CHOICE FUNCTIONS, DYNAMICS, AND EQUAL REPRESENTATION

By

KATHERINE M. DONADIO

A Thesis Submitted to the Graduate Faculty of

WAKE FOREST UNIVERSITY

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF ARTS

in the Department of Mathematics

May 2010

Winston-Salem, North Carolina

Approved By:

Kenneth S. Berenhaut, Ph.D., Advisor ____________________

Examining Committee:

Fredric T. Howard, Ph.D., Chairperson ____________________

Hugh N. Howards, Ph.D. ____________________

Table of Contents

Acknowledgments

There are many people who helped to make this thesis possible. First and foremost, I would like to thank my advisor Dr. Kenneth Berenhaut; his support and advice have been crucial in the development of this thesis. I would also like to thank the faculty in the Department of Mathematics at Wake Forest University, who have all contributed to my mathematical development in varying important ways. Finally, I would like to thank my parents, my family, and Kyle.

Abstract

Katherine M. Donadio

In this thesis, we study dynamics of systems evolving according to choice functions; such scenarios have the potential of being applicable in data and network processing, decision theory, and sampling. In particular, consider n linearly ordered size-one "spots" and k classes from which to fill the spots (where repetitions are allowed). Each time period, there is a shift - a new member enters the system on the right and the left-most member leaves; choice arises in selecting a new member. We are interested in choice functions on subsets of $\{1, 2, \ldots, k\}$ which ensure equal representation among classes both locally and over time. Graph representations encapsulating domination properties of the given choice function have proven to be helpful in obtaining results. Other related matters that could have potential application include determining "optimal" choice functions with minimal period lengths and low burn-in.

Chapter 1: Introduction

The purpose of this thesis is to examine a particular problem which has the potential of being applicable in data and network processing, decision theory, and sampling. In the current section, the issue will first be introduced and discussed, along with various methods of approaching the given problem and their disadvantages compared to the method used in this thesis. Then, a concrete example employing each of the methods previously considered will be given and examined. The section will conclude with possible variations of the problem which may be important or interesting to consider.

Suppose one is given the following general scenario. Presume there are $s + 1$ linearly ordered "spots" of size one, and k classes from which to fill the spots, where repetition is permitted. Each time period, a shift occurs which compels the left-most member of the system to depart and allows a new member to enter on the right. Choice arises in terms of how to fill the newly vacated right-most spot. The problem this thesis seeks to address is how to select a new member in a manner that ensures equal representation of each of the k classes both locally and over time. Often the methods of choosing a new member will rely on choice from the "underrepresented" group in the current s string, where the underrepresented group is composed of the classes who appear the least in the string after the shift. For example, suppose the original string with $s + 1 = 7$ and $k = 4$ was (4 1 1 3 4 2 4). Then the s string used to determine the new class would be (1 1 3 4 2 4), and the underrepresented group would be $\{2, 3\}$ because 2 and 3 only appear one time, while the other classes all appear more than once.

Several methods initially appear reasonable in attempting to determine a good way to procure new entrants in a manner which would result in equal representation.

The first possible approach would be to randomly fill the new spot, since a random system would never intrinsically favor any one item over another; a scenario which would ideally lead to approximately equal representation. There are two basic methods of choosing randomly, either one could randomly pick from the k classes overall, or one could randomly choose from the underrepresented classes in the current string. Both of these scenarios have disadvantages. Choosing randomly from all k classes can result in long strings of one class (especially when the number of classes is small), which completely fails to satisfy the local equal representation requirement. Similarly, choosing randomly from the underrepresented classes could lead to long strings which fail to contain a certain class at all, which additionally fails the local equal representation requirement. Both methods of randomly choosing the new entrant can result in large swings in absolute cumulative total class representations. Thus, random choice fails to satisfactorily achieve the desired equal representation conditions. Finally, it should be mentioned that all methods of choosing randomly, while reasonably accurate, are in actuality pseudo-random.

A second intuitive approach would be determination of the new entrant based on particular events in the past; in other words one could chose the member of the underrepresented group who has also been chosen least recently in some length of time in the past. This idea involves storing a "memory" of what has occurred in the past (ideally beyond the current string), which in itself presents a variety of problems. Firstly one has to decide what size the memory should be, in terms of balancing storage cost and information cost. A larger memory provides a higher degree of information sometimes ensuring better choice of the new entrant; however, a larger memory also uses more storage space and can vastly increase computation time. Storing a sufficient size of memory for large values of k may even be unfeasible. Another disadvantage of this approach is that its highly deterministic nature means that it may exhibit widely different behavior based on varying initial values. Finally, using this approach would require some method for breaking initial ties in starting values, or if one chooses a value for the memory which is smaller than k.

A variation on the idea of using extended memory to determine the new left-most member would be to place weighting on all occurrences of a class in the past, where the weighting is based on the distances at which particular classes have appeared. The selection could then be determined by accumulating the weighted distances for each class and choosing the class with the smallest sum. Thus this method provides a level of priority to classes who have entered least recently. This scenario accommodates a plethora of possibilities for the type of weighting distribution which would be ideal. For example, one could use a weighting that followed a linear decline rate, or perhaps an exponential rate. This scenario is then communicating a certain negative dependence. That is, that the current state is more likely to be dissimilar to recent states than it is to be similar to them, where dissimilar means not of the same category class. One complexity that may arise is the necessity of a method for breaking ties in sums among classes. Like the previous method, this approach could have the disadvantage of requiring the storage of a significant amount of memory, and the aforementioned problems associated with memory storage. While unlimited memory storage may be unfeasible, the use of a finite, small, amount of information from the past will prove useful in the study of the given problem.

Considering the disadvantages of the previous natural first approaches, a more satisfactory solution is still required. If one sought equal representation of these k classes both locally and over time, the following two stipulations seem reasonable to impose.

(i) The new entrant must be from an underrepresented class in the current state of the system.

(ii) Representation should be equal over all classes within any period (orbit) of the system, regardless of the given initial values.

Now consider the choice functions, C, on the non-empty subsets of the set $\{1, 2, \ldots k\}$ as the methods for determining the new entrant from the underrepresented classes. This thesis involves a study on these choice functions to categorize choice functions satisfying condition (ii).

First consider the following simple example to illustrate the differences between the previously given approaches. Suppose that in the stated problem there are nine spots and five classes, in other words $s + 1 = 9$ and $k = 5$. Let the following choice function on subsets of size two of $\{1, \ldots k\}$ be defined as follows.

$$C(\{1,2\}) = 1,\ C(\{1,3\}) = 3,\ C(\{1,4\}) = 1,\ C(\{1,5\}) = 1,\ C(\{2,3\}) = 2,$$
$$C(\{2,4\}) = 4,\ C(\{2,5\}) = 2,\ C(\{3,4\}) = 3,\ C(\{3,5\}) = 5,\ C(\{4,5\}) = 4.$$

The ties of size two are sufficient to examine long term behavior for any choice function with this particular $s+1$ and k since it will be shown that for all starting values given $s + 1 = 9$ and $k = 5$ the underrepresented groups will eventually always be of size one or two (see Theorem 2, below). One calculation of this example using the above defined choice function eventually reaches a period of size ten which looks like (1 1 4 4 2 2 5 5 3 3). A solution with this period obviously satisfies equal representation both locally and globally of each of the five classes. Now, suppose that the new entrant was chosen from the underrepresented group randomly. One calculation of this example using this method leads to the following string (4 1 2 3 4 2 3 1 3 2 2) of size eleven, which fails to contain the class 5 entirely. Thus, it is obvious that random choice, even choosing randomly from the underrepresented classes, can fail to uphold local equal representation. Undoubtedly a string can easily fail both requirements. Now consider the past dependent approach, where the new entrant is the member appearing least often and least recently in the past. One calculation employing this method results in the period (1 2 3 4 5), which clearly results in equal representation locally and globally. However, as discussed previously the employment of this approach still requires method for breaking ties arising from the initial values where no history is available.

Another approach for determining new entrants which would guarantee equal representation both locally and globally would be to take any permutation of $\{1, \ldots, k\}$ and then use this repeated sequence as the sequence of new entrants. This approach clearly satisfies local and global equal representation since each of the k classes will appear exactly once in any k spots. The use of choice functions to procure new entrants works out to be a variant on this idea. The main disadvantage to the former approach is that the first class represented in the permutation obtains an advantage over the others, in other words that class always gets first representation. This may or may not be a desirable situation depending on the specific application of the given problem being practiced.

The solution of choosing new members based on choice functions is potentially applicable in a number of fields. Firstly, in the area of data processing, often multiple tasks need to be processed at any given moment. These tasks could be classified by their type and choice functions could then be used to make the decision of what order in which to process these tasks without giving specific tasks priority ratings. Another potential application would be for any type of draft selection situation in which a number of groups must choose new members in some unbiased order. The predetermined choice functions could be set up to give some priority to lower rated member beating higher rated members if such a scenario was desired.

The given problem considered here allows for several variations on its theme which could be valuable to pursue. One variation would be to consider a shift of length greater than one but less than $s+1$ given $s+1$ linearly ordered size-one spots filled with k classes, thus enabling multiple new entrants at once. This particular scenario may prove important in data and network processing. Another possible important scenario would be to give the various classes priority over each other; this scenario would involve weighting the classes by their importance. Finally, one could attempt to discover yet another, potentially more complex, alternative method of choosing a new entrant which would lead to local and global equal representation. One could also consider requirements in addition to (i) and (ii) such as cut-offs on the time until the eventual period is reached, limits on the length of time between appearances of a given class, or limits on the number of consecutive selections from a given class.

The remainder of this thesis proceeds as follows. Chapter 2 gives background on choice functions (which could be skipped on a first read of the thesis). Chapter 3 includes some preliminary notation and results, while Chapter 4 focuses on results related to graph representations of domination properties for choice functions. Chapter 5 provides some beneficial examples; Chapter 6 concludes the thesis with summary and discussion of further avenues of research. Two appendices which encompass earlier published research of the author are included; these provided important motivation that eventually led to the directions taken in this thesis.

Chapter 2: Background on Choice Functions

This section includes some background on choice functions. Choice functions have proven important in a plethora of fields. Perhaps, one of the most well-known examples relating to choice functions is the infamous Axiom of Choice, which was developed by Ernst Zermelo in 1904. The Axiom of Choice states that given a collection $\mathcal{S}$ of nonempty sets, there is a function C defined on $\mathcal{S}$ which assigns for each set $S \in \mathcal{S}$ an element $C(S) \in S$ (see for instance [8]). In other words, the axiom states that for any collection, a member from each set in the collection can be chosen. Thus, the function C is a choice function. For a finite collection $\mathcal{S}$ there is no difficulty in choosing an element in S for each S, the challenge arises when the collection $\mathcal{S}$ contains an infinite number of sets. While the Axiom of Choice is controversial, most mathematicians believe it to be true and often use it as a basic assumption.

Topology is one of the many areas of mathematics that make use of choice functions. An example of one particular application of choice functions which arose recently in topology is the following. Suppose one is given a function $\varphi : [n]^3 \to [n]$ such that $\varphi(A) \in A$ for all $A \in [n]^3$. Note that φ is choice function since it maps elements of a particular set to some element in that set. Consider $a, b \in [n]$ that satisfy the following stipulation. For each $c \in [n]$ there exist $a_1, b_1, \ldots, a_{k(3)}, b_{k(3)}$ where $\varphi(\{a_i, b_i, a_{i+1}\}) = a_{i+1}, \varphi(\{a_i, b_i, b_{i+1}\}) = b_{i+1}$ for $a_1 = a$, $b_1 = b$, and $a_{k(3)} = c$. The pair $\{a, b\}$ is said to $k(3)$-dominate the element c.

For a set X and a, b, c in X, define $c \leftarrow_{\varphi} \{a, b\}$ ($\{a, b\}$ dominates c) to mean $\varphi\{a, b, c\} = c$ or $c \in \{a, b\}$. In general, if $k > 1$, then $\{a, b\}$ is said to k-dominate c, denoted $c \leftarrow_{\varphi}^{k} \{a, b\}$ if there exist $a', b' \in X$ such that $\{a', b'\}$ $(k-1)$-dominates c and $\{a, b\}$ dominates both a' and b'. Study of this problem in [5] led to proof of the following theorem.

Theorem. *There is for each* $n > 1$ *a number* $k = k(n)$ *such that for any finite set* X *and any choice function* $\varphi : [X]^n \to X$ *there is* $A \in [X]^{n-1}$ *with* $b \xleftarrow[\varphi]{k} A$ *for all* $b \in X$*. We can take* $k(n) = n$*.*

The previously stated problem turns out to be the combinatorial formulation of the following geometric problem. Let $\mathcal{A}$ be a finite family of subsets of the plane such that for any $A_1, A_2, A_3 \in \mathcal{A}$ there exist two A_i, A_j such that an epsilon neighborhood of $A_i \cup A_j$ contains the remaining member. Define the mapping $\varphi : [\mathcal{A}]^3 \to \mathcal{A}$ by choosing for each triple A_1, A_2, A_3 an element $\varphi(\{A_1, A_2, A_3\})$ contained in the epsilon-neighborhood of the union of the other two elements. By the above theorem, this implies there exists a 3-dominating pair $\{A_i, A_j\} \in [\mathcal{A}]^2$. For further explanation and results see [5]. This example illustrates how choice functions can be useful in various fields of mathematics, such as in this instance, topology.

Choice functions also naturally arise in the study of economics, which is dependent on the decisions and choices that people make. One example of this is the Weak Axiom of Revealed Preference, which is inherently based on choice functions. In this axiom, revealed preference is defined as follows [9].

Definition. *Given some vectors of prices and chosen bundles (i.e. quantities of the associated items)* (p^t, x^t) *for* $t = 1, \dots T$*, we say* x^t *is directly revealed preferred to a bundle* x *if* $p^t x^t \geq p^t x$*, denoted* $x^t R_D x$*. We say* x^t *is revealed preferred to* x *if there is some sequence* $r, s, t, \dots u, v$ *such that* $p^r x^r \geq p^r x^s \geq, p^s x^s \geq p^s x^t \geq \dots \geq p^u x^u \geq p^u x$*, denoted* $x^t R x$*. In this case, we say the relation* R *is the transitive closure of the relation* R_D*.*

Thus, an item is directly revealed preferred to another if a person chooses a given product over another when both are available and affordable. The second definition provides transitivity of revealed preference. The above definition of revealed preference led to the following theorem of Samuelson [9].

Theorem. *If* $x^t R_D x^s$ *then it is not the case that* $x^s R_D x^t$*. Algebraically,* $p^t x^t \geq p^t x^s$ *implies* $p^s x^s < p^s x^t$.

This theorem states if an item is revealed preferred over another item, it is not the case that the second item is revealed preferred to the first; in other words revealed preference can only be upheld in one direction. The Weak Axiom of Revealed Preference illustrates a vital component of preference relations in economics, which are characteristically based upon the idea of choice of a consumer.

A number of concepts about choice and preference used in the study of economics have proved useful during the author's study on choice functions. In particular, the binary relation of weak preference can be used to generate choice functions; this symbiotic relationship of weak preference and choice functions is related to the use of graph representations in this thesis. Denote R as weak preference; in other words xRy means that x is at least as good as y. Similarly, P and I can be used to denote strict preference and indifference respectively. Now define the choice set, $\mathcal{C}(S, R)$ where S is a set of best alternatives and R denotes the preference relation, as follows.

Definition. *For a set* S *and a relation* R *on* $S \times S$*, let* $\mathcal{C}(S, R) = \{x : (x \in S)$ *and for all* $y \in S$*,* $xRy\}$.

Let X be the set of all alternatives. Now define a choice function over the choice set as follows.

Definition. *A choice function* C *defined over* X *is a functional relationship such that the choice set* $C(S) = \mathcal{C}(S, R)$ *is non-empty for every non-empty subset* S *of* X.

This definition ensures that the set of best alternatives exists for all subsets S of X (see for instance [7]). Notably, the concepts of weak preference and strict preference were useful for creating graph representations with domination properties to ascertain when choice functions satisfy the given condition (ii).

Another concept from the study of economics which proved useful was the idea of cyclic choice. Using cyclical choice functions helped to ensure that no one class was chosen too regularly by winning over all of the other classes. A cyclic choice function on three variables x, y, and z is defined as follows (see [4])

$$C(x,y) = x;\ C(y,z) = y;\ and\ C(x,z) = z.$$

This concept of cyclicality can easily be applied to subsets of varying sizes. For example a cyclic choice function on four variables w, x, y, and z could be defined via the following equalities

$$C(w,x,y) = w;\ C(x,y,z) = x;\ C(w,x,z) = z;\ and\ C(w,y,z) = y$$

$$C(x,y) = x;\ C(y,z) = y;\ and\ C(x,z) = z$$

$$C(w,x) = w;\ C(x,y) = x;\ and\ C(w,y) = y$$

$$C(w,x) = w;\ C(x,z) = x;\ and\ C(w,z) = z$$

$$C(w,y) = w;\ C(y,z) = y;\ and\ C(w,z) = z.$$

In other words, the stipulation for a choice function to be cyclic is that each element wins exactly once in a subset of a given size which contains that element. Thus, following this method, one could easily extend the definition of cyclic choice to functions on a set of variables of any size. These concepts of cyclical choice functions proved useful in attempting to classify choice functions satisfying or not satisfying (ii) (see Theorems 6, 7, and 9, below).

The study of choice functions is not restricted to conventional mathematical fields, they also arise, for instance, in linguistics during the study of indefinites. Let us consider one treatment of indefinites to examine how choice functions are important in linguistics. One such study of indefinites by Reinhart [10] adopts the three following assumptions.

1. Indefinites lack quantificational force of their own. They are basically predicates.

2. An indefinite Noun-Phrase in an argument position, however, ends up denoting an individual. This is because its semantics involves a free function variable that assigns an individual to the restriction predicate.

3. This function variable is existentially closed, together with the restriction that it is a choice function: a function that chooses a member from any non-empty predicate it gets. This quantificational procedure can apply at any compositional level.

In order to better understand these assumptions and the interplay of choice functions and linguistics, consider the following simple example.

Some woman smiled. (2.1)

From the three above assumptions (1.1) is understood to mean that there exists a choice function which picks an individual from the predicate denoting woman that is in the extension of the predicate denoting smiled. The full development and treatment of indefinites is rich and complex, but this example illustrates how choice functions are integral in their study.

For further discussion on choice functions and their use in various field see for instance [1] and [6].

Chapter 3: Preliminary Notation and Results

This section begins with some preliminary notation. First, suppose $k, s \geq 1$,

$$\tilde{x} = \{x_{-s}, \ldots, x_{-1}, x_0, x_1, \ldots\} \tag{3.1}$$

is a sequence, and let

$$Y = \{1, 2, \ldots, k\}.$$

For a fixed $1 \leq q \leq n + s$, set

$$N_y^q(n) = \|\{1 \leq i \leq q : x_{n-i} = y\}\|, \ n \geq 0, \tag{3.2}$$

i.e. the number of occurrences of class y in the q values preceding x_n. In addition let

$$S(n) = \left\{ y : N_y^s(n) = \min_{z \in Y} N_z^s(n) \right\}, \ n \geq 0 \tag{3.3}$$

denote the set of least represented classes in the previous s. For a given choice function C on the subsets of Y, suppose

$$x_n = C(S(n)), \ n \geq 0, \tag{3.4}$$

and let $P = P_{\tilde{x}}$ denote the eventual (prime) period of $\tilde{x}$. Note that due to (3.4) and the finite nature of Y, $\tilde{x}$ must be eventually periodic.

In what follows, suppose $k, s \geq 1$ are fixed and C is an element of the set, $\mathcal{C}_k$, of all possible choice functions on the subsets of $\{1, 2, \ldots, k\}$.

Definition. *A choice function, C is said to be "good" if for sufficiently large n,*

$$N_i^P(n) = N_j^P(n) \text{ for all } 1 \leq i, j \leq k, \tag{3.5}$$

for all initial values $x_{-s}, \ldots, x_{-1}$. Let $\mathcal{F}_{k,s} \subseteq \mathcal{C}_k$ be the set of all good choice functions.

The following theorem shows that approximately equal distribution in a local sense is unavoidable.

Theorem 1. *Given* $\tilde{x} = \{x_{-s}, \ldots, x_{-1}, x_0, x_1, \ldots\}$ *satisfying (3.4), the following holds for sufficiently large* n

$$|N_i^{s+1}(n) - N_j^{s+1}(n)| \leq 1 \text{ for all } 1 \leq i, j \leq k. \tag{3.6}$$

Any $s+1$ *values* $\{x_{n-(s+1)}, \ldots, x_{n-1}\}$ *resulting in* n *satisfying (3.6) will be referred to as "smooth"; in this case* $N_y^{s+1}(n) \in \{Q, Q+1\}$ *for* $y \in Y$ *where* $Q = \lfloor (s+1)/k \rfloor$*. Note that if* $\{x_{n-(s+1)}, \ldots, x_{n-1}\}$ *is smooth then* $\{x_{n-(s)}, \ldots, x_n\}$ *is as well due to (3.4).*

Proof. First, for $n \geq 1$ set

$$M_n = \max_{y \in Y}\{N_y^{s+1}(n)\}$$

and

$$m_n = \min_{y \in Y}\{N_y^{s+1}(n)\},$$

and define

$$\Delta_n = M_n - m_n. \tag{3.7}$$

In addition, let

$$\begin{aligned} X_1 &= X_1(n) = \{y \in Y : N_y^{s+1}(n) = m_n\}, \\ X_2 &= X_2(n) = \{y \in Y : m_n < N_y^{s+1}(n) < M_n\}, \\ X_3 &= X_3(n) = \{y \in Y : N_y^{s+1}(n) = M_n\}, \end{aligned}$$

partition Y. Then $x_{n-(s+1)}$ is an element of exactly one of the three sets X_1, X_2, and X_3.

1. Suppose $x_{n-(s+1)} \in X_1$, then $x_n = x_{n-(s+1)}$, and therefore $\Delta_{n+1} = \Delta_n$.

2. Suppose $x_{n-(s+1)} \in X_2$, then $x_n \in X_1 \cup \{x_{n-(s+1)}\}$. If $\|X_1\| = 1$ and $N^{s+1}_{x_{n-(s+1)}} - m_n > 1$, then $m_{n+1} = m_n + 1$ and $M_{n+1} = M_n$. Hence, $\Delta_{n+1} = \Delta_n - 1$. Otherwise, $m_{n+1} = m_n$ and $M_{n+1} = M_n$. Thus, $\Delta_{n+1} = \Delta_n$.

3. Suppose $x_{n-(s+1)} \in X_3$, then $x_n \in X_1$. If $\|X_3\| = 1$ and $\|X_1\| = 1$, then $\Delta_{n+1} \leq \Delta_n$. If $\|X_3\| = 1$ and $\|X_1\| > 1$, then $\Delta_{n+1} = \Delta_n - 1$. If $\|X_3\| > 1$ and $\|X_1\| = 1$, then $m_{n+1} = m_n + 1$ and $\Delta_{n+1} = \Delta_n - 1$. Otherwise, $\Delta_{n+1} = \Delta_n$

Thus, the sequence $\{\Delta_n\}$ is decreasing. Since $\{\Delta_n\}$ is decreasing and bounded, it has a limit. Therefore $\Delta_n = A$, a constant, eventually. Without loss of generality, assume $\Delta_n = A > 1$ for all n under consideration. Since Δ_n is constant, if m_n, M_n vary, then they must vary together. In this case,

$$\delta_n \overset{def}{=} m_{n+1} - m_n = M_{n+1} - M_n \in \{0, -1, 1\}.$$

The value δ_n cannot be zero indefinitely, hence assume $\delta_n \in \{-1, 1\}$. Both values for δ_n are impossible; to verify this, consider the two cases separately. If $\delta_n = 1$, then $m_{n+1} - m_n = 1$ and hence

$$\|X_1(n)\| = 1 \text{ and } \|X_2(n+1)\| \geq \|X_2(n)\| + 1.$$

Thus, $M_{n+1} \leq M_n$, a contradiction. If $\Delta_n = -1$, then $M_{n+1} - M_n = -1$ and hence

$$\|X_3(n)\| = 1 \text{ and } \|X_2(n+1)\| \geq \|X_2(n)\| + 1.$$

Thus, $m_{n+1} \geq m_n$, a contradiction. This implies $\Delta_n = A \leq 1$. Returning to (3.7) and comparing with (3.6), the theorem is proven. □

The next theorem proves the fact mentioned in the introduction that when examining long term behavior it is sufficient to consider the values of the choice function at subsets of $\{1, 2, \ldots, k\}$ of one particular size.

Theorem 2. *The cardinality of $S(n)$ is either 1 or $(-s)_k$ for n sufficiently large, where $(-s)_k$ is the least positive residue of $-s$ modulo k. In addition, $S(n) = (-s)_k$ for infinitely many n.*

Proof. Assume smooth initial values (see Theorem 1) and note that $s + 1 = Qk + r$ for some $0 \leq r < k$. Now, let

$$S_1(n) = \{y \in Y : N_y^{s+1} = Q\} \tag{3.8}$$

and

$$S_2(n) = \{y \in Y : N_y^{s+1} = Q + 1\} \tag{3.9}$$

be the set of values appearing Q times in the previous $s + 1$, and the set of values appearing $Q + 1$ times in the previous $s + 1$, respectively. The cardinalities of these sets are given by $\|S_1(n)\| = k - r$ and $\|S_2(n)\| = r$. If $r = 0$, then $x_{n-(s+1)} \in S_1(n)$ and $\|S(n)\| = 1$. Aside from this, there are two cases:

1. If $x_{n-(s+1)} \in S_1(n)$ then $\|S(n)\| = 1$.

2. If $x_{n-(s+1)} \in S_2(n)$ then $\|S(n)\| = k - r + 1$, and

$$(k - r + 1) - (-s) = k - r + 1 + Qk + r - 1 = (Q + 1)k \equiv 0 \bmod k.$$

Thus, $k - r + 1 = -s$ modulo k. Note that $1 \leq k - r + 1 \leq k$ since $1 \leq r < k$. To prove the final statement in the theorem, suppose $z \in Y$ is over-represented in the smooth values $x_n, \ldots, x_{n+s}$ and set

$$i = \min\{0 \leq j \leq s : x_j = z\}.$$

Then

$$N_y^s(i + s + 1) \in \left\{ \left\lceil \frac{s}{k} \right\rceil, \left\lfloor \frac{s}{k} \right\rfloor \right\} \text{ for all } y \in Y,$$

and thus $\|S(i+s+1)\| = (-s)_k$. $\square$

As illustrated by the previous theorem, the quantity $(-s)_k$, is important and will be from hereafter denoted by V.

The following two theorems provide conditions under which a choice function fails to be good.

Theorem 3. *If $A \subseteq Y$ and*

(a) $0 < \|A\| < (-s)_k$

(b) For all sets T such that $\|T\| = (-s)_k$ and $A \subseteq T$, $C(T) \cap A = \emptyset$,

then $C \notin \mathcal{F}_{k,s}$.

Proof. Take smooth initial values with $A \subset S_1(0)$, by (b) $x_n \notin A$ whenever $\|S(n)\| = (-s)_k$. Select $y \in A$ and $z \in S_2(N)$ where N satisfies $x_{n-P} = x_n$ for all $n \geq N$. It follows that

$$N_z^{s+1}(n) \geq Q = N_y^{s+1}(n) \text{ for } n \geq N \text{ and } n \not\equiv N \bmod P$$

and

$$N_z^{s+1}(n) = Q + 1 > Q = N_y^{s+1}(n) \text{ for } n \geq N \text{ and } n \equiv N \bmod P,$$

where P is the period of $\tilde{x}$. Since

$$N_u^P(n) = \frac{\sum_{i=N}^{N+P+1} N_u^{s+1}(i)}{s+1}, \tag{3.10}$$

we have $N_z^P(n) > N_y^P(n)$ for $n \geq N$, and $C \notin \mathcal{F}_{k,s}$. $\square$

Theorem 4. *If $k|s$ then $C \notin \mathcal{F}_{k,s}$.*

Proof. Suppose the given initial values are smooth. Since C is a choice function, $C(Y) = y_1$ for some $y_1 \in Y$ where $Y = \{1, 2, \ldots, k\}$. Since $k|s$, $(-s)_k = k$, and by

Theorem 2, $\|S(n)\| \in \{k,1\}$ for sufficiently large n. Select N such that for $n \geq N$, $x_{n-P} = x_n$ and $\|S(n)\| \in \{k,1\}$. Note that for $z \neq y_1$, $x_n = z$ implies $N_z^s(n) = q-1$ and hence $N_z^{s+1}(n) = q$ for $n \geq N$. Let m be the smallest number such that $\|S(m)\| = k$ where $m > N$, then $N_{y_1}^{s+1}(m+1) = q+1$. As in the proof of Theorem 3, applying (3.10) $N_{y_1}^P(n) > N_z^P(n)$ and $C \notin \mathcal{F}_{k,s}$. □

The next two theorems provide conditions under which equal representation over classes is guaranteed within the given period, P.

Theorem 5. *If $s \equiv -1 \bmod k$, then $C \in \mathcal{F}_{k,s}$.*

Proof. Given smooth initial values, $s \equiv -1 \bmod k$ implies that $\|S(n)\| = 1$ and $x_n = x_{n-(s+1)}$ for all $n > 0$, and hence $\{x_n\}$ is periodic with (not necessarily prime) period $s+1$. The $s+1$ period must have equal representation of classes due to the smooth initial values. It is possible that the prime period P could strictly divide $s+1$; note that in this case $N_{y_1}^P > N_{y_2}^P$ would carry over to $N_{y_1}^{s+1} > N_{y_2}^{s+1}$, which would contradict the equal representation in the period $s+1$. □

Theorem 6. *Suppose that C satisfies*

$$C(\{(i)_k, (i+1)_k, \ldots, (i+V-1)_k\}) = (i)_k \text{ for all } 1 \leq i \leq k, \tag{3.11}$$

where $V = (-s)_k$ is as defined following the proof of Theorem 2. Then there exist initial values $\{x_{-s}, x_{-s+1}, \ldots, x_{-1}\}$ such that $C \in \mathcal{F}_{k,s}(x_{-s}, x_{-s+1}, \ldots, x_{-1})$, the set of good choice functions under the given initial values. In particular, there exist initial values such that the period, P, equals k.

Proof. Express s in the form $s = mk + (k-V)$ and consider initial values

$$\overbrace{(V+1)\ldots(V+(k-V))}^{k-V}\ [\overbrace{(1)\ldots(V)}^{V}\ \overbrace{(V+1)\ldots(V+k-V)}^{k-V}]\ldots$$
$$[(1)\ldots(V)(V+1)\ldots(V+(k-V))]. \tag{3.12}$$

where the string of values $(1)\ldots(V)(V+1)\ldots(V+k-V)$ of length k in (3.12) is repeated m times. It is not difficult to verify via (3.11) that the s initial values under the choice function defined by C in (3.12) begin a periodic sequence with period k, where each class is represented exactly once in the period. □

The main value of Theorem 6 is that it leads to a condition that guarantees $C \notin \mathcal{F}_{k,s}$, as illustrated in the following theorem.

Theorem 7. *Suppose* $1 \leq k_1 < k$ *is fixed and let* $s_1 = s - m(k - k_1)$ *where* $s = mk + (k - V_1)$ *and* $V_1 = (-s_1)_{k_1}$. *In addition, suppose that* C *satisfies*

$$C(\{(i)_{k_1}, (i+1)_{k_1}, \ldots, (i+V_1-1)_{k_1}, (k_1+1), (k_1+2), \ldots, (k)\}) = (i)_{k_1}, \quad (3.13)$$

for all $1 \leq i \leq k_1$. *Then* $C \notin \mathcal{F}_{k,s}$.

Proof. Consider the initial values given in (3.12), inserting the values

$$k_1+1, k_1+2, \ldots, k \quad (3.14)$$

following each of the m repetitions of the sequence of values

$$1, 2, \ldots, k_1, \quad (3.15)$$

resulting in initial values

$$(V_1+1)\ldots(V_1+(k_1-V_1))\ \overbrace{[(1)\ldots k)]\ldots[(1)\ldots k)]}^{m \text{ times}}. \quad (3.16)$$

It is not difficult to verify that for the initial values in (3.16), $X_n = X_{n-s}$ for the inserted $k - k_1$ classes in (3.14) and $x_n = (x_{n-s} + s_1 + 1)_{k_1}$ otherwise. Let t be the least positive integer such that k_1 divides $(s_1+1)t$. The sequence $\{x_n\}$ is then periodic with (not necessarily prime) period $P' = t(s+1)$ with $N_y^{P'}(n) = mt$ for y in the set of classes given in (3.14) and

$$N_y^{P'}(n) = mt + \frac{(s_1+1)_{k_1}t}{k_1} = mt + \frac{(k_1 - V_1 + 1)t}{k_1} > mt$$

otherwise, for $n > P'$. Note that

$$k_1 \left(mt + \frac{(k_1 - V_1 + 1)t}{k_1} \right) + mt(k - k_1) = t(mk + (k_1 - V_1 + 1)) = t(s + 1) = P'$$

Hence, the theorem is proven. □

The reader is referred to Example 2 for an application of Theorem 7 for $k = 5$ and $s = 7$.

The next section includes the introduction of a main result of this thesis which is based on graph representations.

Chapter 4: Graph Representations

In this chapter, the use of graph representations in the context of the given problem will be introduced. Several related results will be proven, followed by numerous illustrative examples.

The following theorem involving class domination will be important for proving the main result of this thesis.

Theorem 8. *Suppose $z_1, z_2 \in Y$. If for any set T the two conditions* (a) $\|T\| = (-s)_k$ *and* (b) $z_1, z_2 \in T$ *imply $C(T) \neq z_2$, then*

$$N^P_{z_1}(n) \geq N^P_{z_2}(n) \tag{4.1}$$

for sufficiently large n.

Proof. Assume the given initial values $\{x_{-(s+1)}, \ldots, x_{-1}\}$ are smooth and recall $Q = \lfloor (s+1)/k \rfloor$ is the number of repetitions from each underrepresented class among any string of $s+1$ values in $\tilde{x}$ (see Theorem 1). Then for $-s \leq i \leq 0$,
$N^{i+(s+1)}_j(i) - N^{i+(s+1)}_l(i) \leq Q + 1$ for all $1 \leq j, l \leq k$. In particular, setting

$$\delta_i = N^{i+(s+1)}_{z_2}(i) - N^{i+(s+1)}_{z_1}(i), \tag{4.2}$$

for $-s \geq i$, it follows that

$$\delta_i \leq Q + 1 \tag{4.3}$$

for $-s \leq i \leq 0$. Assume that δ_i is unbounded above, and consider the smallest n such that $\delta_n = Q + 2$. Note that $n > 0$ by (4.3). Examine $\delta_{n-(s+1)}$. Since $x_{n-1} = z_2$, by the given assumptions, this implies $N^s_{z_2}(n-1) < N^s_{z_1}(n-1)$. Thus

$$\delta_n = \delta_{n-(s+1)} + N^s_{z_2}(n-1) - N^s_{z_1}(n-1) + 1.$$

Therefore,

$$\delta_{n-(s+1)} = \delta_n + N^s_{z_1}(n-1) - N^s_{z_2}(n-1) - 1 \geq \delta_n = Q+2,$$

which contradicts the statement that n is the smallest number such that $\delta_n = Q+2$. Therefore δ_i is bounded above. Suppose that for sufficiently large n, $N^P_{z_2}(n) - N^P_{z_1}(n) = t > 0$. Then $\delta_{n+P} = \delta_n + t$, a contradiction to the fact that $\{\delta_i\}$ is bounded. □

Theorem 9. *Consider a graph G with vertex set Y and directed edge set, $E = \{(i,j)\}$, determined via*

$$(i,j) \in E \text{ if and only if for all } T \text{ with } \|T\| = (-s)_k \text{ and } i,j \in T,\ C(T) \neq j.$$

If the graph G is path connected then $C \in \mathcal{F}_{k,s}$.

Proof. Suppose G is path connected. There exists a path of the form

$$(z_1, z_{i_1})(z_{i_1}, z_{i_2}) \cdots (z_{i_q}, z_1)$$

such that $\{1, i_1, i_2, \ldots, i_q\} = Y$. By Theorem 8,

$$N^P_1(n) \geq N^P_{i_1}(n) \geq N^P_{i_2}(n) \cdots \geq N^P_{i_q}(n) \geq N^P_1(n). \tag{4.4}$$

Hence, the inequalities in (4.4) are equalities and thus

$$N^P_1(n) = N^P_2(n) = \cdots = N^P_k(n).$$

□

Note that the argument employed in Theorem 9 carries over to connected components. This fact is provided in the following theorem.

Theorem 10. *For the construction of G given in Theorem 9 if $Y_1 \subseteq Y$ are the vertices in a connected component of G then for sufficiently large n,*

$$N^P_i(n) = N^P_j(n)$$

for $i, j \in Y_1$.

Proof. The proof of the theorem follows analogously to that of Theorem 9. □

The following theorem shows that the set of good choice functions, $\mathcal{F}_{k,s}$, is nonempty for $k \nmid s$.

Theorem 11. *Suppose $k \nmid s$ and consider the choice function defined as follows. For sets, S, of size $V = (-s)_k$ consider the set $T_S = \{y \in S : y - 1 \notin S\}$ and define*

$$C(S) = \max\{T_S \cup \{1\}\}. \tag{4.5}$$

Then $C \in \mathcal{F}_{k,s}$, and hence $\mathcal{F}_{k,s}$ is nonempty.

Proof. Note that for a given set S, if $y, y+1 \in S$ then $C(S) \neq y+1$ for $y = 1, 2, \ldots, k-1$. In addition, if $1, k \in S$ then $C(S) \neq 1$, since $(-s)_k < k$ and hence $y \in T_s$ for some $y > 1$. Employing Theorem 8 for $n > P$ gives

$$N_1^P(n) \geq N_2^P(n) \geq \cdots \geq N_k^P(n) \geq N_1^P(n)$$

and hence

$$N_1^P(n) = N_2^P(n) = \cdots = N_k^P(n)$$

$C \in \mathcal{F}_{k,s}$. □

See Example 1. for an illustration of Theorem 11.

Theorems 9, 10, and 11 suggest that further study of graphs arising from choice functions may be valuable.

This section concludes with a number of informative examples restricting attention to the case $s = 7$, $k = 5$.

Example 1. Consider the choice function, C, defined via

$$C(1,2,3) = 1,\ C(1,2,4) = 4,\ C(1,2,5) = 5,\ C(1,3,4) = 3,\ C(1,3,5) = 5,$$
$$C(1,4,5) = 4,\ C(2,3,4) = 2,\ C(2,3,5) = 5,\ C(2,4,5) = 4,\ C(3,4,5) = 3. \tag{4.6}$$

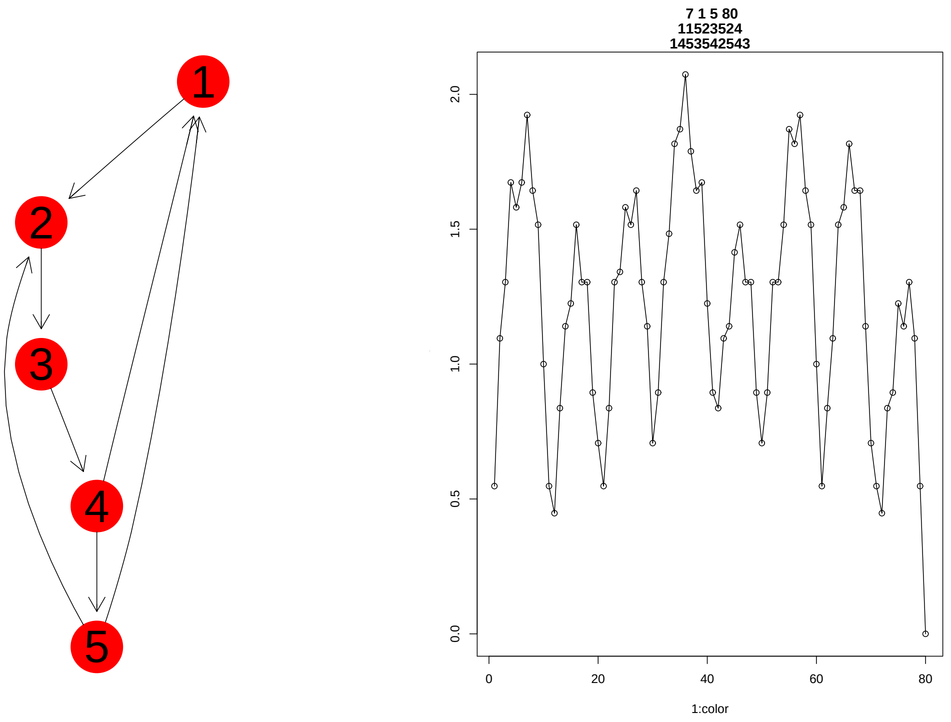
Note that in (4.6), for convenience, the set notation has been dropped. For example $C(1, 2, 3)$ denotes $C(\{1, 2, 3\})$. This C satisfies the requirements of Theorem 11, and hence is a good choice function. One computation with initial values (4 4 2 5 1 1 2) produced the following period of length eighty

$$(3\ 4\ 3\ 4\ 5\ 5\ 1\ 2\ 1\ 2\ 3\ 4\ 3\ 5\ 4\ 5\ 1\ 2\ 1\ 2\ 3\ 3\ 4\ 5\ 4\ 5\ 1\ 2\ 1\ 3\ 2\ 3\ 4\ 5\ 4\ 5\ 1\ 1\ 2\ 3$$
$$2\ 3\ 4\ 5\ 4\ 1\ 5\ 1\ 2\ 3\ 2\ 3\ 4\ 4\ 5\ 1\ 5\ 1\ 2\ 3\ 2\ 4\ 3\ 4\ 5\ 1\ 5\ 1\ 2\ 2\ 3\ 4\ 3\ 4\ 5\ 1\ 5\ 2\ 1\ 2) \tag{4.7}$$

It may be verified that each of the five classes is represented exactly sixteen times in (4.7).

Figure 4.1 (a) gives the path-connected domination graph for C described in Theorem 9. Figure 4.1 (b) is a plot of the sequence of standard deviations of the vector of cumulative totals for the classes at each of the eighty time points in the period. The information above the plot consists of (1) s, (2) the number inserted at each step (in all of the cases considered here this is one), (3) k, and (4) the period in the first row, followed by a string indicating the initial values and a further string with the information contained in (4.6).

Figure 4.1: (a) Domination graph (left) and (b) Standard deviation plot (right) for Example 1.



Note that the standard deviation of the cumulative totals at the conclusion of the period is zero, as expected.

Example 2. Consider the choice function, C, defined via

$$C(1,2,3) = 1,\ C(1,2,4) = 1,\ C(1,2,5) = 1,\ C(1,3,4) = 1,\ C(1,3,5) = 5,$$
$$C(1,4,5) = 4,\ C(2,3,4) = 2,\ C(2,3,5) = 2,\ C(2,4,5) = 2,\ C(3,4,5) = 3. \tag{4.8}$$

Note that

$$C(1,2,5) = 1,\ C(2,3,5) = 2,\ C(3,4,5) = 3,\ C(4,1,5) = 4.$$

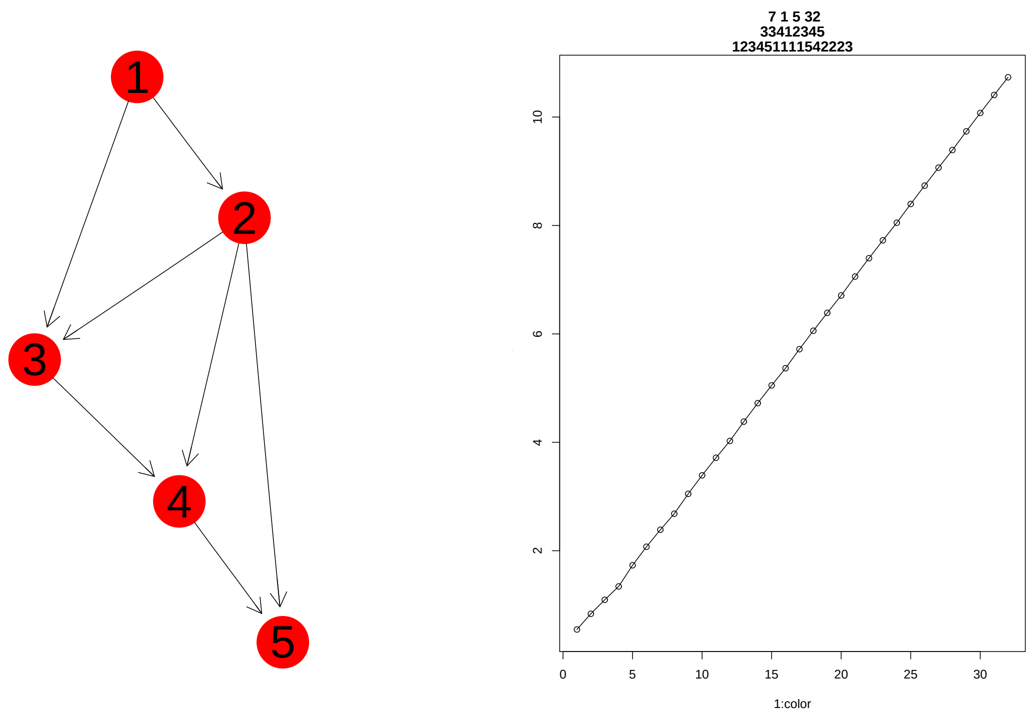
Hence this choice function satisfies the requirements of Theorem 7 with $s = 7$, $k = 5$, $s_1 = 6$, $k_1 = 4$, $m = 1$, and $V_1 = 2$; therefore $C \notin \mathcal{F}_{k,s}$. A run with initial values

(3 4 1 2 3 4 5) as suggested in the proof of Theorem 7 produced the following period of length thirty-two

$$(5\ 4\ 1\ 2\ 3\ 4\ 1\ 2\ 5\ 3\ 4\ 1\ 2\ 3\ 4\ 1\ 5\ 2\ 3\ 4\ 1\ 2\ 3\ 4\ 5\ 1\ 2\ 3\ 4\ 1\ 2\ 3) \tag{4.9}$$

Note that for n sufficiently large, as illustrated in (4.9), $N_y^{32}(n) = 7$ for $1 \leq y \leq 4$ and $N_5^{32}(n) = 4$. In addition, the period thirty-two matches that suggested in the proof of Theorem 7. In this case, $s_1 + 1 = 7$, $k_1 = 4$, and the smallest t such that $4|7t$ is $t = 4$. The period is thus $(s+1)t = 8(4) = 32$. Figure 4.2 (a) gives the domination graph for C described in Theorem 9. Figure 4.2 (b) is a plot of the sequence of standard deviations of the vector of cumulative totals for the classes at each of the thirty-two time points in the period.

Figure 4.2: (a) Domination graph and (b) Standard deviation plot for Example 2.

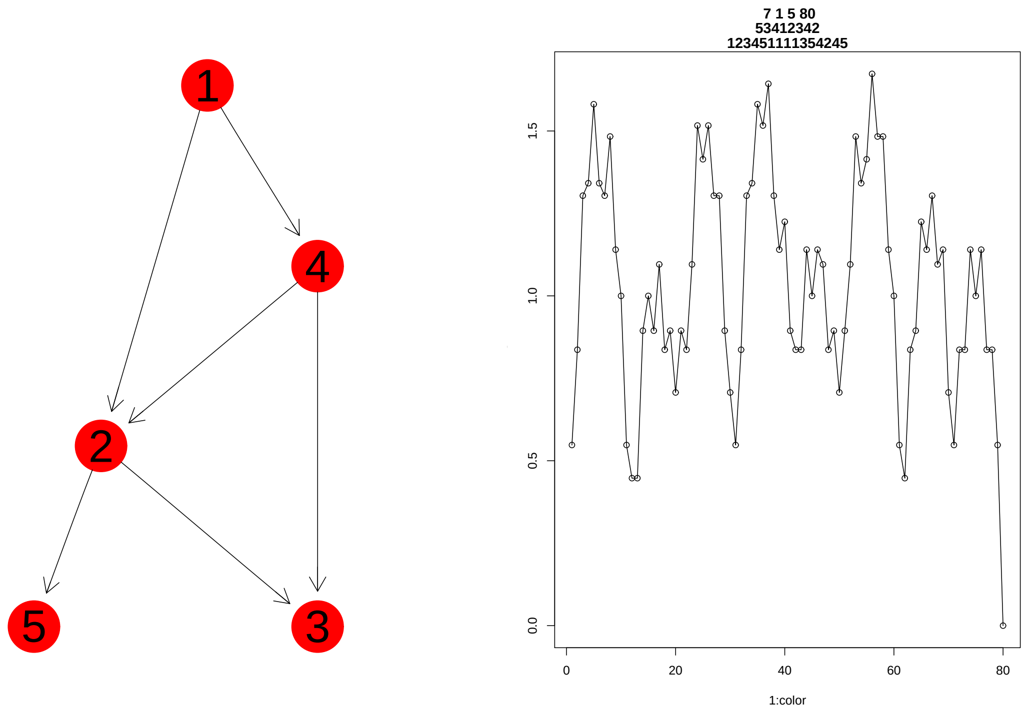


Example 3. Consider the choice function, C, defined via

$$C(1,2,3) = 1,\ C(1,2,4) = 1,\ C(1,2,5) = 1,\ C(1,3,4) = 1,\ C(1,3,5) = 3,$$
$$C(1,4,5) = 5,\ C(2,3,4) = 4,\ C(2,3,5) = 2,\ C(2,4,5) = 4,\ C(3,4,5) = 5. \tag{4.10}$$

Figure 4.3 (a) gives the domination graph for the given C. Figure 4.3 (b) gives a plot of sequence of associated standard deviations of cumulative totals. Note that the graph is not path-connected since there is no edge going into 1 nor any edge leaving 5; however, based on computations, this particular choice function appears to be good (see for instance Figure 4.3 (b) constructed from the initial values (5 3 4 1 2 3 4 2)).

Figure 4.3: (a) Domination graph and (b) Standard deviation plot for Example 3.



Figures 4.4 through 4.7 exemplify further examples of this type, which appear (based on computations) to be arising from good choice functions despite their lack of a path-connected domination graph.

Figure 4.4: (a) Domination graph and (b) Standard deviation plot for Example 4.

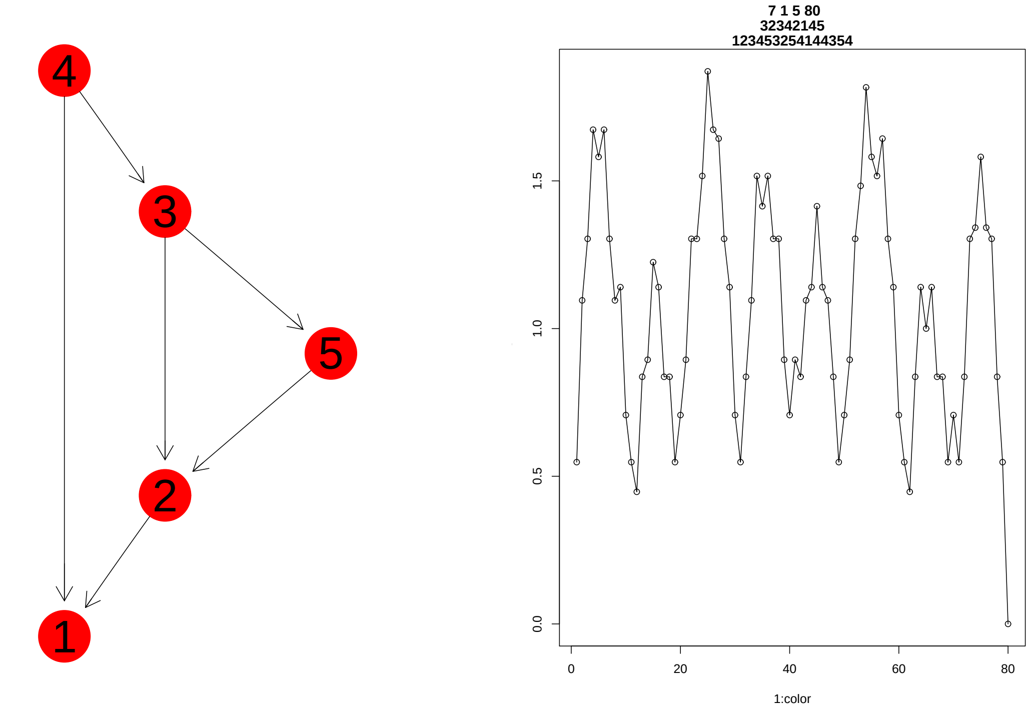


Figure 4.5: (a) Domination graph and (b) Standard deviation plot for Example 5.

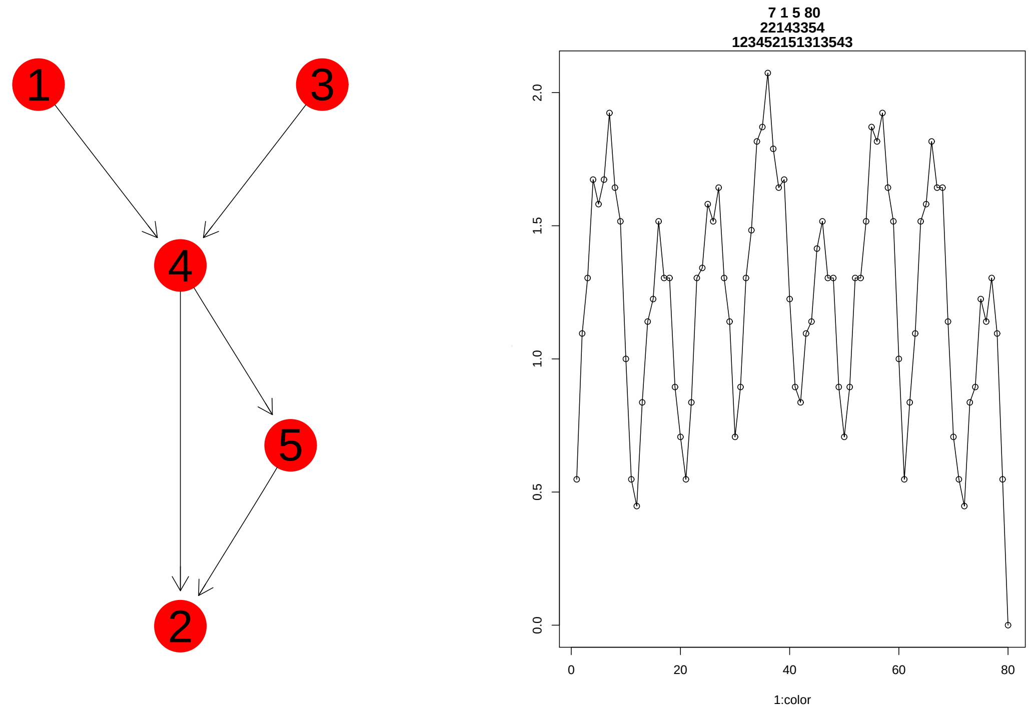


Figure 4.6: (a) Domination graph and (b) Standard deviation plot for Example 6.

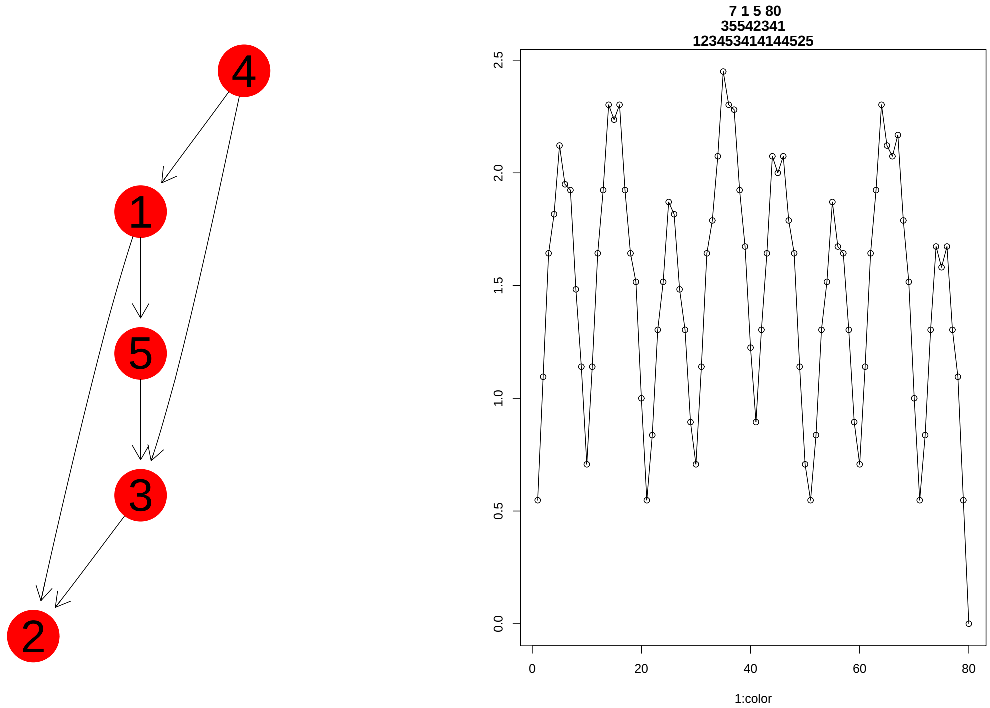
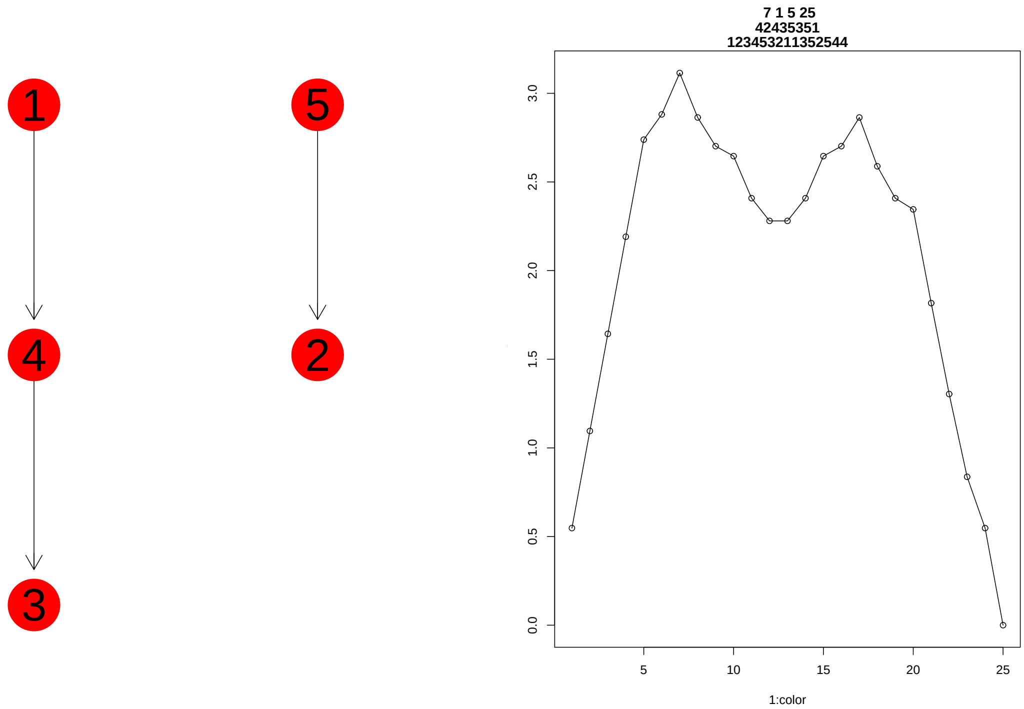


Figure 4.7: (a) Domination graph and (b) Standard deviation plot for Example 7.



Note that computations suggest that good choice functions are, in fact, quite rare, and the vast majority of choice functions are not elements of $\mathcal{F}_{k,s}$.

Chapter 5: Conclusions and Further Research

In summary, this thesis considers systems evolving by the use choice functions. Specifically, a scenario involving choice functions on underrepresented classes in a linearly ordered string of s members comprised of k classes was studied with the goal of achieving local and global equal representation of classes independently of initial values. Given values for k and s, this thesis sought to categorize the choice functions which result in the aforementioned equal representation conditions; such choice functions are classified as "good". A number of results regarding good choice functions, as well as choice functions which fail to be good were proven. Perhaps the most important classification result proved in the thesis is the fact that choice functions which culminate in a connected class domination graph are good choice functions.

The scenario of choosing new members of a system based on choice functions is potentially applicable in a number of fields. Firstly, in the area of data processing, often multiple tasks need to be processed at any given moment. These tasks could be classified by their type and choice functions could then be used to make the decision of what order in which to process these tasks without giving specific tasks priority ratings. Another potential application would be for any type of draft selection situation in which a number of groups must choose new members in some unbiased order. The predetermined choice functions could be set up to give some priority to lower rated members beating higher rated members if such a scenario was desired.

The given problem considered here allows for several variations on its theme which would likely be valuable to pursue. One variation would be to consider a shift of length greater than one, but less than $s+1$, given $s+1$ linearly ordered size-one spots filled with k classes; thus enabling multiple new entrants at once. This particular scenario may prove important in data and network processing where parallel structures are available. Another possible important scenario would be to give various classes priority over others; this scenario would involve weighting the classes by their importance. Finally, one could attempt to discover yet another, potentially more complex, alternative method of choosing a new entrant which would lead to local and global equal representation. One could also consider requirements in addition to Conditions (i) and (ii) such as cut-offs on the time until the eventual period is reached, limits on the length of time between appearances of a given class, or limits on the number of consecutive selections from a given class.

APPENDIX 1

On the rational recursive sequence $y_n = A + \frac{y_{n-1}}{y_{n-m}}$ for small A

Kenneth S. Berenhaut, Katherine M. Donadio, and John D. Foley

The following paper has appeared in the *Applied Mathematics Letters*. Stylistic variations are due to the requirements of the journal.

Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 21 (2008) 906–909
Applied Mathematics Letters
www.elsevier.com/locate/aml

On the rational recursive sequence $y_n = A + \frac{y_{n-1}}{y_{n-m}}$ for small A

Kenneth S. Berenhaut[a,*], Katherine M. Donadio[a], John D. Foley[b]

[a] *Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, United States*
[b] *Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, United States*

Received 22 January 2007; received in revised form 8 July 2007; accepted 27 July 2007

Abstract

This work studies the existence of positive prime periodic solutions of higher order for rational recursive equations of the form

$$y_n = A + \frac{y_{n-1}}{y_{n-m}}, \quad n = 0, 1, 2, \ldots,$$

with $y_{-m}, y_{-m+1}, \ldots, y_{-1} \in (0, \infty)$ and $m \in \{2, 3, 4, \ldots\}$. In particular, we show that for sufficiently small $A > 0$, there exist periodic solutions with prime period $2m + U_m + 1$, for almost all m, where $U_m = \max\{i \in \mathbb{N} : i(i + 1) \le 2(m - 1)\}$.
© 2007 Elsevier Ltd. All rights reserved.

Keywords: Rational difference equation; Periodicity; Existence; Binomial coefficients; Fixed point

1. Introduction

This note studies the existence of prime periodic solutions of higher order for rational recursive equations of the form

$$y_n = A + \frac{y_{n-1}}{y_{n-m}}, \quad n = 0, 1, \ldots, \tag{1}$$

with $A > 0$, $y_{-m}, y_{-m+1}, \ldots, y_{-1} \in (0, \infty)$ and $m \in \{2, 3, 4, \ldots\}$. Eq. (1) has been studied by many authors in the recent past. In [1], conditions for global asymptotic stability of solutions are presented. In [2], some quantitative bounds for solutions are provided. Properties of solutions for $A < 0$ are considered in [3,4]. Results for instances of the more general equation

$$y_n = A + \frac{y_{n-k}}{y_{n-m}}, \quad n = 0, 1, \ldots, \tag{2}$$

$k, m \in \{1, 2, 3, \ldots\}$, can be found in [5–17] and the references therein.

It is known (cf. [1]) that all positive solutions to (1) are bounded and persist, and that a sufficient condition for global asymptotic stability of the positive equilibrium of Eq. (1) is $A > 1$, but little is known regarding the possible

* Corresponding author.
E-mail addresses: berenhks@wfu.edu (K.S. Berenhaut), donakm4@wfu.edu (K.M. Donadio), jfoley@math.ucsd.edu (J.D. Foley).

0893-9659/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2007.07.033

behavior of solutions for small $A > 0$ and large m. Here we will show that for almost all m, for sufficiently small A, there exists a prime period $2m + U_m + 1$ solution to (1), where $U_m = \max\{i \in \mathbb{N} : i(i+1) \le 2(m-1)\}$. In particular, we will prove the following theorem.

Theorem 1. *Set* $\mathcal{V} = \bigcup_{j>0} \left\{\frac{j(j+1)}{2}, \frac{j(j+1)}{2} + 1\right\}$. *If* $m > 1$ *satisfies* $m \notin \mathcal{V}$ *then there exists an* $\epsilon_m > 0$ *such that for all* $0 < A < \epsilon_m$, *there exists a prime period* $2m + U_m + 1$ *solution to* (1).

Proof. First, suppose that

$$y_i \in [A, A + 2A^2] \tag{3}$$

for $-m \le i \le -1$. We will show first that there exist $\{l_0, l_1, \dots, l_{m-1}\}$ and $\{r_0, r_1, \dots, r_{m-1}\}$ such that

$$y_i \in [A^{-i}(1 - l_i A), A^{-i}(1 + r_i A)], \tag{4}$$

for $0 \le i \le m - 1$. Indeed, note that for initial values satisfying (3),

$$y_0 \le A + \frac{A + 2A^2}{A} = 1 + 3A \tag{5}$$

and

$$y_0 \ge A + \frac{A}{A + 2A^2} = A + \frac{1}{1 + 2A} \ge A + (1 - 2A) = 1 - A, \tag{6}$$

for $A > 0$ sufficiently small. Hence, employing (5) and (6) gives

$$y_1 \le A + \frac{1 + 3A}{A} = A^{-1} + 3 + A \le A^{-1}(1 + 4A) \tag{7}$$

and

$$y_1 \ge A + \frac{1 - A}{A(1 + 2A)} \ge A^{-1}(1 - A)(1 - 2A) \ge A^{-1}(1 - 3A), \tag{8}$$

for $A > 0$ sufficiently small. For convenience, throughout this proof, at each instance, $A > 0$ is assumed to be sufficiently small so that the associated inequality holds.

More generally, suppose

$$A^{-j}(1 - l_j A) \le y_j \le A^{-j}(1 + r_j A), \tag{9}$$

for $0 \le j < J$ for some $2 \le J \le m - 1$, where $l_j, r_j > 0$, for $0 \le j < J$. Then,

$$y_J \le A + \frac{A^{-(J-1)}(1 + r_{J-1}A)}{A} \le A^{-J}(1 + (r_{J-1} + 1)A) \tag{10}$$

and

$$y_J \ge A + \frac{A^{-(J-1)}(1 - l_{J-1}A)}{A(1 + 2A)} \ge A^{-J}(1 - (l_{J-1} + 2)A). \tag{11}$$

Setting $r_J = r_{J-1} + 1$ and $l_J = l_{J-1} + 2$, the inequalities in (4) then follow from induction.

Now, consider y_j for $m \le j \le m + U_m$, where $U_m = \max\{i \in \mathbb{N} : i(i+1) \le 2(m-1)\}$. Employing the bounds in (4) gives (for sufficiently small $A > 0$) that

$$\begin{aligned} y_m &\le A + \frac{A^{-(m-1)}(1 + r_{m-1}A)}{(1 - l_0 A)} = A + A^{-(m-1)}(1 + r_{m-1}A)\left(1 + l_0 A + \left(\frac{(l_0^2 A)}{1 - l_0 A}\right) A\right) \\ &\le A + A^{-(m-1)}(1 + r_{m-1}A)(1 + (l_0 + 1)A) \\ &\le A^{-(m-1)}(1 + (r_{m-1} + l_0 + 1 + A^{m-1} + r_{m-1}(l_0 + 1)A)A) \\ &\le A^{-(m-1)}(1 + r_m A), \end{aligned} \tag{12}$$

where $r_m = r_{m-1} + l_0 + 3$. Similarly, we have

$$\begin{aligned} y_m &\geq A + \frac{A^{-(m-1)}(1 - l_{m-1}A)}{(1 + r_0 A)} = A + A^{-(m-1)}(1 - l_{m-1}A)\left(1 - r_0 A + \frac{(r_0 A)^2}{1 - l_0 A}\right) \\ &\geq A^{-(m-1)}(1 - l_{m-1}A)(1 - r_0 A) \geq A^{-(m-1)}(1 - (l_{m-1} + r_0)A) \\ &= A^{-(m-1)}(1 - l_m A), \end{aligned} \tag{13}$$

where $l_m = l_{m-1} + r_0$.

Inductively, employing (4), we obtain

$$y_i \in (A^{v_i}(1 - l_i A), A^{v_i}(1 + r_i A)) \tag{14}$$

for $m \leq i \leq m + U_m$, where $v_{m+j} = -(m-1) + 0 + 1 + 2 + \cdots + j = -(m-1) + j(j+1)/2$. Note that under the assumption that $m \notin \mathcal{V}$, $v_{m+U_m} < 0$ and $v_{m+U_m} + (U_m + 1) > 1$.

Now, suppose $m + U_m + 1 \leq i \leq 2m - 1$. For $i = U_m + 1$, employing (14) and (4), we have

$$\begin{aligned} y_{m+U_m+1} &\leq A + \frac{A^{v_{m+U_m}}(1 + r_{m+U_m}A)}{A^{-(U_m+1)}(1 - l_{U_m+1}A)} \\ &\leq A + A^{v_{m+U_m}+U_m+1}(1 + r_{m+U_m}A)(1 + (l_{U_m+1} + 1)A) \\ &\leq A + A^2(1 + (r_{m+U_m} + l_{U_m+1} + 2)A) \\ &\leq A + A^2(1 + \delta_1), \end{aligned} \tag{15}$$

for sufficiently small $A > 0$, where $0 < \delta_1 < 1$. Inductively, we have

$$\begin{aligned} y_{m+U_m+i} &\leq A + \frac{A(1 + (1 + \delta_{i-1})A)}{A^{-(U_m+i)}(1 - l_{U_m+i}A)} \\ &\leq A + A^{U_m+i+1}(1 + (\delta_{i-1} + l_{U_m+i} + 2)A) \\ &\leq A + A^2(1 + \delta_i) \end{aligned} \tag{16}$$

for $2 \leq i \leq m - 1 - U_m$, where $0 < \delta_j < 1$, for $1 \leq j \leq m - 1 - U_m$.

Hence suppose $2m \leq i \leq 2m + U_m$. Employing (14) and (16), we have

$$\begin{aligned} y_{2m} &\leq A + \frac{A(1 + (1 + \delta_{q-1})A)}{A^{v_m}(1 - l_m A)} \leq A + A^{1+(m-1)}(1 + (\delta_{q-1} + l_m + 2)A) \\ &\leq A + A^2(1 + \delta_q), \end{aligned} \tag{17}$$

where $q = m - U_m$ and $0 < \delta_q < 1$.

Inductively, we obtain

$$\begin{aligned} y_{2m+i} &\leq A + \frac{A(1 + (1 + \delta_{q+i-1})A)}{A^{v_{m+i}}(1 - l_{m+i}A)} \leq A + A^{(m-1)-i(i+1)/2+1}(1 + (\delta_{q+i-1} + l_{m+i} + 2)A) \\ &\leq A + A^2(1 + \delta_{q+i}), \end{aligned} \tag{18}$$

for $1 \leq i \leq U_m$, where we have used the fact that $(m-1) - i(i+1)/2 + 1 \geq 2$.

Note that $y_i \in (A, A + 2A^2)$ for $m + U_m + 1 \leq i \leq 2m + U_m$, and consider the function $F : (\mathbb{R}^+)^m \to (\mathbb{R}^+)^m$ defined via

$$F((x_1, x_2 \ldots, x_m)) = (x_2, x_3 \ldots, x_m, A + x_m/x_1). \tag{19}$$

Recalling (3), we have shown that the $(2m + U_m + 1)$-th iterate of F, F^{2m+U_m+1}, maps $\mathcal{S}_A = [A, A + 2A^2]^m$ into itself. Since $\mathcal{S}_A$ is homeomorphic to a closed disk of dimension m, and F is continuous, Brouwer's fixed point theorem applies and F^{2m+U_m+1} has a fixed point. Thus, a period-$(2m + U_m + 1)$ solution to (1) exists, as desired.

Note that for sufficiently small A, Eqs. (4) and (14) imply that $y_i \notin (A, A + 2A^2)$ for $0 \leq i \leq m + U_m$, which precludes any smaller period, and hence we have a prime periodic solution of the required form. □

Remark. Note that $\{m > 1 | m \notin \mathcal{V}\} = \{5, 8, 9, 12, 13, 14, 17, 18, 19, \ldots\}$. By tracking constants in the above argument, it is possible, for fixed values of m, to obtain explicit ranges of A for which periodic solutions of the form prescribed by the theorem exist.

Acknowledgements

We are very grateful to a referee for comments and insights that improved this manuscript. The first author acknowledges financial support from a Sterge Faculty Fellowship.

References

[1] R.M. Abu-Saris, R. DeVault, Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Lett. 16 (2) (2003) 173–178.

[2] K.S. Berenhaut, J.D. Foley, S. Stevic, Quantitative bounds for the recursive sequence $y_n = A + \frac{y_{n-1}}{y_{n-k}}$, Appl. Math. Lett. 19 (9) (2006) 983–989.

[3] M. Saleh, M. Aloqeili, On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$, Appl. Math. Comput. 176 (1) (2006) 359–363.

[4] M. Saleh, M. Aloqeili, On the rational difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Comput. 177 (1) (2006) 189–193.

[5] A.M. Amleh, E.A. Grove, G. Ladas, D.A. Georgiou, On the recursive sequence $y_{n+1} = \alpha + \frac{y_{n-1}}{y_n}$, J. Math. Anal. Appl. 233 (1999) 790–798.

[6] K.S. Berenhaut, J.D. Foley, S. Stevic, The global attractivity of the rational difference equation $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$, Proc. Amer. Math. Soc. 135 (2007) 1133–1140.

[7] K.S. Berenhaut, J.D. Foley, S. Stevic, The global attractivity of the rational difference equation $y_n = A + \left(\frac{y_{n-k}}{y_{n-m}}\right)^p$, in: Proceedings of the American Mathematical Society, 2006 (in press).

[8] K.S. Berenhaut, S. Stevic, The behaviour of the positive solutions of the difference equation $x_n = A + \left(\frac{x_{n-2}}{x_{n-1}}\right)^p$, J. Difference Equ. Appl. 12 (9) (2006) 909–918.

[9] K.S. Berenhaut, S. Stević, On positive nonoscillatory solutions of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}{}^p}{x_n{}^p}$, J. Difference Equ. Appl. 12 (5) (2006) 495–499.

[10] R. DeVault, C. Kent, W. Kosmala, On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, J. Difference Equ. Appl. 9 (8) (2003) 721–730.

[11] H. El-Metwally, E.A. Grove, G. Ladas, H.D. Voulov, On the global attractivity and the periodic character of some difference equations, J. Difference Equ. Appl. 7 (2001) 837–850.

[12] H.M. El-Owaidy, A.M. Ahmed, M.S. Mousa, On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$, J. Appl. Math. Comput. 12 (1–2) (2003) 31–37.

[13] M.R.S. Kulenović, G. Ladas, Dynamics of Second Order Rational Difference Equations, Chapman & Hall/CRC, 2002.

[14] E.A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall/CRC Press, Boca Raton, 2004.

[15] V. Kocić, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, in: Mathematics and its Applications, vol. 256, Kluwer Academic Publishers Group, Dordrecht, 1993.

[16] W.T. Patula, H.D. Voulov, On the oscillation and periodic character of a third order rational difference equation, Proc. Amer. Math. Soc. 131 (3) (2003) 905–909.

[17] S. Stević, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$, J. Appl. Math. Comput. 18 (1–2) (2005) 229–234.

APPENDIX 2

On a rational recursive sequences with parameter near the boundary

Kenneth S. Berenhaut, Katherine M. Donadio, and John D. Foley

The following paper was submitted to the *International Journal of Difference Equations*. Stylistic variations are due to the requirements of the journal.

International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 3 Number 1 (2008), pp. 53–59
© Research India Publications
http://www.ripublication.com/ijde.htm

On a Rational Recursive Sequence with Parameter near the Boundary

Kenneth S. Berenhaut
Department of Mathematics, Wake Forest University,
Winston-Salem, NC 27109, USA
E-mail: berenhks@wfu.edu

Katherine M. Donadio
Department of Mathematics, Wake Forest University,
Winston-Salem, NC 27109, USA
E-mail: donakm4@wfu.edu

John D. Foley
Department of Mathematics, University of California,
San Diego, La Jolla, CA 92093, USA
E-mail: jfoley@math.ucsd.edu

Abstract

This note studies existence of positive prime periodic solutions of higher order for rational recursive equations of the form $y_n = A + y_{n-k}/y_{n-m}$, $n = 0, 1, 2, \ldots$, with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$, k odd and $m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$. In particular, we show that for $k \geq 5$, odd, $m \geq 1$, $\gcd(k, m) = 1$ and sufficiently small $A > 0$, there exist periodic solutions with prime period $2m^* + U_{m^*}$, for some m^*, where $U_m = \min\{i \in \mathbb{N} : i(i + 1) \geq 2m\}$. A value of $m^* > (k - 1)^2/2 + m$ is given explicitly.

AMS subject classification: 39A10, 39A11.
Keywords: Rational difference equation, periodicity, binomial coefficients, existence.

Received December 19, 2007; Accepted December 26, 2007
Communicated by Gerasimos Ladas

1. Introduction

This note studies existence of prime periodic solutions of higher order for rational recursive equations of the form

$$y_n = A + \frac{y_{n-k}}{y_{n-m}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

with $A > 0$, $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ and $m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$. Equation (1.1) has been studied by many authors in the recent past. In [1], conditions for global asymptotic stability of solutions are presented for $k = 1$. In [4], some quantitative bounds for solutions are provided. Properties of solutions for $A < 0$ are considered in [17] and [18]. Further results for equations of the type in (1.1) can be found in [1–20] and the references therein.

It is known that all positive solutions to (1.1) are bounded (c.f. [1, 6]), and that a sufficient condition for global asymptotic stability of the positive equilibrium of Equation (1.1) is $A > 1$, but little is known regarding possible behavior of solutions for small $A > 0$ and large k and m. One particularly well-known conjecture regarding solutions for $A < 1$ is the following (see for instance, [1]).

Conjecture 1.1. Suppose that $(k, m) = (1, 3)$. Prove that when $A > \sqrt{2}-1$, the unique positive equilibrium of Equation (1.1) is globally asymptotically stable.

In [3], it was shown that for $k = 1$ and almost all m, for sufficiently small A, there exists a prime period $2m + U_m$ solution to (1.1), where $U_m = \max\{i \in \mathbb{N} : i(i + 1) \leq 2(m - 1)\} + 1 = \min\{i \in \mathbb{N} : i(i + 1) \geq 2m\}$. In particular, the following theorem was proven.

Theorem 1.2. Set $\mathcal{V} = \bigcup_{j>0} \left\{ \frac{j(j+1)}{2}, \frac{j(j+1)}{2} + 1 \right\}$. If $m > 1$ satisfies $m \notin \mathcal{V}$ and $k = 1$, then there exists an $\epsilon_m > 0$ such that for all $0 < A < \epsilon_m$, there exists a prime period $2m + U_m$ solution to (1.1).

Here we will generalize the work in [3] to cover the case of general odd k by proving the following theorem.

Theorem 1.3. For $k, m \in \mathbb{N}^+$, set

$$U_{k,m} = \max\{i \in \mathbb{N} : i(i + k - 2) \leq 2m\}. \tag{1.2}$$

If $k \geq 5$ is odd, then **<u>for all</u>** $m \geq 1$ with $\gcd(k, m) = 1$, there exists an $\epsilon_{k,m} > 0$ such that for all $0 < A < \epsilon_{k,m}$, there exists a prime period

$$P_{m^*} = 2m^* + U_{m^*} = k(k - 1) + 2m + kU_{k,m}$$

solution to (1.1), where

$$m^* = \frac{(k-1)^2}{2} + m + \frac{k-1}{2} U_{k,m}. \tag{1.3}$$

Table 1: Prime periods of existing positive solutions to Equation (1.1) for sufficiently small A

k/m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1					13			20	22			29	31	33	
3		13		20	22		29	31			40		44	46	
5	22	29	31	33		42	44	46	53		57	59	61	68	
7	44	46	55	57	59	61		72	74	76	78	87	89		93
9	74	76		89	91		95	97		110	112		116	118	
11	112	114	116	118	131	133	135	137	139	141		156	158	160	162
13	158	160	162	164	166	181	183	185	187	189	191	193		210	212
15	212	214	216	218		222	239	241	243		247	249	251	253	
17	274	276	278	280	282	284	286	305	307	309	311	313	315	317	319
19	344	346	348	350	352	354	356	358	379	381	383	385	387	389	391
21	422	424		428	430			436		461	463		467		
23	508	510	512	514	516	518	520	522	524	526	551	553	555	557	559
25	602	604	606	608		612	614	616	618		622	649	651	653	

Remark 1.4. For the case $k = 3$, see Theorem 2.1, below.

Some periods implied by the results given here are provided in Table 1.4. Theorem 1.3 as well as a result covering the case $k = 3$ are proven in the next section.

2. Proof of the Main Theorem

In this section we prove Theorem 1.3. The essential idea is to show that m^* given in (1.3) (which was initially suggested through computations) satisfies

(i) $U_{m^*} = U_{k,m} + k - 1$

(ii) $m^* \notin \mathcal{V}$

(iiii) $\gcd(k, P_{m^*}) = 1$

(iv) $m^*k = m \mod P_{m^*}$.

The result will then follow upon employing Theorem 1.2 for $(k, m) = (1, m^*)$ to obtain a prime periodic solution, $\{y_i\}$, to the equation $y_n = A + y_{n-1}/y_{n-m^*}$ and verifying that $\{y_i^*\}$ defined via $y_i^* = y_j$ whenever $kj = i \mod P_{m^*}$ is a prime periodic solution, as required.

Proof of Theorem 1.3. From the definition of $U_{k,m}$, we have

$$U_{k,m}(U_{k,m} + (k-2)) \leq 2m \text{ and } (U_{k,m} + 1)(U_{k,m} + (k-1)) \geq 2m + 1. \quad (2.1)$$

Now, note that via (2.1) and (1.3)

$$\begin{aligned}(U_{k,m}+(k-1))(U_{k,m}+k) &= (U_{k,m}+(k-1))(U_{k,m}+1)+U_{k,m}(k-1)+(k-1)^2\\ &\geq 2m+1+U_{k,m}(k-1)+(k-1)^2 > 2m^* \qquad (2.2)\end{aligned}$$

and

$$\begin{aligned}&(U_{k,m}+(k-2))(U_{k,m}+(k-1))\\ &\quad= (U_{k,m}+(k-2))U_{k,m}+U_{k,m}(k-1)+(k-1)(k-2)\\ &\quad= (k-1)^2-(k-1)+U_{k,m}(k-1)+(U_{k,m}+(k-2))U_{k,m}\\ &\quad\leq (k-1)^2+(k-1)U_{k,m}+2m-(k-1)\\ &\quad= 2m^*-(k-1) < 2m^*. \qquad (2.3)\end{aligned}$$

The inequalities in (2.2) and (2.3) and the definition of U_{m^*} give that

$$U_{m^*} = U_{k,m}+(k-1). \qquad (2.4)$$

We then have via (1.3) and (2.4) that

$$\begin{aligned}P_{m^*} &= 2m^*+U_{m^*} = (k-1)^2+2m+(k-1)U_{k,m}+U_{m^*}\\ &= k(k-1)+2m+kU_{k,m}+(U_{m^*}-U_{k,m}-(k-1)) = k(k-1)+2m+kU_{k,m}. \qquad (2.5)\end{aligned}$$

If $(k-1)/2 > 1$ (i.e., $k > 3$), the inequalities in (2.2) and (2.3) also guarantee that $m^* \notin \mathcal{V}$. Hence, suppose that $\{a_i\}$ is a solution to the equation

$$y_n = A + \frac{y_{n-1}}{y_{n-m^*}}, \quad n = 0, 1, \ldots. \qquad (2.6)$$

Note that Equation (2.5) gives that $\gcd(k, P_{m^*}) = 1$ (since k is odd and $\gcd(k, m) = 1$) and define the sequence $\{a_i^*\}$ via $a_i^* = a_j$, whenever $kj = i \mod P_{m^*}$.

Now, for $n > s^* \stackrel{\text{def}}{=} \max\{k, m, m^*\}$, consider a_n^*, a_{n-k}^* and a_{n-m}^*. We have $a_n^* = a_{nk^{-1}}$ where k^{-1} is taken so that $k^{-1}k = 1 \mod P_{m^*}$ and $k^{-1} > 0$. Similarly $a_{n-k}^* = a_{(n-k)k^{-1}} = a_{nk^{-1}-1}$ and $a_{n-m}^* = a_{(n-m)k^{-1}} = a_{nk^{-1}-mk^{-1}}$. Employing (1.3) and (2.5) gives

$$\begin{aligned}m^*k-m &= \frac{k(k-1)^2}{2}+m(k-1)+\frac{k(k-1)}{2}U_{k,m}\\ &= \frac{k-1}{2}(k(k-1)+2m+kU_{k,m}) = \frac{k-1}{2}P_m^* = 0 \mod P_m^*. \qquad (2.7)\end{aligned}$$

Thus, by the definition of $\{a_i^*\}$ and the P_{m^*}-periodicity of $\{a_i\}$, we have

$$\begin{aligned} a_n^* &= a_{nk^{-1}} = A + \frac{a_{nk^{-1}-1}}{a_{nk^{-1}-m^*}} = A + \frac{a_{(n-k)k^{-1}}}{a_{nk^{-1}-m^*}} \\ &= A + \frac{a_{(n-k)k^{-1}}}{a_{nk^{-1}-mk^{-1}}} = A + \frac{a_{(n-k)k^{-1}}}{a_{(n-m)k^{-1}}} \\ &= A + \frac{a_{n-k}^*}{a_{n-m}^*} \end{aligned} \tag{2.8}$$

and $\{a_i^*\}_{i>s^*}$ is a periodic solution of (1.1) with period P_{m^*}. To verify that the constructed solution, $\{a_i^*\}$, has *prime* period P_{m^*}, note that for sufficiently small A, the solution $\{a_i\}$ of period P_{m^*} constructed in [3] has only one value in the interval $[1 - A, 1 + 3A]$ or one in the interval $[1/A - 3, 1/A + 4]$ per each P_{m^*}-cycle. Any P_{m^*} consecutive terms of the solution $\{a_i^*\}$ comprise a simple reordering of the values in the cycle, and hence $\{a_i^*\}$ has prime period P_{m^*}. ■

For the case $k = 3$, we have the following.

Theorem 2.1. Suppose $k = 3$ and set $\mathcal{W} = \bigcup_{j>0} \left\{ \frac{j(j+1)}{2} \right\}$ (the set of positive triangular numbers). If $m > 1$ satisfies $m \notin \mathcal{W}$ and $\gcd(m, 3) = 1$, then there exists an $\epsilon_{3,m} > 0$ such that for all $0 < A < \epsilon_{3,m}$, there exists a prime period $Q = 6+2m+3U_{3,m}$ solution to (1.1).

Proof. Note that for $k = 3$ and m satisfying $\gcd(3, m) = 1$, all parts of the proof of Theorem 1.3 hold except perhaps that m^* could be an element of $\mathcal{V}$. Considering Equation (2.3), this can happen only if $(U_{k,m} + (k-2))U_{k,m} = (U_{3,m} + 1)U_{3,m} = 2m$, or equivalently if $m \in \mathcal{W}$. The proof then follows as in the case of $k \geq 5$. ■

We close with the following conjecture.

Conjecture 2.2. For k, m satisfying the requirements of either Theorem 1.2, 1.3 or 2.1, there exists an $\epsilon_{k,m} > 0$ such that all nontrivial solutions to Equation (1.1) with $0 < A < \epsilon_{k,m}$ are asymptotically periodic with prime period as indicated in the theorem.

References

[1] R.M. Abu-Saris and R. DeVault, Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, *Appl. Math. Lett.*, 16(2):173–178, 2003.

[2] A.M. Amleh, E.A. Grove, G. Ladas, and D.A. Georgiou, On the recursive sequence $x_{n+1} = \alpha + x_{n-1}/x_n$, *J. Math. Anal. Appl.*, 233(2):790–798, 1999.

[3] Kenneth S. Berenhaut, K.M. Donadio, and John D. Foley, On the rational recursive sequence $y_n = a + \frac{y_{n-1}}{y_{n-m}}$ for small a, *Appl. Math. Lett.*, 2008. In press.

[4] Kenneth S. Berenhaut, John D. Foley, and Stevo Stević, Quantitative bounds for the recursive sequence $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, *Appl. Math. Lett.*, 19(9):983–989, 2006.

[5] Kenneth S. Berenhaut, John D. Foley, and Stevo Stević, The global attractivity of the rational difference equation $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$, *Proc. Amer. Math. Soc.*, 135(4):1133–1140 (electronic), 2007.

[6] Kenneth S. Berenhaut, John D. Foley, and Stevo Stević, The boundedness character of the difference equation $y_n = a + \frac{y_{n-k}}{y_{n-m}}$, *Int. J. Comput. Math.*, 2008. In press.

[7] Kenneth S. Berenhaut, John D. Foley, and Stevo Stević, The global attractivity of the rational difference equation $y_n = A + \left(\frac{y_{n-k}}{y_{n-m}}\right)^p$, *Proc. Amer. Math. Soc.*, 136(1):103–110 (electronic), 2008.

[8] Kenneth S. Berenhaut and Stevo Stević, The behaviour of the positive solutions of the difference equation $x_n = A + \left(\frac{x_{n-2}}{x_{n-1}}\right)^p$, *J. Difference Equ. Appl.*, 12(9):909–918, 2006.

[9] Kenneth S. Berenhaut and Stevo Stević, A note on positive non-oscillatory solutions of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$, *J. Difference Equ. Appl.*, 12(5):495–499, 2006.

[10] R. DeVault, C. Kent, and W. Kosmala, On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, *J. Difference Equ. Appl.*, 9(8):721–730, 2003. Special Session of the American Mathematical Society Meeting, Part II (San Diego, CA, 2002).

[11] H. El-Metwally, E.A. Grove, G. Ladas, and H.D. Voulov, On the global attractivity and the periodic character of some difference equations, *J. Differ. Equations Appl.*, 7(6):837–850, 2001. On the occasion of the 60th birthday of Calvin Ahlbrandt.

[12] H.M. El-Owaidy, A.M. Ahmed, and M.S. Mousa, On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + \frac{{x_{n-1}}^p}{{x_n}^p}$, *J. Appl. Math. Comput.*, 12(1-2):31–37, 2003.

[13] E.A. Grove and G. Ladas, *Periodicities in nonlinear difference equations*, volume 4 of *Advances in Discrete Mathematics and Applications*, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[14] V.L. Kocić and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, volume 256 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[15] M.R.S. Kulenović and G. Ladas, *Dynamics of second order rational difference equations*, Chapman & Hall/CRC, Boca Raton, FL, 2002. With open problems and conjectures.

[16] W.T. Patula and H.D. Voulov, On the oscillation and periodic character of a third order rational difference equation, *Proc. Amer. Math. Soc.*, 131(3):905–909 (electronic), 2003.

[17] M. Saleh and M. Aloqeili, On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$, *Appl. Math. Comput.*, 176(1):359–363, 2006.

[18] M. Saleh and M. Aloqeili, On the rational difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, *Appl. Math. Comput.*, 177(1):189–193, 2006.

[19] Stevo Stević, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$, *J. Appl. Math. Comput.*, 18(1-2):229–234, 2005.

[20] Taixiang Sun and Hongjian Xi, On convergence of the solutions of the difference equation $x_{n+1} = 1 + \frac{x_{n-1}}{x_n}$, *J. Math. Anal. Appl.*, 325(2):1491–1494, 2007.

Bibliography

[1] Aizerman, M., and Aleskerov, F. *Theory of Choice*, North-Holland, Amsterdam: Elsevier (1995).

[2] Berenhaut, K.S., Donadio, K.M., and Foley, J.D. On the rational recursive sequence $y_n = A + \frac{y_{n-1}}{y_{n-m}}$ for small A, *Applied Mathematics Letters*, **21** (2008) 906–909.

[3] Berenhaut, K.S., Donadio, K.M., and Foley, J.D. On a rational recursive sequence with parameter near the boundary, *International Journal of Difference Equations*, **3** (2008) 53–59.

[4] Cherepanov, V., Fedderson, T., and Sandroni, A. *Rationalization* available online at

`http://econ.upenn.edu/cgi-bin/mecon/bin/serve.cgi/rationalization`

`82509ecnm10`

[5] Hohti, A. On a combinatorial property of choice functions of finite sets, *Topology and its Applications*, **156** (2009) 1371–1373.

[6] Moulin, H. Choice Functions Over a Finite Set: A Summary, *Social Choice and Welfare*, **2** (1985) 147–160.

[7] Sen, A. Quasi-Transitivity, Rational Choice and Collective Decisions, *The Review of Economic Studies*, **36** (1969) 381–391.

[8] Schechter, E. *Axiom of Choice* available online at

`http://math.vanderbilt.edu/~schectex/ccc/choice.html`

[9] Varian, H.R., Revealed Preference, in M. Szenbarg, L Ramrattan, and A. Gottesman (eds.) *Samuelsonian Economics and the 21st Century*, Oxford, UK: Oxford University Press (2006).

[10] Winter, Y. Choice Functions and the Scopal Semantics of Indefinites, *Linguistics and Philosphy Philadelphia*, **20** (1997), 399–467.

Vita

Katherine M. Donadio

BORN:	September 9, 1986, Winston-Salem, NC
UNDERGRADUATE STUDY:	Wake Forest University Winston-Salem, North Carolina B.S. *Summa Cum Laude*, Mathematics, May 2008
GRADUATE STUDY:	Wake Forest University Winston-Salem, North Carolina M.A., Mathematics, May 2010

Papers

(1) K. S. Berenhaut, K. M. Donadio and J. D. Foley, "On the rational recursive sequence $y_n = A + \frac{y_{n-1}}{y_{n-m}}$ for small A," Applied Mathematics Letters (2008) **21**:906-909.

(2) K. S. Berenhaut, K. M. Donadio and J. D. Foley, "On a rational recursive sequence with parameter near the boundary," International Journal of Difference Equations (2008) **3**:53-59.

Honors and Awards

Freshman Chemistry Achievement Award, 2004.
Dean's List Fall 2004 through Spring 2008 at Wake Forest University.
Walter Low Tatum Scholarship.
Kenneth and Buelah Raynor Scholarship.