1}{2}$.

H	$P(G)$
$\mathbb{Z}_n$ (the cyclic group of order n)	1
V_4	$\frac{5}{8}$
S_3	$\frac{1}{2}$
D_4	$\frac{5}{8}$
Q_8	$\frac{5}{8}$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1

Table 3.1: **Commutativity Degrees for Groups of Order Less than 10**

In the second case, suppose that $|H| \geq 10$. By way of contradiction, let $n = |H|$, let $m = |\{h \in H : |C_H(h)| = \frac{n}{2}\}|$, and let $H_0 = \{h \in H : [H : C_H(h)] \geq 3\}$. Then

$$P(H) = \frac{1}{2} = \frac{|H|^2}{2|H|^2} = \frac{n^2}{2|H|^2}.$$

Hence

$$\begin{aligned}
\frac{n^2}{2} &= |H|^2 P(H) \\
\frac{n^2}{2} &= \sum_{h \in H} C_G(h) \\
\frac{n^2}{2} &\leq n + m(\frac{n}{2}) + (n - m - 1)(\frac{n}{3}) \\
\frac{n^2}{2} &\leq \frac{n^2}{3} + \frac{mn}{6} + \frac{2n}{3} \\
\frac{n^2}{6} &\leq (\frac{n}{6})(m + 4).
\end{aligned}$$

Hence $m \geq n-4$ and this means that there are at least $n-4$ elements of H with conjugacy class of order 2, $\{h_i : |[h_i]| = 2, 1 \leq i \leq n-4\}$. Let $T = \bigcap_{i=1}^{n-4} C_H(h_i)$. Then

$$C_H(T) \supset \{h_i : |[h_i]| = 2, 1 \leq i \leq n-4\}.$$

It follows that

$$|C_H(T)| \geq n-4 \geq \frac{n}{2}+1$$

because $|H| \geq 10$. Thus $[H : C_H(T)] < 2$ so $C_H(T) = H$. Hence $T \subseteq Z(H) = \{e\}$.

As $C_H(h_1) \neq \{e\}$, let j be the smallest positive integer such that

$$\bigcap_{i=1}^{j} C_H(h_i) = \{e\}$$

and let

$$A = \bigcap_{i=1}^{j-1} C_H(h_i)$$

so that $A \cap C_H(h_j) = \{e\}$. Then $|A||C_H(h_j)| \leq |H|$ and so $|A| \leq 2$. By construction, $A \neq \{e\}$, hence $|A| = 2$. Also, since A is the intersection of normal subgroups, $A \lhd H$. Then $A \subseteq Z(H)$, a contradiction. Hence $|H| < 10$, and so $H \cong S_3$.

2. Suppose $P(G) \geq \frac{1}{2}$ and that G is not nilpotent. By the upper degree equation bound,

$$\frac{1}{2} \leq P(G) \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right).$$

Again, solving for $|G'|$ yields $|G'| \leq 3$. Further, $|G'| = 3$ because G is not nilpotent. Let $< \sigma >= G'$. Notice that equality holds in the result of Proposition 3.1.5.

Thus

$$< \sigma Z(G) >= C_{G/Z(G)}(\sigma Z(G)) \cong C_G(\sigma)Z(G)/Z(G).$$

Let $\tau \in G$ be an element of order 2 in G. Then $\tau \notin Z(G)$ because the order of the center is odd. Therefore $\tau Z(G)$ has order 2 in $G/Z(G)$. Next, since G is not nilpotent, $G' \cap Z(G) = \{e\}$. Then

$$\tau \notin Z(G) \cap \sigma Z(G) \cap \sigma^2 Z(G) = C_G(\sigma).$$

Since $<\sigma> \lhd G$, $\tau\sigma\tau^{-1}\sigma^{-1} = \sigma^y \in G'$ for some integer y. However τ and σ do not commute, and neither element is the identity, so $\tau\sigma\tau^{-1} = \sigma^2$. Then $<\sigma, \tau> \cong S_3$ and we conclude that $G =<\sigma, \tau> Z(G)$. Further, $<\sigma, \tau> \cap Z(G) = \{e\}$ since $\tau \notin Z(G)$ and $\sigma \notin Z(G)$. Also, $<\sigma> \lhd G$ implies $<\sigma, \tau> \lhd G$. Therefore,

$$G =<\sigma, \tau> \times Z(G) \cong S_3 \times Z(G).$$

□

Proposition 3.2.4. *If G is a non-Abelian group with $P(G) > \frac{1}{2}$ then $G = P_0 \times P_1 \times P_2 \times .. \times P_k$ where P_0 is a 2-group with $|P_0'| = 2$ and P_i an Abelian p_i-Group for $i > 0$, for some prime $p_i \neq 2$.*

Proof. Let G be a non-Abelian group with $P(G) > \frac{1}{2}$. By Proposition 3.2.2 (2), G is nilpotent because $P(G) > \frac{1}{2}$. Then G is a direct product of p-Sylow subgroups. Observe from Table 2.4 that, for $P(G) > \frac{1}{2}$, G has exactly one non-Abelian factor and this factor must be a 2-Sylow subgroup. Let the 2-Sylow subgroup factor have order 2^n. Then $G = P_0 \times P_1 \times P_2 \times ... \times P_s$ with all P_i, $1 \leq i \leq s$, Abelian factors.

Next, by Proposition 3.2.2 (1), $|G'| = 2$. Since $P_0' \subseteq G'$, and P_0 is non-Abelian, $|P_0'| = 2$. □

3.3 Solvable Groups

In this section, we provide two upper bounds on the commutativity degree of non-solvable groups. Then we show that the lower of these two bounds, $\frac{1}{12}$, is realized by the class of non-solvable groups of the form $A_5 \times C$, where C is Abelian. We conclude by describing all groups with commutativity degree in terms of the solvability of the group. Much of this section is based on Lescot's results in [31], [32],[33], and [34].

Lemma 3.3.1. *If $|G'| < 60$, then G is solvable.*

Proof. Induct on $|G'|$. Suppose $|G'| = 1$. Then G is Abelian, and hence solvable. Suppose $|G'| < 60$. Case by case application of the Sylow Theorems shows that G' has a normal p-Sylow subgroup P. If $P = G'$, then G' is a p-group. Then G' is nilpotent and solvable. If $P \neq G$ then $P \lhd G'$. Then there is a normal series

$$\{e\} \lhd P \lhd G'$$

with Abelian factors and G' is solvable.

Since G' is solvable and G/G' is Abelian, G is solvable. □

Proposition 3.3.2. *If $P(G) > \frac{21}{80}$, then G is solvable.*

Proof. Let $P(G) > \frac{21}{80}$. By the upper degree equation bound,

$$\frac{21}{80} < P(G) \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right).$$

Solving for $|G'|$ yields $|G'| < 60$. Therefore, G is solvable by Lemma 3.3.1. □

Proposition 3.3.2 provides a loose upper bound on the commutativity degree of non-solvable groups. Using the classification of the classes of groups that embed in a degree 2 or 3 representation, we will find a smaller upper bound of $\frac{1}{12}$. Then we will show that $\frac{1}{12}$ is the least upper bound by finding a non-Abelian simple group with commutativity degree $\frac{1}{12}$.

Classification of these groups was first addressed in the 1890s when Jordan proved that if $G \leq GL_n(\mathbb{C})$, then there is a function $f(n)$ such that G has a proper normal Abelian subgroup, M, with index less than $f(n)$. Historically, the known such $f(n)$ functions provided upper bounds on the index of a proper normal subgroup M but were too large for meaningful application. In 2007, Collins [10] published a significantly improved bound of $(n+1)!$ on the index of a proper normal Abelian subgroup of G for a group G with a faithful representation of degree $n > 71$. Collins credited much of his work to the unpublished bounds developed by Weisfeiler in the mid 1980s.

Collins also tabulated the index of a maximum proper normal Abelian subgroup for select n. For $n = 2$, and $n = 3$, he showed that the maximal possible normal Abelian subgroup has index at most 60 and 360, respectively.

To obtain the $\frac{1}{12}$ bound, we will want to identify all simple groups that embed in $GL_2(\mathbb{C})$ or $GL_3(\mathbb{C})$. Suppose that G is a simple group embedding in $GL_2(\mathbb{C})$ or $GL_3(\mathbb{C})$. Then by Collins [10], G has a proper normal subgroup M such that $[G : M] < 360$. Since G is simple, $M = \{e\}$ and $|G| < 360$. Hence if G is a simple group with an image embedding in $GL_2(\mathbb{C})$ or $GL_3(\mathbb{C})$, $|G| < 360$. Then we will want to find all simple groups of order less than 360. In 1893, Cole [8] and [9] classified all simple groups of order less than or equal to 660 using the Sylow Theorems and similar methods. His results include one simple group of order 60, one simple group of order 168, and no other non-Abelian simple groups of order less than 360. The groups of order 60 is A_5 and the group of order 168 is $PSL_2(\mathbb{F}_7)$.

Alternately, we could use more recent classification results to find the relevant simple groups with images that embed in $GL_2(\mathbb{C})$ or in $GL_3(\mathbb{C})$. In 1970, Dixon [13] asked the same question by proposing a problem which required finding all non-Abelian simple groups with $P(G) \geq \frac{1}{12}$. He published a solution to his own problem in [14] by showing that such a group must embed in $GL_2(\mathbb{C})$ or $GL_3(\mathbb{C})$, then appealing to the classification of groups with representation of degree $n = 2$ or $n = 3$ by Blichfeldt [7]. Dixon used the following theorem:

Theorem 3.3.3. *(Blichfeldt) If G is a finite simple non-Abelian subgroup of $GL_2(\mathbb{C})$ or $GL_3(\mathbb{C})$, then $G \cong A_5$, $G \cong A_6$, or $G \cong PSL_2(\mathbb{F}_7)$.*

The commutativity degree of all candidate groups from either method are: $P(A_5) = \frac{1}{12}$, $P(A_6) = \frac{7}{360}$ and $P(PSL_2(\mathbb{F}_7)) = \frac{1}{28}$. (Note that $|A_6| = 360$.) We calculated these values in GAP. Since $P(PSL_2(\mathbb{F}_7)) < \frac{1}{12}$ and $P(A_6) < \frac{1}{12}$, the only non-Abelian simple group with $P(G) \geq \frac{1}{12}$ is A_5.

Proposition 3.3.4. *If $P(G) > \frac{1}{12}$ then G is solvable. Further, the only simple non-Abelian group with commutativity degree $\frac{1}{12}$ is A_5.*

Proof. Suppose G is not solvable. We will show that $P(G) \leq \frac{1}{12}$.

Since G is finite, G has a normal series

$$G = G_0 \rhd G_1 \rhd ... \rhd G_r = \{e\}$$

with each G_i, $1 \leq i \leq r$ maximal normal in G_{i-1}. Then for some j, $1 \leq j \leq r$, G_j/G_{j+1} is a non-Abelian factor because G is not solvable. Since G_{j+1} is maximal in G_j, G_j/G_{j+1} is simple by the correspondence theorem.

By Proposition 3.1.5,

$$P(G) \leq \Pi_{i=1}^{r-1} P(G_i/G_{i+1}).$$

Hence $P(G) \leq P(G_j/G_{j+1})$ and it is sufficient to assume that G is simple and non-Abelian.

Next consider the degree equation for G,

$$|G| = [G : G'] + \sum_{i=[G:G']+1}^{k(G)} n_i^2,$$

where $n_i \geq 2$. Since G is simple and non-Abelian, $[G : G'] = 1$. We will consider the case when all $n_i \geq 4$ and the case when there is some $n_i < 4$.

In the first case, when all $n_i \geq 4$, the degree equation yields the bound

$$|G| \quad \geq \quad 1 + 16(k(G) - 1)$$

and solving for $k(G)$ yields

$$k(G) \leq \frac{|G| + 15}{16}.$$

Then

$$P(G) \leq \frac{|G| + 15}{16|G|} = \frac{1}{16} + \frac{15}{16|G|}. \tag{3.15}$$

Since G is simple and non-Abelian, $|G| \geq 60$. Then

$$P(G) \leq \frac{1}{16} + \frac{15}{16 \cdot 60} = \frac{5}{64} < \frac{1}{12}. \tag{3.16}$$

In the second case, there is an irreducible representation of G with degree $n_i = 2$ or $n_i = 3$, and this yields an embedding of G in $GL_2(\mathbb{C})$ or in $GL_3(\mathbb{C})$. To see this is true, first suppose that G has a degree $n_i = 2$ representation, $\phi : G \to GL_2(\mathbb{C})$. Since G is simple, $\ker(\phi) = \{e\}$ so that ϕ is one to one and faithful. This means that an image of G embeds in $GL_2(\mathbb{C})$. Likewise, if there is an $n_i = 3$, then $G \leq GL_3(\mathbb{C})$. Using either Dixon's solution or the elementary methods discussed prior to this proposition, $P(G) \geq \frac{1}{12}$ implies $G \cong A_5$. Therefore, if $P(G) > \frac{1}{12}$, G is solvable and A_5 is the unique simple non-Abelian group with commutativity degree $\frac{1}{12}$. □

Next we will show that all non-solvable groups of commutativity degree $\frac{1}{12}$ are the direct product of A_5 and an Abelian group. Intuitively, this follows from the fact that A_5 is the unique finite simple non-solvable group with commutativity degree $\frac{1}{12}$, but we require two definitions and two additional lemmas before our proof.

First, we say G is characteristic simple if there is no nontrivial proper characteristic subgroup of G. Secondly, a group H is called a central extension of a group G if there is a normal subgroup $M \lhd H$ such that $M \leq Z(H)$ and $H/M \cong G$.

Lemma 3.3.5. *If G is non-Abelian characteristic simple then there is a simple non-Abelian group X so that $G \cong X^m$ for some $m \geq 1$.*

Proof. First suppose that G is simple. Then $G = X^1$.

Then suppose that G is not simple. Let X be a minimal normal subgroup in G. Since G is finite, $|\mathrm{Aut(G)}| = s$ for some $s \in \mathbb{N}$. Let $\mathrm{Aut}(G) = \{\phi_i : 1 \leq i \leq s\}$. Consider the set S of images of X under the automorphisms of G, $S = \{\phi_i(X) : 1 \leq i \leq s\}$. Let $\phi_1(X)$, $\phi_2(X)$, . . . $\phi_t(X)$ be the distinct images of X.

Let $M = \Pi_{i=1}^{t}(\phi_i(X))$. Notice that M is a normal subgroup of G because each $\phi_i(X)$ is a normal subgroup. For each i, $\phi_i(M) = M$, so M is a characteristic subgroup of G. Since G is characteristic simple, $M = G$. Hence $G \cong X \times X \times ... \times X = X^m$ for some $m \geq 1$. Further, X is simple because G is a direct product of copies of X. Therefore, $G \cong X^m$ where X is a simple, non-Abelian group. □

Lemma 3.3.6. *If G is a non-solvable group with $P(G) = \frac{1}{12}$, then G has two characteristic subgroups, M and N such that*

1. *$N/M \cong A_5$*
2. *G/N is Abelian, and*
3. *$M \subset Z(N)$.*

Proof. Since G is finite, G has a characteristic series

$$G = G_0 \text{ char } G_1 \text{ char } ... \text{ char } G_r = \{e\}$$

with G_i maximal characteristic in G_{i-1} for $1 \leq i \leq r$. Since G is not solvable there is a non-Abelian factor, G_j/G_{j+1} for some j, $1 \leq j \leq r$. By Proposition 3.1.5,

$$P(G) \leq \prod_{i=1}^{r-1} P(G_i/G_{i+1}).$$

Hence $P(G) \leq P(G_j/G_{j+1})$. Since G_{j+1} is maximal characteristic in G_j, G_j/G_{j+1} has no characteristic subgroup by the correspondence theorem. Then G_j/G_{j+1} is also characteristic simple. By Lemma 3.3.5 Therefore, $G_j/G_{j+1} \cong X^m$ where X is a simple, non-Abelian group. Thus $|X| \geq 60$.

Hence $P(X) \leq \frac{1}{12}$ by Proposition 3.3.4, and

$$\frac{1}{12} = P(G) \leq P(G_j/G_{j+1}) \leq P(X)^m \leq \left(\frac{1}{12}\right)^m.$$

Hence $m = 1$ and $P(X) = \frac{1}{12}$. Therefore, $X \cong A_5$.

Let $G_j = N$ and $G_{j+1} = M$. We will show that N and M satisfy properties (1) through (3). First note that M char G and N char G by construction. Also, $N/M \cong A_5$ because $P(N/M) = \frac{1}{12}$ and M/N is simple and non-solvable. This satisfies Property (1). Next, by Proposition 3.1.5,

$$\frac{1}{12} = P(G) \leq P(G/N)P(N/M)P(M) \leq P(N/M) = \frac{1}{12}.$$

Hence $P(G/N) = P(M) = 1$. Therefore G/N is Abelian, satisfying property (2).

Notice that $M \lhd N$ and since

$$\frac{1}{12} = P(G) \leq P(N) \leq P(N/M)P(M) = \frac{1}{12},$$

$P(N) = P(M)P(N/M)$. By Proposition 3.1.5, this is equivalent to

$$C_{N/M}(\sigma M) = C_N(\sigma)M/M$$

for all $\sigma \in N$. Let $\rho \in M$. Since M is Abelian, $M \subseteq C_G(\rho)$. Thus $C_N(\rho) = C_N(\rho)M$. Then

$$N/M = C_{N/M}(\rho M) = C_N(\rho)/M$$

because M is the identity of N/M and $M \subseteq C_G(\rho)$. By the correspondence theorem $C_N(\rho) = N$. Therefore $\rho \in Z(N)$ and $M \subseteq Z(N)$, satisfying (3). □

Proposition 3.3.7. *If G is a non-solvable group with $P(G) = \frac{1}{12}$, then there is an Abelian group C such that $G \cong A_5 \times C$.*

Proof. Let H be a non-solvable subgroup of minimal order in G. Then $H = H'$. By Proposition 3.1.1,

$$\frac{1}{12} = P(G) \leq P(H) \leq \frac{1}{12}$$

because H is non-solvable. Then $P(G) = P(H)$. By Corollary 3.1.6 $H \lhd G$ and G/H is Abelian.

By Proposition 3.3.6 there exist characteristic subgroups M char H and N char H, such that $M \subseteq Z(N)$, $N/M \cong A_5$, and H/N is Abelian. Since $H = H'$, $H = N$ and $H/M \cong A_5$. Also, $M \subseteq Z(N)$, so H is a central extension of A_5. By Aschbacher [3], $(170, 33.15(1))$, $H \cong A_5$ or $H \cong SL_2(\mathbb{F}_5)$. By calculation in GAP $P(SL_2(\mathbb{F}_5)) = \frac{3}{40} < \frac{1}{12}$, and $P(A_5) = \frac{1}{12}$. Thus $H \cong A_5$.

Consider the centralizer $C_G(H)$. Recall from the proof of Corollary 3.1.6 that $P(G) = P(H)$ implies $G = HC_G(H)$. Let $u \in C_G(H)$ and let $g \in G$. Then for some $h \in H$ and $u \in C_G(H)$, $g = hv$. Also,

$$gvg^{-1} = h(vuv^{-1})h^{-1} \in C_G(H)$$

so $C_G(H) \lhd G$. Hence both subgroups H and $C_G(H)$ are normal. Next, notice that $C_G(H)$ commutes with H so

$$H \cap C_G(H) = Z(H) = Z(A_5) = \{e\}.$$

Hence $G = HC_G(H) = H \times C_G(H)$. Finally, $C_G(H) \cong G/H$ and so $C_G(H)$ is Abelian. Therefore, $G = H \times C_G(H) \cong A_5 \times C$ with C Abelian. $\square$

A summary of the preceding propositions describes all groups with commutativity degree greater than $\frac{3}{40}$ in terms of the solvability of the group.

Corollary 3.3.8. *Let $P(G) > \frac{3}{40}$. Then either*

1. *G is solvable or*

2. *$G \cong A_5 \times C$ for some Abelian group C and $P(G) = \frac{1}{12}$.*

Proof. By Proposition 3.3.4, if $P(G) > \frac{1}{12}$, then G is solvable.

If $P(G) = \frac{1}{12}$ and G is not solvable, then by Proposition 3.3.7, $G \cong A_5 \times C$ for some Abelian group C.

If $P(G) < \frac{1}{12}$ and G is not solvable, there are two cases. For the first case, suppose G has no irreducible representation of degree 3 or 2. In the proof of Proposition 3.3.4, we show that $P(G) \leq \frac{5}{64} < \frac{3}{40}$. (Equation 3.16). Secondly, suppose that G does have an irreducible representation of degree 3 or 2. As in Proposition 3.3.7, by Theorem 3.3.3 (Blichfeldt) $G \cong A_5$, $G \cong A_6$, or $G \cong PSL_2(\mathbb{F}_7)$. If $G \cong A_6$ or $G \cong PSL_2(\mathbb{F}_7)$ then $P(G) < \frac{3}{40}$. Otherwise, $G \cong A_5$ and $P(G) = \frac{1}{12}$. $\square$

Chapter 4: Calculations for Specific Groups

It is implied in our discussion of the "$k(GV)$ problem" in Section 3.1 that it is beyond the scope of this thesis to attempt to find the commutativity degree for all finite groups. However, we can look for insight by finding the commutativity degrees of varying types of groups. First, we will explicitly calculate the conjugacy classes and commutativity degree for classes of groups having a presentation with two generators and an order reversing relation. Second, relying heavily on previous results, we will find the commutativity degree of the symmetric group S_n and alternating group A_n with small n. Then we will discuss a class of groups called the 4-property p-groups in order to construct a group having commutativity degree $\frac{1}{2}\left(1+\frac{1}{2m}\right)$ for all $m \in \mathbb{N}$. We generalize this result to find a group with commutativity degree $\frac{p^{s-1}+p^2+p}{p^{s+1}}$ for any prime p and $s \geq 3$. After that, we discuss wreath products and calculate the commutativity degree for two specific types of wreath products.

4.1 Two Generated Groups with an Order Reversing Relation

We say that a group G is a two generated group with an order reversing relation if there is a presentation of G in which G is generated by two elements and in which there is a relation of the form $ab = ba^i$ for some i. The method we use to calculate conjugacy classes for each class of two generated groups with an order reversing relation is outlined in Section 4.1.1, and we use the Rusin pn-groups to demonstrate this method. The conjugacy classes and related properties of the remaining classes of groups are summarized in this section. Calculations for these groups appear in

Appendix 6.2.

4.1.1 Calculation of Conjugacy Classes

If G is a two generated group with an order reversing relation, then any element in the group can be written in the form a^ib^j. In some cases, when $o(b) = 2$, it is more convenient to work with two forms: a^jb and a^i. Before calculating conjugacy classes, we first determine all possible values for i, j. For example, let $R_{pn} =< a, b : a^p = b^n = e, bab^{-1} = a^r >$ such that p is prime, $n|p-1$, and $r^j \equiv 1(\text{mod } p)$ if and only if $n|j$. These conditions may seem restrictive, but such groups do exist. We construct an example in Corollary 4.1.1. The R_{pn} groups were defined by Rusin [43] and we will refer to them as Rusin pn-groups. Notice that any element in R_{pn} can be written in the form a^ib^j for some i and j, and, as a set,

$$R_{pn} = \{a^ib^j : 0 \leq i \leq p-1 \text{ and } 0 \leq j \leq n-1\}.$$

Next we find the form of the inverse of an element by applying the order reversing relation to the form a^ib^j. (If both a^i and a^jb are used, we apply the relation to both forms.) Continuing with the example of R_{pn}, apply the relation $ba = a^rb$ to a^ib^j as follows:

$$(a^ib^j)^{-1} = b^{-j}a^{-i} = a^{\frac{-i}{r^j}}b^{-j}. \tag{4.1}$$

Then the inverse of any element $a^ib^j \in R_{pn}$ is $a^{\frac{-i}{r^j}}b^{-j}$. This will allow us to conjugate elements by the form a^ib^j.

Next we select elements of a specified form from G and conjugate each of them by the form a^ib^j (or a^i and a^jb) the find the conjugacy class of element of the specified form. By process of elimination, we continue to conjugate additional elements of G by the form a^ib^j until all elements of the group are included in some conjugacy class. For the R_{pn} groups, we first found conjugacy classes for elements of the form a^v with $1 \leq v \leq p-1$ and then for elements of the form b^w with $1 \leq w \leq n-1$ as follows:

Let $1 \leq v \leq p-1$. Then

$$\phi_{a^ib^j}(a^v) = a^ib^ja^va^{\frac{-i}{r^j}}b^{-j} = a^{i+(v-\frac{i}{r^j})(r^j)} = a^{vr^j}.$$

Then for each v, $1 \leq v \leq p-1$, the conjugacy class of a^v is

$$[a^v] = \{a^v, a^{vr}, a^{vr^2}, ..., a^{vr^{n-1}}\}.$$

This class has n elements. Further, the $p-1$ elements of the form a^v are partitioned into n element classes. Hence there as $\frac{p-1}{n}$ such classes.

Next let $1 \leq w \leq n-1$. Then

$$\phi_{a^ib^j}(b^w) = a^ib^jb^wa^{\frac{-i}{r^j}}b^{-j} = a^{i+(-\frac{i}{r^j})(r^{j+w})}b^w = a^{i(1-r^w)}b^w.$$

Then for each w, $1 \leq w \leq n-1$, the conjugacy class of b^w is

$$[b^w] = \{b^w, a^{(1-r^w)}b^w, a^{2(1-r^w)}b^w, ..., a^{(p-1)(1-r^w)}b^w\}.$$

This class has p elements because $r^j \equiv 1(\text{mod } p)$. The $p(n-1)$ elements of the form a^ib^w with $w \neq 0$ are partitioned into p element conjugacy classes. Thus there are $n-1$ such classes.

All nonidentity elements are included in a class of the type $[a^v]$ or $[b^w]$. No further computations are necessary, and we have completely determined the conjugacy classes of the group.

4.1.2 Conjugation Tables and Commutativity Degrees

Rusin pn-groups

Let $R_{pn} =< a, b : a^p = b^n = e, bab^{-1} = a^r >$ such that p is prime, $n|p-1$, and $r^j \equiv 1(\text{mod } p)$ if and only if $n|j$. Then R_{pn} is a Rusin pn-group. We list properties of R_{pn}-groups in Table 4.1 and we tabulate the conjugacy classes in Table 4.2.

Presentation	$R_{pn} =< a, b : a^p = b^n = e, bab^{-1} = a^r >$
Order	pn
Element(s)	$\{a^i b^j : 1 \leq i \leq p-1, 1 \leq j \leq n-1\}$
Inverse(s)	$(a^i b^j)^{-1} = a^{\frac{-i}{r^j}} b^{-j}$
Center	$Z(R_{pn}) = \{e\}$

Table 4.1: R_{pn}

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[a^v] = \{a^v, a^{vr^2}, a^{vr^3}, ..., a^{vr^n}\}$	n	$\frac{p-1}{n}$
$[b^w] = \{b^w, a^{(1-r^w)}b^w, a^{2(1-r^w)}b^w, ..., a^{(p-1)(1-r^w)}b^w\}$	p	$n-1$

Table 4.2: **Conjugacy Classes for** R_{pn}

The commutativity degree is

$$P(R_{pn}) = \frac{k(R_{pn})}{|R_{pn}|} = \frac{1 + \frac{p-1}{n} + n - 1}{np} = \frac{n^2 + p - 1}{n^2 p}. \tag{4.2}$$

Corollary 4.1.1. *The value $\frac{1}{p-1}$ is the commutativity degree of some Rusin pn-group.*

Proof. Let p be prime and let $n = p - 1$. Let $(\mathbb{Z}_p^\star, \cdot)$ be the multiplicative group of the field $\mathbb{Z}_p$; this group is cyclic. Let $1 \leq r \leq p-1$ be such that $\overline{r}$ is a generator of $(\mathbb{Z}_p^\star, \cdot)$. Then there is a Rusin pn-group $R_{p(p-1)}$ since $p-1|p-1$ and $r^j \equiv 1 (\mathrm{mod}\ p)$ if and only if $n|j$. By Equation 4.2 the commutativity degree of $R_{p,p-1}$ is

$$R_{p(p-1)} = \frac{(p-1)^2 + p - 1}{(p-1)^2 p} = \frac{1}{p-1}. \tag{4.3}$$

□

A limit point of the set of commutativity degrees can be calculated by taking the limit of the commutativity degrees of some infinite class of groups. When finding a limit point by this method, we also refer to the limit of the commutativity degrees of the class as the asymptotic commutativity degree of that particular class. For instance, we can find the asymptotic commutativity degree of the Rusin pn-groups as follows: Fix a positive integer n. Dirchlet's theorem states that there are infinitely many primes of the form $p_i = 1 + in$, $i \in \mathbb{Z}$. Let r be selected such that r is the generator of the cyclic group $\mathbb{Z}^{\star}_{p_i}$. Then $n|p_i - 1$ and there are infinitely many Rusin $p_i n$-groups of the form $R_{p_i n} =< a, b : a^{p_i} = b^n = e, bab^{-1} = a^r >$. Then the asymptotic commutativity degree of the Rusin $p_i n$-groups is

$$\lim_{i\to\infty} P(R_{p_i n}) = \lim_{i\to\infty}(\frac{n^2 + p_i - 1}{n^2 p_i}) = \frac{1}{n^2}.$$

Hence $\frac{1}{n^2}$ is a limit point of the set of commutativity degrees for each positive integer n.

D_{pq} Groups

Let D_{pq} denote an R_{pn} group with $n = q$ a prime. Then $D_{pq} = R_{pq} =< a, b : a^p = b^q = e, bab^{-1} = a^r >$ such that p is prime, $q|p - 1$, and r has order q mod p. This type of group is called a generalized dihedral group. We list properties of D_{pq}-groups in Table 4.3, and we tabulate the conjugacy classes in Table 4.4.

Presentation	$D_{pq} =< a, b : a^p = b^q = e, bab^{-1} = a^r >$
Order	pq
Element(s)	$\{a^i b^j : 1 \leq i \leq p - 1, 1 \leq j \leq q - 1\}$
Inverse(s)	$(a^i b^j)^{-1} = a^{\frac{-i}{r^j}} b^{-j}$
Center	$Z(D_{pq}) = \{e\}$

Table 4.3: D_{pq}

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[a^v] = \{a^v, a^{vr^2}, a^{vr^3}, ..., a^{vr^n}\}$	q	$\frac{p-1}{q}$
$[b^w] = \{b^w, a^{(1-r^w)}b^w, a^{2(1-r^w)}b^w, ..., a^{(p-1)(1-r^w)}b^w\}$	p	$q-1$

Table 4.4: **Conjugacy Classes for** D_{pq}

The commutativity degree is

$$P(D_{pq}) = \frac{k(D_{pq})}{|D_{pq}|} = \frac{1 + \frac{p-1}{q} + q - 1}{qp} = \frac{q^2 + p - 1}{q^2 p}.$$

Next we find the asymptotic commutativity degree of the D_{pq} groups. Let q be a prime. Similar to the general case of R_{pn} groups, there are infinitely may primes of the form $p_i = 1 + iq$ for some $i \in \mathbb{Z}$. Again let r be selected such that r is the generator of the cyclic group $\mathbb{Z}^{\star}_{p_i}$. Then $q|p_i - 1$ and there are infinitely many $D_{p_i q}$ groups of the form $D_{p_i q} =< a, b : a^{p_i} = b^q = e, bab^{-1} = a^r >$. The asymptotic commutativity degree of the class of $D_{p_i q}$ groups is

$$\lim_{i\to\infty} P(D_{p_i q}) = \lim_{i\to\infty} (\frac{q^2 + p_i - 1}{q^2 p_i}) = \frac{1}{q^2}.$$

Then $\frac{1}{q^2}$ is a limit point of the set of commutativity degrees for each prime q.

$T_{p,q,m,\theta}$ Groups

The $T_{p,q,m,\theta}$ groups are defined in Rusin [43] as follows: Let $m \in \mathbb{N}$ and let p and q be primes so that $q|p-1$. Then $T_{p,q,m,\theta} =< a, b : a^p = b^{q^m} = e, bab^{-1} = a^{\lambda^\theta} >$ such that λ has order q mod p, and $\theta \in \{1, 2, ..q-1\}$. We list properties of $T_{p,q,m,\theta}$ groups in Table 4.5, and we tabulate the conjugacy classes in Table 4.6.

Presentation	$T_{p,q,m,\theta} =< a, b : a^p = b^{q^m} = e, bab^{-1} = a^{\lambda^\theta} >$
Order	pq^m
Element(s)	$\{a^i b^j : 1 \leq i \leq p-1, 1 \leq j \leq q^m - 1\}$
Inverse(s)	$(a^i b^j)^{-1} = a^{\frac{-i}{\lambda^{j\theta}}} b^{-j}$
Center	$Z(T_{p,q,m,\theta}) =< b^q >$
$G/Z(G)$	$< a, b : \bar{a}^p = \bar{b}^q = e, \bar{b}\bar{a}\bar{b}^{-1} = \bar{a}^{\lambda^\theta} >= D_{pq}$ with $\lambda^\theta = r$

Table 4.5: $T_{p,q,m,\theta}$

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[a^v] = \{a^v, a^{v\lambda^\theta}, a^{v\lambda^{2\theta}}, ..., a^{v\lambda^{(p-1)\theta}}\}$	$p-1$	1
$[b^w] = \{b^w, a^{(1-\lambda^{w\theta})}b^w, a^{2(1-\lambda^{w\theta})}b^w, ..., a^{(p-1)(1-\lambda^{w\theta})}b^w\}$ for $q \nmid w$	p	$q^m - q^{m-1}$
$[b^w] = \{b^w\}$ for $q \mid w$	1	$q^{m-1} - 1$
$[a^v b^w] = \{a^v b^w, a^{v\lambda^\theta} b^w, a^{v\lambda^{2\theta}} b^w, ..., a^{v\lambda^{(p-1)\theta}} b^w\}$ for $q \mid w$	$p-1$	$q^{m-1} - 1$

Table 4.6: **Conjugacy Classes for** $T_{p,q,m,\theta}$

The commutativity degree is

$$\begin{aligned} P(T_{p,q,m,\theta}) &= \frac{k(T_{p,q,m,\theta})}{|T_{p,q,m,\theta}|} \\ &= \frac{1 + 1 + (q^m - q^{m-1}) + (q^{m-1} - 1) + (q^{m-1} - 1)}{pq^m} \\ &= \frac{q^m + q^{m-1}}{pq^m} \\ &= \frac{q+1}{pq}. \end{aligned}$$

Again by Dirchlet's theorem, for a fixed q, m, and θ there are infinitely many $T_{p,q,m,\theta}$ with $p_i = 1 + iq$ and with λ a generator of $\mathbb{Z}_{p_i}$, since $q|p_i - 1$ for each $i \in \mathbb{N}$. The asymptotic commutativity degree of the class of $T_{p,q,m,\theta}$ groups is

$$\lim_{i\to\infty}(P(T_{p,q,m,\theta})) = \lim_{i\to\infty}(\frac{q+1}{p_i q}) = 0.$$

We will use the $T_{p,q,m,\theta}$ groups when finding the commutativity degree of groups with $|G/Z| < 12$. Specifically, we will show in Proposition 5.2.13 that if G is a group with $G/Z(G) \cong D_{pq}$ then there is some $\theta \in \{1, 2, ..., q-1\}$, $m \geq 1$, and Abelian group A such that $G \cong T_{p,q,m,\theta} \times A$.

G_m Groups

The G_m groups are a subclass of $T_{p,q,m,\theta}$ groups. Let $p = 3$, $q = 2$, $\theta = 1$, and $\lambda = -1$. Notice that -1 has order $p - 1$ mod $p = 3$, so $\lambda = -1$ is an acceptable λ value. Let $m \in \mathbb{N}$. Then $T_{3,2,m,1} = G_m =< a, b : a^3 = b^{2^m} = e, bab^{-1} = a^{-1} >$. We list properties of G_m-groups in Table 4.7, and we tabulate the conjugacy classes in Table 4.8.

Presentation	$G_m =< a, b : a^3 = b^{2^m} = e, bab^{-1} = a^{-1} >$
Order	$(3)(2^m)$
Element(s)	$\{a^i b^j : 1 \leq i \leq 2, 1 \leq j \leq 2^m - 1\}$
Inverse(s)	$(a^i b^j)^{-1} = a^{\frac{-i}{2^j}} b^{-j}$
Center	$Z(G_m) =< b^2 >$

Table 4.7: G_m

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[a] = \{a, a^2\}$	2	1
$[b^w] = \{b^w, ab^w, a^2b^w\}$ for w odd	3	$2^m - 2^{m-1}$
$[b^w] = \{b^w\}$ for w even	1	$2^{m-1} - 1$
$[ab^w] = \{ab^w, a^2b^w\}$ for w even	2	$2^{m-1} - 1$

Table 4.8: **Conjugacy Classes for** G_m

The commutativity degree is

$$P(G_m) = \frac{2^m + 2^{m-1}}{(3)2^m} = \frac{1}{2}.$$

The G_m groups will also be used when calculating the commutativity degrees of groups with $|G/Z| < 12$. Notice that if $m = 1$ then $G_1 =< a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1} >\cong S_3$. Further, in Proposition 5.2.14 we show that if $G/Z(G) \cong S_3$ then $G \cong G_m \times A$, with $m \geq 1$ and A Abelian.

A Indecomposable Group with $P(G) = \frac{1}{2}$

Examples of groups with commutativity degree $\frac{1}{2}$ are easily constructed using direct products. For example, recall that $P(G_m) = \frac{1}{2}$. Then for every $m \in \mathbb{N}$ and for every nontrivial Abelian group A there is a group $G \cong G_m \times A$ such that $P(G) = P(G_m)P(A) = \frac{1}{2}$. In this section, we provide an example of a group G with commutativity degree $\frac{1}{2}$ that is not a direct product. We list properties of G in Table 4.9, and we tabulate the conjugacy classes in Table 4.10. The commutativity degree is

$$P(G) = \frac{k(G)}{|G|} = \frac{1}{2}.$$

This group has commutativity degree $P(G) = \frac{1}{2}$ but it is not a direct product of a nontrivial Abelian group with another group nor is it a G_m group. In this example, $Z(G) =< a^3 >$ has even order. Recall that we showed in in Proposition 3.2.3 that if G is nilpotent, $|Z(G)|$ is odd, and $P(G) = \frac{1}{2}$, then $G \cong S_3 \times Z(G)$. The group G is an example of a group with $P(G) = \frac{1}{2}$ and $Z(G)$ of even order that is not isomorphic to $S_3 \times Z(G)$.

Presentation	$G =< a, b : a^6 = e, b^2 = a^3, bab^{-1} = a^{-1} >$
Order	12
Element(s)	$\{a^i b^j : 1 \leq i \leq 5, 1 \leq j \leq 1\}$
Inverse(s)	$(a^i b^j)^{-1} = a^{\frac{-i}{5^j}} b^{-j}$
Center	$Z(G) =< a^3 >$

Table 4.9: G

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[a] = \{a, a^5\}$	2	1
$[a^2] = \{a^2, a^4\}$	2	1
$[a^3] = \{a^3\}$	1	1
$[ab] = \{ab, ab^3, ab^5\}$	3	1
and $[a^2b] = \{b, a^2b, a^4b\}$	3	1

Table 4.10: **Conjugacy Classes for** G

Dicyclic Groups

Let $m \in \mathbb{N}$. The general dicyclic group is defined by $D_m =< a, b : a^{2m} = b^4 = e, b^{-1}ab = a^{-1}, a^m = b^2 >$. The notation is similar to that used for the dihedral groups, but the group in question should be clear by context. We list properties of D_m-groups in Table 4.11, and we tabulate the conjugacy classes in Table 4.12. The commutativity degree is

$$P(D_m) = \frac{2 + m - 1 + 2}{4m} = \frac{m + 3}{4m}.$$

Also, the asymptotic commutativity degree for the class of D_m groups is

$$\lim_{m \to \infty} (\frac{m + 3}{4m}) = \frac{1}{4}.$$

Presentation	$D_m =< a, b : a^{2m} = b^4 = e, b^{-1}ab = a^{-1}, a^m = b^2 >$
Order	$4m$
Element(s)	$\{a^i, ba^i : 1 \leq i \leq 2m-1\}$
Inverse(s)	$(a^i)^{-1} = a^{-i}$ and $(ba^i)^{-1} = ba^{i-m}$
Center	$Z(G) =< b^2 >$

Table 4.11: D_m

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[b^2]$	1	1
$[a^t] = \{a^t, a^{-t}\}$	2	$\frac{2m-2}{2} = m-1$
$[ba] = \{ba, ba^3, ..ba^{2m-1}\}$	m	1
$[ba^2] = \{b, ba^2, ..ba^{2m-2}\}$	m	1

Table 4.12: **Conjugacy Classes for** D_m

Generalized Quaternion Groups

Let $n \in \mathbb{N}$. The generalized quaternion group $Q_{2^{n+1}}$ is defined to be the dicyclic group with $m = 2^{n-1}$. Then

$$Q_{2^{n+1}} = D_{2^{n-1}} =< a, b : a^{2^n} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{n-1}} = b^2 > .$$

Notice that when $n = 2$, $Q_{2^{n+1}} = Q_8$ is the usual quaternion group. We list properties of $Q_{2^{n+1}}$-groups in Table 4.13, and we tabulate the conjugacy classes in Table 4.14. (The semidihedral group SD_n is defined in Subsection 4.1.2.)

Presentation	$Q_{2^{n+1}} =< a, b : a^{2^n} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{n-1}} = b^2 >$
Order	2^{n+1}
Element(s)	$\{a^i, ba^i : 1 \leq i \leq 2^n - 1\}$
Inverse(s)	$(a^i)^{-1} = a^{-i}$ and $(ba^i)^{-1} = ba^{i-2^n}$
Center	$Z(G) =< b^2 >$
G/Z	SD_n

Table 4.13: $Q_{2^{n+1}}$

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[b^2]$	1	1
$[a^t] = \{h^t, h^{-t}\}$	2	$\frac{2^n-2}{2} = 2^{n-1} - 1$
$[ba] = \{b, ba^2, ..ba^{2^n-2}\}$	2^{n-1}	1
$[ba^3] = \{ba, ba^3, ..ba^{2^n-1}\}$	2^{n-1}	1

Table 4.14: **Conjugacy Classes for $Q_{2^{n+1}}$**

The commutativity degree is

$$P(Q_{2^{n+1}}) = \frac{2 + 2^{n-1} - 1 + 2}{2^{n+1}} = \frac{2^{n-1} + 3}{2^{n+1}}.$$

The asymptotic commutativity degree is

$$\lim_{n \to \infty} (\frac{2^{n-1} + 3}{2^{n+1}}) = \frac{1}{4}.$$

Dihedral Groups

Let $n \in \mathbb{N}$. The dihedral group, $D_n =< r, \rho : r^2 = e, \rho^n = e, \rho r = r\rho^{n-1} >$ has a different conjugacy class structure depending on whether n is even or odd. We will tabulate the conjugacy classes separately for dihedral groups with even and odd n. We list properties of D_n-groups in Table 4.15, and we tabulate the conjugacy classes in Tables 4.16 and 4.17.

Presentation	$D_n =< r, \rho : r^2 = e, \rho^n = e, \rho r = r\rho^{n-1} >$
Order	$2n$
Element(s)	$\{\rho^i, r\rho^i : 1 \leq i \leq n-1\}$
Inverse(s)	$(\rho^i)^{-1} = \rho^{-i}$ and $(r\rho^i)^{-1} = r\rho^i$
Center	$Z(D_n) = \{e\}$ (n odd), and $Z(D_n) =< \rho^{\frac{n}{2}} >$ (n even)

Table 4.15: D_n

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[r] = \{r\rho^j : 0 \leq j < n\}$	n	1
$[\rho^i] = \{\rho^i, \rho^{n-i}\}$	2	$\frac{n-1}{2}$

Table 4.16: **Conjugacy Classes for D_n with n Odd**

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[r] = \{r\rho^{2j} : 0 \leq j \leq \frac{n-1}{2}\}$	$\frac{n}{2}$	1
$[r] = \{r\rho^{2j+1} : 0 \leq j \leq n$ and j odd $\}$	$\frac{n}{2}$	1
$[\rho^i] = \{\rho^i, \rho^{n-i}\}$	2	$\frac{n-2}{2}$
$[\rho^{\frac{n}{2}}] = \{\rho^{\frac{n}{2}}\}$	1	1

Table 4.17: **Conjugacy Classes for D_n with n Even**

The commutativity degree is

$$P(D_n) = \frac{k(D_n)}{|D_n|} = \frac{n+3}{4n} \text{ (n odd)}$$
$$P(D_n) = \frac{k(D_n)}{|D_n|} = \frac{n+6}{4n} \text{ (n even).}$$

It is interesting that if n is odd then

$$P(D_{2n}) = \frac{2n+6}{4(2n)} = \frac{n+3}{4n} = P(D_n).$$

The asymptotic commutativity degree is

$$\lim_{n\to\infty} P(D_n) = \lim_{n\to\infty} \frac{n+3}{4n} = \frac{1}{4} \text{ (n odd)}$$
$$\lim_{n\to\infty} P(D_n) = \lim_{n\to\infty} \frac{n+6}{4n} = \frac{1}{4} \text{ (n even).}$$

Semidihedral Groups and Quasidihedral Groups

The semidihedral groups

$$SD_n =< a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} >$$

and quasidihedral groups

$$QD_n =< a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} >$$

have similar relations but surprisingly different commutativity degrees. We list properties of SD_n-groups and QD_n-groups in Tables 4.18 and 4.19. We tabulate the conjugacy classes in Tables 4.20 and 4.21. The commutativity degrees are

$$\begin{aligned} P(SD_n) &= \frac{2^{n-1}+3}{2^{n+1}} \\ P(QD_n) &= \frac{1+2^{n-1}-1+2^{n-2}+2^{n-1}}{2^{n+1}} = \frac{2^n+2^{n-2}}{2^{n+1}} = \frac{5}{8}. \end{aligned}$$

Notice that $P(Q_{2^{n+1}}) = P(SD_n)$ and $Q_{2^{n+1}}/Z(Q_{2^{n+1}}) \cong SD_n$. Also, we will show in Section 5.2.2 that $QD_n/Z(QD_n) =< a^2 >\cong V_4$. The asymptotic commutativity degree of SD_n is

$$\lim_{n\to\infty} P(SD_n) = \lim_{n\to\infty} \frac{2^{n-1}+3}{2^{n+1}} = \frac{1}{4}.$$

Presentation	$SD_n =< a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} >$
Order	2^{n+1}
Element(s)	$\{a^i, ba^i : 0 \leq i \leq 2^n - 1\}$
Inverse(s)	$(a^i)^{-1} = a^{-i}$ and $(ba^i)^{-1} = ba^{(-i2^{n-1}+i)}$
Center	$Z(SD_n) =< a^{2^{n-1}} >$

Table 4.18: SD_n

Presentation	$QD_n =< a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} >$
Order	2^{n+1}
Element(s)	$\{a^i, ba^i : 0 \leq i \leq 2^n - 1\}$
Inverse(s)	$(a^i)^{-1} = a^{-i}$ and $(ba^i)^{-1} = ba^{(-i2^{n-1}-i)}$
Center	$Z(QD_n) =< a^2 >$

Table 4.19: QD_n

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[a^i] = \{a^i, a^{-i}\}$ for $i \neq 2^{n-1}$	2	$2^{n-1} - 1$
$[a^{2^{n-1}}]$	1	1
$[ba] = \{ ba^i :$ i odd $\}$	2^{n-1}	1
$[ba^2] = \{ ba^i :$ i even $\}$	2^{n-1}	1

Table 4.20: **Conjugacy Classes for** SD_n

Conjugacy Class Type	No. Elements	No. Classes
$[e]$	1	1
$[a^i] = \{a^i\}$ for i even	1	$2^{n-1} - 1$
$[a^i] = \{a^i, a^{2^{n-1}+i}$ for i odd$\}$	2	2^{n-2}
$[ba^i] = \{ ba^i, ba^{2^{n-1}+i}\}$	2	2^{n-1}

Table 4.21: **Conjugacy Classes for** QD_n

4.2 Symmetric Groups and Alternating Groups

To find the commutativity degree of S_n, we count the number of conjugacy classes of the group. It is well known that two elements of S_n are conjugate if they have the same cycle structure. Dummit and Foote [15] (4.3, Proposition 11) prove the following theorem:

Theorem 4.2.1. *Two elements of S_n are conjugate if and only if they have the same cycle type. The number of conjugacy classes of S_n equals the number of partitions of n.*

Let $p(n)$ denote the number of partitions of n. Hardy and Ramanujan proved another well known result: the number of partitions of n is approximated by

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\frac{2\pi\sqrt{n}}{\sqrt{3}}}.$$

Hence

$$P(S_n) = \frac{p(n)}{n!} \approx \frac{1}{4n\sqrt{3}n!} e^{\frac{2\pi\sqrt{n}}{\sqrt{3}}}. \tag{4.4}$$

and the asymptotic commutativity degree of S_n is $\lim_{n\to\infty} P(S_n) = 0$.

The following formula is given by Dénes, Erdös and Turán in [11]. It counts the number of conjugacy classes of A_n in terms of $p(n)$:

$$\begin{aligned} k(A_n) &= \frac{p(n)}{2} + \frac{3}{2}(-1)^n \sum_{|r|<\sqrt{n}} (-1)^r p\left(\frac{n}{2} - \frac{3r^2+r}{4}\right) \\ &\approx \frac{1}{8n\sqrt{3}n!} e^{\frac{2\pi\sqrt{n}}{\sqrt{3}}} \end{aligned}$$

where the sum is restricted to $r \equiv 2n$ or $(2n+1)$ mod 4. Then

$$P(A_n) = \frac{2k(A_n)}{n!}.$$

Dénes, Erdös and Turán also show that $k(A_n) \approx \frac{p(n)}{2}$. Hence $\lim_{n\to\infty} P(A_n) = 0$, and

$$\lim_{n\to\infty} \frac{P(S_n)}{P(A_n)} = \lim_{n\to\infty} \left(\frac{\frac{p(n)}{n!}}{\frac{2k(A_n)}{n!}} \right) \approx \lim_{n\to\infty} \left(\frac{\frac{p(n)}{n!}}{\frac{2p(n)}{2n!}} \right) = 1.$$

Table 4.22 tabulates $k(S_n)$, $k(A_n)$, $P(S_n)$, $P(A_n)$, and the ratio $\frac{P(S_n)}{P(A_n)}$ for small n, rounded to four decimal places.

n	$k(S_n)$	$k(A_n)$	$P(S_n)$	$P(A_n)$	*Ratio*
2	2	1	1	1	1.0000
3	3	3	0.5	1	0.5000
4	5	4	0.2083	0.3333	0.6250
5	7	5	0.0583	0.0833	0.7000
6	11	7	0.0153	0.0194	0.7858
7	15	9	0.0030	0.0036	0.8333
20	627	324	≈ 0	≈ 0	≈ 1

Table 4.22: **commutativity degree of S_n and A_n**

4.3 4 Property p-Groups

In this section we will describe a class of groups which we call 4-property p-groups; then we will use this class to construct an example of a group having commutativity degree $\frac{1}{2}(1 + \frac{1}{2^{2m}})$ for each $m \in \mathbb{N}$. We generalize the example to show that, for any prime p, there is a group with commutativity degree $\frac{p^{s-1}+p^2-1}{p^{s+1}}$ for $s \geq 3$. Additionally, we will use this class of groups to show that for all $m \in \mathbb{N}$, the value $\frac{1}{m}$ is a limit point of the set of commutativity degrees.

We define a 4-property p-group G_n as a group satisfying the following four properties:

1. $|G_n| = p^n$ for some prime p

2. $|G'_n| = p$, and $G'_n =< a >$ for some $a \in G$

3. $Z(G_n) = G'_n$

4. $G_n/Z(G_n) = \Pi_{i=1}^{j}\mathbb{Z}_{p_i}$ for some $j \in \mathbb{N}$.

Before constructing our example class of groups, we will compute the commutativity degree of a 4-property p-group.

Proposition 4.3.1. *Let p be a prime and let G_n be a 4-property p-group with $|G_n| = p^n$. Then*

$$P(G_n) = \frac{1}{p}\left[1 + \frac{1}{p^{n-2}} - \frac{1}{p^{n-1}}\right].$$

Proof. Assume $G_n' =< a >$ for $a \in G_n$. Let $b \in G$ and suppose $b \notin Z(G_n)$. Then $1 < |[b]|$ and there is some $g \in G_n$ such that $gbg^{-1} = x$ with $x \neq b$. Also

$$[g, b] = gbg^{-1}b^{-1} = xb^{-1} = a^i \in G_n' \tag{4.5}$$

for some i. Then $x = a^i b$. Since $a^p = e$, there are at most p distinct possible values for x. Hence b has at most p distinct conjugates and $|[b]| \leq p$. Thus $|[b]| = p$.

Next we will the number of conjugacy classes in G. The center is partitioned into $|Z(G_n)| = p$ many conjugacy classes. The set of noncentral elements of G is partitioned into conjugacy classes containing p elements each. There are

$$|G_n| - |Z(G_n)| = p^n - p$$

noncentral elements in G_n. Hence the noncentral elements are partitioned into $\frac{p^n-p}{p}$ many conjugacy classes. Then

$$k(G_n) = p + \frac{p^n - p}{p} = \frac{p^n + p^2 - p}{p},$$

and

$$\begin{aligned} P(G_n) &= \frac{p^n + p^2 - p}{p^{n+1}} \\ &= \frac{1}{p}\left[1 + \frac{1}{p^{n-2}} - \frac{1}{p^{n-1}}\right]. \end{aligned} \tag{4.6}$$

□

For a given p the asymptotic commutativity degree of the G_n groups is

$$\lim_{n\to\infty} P(G_n) = \lim_{n\to\infty} \left(\frac{1}{p}\left[1 + \frac{1}{p^{n-2}} - \frac{1}{p^{n-1}}\right]\right) = \frac{1}{p}.$$

We show in Example 4.3.6 that there are infinitely many G_n groups for each prime p. Then we can generalize this result as follows:

Corollary 4.3.2. *For each $m \in \mathbb{N}$, the value $\frac{1}{m}$ is a limit point for the set of commutativity degrees.*

Proof. Let $m \in \mathbb{N}$ and suppose $m = p_1 p_2 .. p_s$ for (not necessarily distinct) primes p_i, $1 \leq i \leq s$. Let $n \in \mathbb{N}$ and let $G_n = \prod_{i=1}^{s} G_i$ where each G_i is a 4-property p-group with $|G_i| = p_i^{n_i}$. By Proposition 1.2.3,

$$P(G_n) = \prod_{i=1}^{s} P(G_i),$$

and it follows that

$$P(G_n) \leq \prod_{i=1}^{s} \left(\frac{1}{p_i}\left[1 + \frac{1}{p_i^{n-2}} - \frac{1}{p_i^{n-1}}\right]\right).$$

Then

$$\lim_{n\to\infty} P(G_n) = \lim_{n\to\infty} \prod_{i=1}^{s} \left(\frac{1}{p_i}\left[1 + \frac{1}{p_i^{n-2}} - \frac{1}{p_i^{n-1}}\right]\right) = \prod_{i=1}^{k} (\frac{1}{p_i}) = \frac{1}{m}.$$

□

Lemmas 4.3.3 and 4.3.4 are used in the construction of a class of groups with commutativity degree $P(G) = \frac{1}{2} + \frac{1}{2^{2m}}$ for $m \in \mathbb{N}$.

Lemma 4.3.3. *If G_n is a 4-property 2-group of order 2^n then the commutativity degree of $G_n = \frac{1}{2}(1 + \frac{1}{2^{2m}})$ for some $m \in \mathbb{N}$.*

Proof. Let $p = 2$ so that $|G_n| = 2^n$. Then by Equation 4.6,

$$\begin{aligned} P(G_n) &= \frac{1}{2}\left[1 + \frac{1}{2^{n-2}} - \frac{1}{2^{n-1}}\right] \\ &= \frac{1}{2}\left[1 + \frac{2-1}{2^{n-1}}\right] \\ &= \frac{1}{2}\left(1 + \frac{1}{2^{n-1}}\right). \end{aligned} \tag{4.7}$$

The 4-property 2-group G_n has $|Z(G_n)| = 2$ and thus $|G_n/Z(G_n)| = 2^{n-1}$. Additionally, Proposition 5.1.2 applies to the 4-property 2-groups so that

$$|G_n/Z(G_n)| = \prod_{i=1}^{r} (\mathbb{Z}_2 \times \mathbb{Z}_2)_i$$

for some $r \in \mathbb{N}$. Then $|G_n/Z(G_n)| = 2^{2m}$ for some $m \in \mathbb{N}$. It follows that $2^{2m} = 2^{n-1}$ with $m \in \mathbb{N}$ and so $n-1$ is even. Therefore,

$$P(G) = \frac{1}{2}\left(1 + \frac{1}{2^{2m}}\right)$$

for some $m \in \mathbb{N}$. □

Lemma 4.3.4. *Let G_1 and G_2 be 4-property p-groups such that $|G_1| = p^{n_1}$, $|G_2| = p^{n_2}$, $G_1' =< a_1 >$, and $G_2' =< a_2 >$. Then the group $G = G_1 \times G_2/ < (a_1, a_2) >$ is also a 4-property p-group.*

Proof. Let $< (a_1, a_2) >= N$ so that

$$G = (G_1 \times G_2)/ < (a_1, a_2) >= (G_1 \times G_2)/N.$$

We will show that G satisfies properties (1) through (4) of a 4-property p-group.

1. Property (1) : Clearly,

$$|G| = \frac{|G_1||G_2|}{| < (a_1, a_2) > |} = \frac{p^{n_1} p^{n_2}}{p} = p^{n_1+n_2-1}.$$

   Hence G is a p-group.

2. Property (2) : Let $A = (u_1, u_2)N \in G'$ and let $B = (v_1, v_2)N \in G'$. The commutator of A and B is

$$[A, B] = ([u_1, v_1], [u_2, v_2])N = (a_1^i, a_2^j)N$$

for some $a_1^i \in G_1'$ and $a_2^j \in G_2'$ Then

$$[A, B] = (a_1^{i-j}, e_2)N = (e_1, a_2^{j-i})N.$$

Hence $G' =< (a_1, e_2)N >$ and $|G'| = | < (a_1, e_2)N > | = | < a_1 > | = p$.

3. Property (3) : First we will show that $Z(G) \subseteq G'$. Let $A = (u, v)N \in Z(G)$ and $g \in G_1$. Then A commutes with all elements of G, specifically with $(g, e_2)N$. Then $ugu^{-1}g^{-1} = e_1 = a^p$. Further,

$$[A, (g, e_2)N] = (ugu^{-1}g^{-1}, e_2)N = N,$$

and so $(ugu^{-1}g^{-1}, e_2) \in N$. This implies that $(ugu^{-1}g^{-1}, e_2) = (a_1^i, a_2^i)$ for some i. Consequently, p/i, so $a_1^i = e_1$ and $u \in Z(G_1)$. Similarly, $v \in Z(G_2)$. Therefore,

$$(u, v) = (a_1^i, a_2^j) \in< (a_1, e_2)N >= G'$$

and $Z(G) \subseteq G'$.

Next we will show that $G' \subseteq Z(G)$. Let $a \in G'$. By the proof of property (2), $a \in G' =< (a_1, e_2) > N$ where $N =< (a_1, a_2) >$. Then for some positive integers i, l, s, t,

$$a = (a_1^i, e_2)(a_1^s, a_2^t) = (a_1^l, a_2^t) \in Z(G).$$

Hence $G' \subseteq Z(G)$ and we conclude that $G' = Z(G)$.

4. Property (4) : Since $G/Z(G) = G/G'$, $G/Z(G)$ is Abelian. Also, $|G/Z(G)| = |G/G'| = p^{n_1+n_2-2}$ so $G/Z(G)$ is a p-group. Hence $G/Z(G) \cong \Pi_{i=1}^{s}\mathbb{Z}_{p_i}$ for some $s \in \mathbb{N}$.

Therefore, G satisfies the four properties of a 4-property p-group. $\square$

Finally, we are ready to construct our example of a group with commutativity degree $\frac{1}{2}(1+\frac{1}{2^{2m}})$ for $m \in \mathbb{N}$.

Example 4.3.5. *The class of Q_8^s groups.* First we will verify that the quaternion group,

$$Q_8 =< a, b : a^4 = b^4 = e, a^2 = b^2, ba = ab^3 >,$$

is a 4-property 2-group. Notice that $|Q_8| = 2^3$. Also, $Q_8' = Z(Q_8) = \{e, b^2\} \cong \mathbb{Z}_2$ and so $|Q_8'| = 2$. This verifies properties (1) through (3). In addition, $Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This verifies property (4). Hence Q_8 is a 4-property 2-group. By Equation 4.7 the commutativity degree of Q_8 is

$$P(Q_8) = \frac{1}{2}(1 + \frac{1}{2^{3-1}}) = \frac{5}{8}.$$

(Note that this value is also calculated in Table 1.2.)

Next let $< (b^2, b^2) >= N_2$ and then let $Q_8^2 = Q_8 \times Q_8/N_2$. By Lemma 4.3.4, Q_8^2 is a 4-property 2-group. Notice that $|Q_8^2| = \frac{2^3 \cdot 2^3}{2} = 2^5$.Then by Equation 4.7

$$P(Q_8^2) = \frac{1}{2}(1 + \frac{1}{2^4}) = \frac{17}{32}.$$

Let $j > 2$ and assume that for $i = 1, 2, ..j-1$, Q_8^i is a 4-property 2-group such that $|Q_8^i| = p^n$ with $n-1$ even. Then set $< (b^2, N^{j-1}) >= N^j$ and define

$$Q_8^j = Q_8 \times Q_8^{j-1}/N^j.$$

Notice that

$$|Q_8^j| = \frac{|Q_8| \cdot |Q_8^{j-1}|}{N^j} = \frac{2^3 \cdot 2^n}{2} = 2^{2+n}$$

with $n+1$ even. By Lemma 4.3.4, Q_8^j is a 4-property 2-group and

$$P(Q_8^j) = \frac{1}{2}\left(1 + \frac{1}{2^{j-1}}\right).$$

By induction, for each odd integer $s > 2$, the group Q_8^s is a 4-property 2-group and has commutativity degree

$$P(Q_8^s) \quad = \quad \frac{1}{2}\left(1 + \frac{1}{2^{s-1}}\right).$$

Since s is odd, $s - 1 = 2m$ for some $m \in \mathbb{N}$, and

$$P(Q_8^s) \quad = \quad \frac{1}{2}\left(1 + \frac{1}{2^{2m}}\right).\blacksquare$$

Example 4.3.6. *Generalization of the Q_8^s Groups* To generalize the Q_8^s example to any prime $p > 2$, we begin with an analogous group to Q_8,

$$Q_{p^3} = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z}_p \right\}.$$

Like Q_8 this group is a 4-property p-group and $Q'_{p^3} =< b >$. By Equation 4.6

$$P(Q_{p^3}) = \frac{p^2 + p - 1}{p^3}.$$

Like the Q_8^s example, it can be shown that $Q_{p^3}^2 = Q_{p^3} \times Q_{p^3}/N_2$, where $N_2 =< (b, b) >$, is a 4-property p-group. Similarly, it can also be shown that for odd integers $s > 2$, the group

$$Q_{p^3}^s = Q_{p^3} \times Q_{p^3}^{s-1}/ < (b, N^{s-1}) >= Q_{p^3} \times Q_{p^3}^{s-1}/N^s$$

has commutativity degree

$$P(Q_{p^3}) = \frac{p^{s-1} + p^2 - p}{p^{s+1}}.\blacksquare$$

4.4 Wreath Products

Let H be a group and let $N = H \times H \times ... \times H$ be the product of n copies of H. Let $A \leq S_n$. Recall from Section 3.1 that the wreath product of N by A with respects to

n is the semidirect product of N by A and is denoted $G = N$ Wr A. Recall that the order of G is $|G| = |N||A| = |H|^n|A|$.

We will calculate the commutativity degree for select types of wreath products. We will also use a type of wreath product as an example showing that nilpotence class is not a good measure of commutativity degree.

4.4.1 The Wreath Product of $\Pi_{i=1}^{p}\mathbb{Z}_q$ and a p-Cycle

Let q, p be primes, $\mathbb{Z}_q$ the cyclic group of order q, and A be the subgroup of S_p generated by a p-cycle. Let $G = \Pi_{i=1}^{p}\mathbb{Z}_q$ Wr A. We will compute the commutativity degree of G using the class equation.

First we compute the inverse of an element. Let $(n, \sigma) \in G$ and suppose that $(m, \tau) = (n, \sigma)^{-1}$. Then

$$(\bar{e}, e) = (n, \sigma)(m, \tau) = (n\sigma m, \sigma\tau) \tag{4.8}$$

so that $m = \sigma^{-1}n^{-1}$ and $\tau = \sigma^{-1}$. Hence $(n, \sigma)^{-1} = (\sigma^{-1}n^{-1}, \sigma^{-1})$.

Next we find $Z(G)$. Suppose that $y = (n_1, \sigma_1) \in Z(G)$ such that $n_1 = (h_1, h_2, ..h_p)$. Let $x = (n_2, \sigma_2) \in G$ so that $n_2 = (h'_1, h'_2...h'_p)$. Then

$$\begin{aligned} yx &= (n_1\sigma_1 n_2, \sigma_1\sigma_2) \\ xy &= (n_2\sigma_2 n_1, \sigma_2\sigma_1), \end{aligned}$$

and setting $yx = xy$ yields

$$(n_1\sigma_1 \cdot n_2, \sigma_1\sigma_2) = (n_2\sigma_2 n_1, \sigma_2\sigma_1). \tag{4.9}$$

Expanding the first coordinate of Equation 4.9,

$$(h_1, h_2, ..h_p)(h'_{\sigma_1(1)}, h'_{\sigma_1(2)}, ...h'_{\sigma_1(p)}) = (h'_1, h'_2...h'_p)(h_{\sigma_2(1)}, h_{\sigma_2(2)}..., h_{\sigma_2(p)}). \tag{4.10}$$

Suppose that $\sigma_2 = e$. Equating the entries in each coordinate in Equation 4.10 yields

$$h_i h'_{\sigma_1(i)} = h'_i h_i$$

for $1 \leq i \leq p$. Then

$$h'_{\sigma_1(i)} = h'_i$$

because N is Abelian. Hence $\sigma_1 = e$.

Next suppose that $\sigma_2 = (1, 2, .., p)$. Again by equating the entries in the each coordinate of Equation 4.10,

$$h_i h'_i = h'_i h_{i+1}$$

for $1 \leq i < p$. Then

$$h_i = h_{i+1}$$

for $1 \leq i < p$, again because N is Abelian. (Similarly, if $i = p$ then $h_p = h_1$). Thus $h_1 = h_2 = ... = h_p$.

Finally, if $\sigma_2 = (1, 2, .., p)^s$ with $1 < s < p$, then similar calculation yields that for each i, $1 \leq i \leq p$, there is a $j \neq i$, $1 \leq j \leq p$, such that $h_i h'_i = h'_i h_j$. Then $\sigma_2(h_j) = h_i$ and it follows that $h_1 = h_2 = h_3 = .. = h_p$. Hence

$$Z(G) = \{((h_1, h_2, ..h_p), e) : h_1 = h_2 = ... = h_p\}$$

and $|Z(G)| = q$ since there are q choices for h_1.

As a side note, notice that

$$Z(N \text{ Wr } A) = \{((h_1, h_2, ..h_p), e) : h_1 = h_2 = ... = h_p\}$$

whenever $A \leq S_n$ is a primitive subgroup of S_n. (A primitive subgroup has the property that given $1 \leq i \leq j \leq n$, there is a $\sigma \in A$ such that $\sigma(i) = j$).

Next we calculate the conjugacy class of a noncentral element that is in the subgroup (N, e). Let $(n, e) \in (N, e)$ such that $(n, e) \notin Z(G)$. Then

$$(N, e) \subset C_G((n, e)) \subsetneq G,$$

and

$$q^p = |(N,e)| \leq |C_G((n,e))| < |G| = pq^p.$$

Then $q^p = |C_G((n,e))|$. There is a conjugacy class $[(n,e)] = [G : C_G((n,e))]$ with $|[(n,e)]| = \frac{|G|}{q^p} = p$ for each such (n,e). There are $q^p - q$ elements in (N,e) that are not in the center of G (because $|Z(G)| = q$ and $Z(G) \subset (N,e)$). Then there are

$$\frac{q^p - q}{p} \tag{4.11}$$

conjugacy classes of order p of this type.

All that remains is to partition the noncentral elements that are not in (N,e) into conjugacy classes. Let $(n,\sigma) \in G$ with $n = (h_1, h_2, ..h_p)$ such that $(n,\sigma) \notin (N,e)$ and let $(m,\rho) \in G$ with $m = (a_1, a_2, ..a_p)$. (Since $Z(G) \subset (N,e)$, $(n,\sigma) \notin Z(G)$). Then

$$\begin{aligned}
\phi_{(m,\rho)}(n,\sigma) &= (m,\rho)(n,\sigma)(\rho^{-1}m^{-1},\rho^{-1}) \\
&= (m\rho n, \rho\sigma)(\rho^{-1}m^{-1},\rho^{-1}) \\
&= (m\rho n\rho\sigma\rho^{-1}m^{-1}, \rho\sigma\rho^{-1}) \\
&= (m\sigma m^{-1}\rho n, \sigma) \\
&= ((a_1, a_2, .., a_p)(a_{\sigma(1)}^{-1}, a_{\sigma(2)}^{-1}, .., a_{\sigma(p)}^{-1})\rho n, \sigma).
\end{aligned}$$

Suppose $\rho = e$. Then

$$\begin{aligned}
\phi_{(m,\rho)}(n,\sigma) &= ((h_1 a_1 a_{\sigma(1)}^{-1}, h_2 a_2 a_{\sigma(2)}^{-1}, .., h_p a_p a_{\sigma(p)}^{-1}), \sigma) \\
&= ((h_1 y_1, h_2 y_2, ..., h_p y_p), \sigma) \qquad (4.12)
\end{aligned}$$

for some $y_1 \in N$ such that $y_1 = y_p^{-1} y_{p-1}^{-1} ... y_3^{-1} y_2^{-1}$. Then $1 = y_1 y_2 ... y_p$ and

$$h_1 y_1 h_2 y_2, ...h_p y_p = h_1 h_2 ... h_p,$$

since N is Abelian. There are q^{p-1} elements of the type described in Equation 4.12 because there are q choices for $y_2, ..., y_p$ and once selected, these fix y_1. Let

$M = \{((h_1y_1, h_2y_2, h_3y_3..., h_py_p), \sigma) : 1 = y_1y_2...y_p\}$. Then $[(n, \sigma)] \supseteq M$. Let $M' = \{((b_1, b_2, ..b_p), \sigma) : b_1b_2...b_p = h_1h_2...h_p\}$. Then $|M| = |M'|$, and it follows that $M = M'$.

Next suppose $\rho \neq e$. Then

$$\begin{aligned} \phi_{(m,\rho)}(n, \sigma) &= ((h_{\rho(1)}a_1a^{-1}_{\sigma(1)}, h_{\rho(2)}a_2a^{-1}_{\sigma(2)}, .., h_{\rho(p)}a_pa^{-1}_{\sigma(p)}), \sigma) \\ &= ((h_{\rho(1)}y_1, h_{\rho(2)}y_2, ..., h_{\rho(p)}y_p), \sigma) \end{aligned} \tag{4.13}$$

for some $y_1 \in N$ such that $y_1 = y_p^{-1}y_{p-1}^{-1}...y_3^{-1}y_2^{-1}$. Then

$$h_{\rho(1)}y_1h_{\rho(2)}y_2...h_{\rho(p)}y_p) = h_{\rho(1)}h_{\rho(2)}..h_{\rho(p)} = h_1h_2...h_p$$

since N is Abelian. Then $((h_\rho(1)y_1, h_{\rho(2)}y_2, ..., h_{\rho(p)}y_p), \sigma) \in M'$ and $|[(n, \sigma)]| = |M'| = q^{p-1}$. There are

$$|G| - |N| = pq^p - q^p$$

noncentral elements of G that are not in N and we partition these into

$$\frac{|G| - |N|}{q^{p-1}} = \frac{pq^p - q^p}{q^{p-1}} = qp - q$$

many conjugacy classes of this type. Summing the center, conjugacy classes contained in N, and conjugacy classes disjoint from N yield the class equation:

$$|G| = q + \sum_{i=1}^{(\frac{q^p-q}{p})} p + \sum_{j=1}^{(qp-q)} q^{p-1}.$$

Then the number of conjugacy classes is

$$k(G) = q + \frac{q^p - q}{p} + (qp - q),$$

and the commutativity degree is

$$\begin{aligned} P(G) &= \frac{k(G)}{|G|} = \frac{qp + q^p - q + qp^2 - qp}{p(pq^p)} \\ &= \frac{q^p + qp^2 - q}{p(pq^p)} \\ &= \frac{q^{p-1} + p^2 - 1}{p^2q^{p-1}}. \end{aligned}$$

Therefore, for a wreath product of the type $G = \Pi_{i=1}^{p}\mathbb{Z}_q$ Wr A for primes p and q, the commutativity degree is

$$P(G) = \frac{q^{p-1} + p^2 - 1}{p^2 q^{p-1}}. \tag{4.14}$$

4.4.2 The Wreath Product of $\Pi_{i=1}^{p}\mathbb{Z}_p$ and a p-Cycle

In the case of $q = p$ the wreath product,

$$G = \Pi_{i=1}^{p}\mathbb{Z}_p \text{ Wr } < (1, 2, ..p) >$$

is a p-group and hence nilpotent. By Equation 4.14, the commutativity degree is

$$P(G) = \frac{p^{p-1} + p^2 - 1}{p^{p+1}}. \tag{4.15}$$

We also can calculate the asymptotic commutativity degree as follows:

$$\lim_{p\to\infty} \frac{p^{p-1} + p^2 - 1}{p^{p+1}} = 0.$$

One might expect that, as the nilpotence class increases the commutativity degree of the group decreases. To investigate this conjecture, we calculate the nilpotence class of G.

Lemma 4.4.1. *The commutator subgroup of* $G = \Pi_{i=1}^{p}\mathbb{Z}_p$ *Wr* $< (1, 2, .., p) >$ *is* $G' = \{((a_1, a_2, ..a_p), e) : 1 = a_1 a_2 a_3 ... a_p\}$ *and* $|G'| = p^{p-1}$.

Proof. We will find the commutators of select elements of G, then show that this set of commutators is the commutator subgroup. First suppose that $n_1, n_2 \in N$. Then

$$\begin{aligned}
[(n_1, e), (n_2, e)] &= (n_1, e)(n_2, e)(n_1^{-1}, e)(n_2^{-1}, e) \\
&= (n_1 n_2 n_1^{-1} n_2^{-1}, e) \\
&= (e, e)
\end{aligned}$$

Next suppose $\sigma = (1, 2, ..., p)$ and $n_1, n_2 \in N$. Then

$$\begin{aligned} [(n_1, e), (n_2, \sigma)] &= (n_1, e)(n_2, \sigma)(n_1^{-1}, e)(\sigma^{-1} n_2^{-1}, \sigma) \\ &= (n_1 n_2, \sigma)(n_1^{-1} \sigma^{-1} n_2^{-1}, \sigma^{-1}) \\ &= (n_1 n_2 \sigma n_1^{-1} \sigma^{-1} n_2^{-1}, \sigma\sigma^{-1}) \\ &= (n_1 \sigma_2 n_1^{-1}, e) \end{aligned} \tag{4.16}$$

where the last simplification occurs because N is Abelian. By expanding $n_1 = (h_1, h_2, ..., h_p)$ and $n_2 = (h_1^{-1}, h_2^{-1}, ..., h_p^{-1})$, in Equation 4.16,

$$\begin{aligned} [(n_1, e), (n_2, \sigma)] &= ((h_1 h_2^{-1}, h_2 h_3^{-1}, h_3 h_4^{-1} ... h_p h_1^{-1}), e) \\ &= ((a_1, a_2, a_3 ... a_p), e) \end{aligned} \tag{4.17}$$

for some $a_1 \in N$ such that $a_1 = a_p^{-1} a_{p-1}^{-1} ... a_3^{-1} a_2^{-1}$. Then $1 = a_1 a_2 ... a_p$ and there are p^{p-1} elements of this type (as arbitrarily chosen $a_2, a_3, ... a_p$ determine a_1). Let $M = \{((a_1, a_2, a_3 ... a_p), e)\}$. Then $G' \supseteq M$. Next we claim that $G' = M$. To verify the claim, let $(n, \sigma) \in G$ and $(m, e) \in M$ and then conjugate (m, e) by (n, σ) as follows:

$$(n, \sigma)(m, e)(\sigma^{-1} n^{-1}, \sigma^{-1}) = (n \sigma m \sigma^{-1} n^{-1}, e).$$

Since, σ^{-1} permutes the entries of n^{-1} and since N is Abelian

$$(n \sigma m \sigma^{-1} n^{-1}, e) = (n \sigma^{-1} n^{-1} \sigma m, e) = (\sigma m, e).$$

Also, σm permutes the entries of m, so $(\sigma m, e) \in M$. Thus $M \lhd G$. Then, since $|M| = p^{p-1}$, $|G/M| = p^2$ so G/M is Abelian. It follows that $G' \leq M$. Therefore, $G' = M$ and

$$G' = \{((a_1, a_2, ..a_p), e) : 1 = a_1 a_2 ... a_p\}$$

where $|G'| = p^{p-1}$. $\square$

Proposition 4.4.2. *The nilpotence class of* $G = \Pi_{i=1}^{p} \mathbb{Z}_p \ Wr \ < (1, 2, .., p) >$ *is* p.

Proof. Recall that the lower central series of G is

$$G = L^{(0)} \rhd L^{(1)} \rhd L^{(2)} \rhd ... \rhd L^{(n)} = \{e\}$$

where $L^{(i)} = [G, L^{(i-1)}]$ and n is the nilpotence class. First we find the maximal possible nilpotence class of G. Since $L^{(i)}$ is a proper subgroup of $L^{(i-1)}$, $|L^{(i)}| \leq \frac{|G^{(i-1)}|}{p}$. By Lemma 4.4.1 $|L^{(1)}| = |G'| = p^{p-1}$, so the maximal length of the lower descending series is $i = p$. Hence the maximal possible nilpotence class is $n = p$.

Next we find an element in each $L^{(i)}$ for $2 \leq i \leq p$. Let $((h^{-1}, h, e, ..e), e) \in L^{(1)}$ and let $(e, \sigma) \in G$ where $\sigma = (1, 2..p)$. Then $[(e, \sigma)((h^{-1}, h, e, ..e), e)] \in L^{(2)}$, and

$$\begin{aligned} [(e,\sigma)((h^{-1},h,e,..e),e)] &= (e,\sigma)((h^{-1},h,e,..e),e)(e,\sigma^{-1})((h,h^{-1},e,..e),e) \\ &= (\sigma(h^{-1},h,e,..e),\sigma)(\sigma^{-1}(h,h^{-1},e,..e),\sigma^{-1}) \\ &= ((e,h^{-1},h,e,..e)(h,h^{-1},e,..e),e) \\ &= ((h,h^{-2},h,e,..e),e). \end{aligned} \tag{4.18}$$

Then $[(e, \sigma)((h^{-1}, h, e, ..e), e)] \neq (e, e)$ and so $L^{(2)} \neq e$. We can continue computing a nonidentity element in each L^{i+1} for $i + 1 < p$ as follows. Let

$$(n, e) = ((h, h^{-C(i,1)}, h^{-C(i,2)}, ..., h^{-C(i,(i-1))}, h, e, .., e), e) \in L^{(i)}.$$

a similar calculation to Equation 4.18 yields that

$$[(e,\sigma)(n,e)] \;=\; ((h, h^{-C(i+1,1)}, h^{-C(i+1,2)}, ..., h^{-C(i+1,(i-1))}, h, e, .., e), e).$$

Hence the length of the lower central series is p and G has nilpotence class $n = p$. □

As p increases, the commutativity degree asymptotically decreases like $\frac{1}{p^2}$ even though the nilpotence class, which equals p, increases. This illustrates that the nilpotence class of a group does not measure commutativity degree. Rather, notice the derived length of G is $d = 2$ since $[G', G'] = e$. In this case the derived length may be thought of as a measure of how fast the commutativity degree decreases, as $d = 2$ corresponds to the exponent of p in $\frac{1}{p^2}$. In general, we use the derived length to compute bounds on the commutativity degree, such as the derived length bounds in Section 2.2.3 and the Pyber Bound. In this case ,perhaps the most we can say more about the commutativity degree by using either the derived length or the nilpotence class would be to compute bounds using the derived length.

4.4.3 Summary of Commutativity Degree Values for Groups in Chapter 4

Tables 4.23 and 4.24 summarize the commutativity degree values computed in this chapter.

Group	Commutativity Degree
Symmetric Group S_n	$\frac{p(n)}{n!}$
4-Property p-group	$\frac{p^{s-1}+p^2-p}{p^{s+1}}$
Q_8^s groups	$\frac{1}{2}(1+\frac{1}{2^{2m}})$
$\prod_{i=1}^{p} \mathbb{Z}_q$ Wr $(1, 2...p)$	$\frac{q^{p-1}+p^2-1}{p^2q^{p-1}}$
$\prod_{i=1}^{p} \mathbb{Z}_p$ Wr $(1, 2...p)$	$\frac{p^{p-1}+p^2-1}{p^{p+1}}$

Table 4.23: **Commutativity Degree of Groups in Sections 4.2, 4.3 and 4.4**

Group/Presenation	Commutativity Degree
Rusin pn-groups $R_{pn} =< a, b : a^p = b^n = e, bab^{-1} = a^r >$	$\frac{n^2+p-1}{n^2p}$
Generalized Dihedral $D_{pq} = R_{pq} =< a, b : a^p = b^q = e, bab^{-1} = a^r >$	$\frac{q^2+p-1}{q^2p}$
$T_{p,q,m,\theta}$ Groups $T_{p,q,m,\theta} =< a, b : a^p = b^{q^m} = e, bab^{-1} = a^{\lambda^\theta} >$	$\frac{q+1}{pq}$
G_m groups $G_m =< a, b : a^3 = b^{2^m} = e, bab^{-1} = a^{-1} >$	$\frac{1}{2}$
Indecomposable Group with $P(G) = \frac{1}{2}$ $G =< a, b : a^6 = e, b^2 = a^3, bab^{-1} = a^{-1} >$	$\frac{1}{2}$
Dicyclic Groups $D_m =< a, b : a^{2m} = b^4 = e, b^{-1}ab = a^{-1}, a^m = b^2 >$	$\frac{m+3}{4m}$
Generalized Quaternion Groups $Q_{2^{n+1}} =< a, b : a^{2^n} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{n-1}} = b^2 >$	$\frac{2^{n-1}+3}{2^{n+1}}$
Dihedral Groups $D_n =< r, \rho : r^2 = e, \rho^n = e, \rho r = r\rho^{n-1} >$	$\frac{n+3}{4n}$ n odd $\frac{n+6}{4n}$ n even
Semidihedral Groups $SD_n =< a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} >$	$\frac{2^{n-1}+3}{2^{n+1}}$
Quasidihedral Groups $QD_n =< a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} >$	$\frac{5}{8}$

Table 4.24: **Commutativity Degree of Order Reversing Groups**

Chapter 5: Possible Values of Commutativity Degrees

In Chapter 3 we discussed commutativity degree in terms of the structure of a group, and in Chapter 4 we built a library of commutativity degrees for given groups. In this chapter, we will focus on the set of possible values of commutativity degree.

Perhaps the most straightforward, but computationally intense, way to explore possible commutativity degrees is to calculate the commutativity of every group G starting with $|G| = 1$ and continuing to $|G| = n$ for some fixed integer n. In GAP, we calculated the commutativity degree of all groups of order less than 101 by finding the number of conjugacy classes of each group of each order. Tables 5.1 and 5.2 list the possible commutativity degrees of groups of each order.

As expected, no values in Table 5.1 are in the interval $(\frac{5}{8}, 1)$. The only values greater than $\frac{1}{2}$ are 1, $\frac{5}{8}$, and $\frac{17}{32}$. What additional values of commutativity degree greater than $\frac{1}{2}$ are possible? Notice that $\frac{2}{5}$ is the commutativity degree of a group whose order is a multiple of 10. This occurs because, if $|H| = 10$ and $P(H) = \frac{2}{5}$, then the direct product $G = A \times H$, where A is Abelian, has $P(G) = \frac{2}{5}$. What can we say about the commutativity degree of G if $|G/Z| \cong H$? The values $\frac{1}{2}$, $\frac{1}{3}$,and $\frac{1}{7}$ appear in the Table. Given a prime p, can the value $\frac{1}{p}$ always be realized? In this chapter, we will investigate these types of questions.

5.1 Possible Values of $P \in (\frac{1}{2}, 1)$

Recall from Section 3.2 that all groups with $P(G) > \frac{1}{2}$ are nilpotent. Also, if G is a non-Abelian group with $P(G) > \frac{1}{2}$ then $G = P_0 \times P_1 \times P_2 \times .. \times P_k$ where P_0 is a 2-group with $|P_0'| = 2$ and, for each $i > 0$, P_i an Abelian p_i-group for some prime $p_i \neq 2$ (Proposition 3.2.4). In this section we extend this result by showing

$\|G\|$	6	8	10	12	14	16	18	20	21	22
	1/2	5/8	2/5	1/2	5/14	5/8	1/2	2/5	5/21	7/22
				1/3		7/16	1/3	1/4		

$\|G\|$	24	26	27	28	30	32	34	36	38	39
	5/8	4/13	11/27	5/8	1/2	5/8	5/17	1/2	11/38	7/39
	1/2			5/14	2/5	17/32		1/3		
	3/8				3/10	7/16		1/4		
	1/3					11/32		1/6		
	7/24									
	5/24									

$\|G\|$	40	42	44	46	48	50	52	54	55	56
	5/8	1/2	7/22	13/46	5/8	2/5	4/13	1/2	7/55	5/8
	2/5	5/14			1/2	7/25	7/52	11/27		5/14
	13/40	2/7			7/16			1/3		17/56
	1/4	5/21			3/8			5/18		1/7
		1/6			1/3			5/27		
					5/16					
					7/24					
					1/4					
					5/24					
					1/6					

$\|G\|$	57	58	60	62	63	64	66	68	70	72
	3/19	8/29	1/2	17/62	5/21	5/8	1/2	5/17	2/5	5/8
			2/5			17/32	7/22	2/17	5/14	1/2
			1/3			7/16	3/11		19/70	3/8
			3/10			25/64				1/3
			1/4			11/32				7/24
			1/5			19/64				1/4
			3/10			1/4				5/24
			1/12			13/64				1/6
						1/32				1/8
										1/12

Table 5.1: **Commutativity Degrees of Groups of Order Less Than** 101 **(a)**

$\|G\|$	57	58	60	62	63	64	66	68	70
	3/19	8/29	1/2	17/62	5/21	5/8	1/2	5/17	2/5
			2/5			17/32	7/22	2/17	5/14
			1/3			7/16	3/11		19/70
			3/10			25/64			
			1/4			11/32			
			1/5			19/64			
			3/10			1/4			
			1/12			13/64			
						1/32			

$\|G\|$	72	74	75	76	78	80	81	82	84
	5/8	10/37	11/75	11/38	1/2	1/2	11/27	11/41	1/2
	1/2				4/13	7/16	17/81		5/14
	3/8				7/26	2/5			1/3
	1/3				7/39	13/40			2/7
	7/24				4/39	23/80			5/21
	1/4					1/4			5/28
	5/24					17/80			1/6
	1/6					7/40			1/7
	1/8								
	1/12								

$\|G\|$	86	88	90	92	93	94	96	98	100
	23/86	5/8	1/2	13/46	13/93	25/94	5/8,17/32	5/14	2/5
		7/22	2/5				1/2,7/16	13/94	7/25
		25/88	1/3				3/8,11/32		1/4
			3/10				1/3,5/16		4/25
			4/15				7/24,9/32		13/100
							1/4, 7/32		1/10
							5/24,19/96		
							3/16,1/6		
							7/48,13/96		
							1/8, 11/96		
							5/48		

Table 5.2: **Commutativity Degrees of Groups of Order Less Than** 101 **(b)**

that if a commutativity degree is in the interval $(\frac{1}{2}, 1)$ then it equals $\frac{1}{2}(1 + \frac{1}{2^{2m}})$, for some $m \in \mathbb{N}$. Recall that in Section 4.3 we constructed a class of examples with commutativity degree realizing the values $\frac{1}{2}(1+\frac{1}{2^{2m}})$, for all $m \in \mathbb{N}$. We conclude this section by describing all groups with commutativity degree greater than or equal to $\frac{1}{2}$.

Lemma 5.1.1. *Let $a, b \in G$, $G' \subseteq Z(G)$, and $m \in \mathbb{Z}$. Then $[a^m, b] = [a, b]^m$.*

Proof. Let $a, b \in G$, $G' \subseteq Z(G)$, and $m \in \mathbb{Z}$. Since $[a, b] \in G' \subseteq Z(G)$,

$$\begin{aligned}
[a^m, b] &= a^m b a^{-m} b^{-1} \\
&= a^{m-1} a b a^{-1} a^{-(m-1)} b^{-1} \\
&= a^{(m-1)} a b a^{-1} b^{-1} b a^{-(m-1)} b^{-1} \\
&= a^{(m-1)} [a, b] b a^{-(m-1)} b^{-1} \\
&= [a, b] a^{m-1} b a^{-(m-1)} b^{-1} \\
&= [a, b][a^{m-1}, b].
\end{aligned}$$

Repeating this process m times yields

$$[a^m, b] = [a, b]^m.$$

□

Proposition 5.1.2 (Rusin [43]). *If H is a non-Abelian 2-group with $H' = \{e, c\}$ for some $c \in H$, then $H/Z(H) \cong \prod_{i=1}^{r-1} (\mathbb{Z}_2 \times \mathbb{Z}_2)_i$ for some $r \in \mathbb{N}$.*

Proof. By Lemma 3.2.1, $H' \leq Z(H)$ because $|H'| = 2$. Then $H/Z(H)$ is Abelian. Then

$$H/Z(H) = < a_1 Z(H) > \times < a_2 Z(H) > \times ... \times < a_r Z(H) >$$

where r is called the rank of the Abelian group $H/Z(H)$ and indicates the number of cyclic groups in the direct product. We will induct on the rank r of $H/Z(H)$. If

$r = 0$, then $Z(H) = H$ and H is Abelian. If $r = 1$ then $H/Z(H)$ would be cyclic, which cannot happen.

Consider $H/Z(H) =< a_1Z(H) > \times < a_2Z(H) >$. By definition, $H' =< \{[a_i, a_j] : 1 \leq i, j \leq r\} >$ and by assumption, $H' = \{e, c\}$. Then without loss of generality, we can assume that $[a_1, a_2] = c$. By Lemma 5.1.1, $[a_1^2, a_j] = [a_1, a_j]^2 = e$ for all j. Hence $a_1^2 \in Z(H)$, but $a_1 \notin Z(H)$. Therefore, $< a_1 >\cong \mathbb{Z}_2$. Similarly, $[a_i, a_2^2] = [a_i, a_2]^2$ and $< a_2 >\cong \mathbb{Z}_2$. Then $H/Z(H) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Next let $r > 2$ and assume that the result is true for all ranks less than r.Then suppose that $H/Z(H) =< a_1Z(H) > \times < a_2Z(H) > \times ... \times < a_rZ(H) >$. Again note that $H' =< \{[a_i, a_j] : 1 \leq i, j \leq r\} >$ and $H' = \{e, c\}$. Since $H' =< c >$, $[a_i, a_j] = c^{e_{ij}}$. Let $b_i = a_i a_1^{e_{1i}} a_2^{-e_1 i}$ for $i > 2$. Then since $c \in Z(G)$,

$$\begin{aligned}
[a_1, b_i] &= a_1 b_i a_1^{-1} b_i^{-1} \\
&= a_1 (a_i a_1^{e_{1i}} a_2^{-e_1 i})(a_1^{-1})(a_i a_1^{e_{1i}} a_2^{-e_1 i})^{-1} \\
&= a_1 a_i a_1^{e_{1i}} (a_1^{-1} a_1) a_2^{-e_1 i} a_1^{-1} a_2^{e_1 i} a_1^{-e_{1i}} (a_1 a_1^{-1}) a_i^{-1} \\
&= a_1 a_i a_1^{e_{1i}} a_1^{-1} [a_1, a_2^{-e_1 i}] a_1^{-e_{1i}} a_1 a_1^{-1} a_i^{-1} \\
&= a_1 a_i a_1^{e_{1i}} a_1^{-1} c^{-e_1 i} a_1^{-e_{1i}} a_1 a_1^{-1} a_i^{-1} \\
&= c^{-e_1 i} a_1 a_i a_1^{e_{1i}} a_1^{-1} a_1^{-e_{1i}} a_1 a_1^{-1} a_i^{-1} \\
&= c^{-e_1 i} a_1 a_i [a_1^{e_{1i}}, a_1^{-1}] a_1 a_i^{-1} \\
&= c^{-e_1 i} a_1 a_i e a_1^{-1} a_i^{-1} \\
&= c^{-e_1 i} [a_1, a_i] \\
&= c^{-e_1 i} c^{e_{1i}} = e
\end{aligned}$$

Similarly, $[a_2, b_i] = e$. For $i > 2$, $< a_i > \cap < a_1, a_2 > \leq Z(H)$, and $|b_iZ(H)| = |a_iZ(H)|$. Rewriting the partition yields

$$H/Z(H) =< a_1Z(H) > \times < a_2Z(H) > \times < b_3Z(H) > \times ... \times < b_rZ(H) > .$$

Next let K be the following subgroup of H:

$$K =< Z(H), b_3, b_4, ...b_r >$$

Clearly, $Z(H) \subseteq Z(K)$. Also, since $H =< K, a_1, a_2 >$ and $[a_1, b_i] = [a_2, b_i] = e$, $Z(K) \subseteq Z(H)$. Thus $Z(K) = Z(H)$. The following properties of K allow us to apply the inductive hypothesis on K:

1. $K' \subseteq H'$, so K' is cyclic.
2. $K' \subseteq H' \leq Z(H) = Z(K)$
3. Since $K \leq H$, K is a 2-group.
4. $K/Z(K) =< b_3Z > \times < b_4Z > \times ... \times < b_rZ >$ has rank $r-2$.

Then for $H/Z(H)$ of rank r,

$$\begin{aligned} H/Z(H) &= < a_1Z(H) > \times < a_2Z(H) > \times < b_3Z > \times ... \times < b_rZ > \\ &= (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \prod_{i=1}^{r-2} (\mathbb{Z}_2 \times \mathbb{Z}_2)_i \\ &= \prod_{i=1}^{r-1} (\mathbb{Z}_2 \times \mathbb{Z}_2)_i \end{aligned}$$

by induction. □

Proposition 5.1.3. *If G is a non-Abelian group such that $P(G) > \frac{1}{2}$, then G has commutativity degree $P(G) = \frac{1}{2}(1 + \frac{1}{2^{2m}})$ with $m \in \mathbb{N}$.*

Proof. By Proposition 3.2.4, $P(G) > \frac{1}{2}$ implies that $G = H \times A$ with A Abelian and $|H| = 2^s$, for some $s \in \mathbb{N}$. Then $P(G) = P(H)P(A) = P(H)$ and it suffices to show that $P(H) = \frac{1}{2}(1 + \frac{1}{2^m})$.

By Proposition 3.2.2, $G' = \{e, a\}$ for some $a \in G$. Then $H' = \{e, a\}$ as well because H is non-Abelian and $H' \subseteq G'$. Suppose $x_i \in H$ and $x_i \notin Z(H)$. We will show that $|[x_i]| = 2$. Since $x_i \notin Z(H)$, $|[x_i]| \geq 2$. Then there is a $h \in H$ such that $hx_ih^{-1} = c$ and $c \neq x_i$. Then $hx_ih^{-1}x_i^{-1} = cx_i^{-1} \in G'$. Since $x_i \notin Z(H)$, $cx_i^{-1} = a$ and so $c = ax_i$. Therefore $[x_i] = \{x_i, ax_i\}$. Then by the class equation

$$|H| = |Z(H)| + 2(k(H) - |Z(H)|),$$

and solving for $k(H)$,

$$k(H) = \frac{|H| + |Z(H)|}{2}.$$

Therefore

$$P(H) = \frac{|H| + |Z(H)|}{2|H|} = \frac{1}{2}\left(1 + \frac{1}{|H/Z(H)|}\right).$$

Since H is a 2-group, $|H'| = 2$, and $H' \subseteq Z(H)$, $H/Z(H) \cong \prod_{i=1}^{r-1}(\mathbb{Z}_2 \times \mathbb{Z}_2)_i$ by Proposition 5.1.2. Then

$$P(H) = \frac{1}{2}\left(1 + \frac{1}{2^{2m}}\right). \tag{5.1}$$

with $m \in \mathbb{N}$. □

5.1.1 Groups with Commutativity Degree $P(G) \geq \frac{1}{2}$

All possible values of commutativity degree greater than or equal to $\frac{1}{2}$ are given by $P(G) = \frac{1}{2}$, $P(G) = 1$ or $P(G) = \frac{1}{2}(1 + \frac{1}{2^{2m}})$, for some $m \in \mathbb{N}$. In this section we will describe all groups realizing these commutativity degrees. First, note that all groups of the form $G \cong P_0 \times A$ where P_0 is a 2-group, A Abelian, and $|G'| = 2$ are classified by Blackburn in [6]. Also note that a G_m group is given by the presentation $G_m =< a, b : a^3 = b^{2^m} = e, bab^{-1} = a^{-1} >$. Recall that we showed $P(G_m) = \frac{1}{2}$ for $m \geq 1$ in Section 4.1.2.

Proposition 5.1.4. *A group G has $P(G) \geq \frac{1}{2}$ if and only if one of the following conditions holds:*

1. *G is Abelian.*

2. $G \cong P_0 \times A$ *where* P_0 *is a* 2*-group,* A *is Abelian, and* $|G'| = 2$*.*

3. $G \cong G_m \times A$*, with* $m \geq 1$ *and* A *Abelian.*

Proof. Suppose $P(G) \geq \frac{1}{2}$. By the upper degree equation bound,

$$P(G) \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right). \tag{5.2}$$

Solving this equation for $|G'|$ yields $|G'| \leq 3$. Consider the following cases:

1. $|G'| = 1$.

2. $|G'| = 2$.

3. $|G'| = 3$.

Case (1) corresponds to condition (1) in which $G' = \{e\}$ and G is Abelian.

In Case (2), G is non-Abelian and $|G'| = 2$. It follows that $P(G)$ is strictly greater than $\frac{1}{2}$ by the upper degree equation bound (Equation 5.2). Further, by Proposition 3.2.4, $G = P_0 \times A$ with P_0 a 2-group and A Abelian. By Proposition 5.1.3, the commutativity degree of G is $P(G) = \frac{1}{2}(1 + \frac{1}{2^{2m}})$, for some $m \in \mathbb{N}$. As noted above, this extensive collection of groups was classified by Blackburn in [6].

Next consider Case (3). First suppose G is nilpotent. Then G is the product of p-Sylow subgroups. Observe from Table 2.4 that $P(G) \leq \frac{1}{2}$ implies $G = H \times P_2 \times .. \times P_s$, such that for $2 \leq i \leq s$, P_i is an Abelian p-group and H is a non-Abelian 2-group. Then $H' \neq \{e\}$. Also, since H is a 2-group, $|H'| = 2^t$ for some t, $1 < t \leq s$. However, $H' \leq G'$ and so $|H'| = 3$, a contradiction. Hence G is not nilpotent.

Assume G is not nilpotent. By Proposition 3.2.3 $G/Z(G) \cong S_3$. It follows from Corollary 5.2.14 that $G \cong G_m \times A$, which $m \geq 1$ and A Abelian. $\square$

5.2 Possible Values for Groups with $|G/Z| < 12$

If $|G/Z| = n$ for a given integer n, then can we determine the possible commutativity degrees of G from the commutativity degree of the groups of order n? We will determine for which groups H, $|H| < 12$, it is possible for $G/Z \cong H$ for some group G. We address $|H| < 12$ because of the increasing complexity and number of groups with order 12. Then, with one exception, for those groups H we will find the commutativity degree of all groups G with $G/Z \cong H$. First we will address the cases when H is Abelian. Then we will consider the remaining cases, when H is non-Abelian.

5.2.1 Groups with G/Z Abelian

We begin with the simple, but important, result for the case when G/Z is cyclic.

Proposition 5.2.1. *If G/Z is cyclic, then $P(G) = 1$.*

Proof. Suppose G/Z is cyclic and generated by gZ. Let $a, b \in G$. Then $a = g^i z_1$ and $b = g^j z_2$, for some $z_1, z_2 \in Z$. Hence

$$ab = g^i z_1 g^j z_2 = g^i g^j z_1 z_2 = g^j g^i z_2 z_1 = g^j z_2 g^i z_1 = ba,$$

and G is Abelian. □

Next we consider the case when $|G/Z| \cong V_4$, the Klein 4-group. This result is referenced in a number of our sources, including [18], [21], [37], and [43]. Alternate versions of our proof appear in Joseph [26]and Lescot [32].

Proposition 5.2.2. *If G is a finite group, then $P(G) = \frac{5}{8}$ if and only if $G/Z(G) \cong V_4$,the Klein 4-group.*

Proof. Suppose $G/Z(G) \cong V_4$. Also suppose $a \in G$ and $a \notin Z(G)$. Since $Z(G) \subseteq C_G(a)$ and $a \notin Z(G)$, $Z(G) \subsetneq C_G(a)$. Also, $C_G(a) \subsetneq G$ because $a \in C_G(a)$ but $a \notin Z(G)$. Hence

$$|Z(G)| < |C_G(a)| < |G|.$$

Also notice that $|C_G(a)|$ divides $|G|$. Thus $|C_G(a)| = 2|Z(G)|$, and it follows that $|[a]| = [G : C_G(a)] = 2$.

Since $|[x_i]| = 2$ for all $x_i \notin Z(G)$ the class equation is

$$\begin{aligned} |G| &= |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} |[x_i]| \\ &= |Z(G)| + 2(k(G) - |Z(G)|). \end{aligned} \tag{5.3}$$

Solving for $k(G)$ yields

$$k(G) = \frac{|G| + |Z(G)|}{2},$$

and then

$$P(G) = \frac{|G| + |Z(G)|}{2|G|}.$$

Finally,

$$P(G) = \frac{4|Z(G)| + |Z(G)|}{8|Z(G)|} = \frac{5}{8}.$$

Conversely, assume $P(G) = \frac{5}{8}$ and let $|G/Z(G)| = l$. Suppose $a \in G$ and $a \notin Z(G)$. First we will show that $|[a]| = 2$. Suppose $|[a]| > 2$. Then $[a] = [x_i]$ for some x_i in the class equation, and

$$\begin{aligned} |G| &= |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} |[x_i]| \\ &> |Z(G)| + 2(k(G) - |Z(G)|). \end{aligned}$$

Solving for $k(G)$ yields

$$k(G) < \frac{|G| + |Z(G)|}{2},$$

and then

$$P(G) < \frac{|G| + |Z(G)|}{2|G|}.$$

Since $|G| = l|Z(G)|$,

$$P(G) < \frac{(l+1)|Z(G)|}{2l|Z(G)|} = \frac{l+1}{2l}.$$

Since $G/Z(G)$ is not cyclic, $l = |G/Z(G)| \geq 4$. Also, the function $f(l) = \frac{l+1}{2l}$ is a decreasing function. Consequently

$$P(G) < \frac{l+1}{2l} \leq \frac{5}{8},$$

a contradiction. Therefore if $a \notin Z(G)$, then $|[a]| = 2$.

Now we will show that $G/Z \simeq V_4$. Since $|[x_i]| = 2$ for all $x_i \notin Z(G)$, the class equation is

$$|G| = |Z(G)| + 2(k(G) - |Z(G)|),$$

Solving for $k(G)$ yields

$$k(G) = \frac{|G| + |Z(G)|}{2}.$$

Then

$$\frac{5|G|}{8} = \frac{|G| + |Z(G)|}{2}$$

because $\frac{5}{8} = \frac{k(G)}{|G|}$. Next, since $|G| = l|Z(G)|$,

$$5l|Z(G)| = 4l|Z(G)| + 4|Z(G)|.$$

Finally, solving for l yields $l = 4$. Hence $|G/Z(G)| = 4$. Since $G/Z(G)$ is not cyclic, $G/Z(G) \cong V_4$. $\square$

Next recall the p-upper bound: if p is the smallest prime dividing $|G/Z(G)|$, then $P(G) \leq \frac{p^2+p-1}{p^3}$. We generalize Proposition 5.2.2 to show that the p-upper bound is realized when $|G/Z(G)| = p^2$. An alternate proof also appears in Joseph [26].

Proposition 5.2.3. *Let p be the smallest prime dividing $|G/Z(G)|$. Then $P(G) = \frac{p^2+p-1}{p^3}$ if and only if $|G/Z(G)| = p^2$.*

Proof. Suppose $|G/Z(G)| = p^2$. Let $x \in G$ such that $x \notin Z(G)$. Since $x \in C_G(x)$ and $x \notin Z(G)$, $C_G(x) \subsetneq G$. Also, $Z(G) \subsetneq C_G(x)$ because $Z(G) \subseteq C_G(x)$ but $x \notin Z(G)$. Thus

$$|Z(G)| < |C_G(x)| < |G|.$$

Then $|C_G(x)| = p|Z(G)|$ because $|C_G(x)|$ divides $|G|$. Therefore $|[x]| = [G : C_G(x)] = p$. Since $|[x_i]| = p$ for all $x_i \notin Z(G)$, the class equation is

$$|G| = |Z(G)| + p(k(G) - |Z(G)|).$$

Solving for $k(G)$ yields

$$k(G) = \frac{|G| + (p-1)|Z(G)|}{p}.$$

Since $|G| = p^2|Z(G)|$,

$$k(G) = \frac{p^2|Z(G)| + p|Z(G)| - |Z(G)|}{p},$$

and

$$P(G) = \frac{(p^2+p-1)|Z(G)|}{p^3|Z(G)|} = \frac{p^2+p-1}{p^3}.$$

Conversely, assume that $P(G) = \frac{p^2+p-1}{p^3}$ and let $|G/Z(G)| = l$. Let $a \in G$. If $a \notin Z(G)$, $|[a]| \geq p$ because $|[a]|$ divides $|G/Z|$. Then the class equation yields

$$|G| \geq |Z(G)| + p(k(G) - |Z(G)|),$$

and solving for $k(G)$,

$$k(G) \leq \frac{|G| + (p-1)|Z(G)|}{p}.$$

The commutativity degree is

$$\begin{aligned} P(G) &\leq \frac{|G| + (p-1)|Z(G)|}{p|G|} \\ &= \frac{(p-1)|Z(G)| + l|Z(G)|}{pl|Z(G)|} \\ &= \frac{(p-1)+l}{pl}. \end{aligned}$$

By assumption

$$\frac{p^2+p-1}{p^3} \leq \frac{(p-1)+l}{pl}.$$

Then we solve for l in terms of p as follows

$$\begin{aligned} (p^3+p^2-p)l &\leq p^4 - p^3 + p^3 l \\ (p^2-p)l &\leq p^4 - p^3 \\ l &\leq \frac{p^4-p^3}{p^2-p} \\ l &\leq \frac{p^2(p-1)}{(p-1)} \\ l &\leq p^2. \end{aligned}$$

Hence $|G/Z(G)| \leq p^2$. If $|G/Z(G)| = q$ for some q such that $p \leq q < p^2$, then $q = p$ because p is the smallest prime dividing the order of $G/Z(G)$. Then $G/Z(G)$ is cyclic and G is Abelian. Consequently, $|G/Z(G)| = p^2$. □

Corollary 5.2.4 describes the structure of a group having commutativity degree $\frac{p^2+p-1}{p^3}$.

Corollary 5.2.4. *If $P(G) = \frac{p^2+p-1}{p^3}$ where p is the smallest prime dividing $|G/Z(G)|$, then $G \cong P \times A$ such that P is a p-group and A is Abelian.*

Proof. Let p be the smallest prime dividing $|G/Z(G)|$. By Proposition 5.2.3 $|G/Z(G)| = p^2$, so G is nilpotent (of nilpotence degree 2). Then for some m,

$$G = P \times P_1 \times P_2 \times ... \times P_m$$

where each P_i is a p_i-Sylow subgroup of order $p_i^{n_i}$ and P is a p-Sylow subgroup of order p^n. Since $|G/Z(G)| = p^2$,

$$|Z(G)| = p^{n-2} p_1^{n_1} p_2^{n_2} ... p_m^{n_m}.$$

Also, for each i, $P_i \subset Z(G)$ and each p_i-Sylow subgroup is Abelian. Therefore

$$G \cong P \times A$$

such that $A = P_1 P_2 .. P_m$ is Abelian. □

Next we will find all groups H such that if H is two-generated and Abelian, then it is possible for $G/Z \cong H$ for some group G. For each such two-generated group H with $|H| < 12$, we will find the commutativity degree for those groups G with $G/Z \cong H$. We will apply the following theorem from Rusin (Proposition (2), [43]) which is a more general form of Proposition 5.1.2. The proof is similar and we omit it here.

Theorem 5.2.5 (Proposition 2). *If H is a p-group with $H' \leq Z(H)$ and H' cyclic then $H/Z \cong \prod_i (\mathbb{Z}_{p_i{}^{n_i}} \times \mathbb{Z}_{p_i{}^{n_i}})$ with $n_1 = k$ and all $n_i \leq k$ (where $|H| = p^k$). In particular, H/Z is a square and is at least $|H'|^2$.*

Additionally, the following two properties of commutators will be applied. First, recall the definition that $[a, b] = c$ if and only if $ab = cba$. We will use this to rewrite group elements in a consistent form via a "turn around rule". Secondly, we will calculate commutators of noncentral elements using the property that if $z \in Z(G)$ and $a, b \notin G$ then

$$[a, bz] = [az, b]aba^{-1}b^{-1}zz^{-1} = [a, b].$$

Lemma 5.2.6. *Let G be a finite group and denote G/Z by $\overline{Z}$.*

(1) Suppose that $\overline{G}$ is generated by $\overline{x}_0, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$ such that $[x_i, x_j] = x_0^{n_{ij}}$ for $0 <$

$i, j \leq k$. Then G' is cyclic.

(2) Suppose that G/Z is two generated and Abelian. Then G' is cyclic.

Proof. (1) Let $\overline{G}$ be generated by $\overline{x}_0, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$ such that $[x_i, x_j] = x_0^{n_{ij}}$ for $0 < i, j \leq k$.

First we will show that

$$[x_i, x_0^m] = x_0^{mn_{i0}} \tag{5.4}$$

for each $i \leq k$ by inducting on m. Since $[x_i, x_0] = x_0^{n_i0}$, $x_i x_0 = x_0^{n_{i0}} x_0 x_i$. Then, applying the "turn around rule",

$$\begin{aligned} x_i x_0^m &= x_0^{m-1} x_0^{n_{i0}} x_0 x_i \\ &= (x_0^{n_{i0}})^m x_0^m x_i \\ &= (x_0^{mn_{i0}}) x_0^m x_i \end{aligned}$$

with the last equation resulting from the induction hypothesis. Hence $[x_i, x_0^m] = x_0^{mn_{i0}}$.

Next let $a \in G$. Then $a = w_1 w_2 ... w_{l_a} z_a$ with $w_j = x_{i_j}$ for some i_j and $z_a \in Z(G)$. For each i, $0 \leq i \leq k$, we will show that $[x_i, a] \in < x_0 >$ by inducting on l_a. If $l_a = 1$, then $i = 0$ and $[x_i, w_1] = x_0^{n_i1}$.Assume that, for $l_a < k$ and for each i, $[x_i, a] = x_0^{n_{ia}}$ for some n_{ia}. Then, supposing $l_a = k$, recall

$$[x_i, a] = [x_i, w_1 w_2 ... w_{l_a} z_a] = [x_i, w_1 w_2 ... w_{l_a}]$$

By the induction hypothesis and Equation 5.4

$$\begin{aligned} x_i w_1 w_2 ... w_{l_a} &= (x_0^{n_{i1}} w_1 x_i)(w_2 ... w_{l_a}) \\ &= x_0^{n_{i1}} w_1 (x_0^{n_{ia}})(w_2 ... w_{l_a} x_i) \\ &= (x_0^{n_{i1}} x_0^{n_{ia}}) w_1 w_2 ... w_{l_a} x_i. \end{aligned}$$

Then by induction $[x_i, a] = x_0^{n_{i1} + n_{ia}} \in < x_0 >$.

Let $b = w_1 w_2 ... w_{l_b} z_b \in G$. By a second similar induction on l_b, we can show that $[b, a] \in < x_0 >$. Since G' is generated by commutators, $G' \leq < x_0 >$. Therefore G' is cyclic.

(2) Let x_1 and x_2 be generators of G/Z and suppose $[x_1, x_2] = x_0$. Since G/Z is Abelian, $x_0 \in Z(G)$. Then all commutators of pairs of x_1, x_2, x_0 are powers of x_0 and by (1) G' is cyclic. □

Proposition 5.2.7. *Let G be a p-group and let $G/Z(G)$ be a 2-generated Abelian group. Then $G/Z(G) \cong \mathbb{Z}_{p^i} \times \mathbb{Z}_{p^i}$ for some prime p.*

Proof. Since $G/Z(G)$ is Abelian, G is nilpotent and $G' \subset Z(G)$. By Lemma 5.2.6, G' is cyclic. Also, because $G/Z(G)$ is two generated and Abelian, $G/Z(G)$ is of the form $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}}$ for primes p_1, p_2. Hence by Theorem 5.2.5 (Rusin's Proposition 2) , $G/Z(G) \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$ for some prime p and some $n \in \mathbb{N}$. □

Corollary 5.2.8. *Let p be prime. Then $G/Z \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $P(G) = \frac{p^2+p-1}{p^3}$ where p is the smallest prime dividing $|G/Z|$.*

Proof. Suppose $P(G) = \frac{p^2+p-1}{p^3}$. By Corollary 5.2.4, $G \cong P \times A$ where P is a p-group and A is Abelian so that $P(G) = P(P)$. Hence, with no loss of generality, assume G is a p-group. Then G is nilpotent. It follows that $G/Z = \prod_{i=1}^{n} \mathbb{Z}_p$ for some $n \in \mathbb{N}$. By Proposition 5.2.3, $|G/Z| = p^2$, so $G/Z \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Conversely, by Proposition 5.2.3, since $|G/Z| = p^2$ then $P(G) = \frac{p^2+p-1}{p^3}$. □

Corollary 5.2.9. *If $G/Z \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then $P(G) = \frac{11}{27}$.*

Proof. The result follows from Corollary 5.2.8. □

Corollary 5.2.10. *For a group G, G/Z cannot be isomorphic to any group of the form $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ for a prime p and positive integers $i \neq j$.*

Proof. Suppose $G/Z \cong \mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ and $i \neq j$. Then G is nilpotent and

$$G = P \times P_1 \times P_2 \times .. \times P_n$$

where P is a p-group and each P_i is a p_i-group for some $p_i \neq p$. Further each group $P_i \in Z(G)$, and the product $P_1 \times P_2 \times .. \times P_n$ is Abelian. Hence we may assume G is a p-group with no loss of generality.

By Proposition 5.2.7, $i = j$, a contradiction. Therefore, there is no group G with $G/Z \cong \mathbb{Z}_p^i \times \mathbb{Z}_p^j$ if $i \neq j$. □

Corollary 5.2.11. *There is no group G with G/Z isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.*

Proof. Follows directly from Corollary 5.2.10. □

For those groups G with $|G/Z| < 12$, there is only one group H such that if $G/Z \cong H$ then the commutativity degree of G cannot be determined from H. This group is $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ We provide an example of two groups having $G/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ but with different commutativity degrees.

Example 5.2.12. *Two Groups with $G/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

In GAP, we identified all groups G_i with $|G_i| = 64$ satisfying the property that $|G_i/Z| = 8$. We calculated G_i/Z for each such group G_i and found that $G_i/Z \cong D_4$ or $G_i/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for each group. We located these groups by comparing the SmallGroup ID Tag of G_i/Z with the ID Tags of groups of order 8.

We began calculating the commutativity degree of each group satisfying $G_i/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by reading the number of conjugacy classes from the Character Tables. By systematically testing these groups, we found the ID Tags for two groups, $G_1 = [64, 75]$ and $G_2 = [64, 63]$, with $P(G_1) = \frac{11}{32}$ and $P(G_2) = \frac{7}{16}$. Presentations of G_1 and G_2 are:

$$G_1 =< a, b, c : a^2 = b^2 = c^4 = e, (ac^{-2})^2 = (bc^{-2})^2, (ac^{-1})^4 = (bc^{-1})^4,$$

$$(bac^{-1})^3 = ca^{-1}b^{-1}, (bab)^{-1} = ac^2, (cbca)^{-1} = c^{-1}bca >$$

and

$$G_2 =< a, b, c : a^4 = b^4 = c^4 = e, ab^2 = b^2a, ba^2 = a2b, ac^2 = c^2a, ca^2 = a^2c, bc = cb,$$

$$abc^{-1} = bac, acb^{-1} = cab >.$$

For more information about GAP and descriptions of the methods and notation in this example see Appendix 6.1.

5.2.2 Groups with non-Abelian G/Z

There are four non-Abelian groups of order less than 12. These groups are the symmetric group S_3, the dihedral groups D_4 and D_5, and the quaternion group Q_8. We will show that if a group G has $G/Z \cong S_3$, then G can be written as a direct product of S_3 with an Abelian group and that $P(G) = \frac{1}{2}$. Then we will find a formula for a group G with $G/Z \cong D_n$ for any $n \in \mathbb{N}$. We will calculate the commutativity degree of D_4 and D_5 from this formula. Finally, we will show that there is no group G with $G/Z \cong Q_8$.

First, we address the case when $|G/Z| \cong S_3$. We begin with several definitions. First, Taunt [47] defines an A-group as a solvable group whose Sylow subgroups are all Abelian. He proved that if A is an A-group then $A' \cap Z(A) = \{e\}$. Secondly, let q be a prime and let G be a group such that $G = Q \times B$ for a q-group Q and subgroup B such that $q \nmid |B|$. Then we call the subgroup Q the qth-component q-group of G and denoted it by $Q = G_q$. Also, B is called the q'th- component of G and is denoted $G_{q'}$. Hence G is the direct product $G = G_q \times G_{q'}$.

Next, recall the following classes of groups discussed in Sections 4.1.2, 4.1.2, and 4.1.2. Let q and q be primes so that p is prime and $q|p-1$ and let $m \in \mathbb{N}$. A group D_{pq} is called a generalized dihedral Group and has a presentation of

$$D_{pq} =< a, b : a^p = b^q = e, bab^{-1} = a^r >$$

where r has order q mod p. The group $T_{p,q,m,\theta}$ had a presentation

$$T_{p,q,m,\theta} =< a, b : a^p = b^{q^m} = e, bab^{-1} = a^{\lambda^\theta} >$$

where λ has order q mod p and $\theta \in \{1, 2, ..q-1\}$. A subclass of the $T_{p,q,m,\theta}$ groups are called the G_m groups. Each G_m group has the presentation

$$G_m = T_{3,2,m,1} =< a, b : a^3 = b^{2^m} = e, bab^{-1} = a^{-1} > .$$

Proposition 5.2.13 (Lescot [35]). . *If G is a group with $G/Z(G) \cong D_{pq}$ then there is some $\theta \in \{1, 2, ..., q-1\}$, $m \geq 1$, and Abelian group A such that $G \cong T_{p,q,m,\theta} \times A$.*

Proof. Let G be a group with $G/Z(G) \cong D_{pq}$, let r be prime, and let R be an r-Sylow subgroup of G. Then $RZ(G)/Z(G)$ is a Sylow subgroup of $G/Z(G) \cong D_{pq}$. If $r \neq p, q$ then $R \leq Z(G)$. The only nontrivial Sylow subgroups of D_{pq} are of order p or q and hence cyclic, so $RZ(G)/Z(G)$ is cyclic. It follows that $R/R \cap Z(G)$ is cyclic by the diamond isomorphism theorem. Since $R \cap Z(G) \subseteq Z(R)$, $R/Z(R)$ is also cyclic. Therefore R is Abelian. Also note that there will be an Abelian Sylow subgroup for primes $r = p$ and $r = q$.

Next, D_{pq} is solvable by Burnside's Theorem because $|D_{pq}| = pq$. Since $G/Z(G) \cong D_{pq}$ and D_{pq} is solvable, G is also solvable.

As noted above, all Sylow subgroups of G are Abelian. Then G is an A-group and by Taunt's result $G' \cap Z(G) = \{e\}$. Applying Taunt's result and the diamond isomorphism theorem,

$$G' = \frac{G'}{G' \cap Z(G)} \cong \frac{G'Z(G)}{Z(G)} = (\frac{G}{Z(G)})' \cong (D_{pq})' =< \bar{a} > .$$

Let $\overline{a} = aZ(G)$ for $a \in G'$. Since $G' \cap Z(G) = e$, $|G'| = | < \overline{a} > | = p = \circ(a)$.

Recall $G' =< a >$. Then $|G'| = | < a > | = | < \bar{a} > | = p$.

Next, let $t \in G$ such that $tat^{-1} \neq a$. Since $< a >= G' \lhd G$, $tat^{-1} = a^n$ for some $n \in \mathbb{N}$, $n \not\equiv 1(\text{mod } p)$. Projecting to $G/Z(G) = D_{pq}$ yields

$$\bar{t}\bar{a}\bar{t}^{-1} = \bar{a}^n.$$

The conjugacy class of $[\bar{a}] = \{\bar{a}^{\lambda^j} : 0 \leq j \leq q-1\}$ in D_{pq} so $n = \lambda^j$ and

$$tat^{-1} = a^n = a^{\lambda^j}$$

with $j > 0$ (as $j = 0$ implies $a \in Z(G)$). Thus $q \nmid j$. Let $o(t) = q^m\gamma$, so that $q^m \nmid \gamma$ and q^m and t are relatively prime. Then

$$(t^\gamma)a(t^\gamma)^{-1} = a^{\lambda^{j\gamma}} \tag{5.5}$$

and $j\gamma \not\equiv 0(\text{mod } q)$. Let $\theta \equiv j\gamma(\text{mod } q)$ be the least residue of $j\gamma$ and set $t^\gamma = b$. Then

$$bab^{-1} = a^{\lambda^\theta}$$

and $o(b) = q^m$.

Next suppose $m = 0$. Then $b = 1$, $a = a^{\lambda^\theta}$, and $\lambda^\theta \equiv 1(\text{mod } p)$. Then $q|\theta$, a contradiction. Therefore, $m \geq 1$.

Select $b \in G$ of order q^m such that Equation 5.5 holds for the minimal possible value of m. Then $H =< a, b >\cong T_{p,q,m,\theta}$. Observe via the diamond isomorphism theorem that

$$\begin{aligned}
pq &= |D_{pq}| = |\frac{G}{Z(G)}| \\
&\geq |\frac{HZ(G)}{Z(G)}| = |\frac{H}{H \cap Z(G)}| = |\frac{H}{Z(H)}||\frac{Z(H)}{H \cap Z(G)}| \\
&= |\frac{T_{p,q,m,\theta}}{Z(T_{p,q,m,\theta})}||\frac{Z(H)}{H \cap Z(G)}| = pq|\frac{Z(H)}{H \cap Z(G)}| \geq pq
\end{aligned} \tag{5.6}$$

Equality follows through all of Equation 5.6 so that $G = HZ(G)$ and $Z(H) = H \cap Z(G)$. Since $b^q \in Z(H)$ as recorded in Table 4.5, $b^q \in Z(G)$. Also notice that $o(b^q) = \frac{1}{q}o(b) = q^{m-1}$.

Next let $R = Z(G)_q$ be the qth component q-group of $Z(G)$. By Lescot ([35], Proposition 1.1), either

1. $< b^q >$ is a direct factor of $Z(G)_q$ or
2. for some $v \in Z(G)_q$, $o(b^q v^q) < o(b^q)$.

First suppose that the second case occurs. Then

$$o(bv) \leq qo(b^q v^q) \leq o(b^q) = q^{m-1}$$

and since R is a q-group, $o(bv) = q^y$ for $1 \leq y \leq m-1$. Then

$$(bv)a(bv)^{-1} = bab^{-1} = a^{\lambda^\theta}$$

since $v \in Z(G)$. This contradicts the minimality of m because $o(b) = q^r$ and $r < m$. Therefore, case (1) holds and $Z(G)_q =< b^q > \times B$ for some Abelian group B.

Finally consider the generating factors of the group:

$$\begin{aligned} G &= HZ(G) \\ &= < H, Z(G) > \\ &= < a, bZ(G)_q, Z(G)_{q'} > \\ &= < a, b, b^q, B, Z(G)_{q'} > \\ &= < a, b > \times B \times Z(G)_{q'} \\ &\cong T_{p,q,m,\theta} \times B \times Z(G)_{q'} \\ &= T_{p,q,m,\theta} \times A \end{aligned} \tag{5.7}$$

with $A = B \times Z(G)_{q'}$ Abelian. $\square$

Corollary 5.2.14. *If $G/Z(G) \cong S_3$ then $G \cong G_m \times A$, with $m \geq 1$ and A Abelian.*

Proof. We apply Proposition 5.2.13. Suppose

$$|G/Z| \cong S_3 \cong D_3 = D_{pq}$$

with $p = 3$, $q = 2$. Then $G \cong T_{3,2,m,\theta} \times A$ for some $m \geq 1$,$\theta = 1$, and an Abelian group A by Proposition 5.2.13. Hence

$$G \cong T_{3,2,m,1} \times A \cong G_m \times A.$$

□

Corollary 5.2.15. *If $G/Z(G) \cong S_3$, then $P(G) = \frac{1}{2}$.*

Proof. By Corollary 5.2.14 $G \cong G_m \times A$ for some $m \geq 1$. Then

$$P(G) = P(G_m)P(A) = \frac{1}{2}.$$

□

Suppose G is a group such that $G/Z \cong D_n$. Next we derive a formula for $P(G)$. Then we will use the formula to find the commutativity degree of the dihedral groups D_4 and D_5.

Proposition 5.2.16. *If $G/Z(G) \simeq D_n$ then $P(G) = \frac{n+3}{4n}$.*

Proof. Suppose $G/Z(G) \cong < rZ, \rho Z : r^2 Z = e, \rho^n Z = e, r^{-1}\rho r Z = r\rho^{n-1}Z >$. Then we can partition G into cosets as follows:

$$G = Z \cup rZ \cup \rho Z \cup \rho^2 Z \cup ..\rho^{n-1}Z \cup r\rho Z \cup r\rho^2 Z \cup .. \cup r\rho^{n-1}Z.$$

The strategy of this proof is to partition the cosets into conjugacy classes for D_n then to count the conjugacy classes. We will consider the cases when n in odd and even separately.

First, if n is even or if n is odd, there are $|Z|$ conjugacy classes in the coset Z.

Next, in order to partition cosets of the form $\rho^i Z$, we will find $|[\rho]|$ for both n odd and n even. Since $C_G(\rho) \supseteq \{Z, \rho\}$, then $C_G(\rho) =< \rho > Z$ and $|[\rho]| = [G : C_G(\rho)] = 2$. Then $[\rho] = \{\rho, \rho^{-1}u\}$ for some $u \in Z$ because $[\rho Z] = \{\rho Z, \rho^{-1}Z\}$. To extend this result, let $z_1 \in Z$. Then $[\rho z_1] = \{\rho z_1, \rho^{-1}z_1 u\}$ by the same reasoning. Additionally, for $1 \leq i < n$, $[\rho^i Z] = \{\rho^i, \rho^{-i}w\}$ some $w \in Z$ and $[\rho^i z_1] = \{\rho^i z_1, \rho^{-i}z_1 w\}$ for each $z_1 \in Z$.

Now we will count the number of conjugacy classes in cosets of the form $\rho^i Z$ for odd and even n. If n is odd there are $n-1$ cosets of the form $\rho^i Z$ and these are partitioned into two element conjugacy classes. Hence there are $\frac{n-1}{2}|Z|$ conjugacy classes of this form for odd n. If n is even, the coset $\rho^{\frac{n-1}{2}}|Z|$ is partitioned into $\frac{|Z|}{2}$ conjugacy classes and the remaining $n-2$ cosets of the form $\rho^i Z$ (for $1 \leq i < n$ and $i \neq \frac{n-1}{2}$) are partitioned into two element conjugacy classes. These remaining cosets yield $\frac{n-2}{2}|Z|$ conjugacy classes. For n even, cosets of the form $\rho^i Z$ are partitioned into a total of $\frac{n-2}{2}|Z| + \frac{|Z|}{2}$ conjugacy classes.

Next, in order to partition cosets of the form $r\rho^i Z$, we will find $|[r]|$ for n odd and n even. First suppose n is odd. Then $C_G(r) \supseteq \{Z, r\}$ and $C_G(r) =< r > Z$. Hence $|[r]| = [G : C_G(r)] = n$. Since $[rZ] = \{rZ, r\rho Z, r\rho^2 Z, ..r\rho^{n-1}Z\}$, it follows that

$$[r] = \{r, r\rho v_1, r\rho^2 v_2, ..r\rho^{n-1}v_{n-1}\}$$

with $v_i \in Z$. This result can be extended as follows. Let $z_1 \in Z$. Then

$$[rz_1] = \{rz_1, r\rho z_1 v_1, r\rho^2 z_1 v_2, ..r\rho^{n-1}z_1 v_{n-1}\}.$$

For n odd, there are n cosets of the form $r\rho^i Z$, $0 \leq i < n$ partitioned into n element conjugacy classes. There are $\frac{n|Z|}{n} = |Z|$ conjugacy classes of this form for n odd.

Next suppose n is even. Then $C_G(r) \supseteq \{Z, r\}$ and so $C_G(r) \subseteq< r > Z$. Since

$[rZ] = \{rZ, r\rho^2 Z, ..r\rho^{n-2}Z\}$, it follows that

$$[r] \supseteq \{r, r\rho^2 v_2, ..r\rho^{n-2}v_{n-2}\}.$$

However, because $\rho^{\frac{n-1}{2}}Z \in Z(Z)$, conjugation by $\rho^2 Z$ results in

$$\rho^{\frac{n-1}{2}}Zr\rho^{\frac{n-1}{2}}Z = rZ,$$

and then for some $z' \in Z$, $rz' \in [r]$. Then each v_i, $rz'v_i \in [r]$, and

$$[r] = \{r, rz', r\rho^2 v_2, r\rho^2 z' v_2, .., r\rho^{n-2}v_{n-2}, r\rho^{n-2}z'v_{n-2}\}.$$

Before counting classes, notice that finding $[r\rho]$ for n even is similar to finding $[r]$, and

$$[r\rho] = \{r\rho, r\rho z', r\rho^3 v_3, r\rho^3 z' v_3, .., r\rho^{n-1}v_{n-1}, r\rho^{n-1}z'v_{n-1}\}.$$

Then $|[r]| = |[r\rho]| = n$, and there are n cosets of the form $r\rho^i Z$, $0 \leq i < n$ partitioned into n element conjugacy classes. These cosets are partitioned into $\frac{n|Z|}{n} = |Z|$ conjugacy classes for n even.

Suppose n is odd. Then we add the conjugacy classes as follows:

$$k(G) = |Z| + \frac{n-1}{2}|Z| + |Z| = \frac{n+3}{2}|Z|,$$

and since $|G| = |D_n||Z|$,

$$P(G) = \frac{\frac{n+3}{2}|Z|}{2n|Z|} = \frac{n+3}{4n}.$$

Suppose n is even. Then

$$k(G) = |Z| + \frac{n-2}{2}|Z| + \frac{|Z|}{2} + |Z| = \frac{n+3}{2}|Z|,$$

and since $|G| = |D_n||Z|$,

$$P(G) = \frac{\frac{n+3}{2}|Z|}{2n|Z|} = \frac{n+3}{4n}.$$

$\square$

Notice that if $G/Z \cong D_n$, then $P(G) = \frac{n+3}{4n}$ regardless of whether n is odd or even. This is interesting because we showed in Section 4.1.2 that if $G \cong D_n$ then $P(G) = \frac{n+3}{4n}$ if n is odd and $P(G) = \frac{n+6}{4n}$ if n is even.

Corollary 5.2.17. *If $G/Z \cong D_4$ then $P(G) = \frac{7}{16}$ and if $G/Z \cong D_5$ then $P(G) = \frac{2}{5}$*

Proof. The result follows from Proposition 5.2.16. □

Finally, we will address the quaternion group Q_8.

Proposition 5.2.18. *There is no group G with $G/Z \simeq Q_8$, the quaternion group.*

Proof. Suppose $G/Z =< aZ, bZ : a^2Z = b^2Z, a^4Z = Z, baZ = ab^3Z >$. We will show that $a^2 \in Z(G)$ by showing that a^2 commutes with each element in the group. Notice that G is generated by $\{Z, a, b\}$, so it suffices to show that Z, a and b commute with a^2. Clearly, if $z \in Z$ then $za = az$ and $aa^2 = a^3 = a^2a$. Since $a^2Z = b^2Z$, $a^2 = b^2z_1$ for some $z_1 \in Z$. It follows that $ba^2 = b(b^2z_1) = (b^2z_1)b = a^2b$. Then $a^2 \in Z$. This contradicts the implicit assumption that $a^2 \in a^2Z \neq Z$. Therefore, there is no group G with $G/Z \simeq Q_8$. □

We can generalize the result of Proposition 5.2.18 to the dicyclic groups. Let $m \in \mathbb{N}$. Recall that the dicyclic group D_m has the presentation

$$D_m =< a, b : a^{2m} = b^4 = e, b^{-1}ab = a^{-1}, a^m = b^2 > .$$

Then there is no group G such that $G/Z \cong D_m$ for any $m \in \mathbb{Z}$. The proof of this generalization is similar to the proof in Proposition 5.2.18: because of the relation $a^m = b^2$, we can show that $a \in Z$ which contradicts $a^m \in a^mZ \neq Z$.

5.2.3 Summary of Commutativity Degrees

The commutativity degrees of all groups G with $|G/Z| < 12$ are summarized in Table 5.3. In order to find the commutativity degree of groups with $|G/Z| < 12$, we have calculated formulas for the commutativity degree of a number of additional groups. Table 5.4 summarizes the formula for the commutativity degree of each class of groups discussed in this section.

G/Z	$P(G)$	G/Z	$P(G)$
$\mathbb{Z}_1$	1	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Not determined
$\mathbb{Z}_2$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	DNE
$\mathbb{Z}_3$	1	D_4	$\frac{7}{16}$
$\mathbb{Z}_4$	1	Q_8	DNE
V_4	$\frac{5}{8}$	$\mathbb{Z}_9$	1
$\mathbb{Z}_5$	1	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\frac{11}{27}$
$\mathbb{Z}_6$	1	$\mathbb{Z}_{10}$	1
S_3	$\frac{1}{2}$	D_5	$\frac{2}{5}$
$\mathbb{Z}_7$	1	$\mathbb{Z}_{11}$	1
$\mathbb{Z}_8$	1		

Table 5.3: **Commutativity Degrees for Groups with** $|G/Z| < 12$

G/Z	G (if avaliable)	$P(G)$
$\mathbb{Z}_n$ (cyclic, order n)		1
$\mathbb{Z}_p \times \mathbb{Z}_p$	$P \times A$ ($\|P\| = p^m$ and A Abelian)	$\frac{p^2+p-1}{p^3}$
D_{pq}	$T_{p,q,m,\theta} \times A$ (A Abelian)	$\frac{q^{m+1}+m(p-1)}{pq^{m+1}}$
D_n		$\frac{n+3}{4n}$

Table 5.4: **Groups with Given** G/Z

5.3 On the Value $\frac{1}{p}$, for a prime p

We show in Section 4.1.2 that the value $\frac{1}{2}$ is realized as the commutativity degree by the class of G_m groups. In Chapter 1, we found that $P(A_4) = \frac{1}{3}$. In this section, we will show that there is a group with commutativity degree $\frac{1}{p}$ for every prime p. Then we will extend the result to show there is a group with commutativity degree $\frac{1}{n}$ for each $n \in \mathbb{N}$. Although we do not know whether it is possible to construct an indecomposable group with commutativity degree $\frac{1}{p}$ for every prime p, we will construct an example of an indecomposable group with commutativity degree $\frac{1}{p}$ for each Mersenne Prime, p. A Mersenne prime is a prime of the form $p = 2^k - 1$ for some $k \in \mathbb{N}$.

We already know some types of groups that cannot have $\frac{1}{p}$ as a commutativity degree. Recall from Section 4.3 that for each prime p_i, the value $\frac{1}{p_i}$ is a limit point of the commutativity degrees of the class of 4-property p_i-groups. However, no 4-property p_i-group has commutativity degree $\frac{1}{p_i}$. In this section, we will also show that if G is nilpotent, then $P(G) \neq \frac{1}{p}$ for any prime p.

Proposition 5.3.1. *Let p be prime. Then there is a group with commutativity degree $\frac{1}{p}$. This group is a direct product of solvable groups.*

Proof. Let n be an odd positive integer and recall that the dihedral group $D_n = < r, \rho : r^2 = \rho^n = e, \rho r = r\rho^{n-1} >$ has commutativity degree $P(D_n) = \frac{n+3}{4n}$. We induct on p. Suppose $p = 2$. In Table 1.1, we showed by direct computation that $P(D_3) = \frac{1}{2}$. Suppose $p = 3$. In Table 1.3, we showed by direct computation that $P(A_4) = \frac{1}{3}$. Next assume that for all primes p_i less than p, there is some product of solvable groups H_i with $P(H_i) = \frac{1}{p_i}$. Now consider the prime p (with $p > 3$). First suppose that $p \equiv 3 (\text{mod } 4)$. Then $p = 4m + 3$ for some $m \in \mathbb{N}$, and

$$P(D_{3p}) = \frac{3(4m+3)+3}{4(3p)} = \frac{m+1}{p}$$

. Then $m+1 = p_1p_2...p_s$, with each $p_i < p$. By the induction hypothesis and Proposition 1.2.3, there is some group $M = \prod_{i=1}^{s} H_i$ with

$$P(M) = P\left(\prod_{i=1}^{s} H_i\right) = \frac{1}{(p_1p_2..p_s)} = \frac{1}{m+1}.$$

We will apply Proposition 1.2.3 once more to construct the group $G = D_{3p} \times M$. The commutativity degree of G is

$$P(G) = P(D_{3p})P(M) = \left(\frac{m+1}{p}\right)\left(\frac{1}{m+1}\right) = \frac{1}{p}.$$

Secondly suppose that $p \equiv 1(\text{mod } 4)$. Then $p = 4m+1$ for some $m \in \mathbb{N} \cup \{0\}$, and

$$P(D_p) = \frac{(4m+1)+3}{4(p)} = \frac{m+1}{p}.$$

If $m = 0$, we are done. Otherwise, notice that similar to the first case, there is a group $M = \prod_{i=1}^{s}(H_i)$ with $P(M) = \frac{1}{m+1}$ and we construct the group $G = D_p \times M$. Again, the commutativity degree of G is

$$P(G) = P(D_p)P(M) = \frac{1}{p}$$

.

Therefore, by induction, for every prime p there is a group G such that $P(G) = \frac{1}{p}$, and G is a direct product of solvable groups. □

Example 5.3.2. *A group with Commutativity Degree* $\frac{1}{27109}$. The prime $27109 = 4(6777)+1$. Then

$$P(D_{27109}) = \frac{6778}{27109} = \frac{(2)(3)(461)}{27109}.$$

Recall that $P(D_3) = \frac{1}{2}$. Since $3 = 4(0)+3$, $P(D_9) = \frac{1}{3}$. Also, $461 = 4(115)+1$, so

$$P(D_{461}) = \frac{116}{461} = \frac{(2^2)(29)}{461}.$$

Then

$$P(D_{27109} \times D_3 \times D_9 \times D_{461}) \quad = \quad \frac{(2^2)(29)}{27109}.$$

Repeat the process again for $29 = 4(7) + 1$:

$$P(D_{29}) = \frac{8}{29} = \frac{2^3}{29}.$$

Then

$$P(D_{27109} \times D_9 \times D_{461} \times D_{29} \times (D_3)^4) \quad = \quad \frac{1}{27109}$$

Hence the group

$$G = D_{27109} \times D_9 \times D_{461} \times D_{29} \times (D_3)^4$$

has commutativity degree $\frac{1}{27109}$. ∎

Corollary 5.3.3. *For each $n \in \mathbb{N}$ there is a solvable group G with commutativity degree $\frac{1}{n}$.*

Proof. Let $n \in \mathbb{N}$. Then for $1 \le i \le r$ for some r, $n = p_1 p_2 ... p_r$ for (not necessarily distinct) primes p_i. By Proposition 5.3.1 for each i there is a group H_i so that $P(H_i) = \frac{1}{p_i}$ and H_i is a direct product of solvable groups. Then the group $G = \prod_{i=1}^{r} (H_i)$ has commutativity degree $P(G) = \frac{1}{n}$. □

Next we will show that if p is a prime of the form $2^k - 1$ (a Mersenne prime), then there is an indecomposable group G with $P(G) = \frac{1}{p}$.

Example 5.3.4. *If p is a Mersenne prime, then there is an indecomposable group G with $P(G) = \frac{1}{p}$.* We will construct the group G and call this group a Mersenne Group. Let $p = 2^k - 1$ be a Mersenne prime and let $K = \prod_{i=1}^{k} \mathbb{Z}_2$. Consider $GL_k(\mathbb{Z}_2)$. To find $|GL_k(\mathbb{Z}_2)|$, let R_i denote the number of possible combinations of entries in row i of an invertible $k \times k$ matrix. (We count the rows in order; possible entries row R_{i+1}

are counted after entries in row R_i are determined.) Then $|GL_k(\mathbb{Z}_2)| = \prod_{i=1}^{k}(R_i)$. There are $R_1 = 2^k - 1$ possible entries in the first row since a 0 or 1 may appear in each entry and the zero vector is not allowed. Then the prime $p = 2^k - 1$ divides $|GL_k(\mathbb{Z}_2)|$ and there is an element of order $2^k - 1$ in $GL_k(\mathbb{Z}_2)$ by Cauchy's Theorem. Call this element p and let $H =< p >$. Then H acts on K because each element in H is invertible and thus is an automorphism of K. Hence $K \rtimes H$ is a semidirect product. Let G be the semidirect product $G = K \rtimes H$.

Next we will show that $P(G) = \frac{1}{p}$ by counting conjugacy class of G. Since H is not normal in G, H is not a unique p-Sylow subgroup. Then H has 2^k conjugate Sylow subgroups, because the number of $(2^k - 1)$-Sylow subgroups is equivalent to 1 mod $2^k - 1$. Thus each of the $p^k - 2$ nonidentity elements of H has 2^k conjugates; one from each conjugate $(2^k - 1)$-Sylow subgroup.

Notice that K is the unique 2-Sylow subgroup because it is normal in G. Consider H acting on K as a set. Let $v \in K$ and let O_v be the orbit of v and let S_v be the stabilizer of v. Recall that $|O_v| = [H : S_v]$. Since $S_v \subset H$, $S_v \subset K$, and $H \cap K = \{e\}$, it follows that $S_v = \{e\}$. Hence $|O_v| = p$ and so v has $2^k - 1$ conjugates. Then the class equation is

$$|G| = 1 + \sum_{i=1}^{2^k-2} 2^k + (2^k - 1)$$

and there are $k(G) = 1 + 2^k - 2 + 1 = 2^k$ conjugacy classes. Therefore, the commutativity degree is

$$P(G) = \frac{2^k}{2^k p} = \frac{1}{p}. \blacksquare$$

We will use Lemmas 5.3.5 and 5.3.6 in order to prove that if G is a non-Abelian nilpotent group, then $P(G) \neq \frac{1}{p}$ for any prime p.

Lemma 5.3.5. *If G is nilpotent with a non-Abelian p-Sylow subgroup for some prime p, then $P(G) < \frac{1}{p}$ except perhaps in the case when $|G'| = p$ and there is exactly one p-Sylow subgroup.*

Proof. Since G is nilpotent, $G = P_1 \times P_2 \times .. \times P_s$ for some $s \in \mathbb{N}$ such that each factor P_i is a p_i-Sylow subgroup.

If P_1 is the unique non-Abelian Sylow subgroup and $|G'| \neq p_1$, then $|G'| \geq p_1^2$. By Proposition 2.2.3,

$$P(G) \leq \frac{1}{d^2} + \left(1 - \frac{1}{d^2}\right)\left(\frac{1}{p_1^2}\right)$$

where d is the smallest degree of a nonlinear representation of G. Since $d > 1$ and d divides $|G|$, $d \geq p_1$. Hence

$$P(G) \leq \frac{1}{p_1^2} + \left(1 - \frac{1}{p_1^2}\right)\left(\frac{1}{p_1^2}\right) \leq \frac{2}{p_1^2} \leq \frac{1}{p_1}.$$

Next suppose that for some r, $1 < r < s$, $P_1, P_2, ...P_r$ are non-Abelian Sylow subgroups of G such that $p_1 < p_2 < ... < p_r$. Then $P(G) \leq P(P_1)P(P_2)...P(P_r)$. For each group P_i with $1 \leq i \leq r$, $P(P_i) < 1$. Hence $P(G) \leq P(P_1)P(P_r)$. Then by Proposition 2.1.3,

$$P(G) \leq \left(\frac{p_1^2 + p_1 - 1}{p_1^3}\right)\left(\frac{p_r^2 + p_r - 1}{p_r^3}\right) < \left(\frac{p_1^2 + p_1}{p_1^3}\right)\left(\frac{p_r^2 + p_r}{p_r^3}\right),$$

and since $p_1 \geq 2$,

$$P(G) \leq \left(\frac{p_r^2 + p_r}{p_r^3}\right)\left(\frac{5}{8}\right) = \frac{5p_r + 5}{8p_r^2}.$$

Finally since $p_r \geq 3$,

$$P(G) \leq \frac{18}{24p_r} < \frac{1}{p_r}.$$

Therefore, for each p_i, $1 \leq i \leq r$,

$$P(G) < \frac{1}{p_i}.$$

$\square$

Lemma 5.3.6. *Let p be prime. If G is nilpotent and $|G'| = p$, then $P(G) > \frac{1}{p}$.*

Proof. Let p be a prime. Assume $|G'| = p$. Since G is nilpotent, $G = P \times \prod_{i=1}^{s} P_i$ for some $s \in \mathbb{N}$ such that P is a p-Sylow subgroup and each factor P_i is a p_i-Sylow subgroup for some $p_i \neq p$. Assume $|P| = p^m$. Then $G' \subseteq P$, and $|G/G'| = p^{m-1}|\prod_{i=1}^{s} P_i|$. by Lemma 2.3.5 $k(G) \geq |G/G'|$. Since $|G| - |G/G'| > 0$, $k(G) \geq |G/G'| + 1$, and

$$P(G) \geq \frac{p^{m-1}|\prod_{i=1}^{s} P_i| + 1}{p^m|\prod_{i=1}^{s} P_i|} > \frac{1}{p}.$$

$\square$

Proposition 5.3.7. *Let p be prime. If G is a non-Abelian nilpotent group, then $P(G) \neq \frac{1}{p}$.*

Proof. Since G is nilpotent $G = P \times \prod_{i=1}^{s} P_i$ for some $s \in \mathbb{N}$, where each factor P_i is a p_i-Sylow subgroup.

Suppose p divides $|G|$. Then $p = p_i$ for some i. If $|G'| > p$, then by Lemma 5.3.5, $P(G) < \frac{1}{p}$. If $|G'| = p$, then by Lemma 5.3.6,$P(G) > \frac{1}{p}$.

Next assume that p does not divide $|G|$. Suppose that $P(G) = \frac{1}{p}$. Then

$$\frac{1}{p} = \frac{k(G)}{|G|}$$

and this implies that p divides $|G|$, a contradiction.

Therefore, for any prime p, $P(G) \neq \frac{1}{p}$. $\square$

5.4 Concluding Remarks and Additional Questions

There is a group G with commutativity degree $P(G) = \left(1 + \frac{1}{2^{2m}}\right)$ for all $m \in \mathbb{N}$, and all possible commutativity degrees in the interval $(\frac{1}{2}, 1)$ are of this form. If $P(G) = \frac{1}{2}$ and G is not nilpotent, then $G/Z \cong G_m \times C$ for a G_m group and some Abelian group C. Further, $\frac{1}{2}$ is the least upper bound on the commutativity degree of non-nilpotent groups and $\frac{1}{12}$ the least upper bound on the commutativity degree of non-solvable groups. If $P(G) = \frac{1}{12}$, then $G \cong A_5 \times C$ for some Abelian group C. Rusin [43] found all possible commutativity degrees greater than $\frac{11}{32}$, but all possible commutativity degrees greater than $\frac{1}{12}$ are unknown.

In general, it becomes more difficult to classify groups that have smaller commutativity degrees because fewer properties are known as the groups become less Abelian. In light of this, rather than attempt to explicitly calculate the commutativity degree of all groups, we would like to investigate properties of the set of possible commutativity degrees. For instance, gaps do occur in commutativity degree values, such as the interval $(7/16, 5/8)$. Are there gaps of arbitrarily small intervals in the set of possible commutativity degrees as the values get close to 0? For all $n \in \mathbb{N}$, $\frac{1}{n}$ is a limit point of the set of commutativity degrees. Does the set of commutativity degrees have any irrational limit points? We proved inductively that there is a decomposable group with commutativity degree $\frac{1}{n}$ for all $n \in \mathbb{N}$. For which m, $m < n$, can we find a group with commutativity degree $\frac{m}{n}$? We also showed that there are no nilpotent groups with commutativity degree $\frac{1}{p}$ for any prime p. Then we found an indecomposable group with commutativity degree $\frac{1}{p}$ for each prime of the form $p = 2^k - 1$. Is there an indecomposable group with commutativity degree $\frac{1}{p}$ for every prime p?

Chapter 6: Appendices

6.1 Appendix A: Computations Using GAP

6.1.1 The Small Group Package, Conjugacy Class Computation, and Additional Commands

GAP stands for "Groups, Algorithms, Programming" and is a program that runs in D.O.S. that is used for computation in algebra. We used GAP and the Small Group Library package available for GAP in Examples 2.1.6 and 5.2.12, Tables 5.1 and 5.2, and Proposition 3.3.7.

The Small Groups Library in GAP was developed by Besche, Eick, and O'Brian. The library classifies all groups of order less than 2000 (except order 1024). This GAP package contains varying properties of the groups, depending on the classification and complexity of the group. For more information, see the File [4]. This file also describes how to get the package.

Each group in the Small Groups Library is given an ID Tag of the form $[n, x]$ where n is the order of the group and x, with $1 \leq x \leq m$, is a reference number given to each of m non-isomorphic groups of order n. Table 6.1 lists common commands we used from the SmallGroups package in GAP.

We computed the number of conjugacy classes for a given group in GAP using two different methods. We found the classes from the character table and computed the classes directly. The second method is easier. The advantage of the first method is that the class equation can easily be computed from the list of conjugacy class sizes. Both methods are outlined in Table 6.2.

Table 6.3 lists additional commands that we found useful.

COMMAND	PURPOSE
G:=SmallGroup(n,x);	Defines G as the small group $[n, x]$.
H:=OneSmallGroup(n);	Defines H as a small group of order n.
IdSmallGroup(G);	Returns the ID Tag of Group G.
IdGroup(G);	Returns the ID Tag of Group G.
NumberSmallGroups(n);	Returns the number of Small Groups of order n.
SmallGroupsInformation(n);	Returns a list of groups and properties of groups of order n.
P:=PresentationViaCosetTable(G);	Saves P as the presentation of SmallGroup G.
SimplifyPresentation(P);	Reduces the number of generators and/or relations in the presentation P.
TzPrintRelators(P);	Prints the generators and relations of P.

Table 6.1: **The SmallGroup Package**

COMMAND	PURPOSE
c:=CharacterTable(G);	Defines c as the character table for group G. (For more information about character tables in GAP, see [23]).
Display(c);	Prints the character table c.
s:=SizesConjugacyClasses(c);	Defines s as a list of the sizes of the conjugacy classes of G
Size(s);	Returns the number of conjugacy classes of G.
t:=ConjugacyClasses(G);	Defines t as a list of conjugacy classes of group G.
Size(t);	Returns the number of conjugacy classes of G.

Table 6.2: **Finding the Number of Conjugacy Classes**

COMMAND	PURPOSE
CyclicGroup(IsPermGroup, n);	Returns the cyclic group of order n.
DihedralGroup(IsPermGroup, n);	Returns the Dihedral Group of order n.
SymmetricGroup(n);	Returns the symmetric Group of order n.
DirectProduct(G, H);	Returns the direct product of groups G and H.
Center(G);	Returns the center of group G.
DerivedSubgroup(G);	Returns the commutator subgroup of group G.
FactorGroup(G,H);	Returns the quotient group G/H.
FactorGroup(G, Center(G));	Returns $G/Z(G)$.
IsCyclic(G);	Returns ”true” if group G is cyclic and ”false” if it is not.
IsAbelian(G);	Returns ”true” is group G is Abelian and ”false” if it is not.
Size(L);	Returns the number of entries in an object L.
Order(G);	Returns the order of a group G.
Exponent(g);	Returns the order of an element.
LogTo(”filename”);	Saves a file.
LogTo();	Ends the file.
quit;	Returns to GAP from a break.

Table 6.3: **Common Groups and Commands in GAP.**

6.1.2 A Simple Sample Program

Note that before using a program, it must be read via the command: Read("filename");. Often, rather than type a program in GAP, I created an empty file by typing the command " InputLogTo("filename"); ", and then immediately ending the file with " InputLogTo();". then type the program in another editor. The following program requires "noGroups", the number of groups of order n, and the "order" n as input. It loops though all groups of the given order and adds the ID Tags of all the groups satisfying the property $|G/Z| = 8$ to a list. A copy of the program in included.

```
InputLogTo("GZorder8");
GZorder8:=function(order, noGroups)
local s,j,GZorder8;

GZorder8:=[];

s:=AllSmallGroups(order);

for j in [1..noGroups] do
if Size(FactorGroup(s[j],Center(s[j])))=8 then
Add(GZorder8,IdGroup(s[j]));
Add(GZorder8, IdGroup(FactorGroup(s[j],Center(s[j]))));
fi;

od;
return GZorder8;
end;
InputLogTO();
```

6.2 Appendix B: Calculations of Inverse Elements and Conjugacy Classes for Order Reversing Groups

Rusin pn-groups

Calculations for the Rusin pn-groups are included in Section 4.1.1.

D_{pq} Groups

Since D_{pq} groups are a subclass of Rusin pn-groups, calculations are identical to those for R_{pn}-groups.

$T_{p,q,m,\theta}$ Groups

Notice that, as a set

$$T_{p,q,m,\theta} = \{a^i b^j : 0 \leq i < p, 0 \leq j < q^m\}.$$

To find the inverse of the form $a^i b^j$, we apply the relation $bab^{-1} = a^{\lambda^\theta}$ as follows:

$$\begin{aligned}(a^i b^j)^{-1} &= b^{-j} a^{-i} \\ &= a^{(-i)(\lambda^{-j\theta})} b^{-j} \\ &= a^{\frac{-i}{\lambda^{j\theta}}} b^{-j}.\end{aligned}$$

Next we conjugate elements of $T_{p,q,m,\theta}$ by $a^i b^j$ to find the conjugacy classes. Let $1 \leq v \leq p-1$. Then

$$\phi_{a^i b^j}(a^v) = a^i b^j (a^v) a^{\frac{-i}{\lambda^{j\theta}}} b^{-j} = a^{i+(v-\frac{i}{\lambda^{j\theta}})(\lambda^{j\theta})} = a^{v\lambda^{j\theta}}. \tag{6.1}$$

Since $o(a) = p$, there are $p-1$ elements in each class of the type $[a^v]$. There are $p-1$ nonidentity elements in the subgroup $< a >$ partitioned into this type of class. Hence there is one conjugacy class $[a^v]$ with $p-1$ elements.

Let $1 \leq w \leq p^m$. Then

$$\phi_{a^ib^j}(b^w) = a^ib^jb^wa^{\frac{-i}{\lambda^{j\theta}}}b^{-j} = a^{i+(-\frac{i}{\lambda^{j\theta}})(\lambda^{(j+w)\theta})}b^w = a^{i-i\lambda^{w\theta}}b^w = a^{i(1-\lambda^{w\theta})}b^w. \qquad (6.2)$$

Notice that if $q \mid w$, for $1 \leq w \leq 2^m$, then $(1-\lambda^{w\theta}) \equiv 0(\text{mod } p)$, and

$$[b^w] = \{a^0b^w\} = \{b^w\}.$$

There are $\phi(q^m) = q^m - q^{m-1}$ choices for w that are relatively prime to q^m. There are $q^m - 1$ nonidentity elements in $< b >$. Of these, there are $q^m - 1 - \phi(q^m) = q^{m-1}$ choices for w such that $q \mid w$ and $[b^w] = \{b^w\}$.

There are additional conjugacy classes containing elements of the form a^vb^w in the case that $q \mid w$ and $v \neq 0$. In this case,

$$\begin{aligned}
\phi_{a^ib^j}(a^vb^w) &= a^ib^ja^vb^wa^{\frac{-i}{\lambda^{j\theta}}}b^{-j} \\
&= a^ia^{v\lambda^{j\theta}}a^{(\frac{-i}{\lambda^{j\theta}})(\lambda^{(j+w)(\theta)})}b^w \\
&= a^{i(1-\lambda^w)+v\lambda^{j\theta}}b^w \\
&= a^{v\lambda^{j\theta}}b^w.
\end{aligned}$$

Since $o(a) = p$, there are $p-1$ elements in each of these classes. There are $q^m - q = q^{m-1}$ such classes because $q \mid w$.

If $q \nmid w$, then

$$[b^w] = \{b^w, a^{(1-\lambda^{w\theta})}b^w, a^{2(1-\lambda^{w\theta})}b^w, ..., a^{(p-1)(1-\lambda^{w\theta})}b^w\}.$$

There are $\phi(q^m) = q^m - q^{m-1}$ such choices for w, partitioned into classes containing p elements each. All nonidentity elements of $T_{p,q,m,\theta}$ are included in one of the classes described above.

G_m Groups

Calculations are identical to those for the $T_{p,q,m,\theta}$ groups since G_m groups are subclass of the $T_{p,q,m,\theta}$ groups.

A Group with $P(G) = \frac{1}{2}$.

AS a set,

$$G = \{a^i b^j : 0 \leq i \leq 5, 0 \leq j \leq 1\}.$$

To find the inverse of the form $a^i b^j$, we apply the relation $bab^{-1} = a^{-1} = a^5$ as follows:

$$\begin{aligned}(a^i b^j)^{-1} &= b^{-j} a^{-i} \\ &= a^{(-i)(5^{-j})} b^{-j} \\ &= a^{\frac{-i}{5^j}} b^{-j}.\end{aligned}$$

Next we conjugate elements of G by $a^i b^j$ to find each conjugacy class. Let $1 \leq v \leq 5$. Then

$$\phi_{a^i b^j}(a^v) = a^i b^j (a^v) a^{\frac{-i}{5^j}} b^{-j} = a^{i+(v-\frac{i}{5^j})(5^j)} = a^{v(5^j)}.$$

This yields that following conjugacy classes:

$$[a] = \{a, a^5\}, [a^2] = \{a^2, a^4\}, \text{ and } [a^3] = \{a^3\}.$$

Next consider $a^v b$:

$$\phi_{a^i b^j}(a^v b) = a^i b^j a^v b a^{\frac{-i}{5^j}} b^{-j} = a^i a^{v(5^j)} a^{(\frac{-i}{5^j})(5^{j+1})} b = a^{-4i+v(5^j)} b$$

This yields the following conjugacy classes:

$$[ab] = \{ab, a^3 b, a^5 b\}, \text{ and } [a^2 b] = \{b, a^2 b, a^4 b\}.$$

All nonidentity elements are included in one of the classes listed above.

Dicyclic Groups

As set

$$D_m = \{a^i : 0 \leq i \leq 2m\} \cup \{ba^i : 0 \leq i \leq 2m\},$$

because of the relation $a^m = b^2$.

Now we find the inverse forms of the forms a^i and ba^i. If $a^i \in D_m$, then $(a^i)^{-1} = a^{-i}$. Secondly, let $ba^i \in D_m$ and apply the relation $ab = ba^{-1}$ (ie. $ab = ba^{2m-1}$) as follows:

$$\begin{aligned} (ba^i)^{-1} &= a^{-i}b^3 \\ &= a^{(-i+m)}b \\ &= ba^{(-i+m)(2m-1)} \\ &= ba^{i-m}. \end{aligned}$$

Next we conjugate elements of each form by both a^j and ba^j to find conjugacy classes. First consider an element of the form $a^t \in D_m$. Then

$$\phi_{a^i}(a^t) = a^t \tag{6.3}$$

and

$$\begin{aligned} \phi_{ba^i}(a^t) &= ba^i a^t ba^{i-m} \\ &= b^2 a^{(i+t)(2m-1)} a^{i-m} \\ &= a^m a^{(-i-t)} a^{i-m} \\ &= a^m a^{(-i-t)} a^{i-m} \\ &= a^{-t}. \end{aligned}$$

If $t = m$, then $[a^m] = [b^2] = \{b^2\}$ and this element is in the center. For $t \neq m$, the conjugates yield the two element conjugacy class $[a^t] = \{a^t, a^{-t}\}$. Hence there are $2m-2$ noncentral elements in $< a >$ partitioned into two element conjugacy classes.

Next consider an element of the second form, $ba^t \in D_m$. Then

$$\begin{aligned} \phi_{a^i}(ba^t) &= a^i ba^t a^{-i} \\ &= ba^{i(2m-1)} a^t a^{-i} \\ &= ba^{t-2i}, \end{aligned}$$

and

$$
\begin{aligned}
\phi_{ba^i}(ba^t) &= ba^i ba^t ba^{i-m} \\
&= b^2 a^{i(2m-1)} a^t ba^{i-m} \\
&= a^m a^{-i} a^t ba^{i-m} \\
&= ba^{(m-i+t)(2m-1)} a^{i-m} \\
&= ba^{-m+i-t+i-m} \\
&= ba^{2i-t}.
\end{aligned}
$$

Then $[ba] = \{ba, ba^3 ... ba^{2m-1}\}$ and $[ba^2] = \{b, ba^2 ... ba^{2m-2}\}$ are the two conjugacy classes consisting of all elements of the type ba^t, with $0 \leq t \leq 2^n - 1$. All nonidentity elements are included in one of the classes calculated above.

Generalized Quaternion Groups

Calculations for the generalized quaternions are identical to those for dicyclic qroups.

Dihedral Groups

First notice that as a set

$$D_n = \{\rho^i : 0 \leq i \leq n-1\} \cup \{r\rho^i : 0 \leq i \leq n-1\}.$$

First we find the inverses of the forms ρ^i and $r\rho^i$. If $\rho^i \in D_n$, then $(\rho^i)^{-1} = \rho^{-i}$. Secondly, let $r\rho^i \in D_n$ and apply the relation $\rho r = r\rho^{n-1}$ as follows:

$$
\begin{aligned}
(r\rho^i)^{-1} &= \rho^{-i} r^{-1} \\
&= r^{-1} \rho^{-i(n-1)} \\
&= r\rho^i.
\end{aligned}
$$

Next we will conjugate elements of D_n by ρ^i and $r\rho^i$ to find the conjugacy classes.

First notice that for $1 \leq v < n$, $\phi_{\rho^i}(\rho^v) = \rho^v$ and $\phi_{r\rho^i}(\rho^v) = \rho^{-v}$. If n is odd, then the $n-1$ noncentral elements of $< \rho >$ are partitioned into 2-element conjugacy classes of the form $\{\rho^v, \rho^{-v}\}$. If n is even, the $[\rho^{\frac{n}{2}}] = \{\rho^{\frac{n}{2}}\}$ and the remaining $n-2$ noncentral elements of $< \rho >$ are partitioned into 2-element conjugacy classes of the form $\{\rho^v, \rho^{-v}\}$.

Next,

$$\phi_{\rho^{-j}}(r) = \rho^{-j} r \rho^j = r\rho^{-j(n-1)+j} = r\rho^{2j},$$

and conjugation of r by $r\rho^j$ produces no additional elements since

$$\phi_{r\rho^j}(r) = r\rho^j r (r\rho^j)^{-1} = r\rho^j r (r\rho^j) = r\rho^{2j}.$$

If n is odd then

$$< \rho^2 >=< \rho >= \{\rho^i : 1 \leq i < n\},$$

and it follows that

$$[r] = [r\rho^{2j}] = \{r\rho^j : 0 \leq j < n\}.$$

Hence $[r]$ contains all n elements of the form $r\rho^j$ if n is odd. If n is even then

$$< \rho^2 >= \{\rho^{2i} : 1 \leq i < \frac{n-1}{2}\},$$

and then

$$[r] = [r\rho^{2j}] = \{r\rho^{2j} : 0 \leq j < \frac{n-1}{2}\}.$$

Thus $[r]$ contains half of the elements of the form $r\rho^j$ if n is even. For even n, we will find all additional conjugacy classes containing elements of the form $r\rho^j$ beginning with $[r\rho]$. Let $1 \geq j < n-1$. Then

$$\phi_{\rho^{-j}}(r\rho) = \rho^{-j} r\rho\rho^j = r\rho^{-j(n-1)+j+1} = r\rho^{2j+1}.$$

Conjugation of $r\rho$ by $r\rho^j$ produces no additional elements since

$$\phi_{r\rho^j}(r\rho) = r\rho^j r\rho(r\rho^j)^{-1} = r\rho^j r\rho(r\rho^j) = r\rho^{j+(n-1)}\rho^j = r\rho^{2j+1}.$$

Since

$$< \rho^2 i + 1 >= \{\rho^i : 1 \leq i < n \text{ and } i \text{ is odd } \},$$

$$[r\rho] = [r\rho^{2j+1}] = \{r\rho^j : 0 \leq j < n \text{ and } j \text{ is odd } \}.$$

Thus $[r\rho]$ contains the other half of the elements of the form $r\rho^i$ if n is even. We have found all classes containing elements of the form $r\rho^j$, with $0 \leq j < n$ for both even and odd n. All nonidentity elements of D_n are included in one of the classes described above.

Semidihedral and Quasidihedral Groups

Because the relations are similar, we compute the different conjugacy classes similarly for both groups. First notice that each element of SD_n or QD_n can be written in the form a^i or ba^i with $0 \leq i \leq 2^n$.

Next we find the inverse of both forms: If $a^i \in SD_n$ or $a^i \in QD_n$, then $(a^i)^{-1} = a^{-i}$. Secondly, let $ba^i \in SD_n$ and apply the relation $ab = ba^{(2^{n-1}-1)}$ as follows:

$$(ba^i)^{-1} = a^{-i}b = ba^{(-i2^{n-1}+i)}.$$

Similarly, if $ba^i \in QD_n$, then

$$(ba^i)^{-1} = ba^{(-i2^{n-1}-i)}.$$

Next we conjugate elements of each form by both a^j and ba^j to find conjugacy classes. Suppose $a^t \in SD_n$. Then

$$\phi_{a^i}(a^t) = a^t \tag{6.4}$$

and

$$\begin{aligned}
\phi_{ba^i}(a^t) &= ba^i a^t ba^{(-i2^{n-1}+i)} \\
&= b^2 a^{(i+t)(2^{n-1}-1)} a^{(-i2^{n-1}+i)} \\
&= a^{(i2^{n-1}-i+t2^{n-1}-t-i2^{n-1}+i)} \\
&= a^{t2^{n-1}-t}.
\end{aligned}$$

Likewise, if $a^t \in QD_n$, then

$$\phi_{a^i}(a^t) = a^t \tag{6.5}$$

and

$$\phi_{ba^i}(a^t) = a^{t2^{n-1}+t}. \tag{6.6}$$

In SD_n, if t is even then $[a^t] = \{a^t, a^{-t}\}$, and if t is odd then $[a^t] = \{a^t, a^{2^{n-1}-t}\}$. This counts the single element conjugacy class $[a^{2^{n-1}}]$ and $\frac{|<a>|-2}{2} = 2^{n-1}-1$ conjugacy classes of order 2. (This does not count $[e]$.)

In QD_n, if t is even then $[a^t] = \{a^t\}$, and if t is odd then $[a^t] = \{a^t, a^{2^{n-1}+t}\}$. This counts $\frac{|<a>|}{2} - 1 = 2^{(n-1)} - 1$ conjugacy classes of order 1 and $\frac{|<a>|}{2^2} = 2^{(n-2)}$ conjugacy classes of order 2. (This does not count $[e]$.)

Next consider an element of the second form, $ba^t \in SD_n$. Then

$$\begin{aligned}
\phi_{a^i}(ba^t) &= a^i ba^t a^{-i} \\
&= ba^{i(2^{n-1}-1)+t-i} \\
&= ba^{i2^{n-1}-2i+t}
\end{aligned}$$

and

$$\begin{aligned}
\phi_{ba^i}(ba^t) &= ba^i ba^t ba^{(-i2^{n-1}+i)} \\
&= b^2 a^{i(2^{n-1}-1)} a^t ba^{(-i2^{n-1}+i)} \\
&= a^{i2^{n-1}-i+t} ba^{(-i2^{n-1}+i} \\
&= ba^{(i2^{(n-1)}-i+t)(2^{(n-1)}-1)} a^{(-i2^{(n-1)}+i)} \\
&= ba^{i2^{n-1}-i2^{n-1}+i+t2^{n-1}-t-i2^{n-1}+i} \\
&= ba^{(i+t)(2^{n-1})+2i-t}.
\end{aligned}$$

Likewise, if $a^t \in QD_n$, then

$$\phi_{a^i}(ba^t) = ba^{i2^{n-1}+t} \tag{6.7}$$

and

$$\phi_{ba^i}(a^t) = ba^{(i+t)(2^{n-1}+t}. \tag{6.8}$$

In SD_n,

$$\begin{aligned}
[ba] &= \{ba, ba^{2^{n-1}-1}, ba^{-3}, ba^{2^{n-1}+3}, ...\} = \{ba^j : \text{ j odd }\} \\
[ba^2] &= \{ba^2, ba^{2^{n-1}-2}, ba^{-4}, ba^{2^{n-1}+4}, ...\} = \{ba^j : \text{ j even }\}.
\end{aligned}$$

There are 2 conjugacy classes with 2^{n-1} elements each.

In QD_n, $[ba^t] = \{ba^t, ba^{2^{n-1}+t}\}$. There are $\frac{2^n}{2}$ conjugacy classes with 2 elements of this type. All nonidentity elements are included in one of the classes listed above.

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Vita

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