HIGHER DIMENSIONAL CONVEX BRUNNIAN LINKS AND OTHER EXPLORATIONS IN KNOTS

By

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Table of Contents

List of Figures ................................................................. iii
Acknowledgments .............................................................. iv
Abstract ................................................................. v
Chapter 1 Introduction .......................................................... 1
Chapter 2 The Handedness of Vectors in Knots ...................... 6
  2.1 A Method for Constructing Knots Entirely of Right-Handed Vectors 7
  2.2 Implementing the Construction ...................................... 16
  2.3 An Upper Bound on the Number of Vectors Required ............ 18
  2.4 A Better Upper Bound ............................................... 21
Chapter 3 Higher Dimensional Convex Brunnian Links ............. 25
  3.1 An Infinite Family of Convex Brunnian Links ................... 26
  3.2 A Special Case of the Brunnian Links ......................... 46
  3.3 Proof of the Symmetry of Our Knots .......................... 48
  3.4 Open Questions .................................................. 49
Bibliography ............................................................... 51
Vita ................................................................. 52
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>The $V_1$ unit</td>
<td>8</td>
</tr>
<tr>
<td>2.2</td>
<td>The $V_2$ unit attached to the $V_1$ unit</td>
<td>9</td>
</tr>
<tr>
<td>2.3</td>
<td>The $V_3$ unit attached to the $V_1$ unit</td>
<td>11</td>
</tr>
<tr>
<td>2.4</td>
<td>The $V_4$ unit attached to the $V_1$ unit</td>
<td>12</td>
</tr>
<tr>
<td>2.5</td>
<td>The $V_5$ unit attached to the $V_1$ unit</td>
<td>13</td>
</tr>
<tr>
<td>2.6</td>
<td>The $V_6$ unit attached to the $V_1$ unit</td>
<td>15</td>
</tr>
<tr>
<td>2.7</td>
<td>The unknot constructed of one $V_1$ units and seven $V_3$ units</td>
<td>17</td>
</tr>
<tr>
<td>2.8</td>
<td>Replacing $L_2$ with $L_a$ and $L_b$</td>
<td>20</td>
</tr>
<tr>
<td>2.9</td>
<td>Replacing $L_2$ with $L_a$ and $L_b$</td>
<td>22</td>
</tr>
<tr>
<td>3.1</td>
<td>Brunnian links of three components in $\mathbb{R}^3$</td>
<td>25</td>
</tr>
<tr>
<td>3.2</td>
<td>The deformation retract from $S^1 - S^0$ to $S^0$</td>
<td>28</td>
</tr>
<tr>
<td>3.3</td>
<td>The deformation retract from $S^2 - S^0$ to $S^1$</td>
<td>29</td>
</tr>
<tr>
<td>3.4</td>
<td>The deformation retract from $S^2 - S^1$ to $S^0$</td>
<td>31</td>
</tr>
<tr>
<td>3.5</td>
<td>$K_2$ and $B_1$</td>
<td>38</td>
</tr>
<tr>
<td>3.6</td>
<td>$B_1$ and $K_3$</td>
<td>39</td>
</tr>
<tr>
<td>3.7</td>
<td>$K'_3$ in $\mathbb{R}^3$</td>
<td>40</td>
</tr>
<tr>
<td>3.8</td>
<td>The Hopf link in $\mathbb{R}^3$</td>
<td>41</td>
</tr>
<tr>
<td>3.9</td>
<td>Examples of Innermost in $\mathbb{R}^3$ and $\mathbb{R}^4$</td>
<td>44</td>
</tr>
<tr>
<td>3.10</td>
<td>Process in the proof of Lemma 8</td>
<td>45</td>
</tr>
</tbody>
</table>
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Abstract

Jonathan Newman

Higher Dimensional Convex Brunnian Links and Other Explorations in Knots.
Thesis under the direction of Hugh N. Howards, Ph.D., Associate Professor of Mathematics.

This thesis will examine two areas of knot theory. The first involves stick numbers of knot, or the minimum number of straight line segments required to make a knot. The triple products of three consecutive sticks can be right or left-handed, which is an important property to the structure of a stick representation of a knot. We will examine a few ways to construct a stick representation of a knot so that all the vectors in the stick representation are right-handed, or all the vectors in the stick representation are left-handed. We will finish this area with a newly discovered lower bound on the number of sticks it required to make a knot out of entirely right or left-handed vectors.

The second area involves convex Brunnian links and their analogs in higher dimensions. Using a proof constructed in collaboration with Dr. Hugh Howards, Mr. Robert Davis, and Dr. Jason Parsley, we demonstrate that certain higher dimensional knots are indeed convex Brunnian links. This will lead to the construction of an infinite family of these convex Brunnian links. We will also examine a special case of convex Brunnian links and provide a proof that these are convex Brunnian links.
Chapter 1: Introduction

This thesis focuses on two areas of knot theory. The first area is concerned with the stick number of a knot, which is the number of line segments required to make a knot in $\mathbb{R}^3$. A stick, or vector once we have an orientation, is right-handed if it is the middle vector of three consecutive vectors whose triple product is positive. We demonstrate how to construct any knot entirely of right-handed vectors, or, by symmetry, entirely of left-handed vectors. We then provide a new upper bound on the number of vectors required to make a given knot entirely of right-handed vectors.

The second area is related to a masters thesis by Robert M. Davis [1] and is an extension of two papers given by Hugh N. Howards [4],[5]. Davis and Howards demonstrated that the Borromean rings are the only convex Brunnian links in $\mathbb{R}^3$ of three, four, or five components. Howards demonstrated that a convex Brunnian link exists in $\mathbb{R}^4$ in the form of two 2-spheres and one 1-sphere. We prove the existence of an infinite family of convex Brunnian links of three components.

We begin by giving some preliminary definitions and notations.

- $\mathbb{R}^n = \{x = (x_1, ..., x_n) : x_i \in \mathbb{R}\}$ is the Euclidean n-dimensional space with the usual norm $|x| = (\Sigma(x_i)^2)^{\frac{1}{2}}$ and metric $d(x, y) = |x - y|$.
- $B^n$ is the closed unit ball (or unit disk) of $\mathbb{R}^n$ defined by $|x| \leq 1$ in $\mathbb{R}^n$.
- $S^{(n-1)} = \partial B^n$ is the unit $(n - 1)$-sphere defined by $|x| = 1$ in $\mathbb{R}^n$.
- $I = [0, 1]$ is the unit interval of $\mathbb{R}^1$.
- $\text{Im}(f)$ is the image of the map $f$. 
A **homeomorphism** is a function $f : X \rightarrow Y$ of topological spaces $X$ and $Y$ which is bijective and where both $f$ and its inverse $f^{-1}$ are continuous. A subset $K$ of a space $X$ is a **knot** if $K$ is homeomorphic to a sphere $S^n$. More generally, a subset $K$ of a space $X$ is a **link** if $K$ is homeomorphic to a disjoint union of spheres, possibly of different dimensions.

If $f_0$ and $f_1$ are continuous maps from space $X$ into space $Y$, then

$$F : X \times I \rightarrow Y$$

is a **homotopy** if $F$ is continuous and $\forall x \in X$,

$$F(x, 0) = f_0(x) \text{ and } F(x, 1) = f_1(x)$$

and $f_0$ and $f_1$ are called **homotopically equivalent** in $X$. An **isotopy** between two knots is a homotopy $F$ where at each $t \in I$, $F(x, t) = f_t(x)$ is a homeomorphism. A homotopy

$$F : X \times I \rightarrow X$$

is called an **ambient isotopy** if $F(x, 0)$ is the identity function and for each $t \in I$, $F(x, t) = f_t(x)$ is a homeomorphism. An ambient isotopy is the mathematical definition for the common sense notion of moving a string about in $\mathbb{R}^3$ where you cannot cut the string or pass the string through itself. Two knots or links are considered to be the same when there exists an ambient isotopy from one to the other.

The **stick number** of a knot $K$ in $\mathbb{R}^3$ is the minimum number of connecting line segments required to construct a knot. The union of these line segments are called the **stick representation** of the knot. We can give an **orientation** to a knot in $\mathbb{R}^3$ by defining a positive direction along the knot. The stick representation of an
oriented knot can be viewed as a series of vectors where each line segment becomes a vector pointed in the direction of orientation.

The cross product of two vectors in \( \mathbb{R}^3 \), \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \), is shown as \( u \times v \) and results in a vector in \( \mathbb{R}^3 \) given by the equation:

\[
u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\]

The dot product of two vectors in \( \mathbb{R}^3 \), \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \), is shown as \( u \cdot v \) and results in a scalar given by the equation:

\[
u \cdot v = u_1v_1 + u_2v_2 + u_3v_3\]

In an oriented knot represented entirely by vectors, let \( S_1, S_2, \) and \( S_3 \) be three consecutive vectors. The expression \((S_1 \times S_2) \cdot S_3\) is the triple product of \( S_1, S_2, \) and \( S_3 \). We say that vector \( S_2 \) is right-handed if \((S_1 \times S_2) \cdot S_3 > 0\) and left-handed if \((S_1 \times S_2) \cdot S_3 < 0\). If \((S_1 \times S_2) \cdot S_3 = 0\), then \( S_1, S_2, \) and \( S_3 \) lie in the same plane and \( S_2 \) is neither right nor left-handed.

If we take two manifolds, \( X \) and \( Y \), contained in ambient space \( Z \), then we say that \( X \) and \( Y \) are transversal if \( \forall x \in X \cap Y \),

\[
T_x(X) + T_x(Y) = T_x(Z)
\]

where \( T_x(X) \) is the tangent space of \( X \) at point \( x \). If \( X \) and \( Y \) intersect transversally, and the sum of their codimension is greater than the dimension of the ambient space, then \( X \cap Y \) is null. If not, then \( X \cap Y \) is a submanifold whose codimension is equal to the sum of the codimension of \( X \) and the codimension \( Y \) in the ambient space. If \( f \) is a smooth map from a compact manifold \( X \) into a manifold \( Z \) which is transversal to a closed submanifold \( Y \subset Z \), then \( X \) may be perturbed by a sufficiently small
amount and $X$ will remain transversal to $Y$ in $Z$. Furthermore, if the above $X$ and $Y$ are not transversal in $Z$, then $X$ may be perturbed by an arbitrarily small amount in a certain direction and the result will be that $X$ and $Y$ will be transversal in $Z$ \[2\].

If $\dim(X) + \dim(Y) < \dim(Z)$ and do not intersect transversally, then either $X$ or $Y$ may be perturbed by a very small amount so as to give a transverse intersection.

A knot $K$ is the **unknot** if there exists an ambient isotopy which takes the sphere $K$ to the unit sphere. A link $L$ is the **unlink** if for each link component, $K_i$, there exists a ball disjoint from the other link components whose boundary is $K_i$.

For the purposes of this thesis, we will use $S^n$ and $\mathbb{R}^n$ as ambient spaces interchangeably since we can get from one to the other via stereographic projection. In other words, $S^n - \{x\}$ is homeomorphic to $\mathbb{R}^n$ where $x$ is just a point disjoint from any knots under consideration. We know that such a point always exists because the ambient space $X$ is always at least two dimensions greater than any knot $K$ in $X$.

Let $X$ be a topological space and let $x_0$ be a point in that space. Let $f$ be a continuous map, $f : I \rightarrow X$, such that $f(0) = x_0 = f(1)$. Now let $[f]$ be the class of all such maps which are homotopically equivalent to one another. We call $[f]$ the **homotopy class** of $f$. The **fundamental group** is the group with these homotopy classes as elements with the product $[f][g] = [f \cdot g]$ where $f \cdot g$ is the composition path, or the loop which transverses $f$ first and then $g$. The fundamental group is notated $\pi_1(X, x_0)$ for base point $x_0$ or abbreviated to $\pi_1(X)$ when $X$ is path connected and the selection of the base point is independent of the resulting algebraic group. \[3\]

The higher dimensional analog of fundamental groups are homotopy groups, which map $n$-spheres, $S^n$, for $n \geq 2$ into our topological space $X$ instead of loops, or $S^1$. These groups are written $\pi_n(X)$. For the purposes of this paper it is only necessary that we know that $\pi_n(S^n) = \mathbb{Z}$, which is true for all $n \geq 1$ \[3\].

A set $X$ is **convex** if between any two points in the set, there exists a straight
line segment connecting the two points which is entirely contained in $X$. We call a sphere convex if the standard ball bounded by that sphere is convex. For example, in $\mathbb{R}^2$, $\{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ is convex because the standard ball it bounds is $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ and is convex.
Chapter 2: The Handedness of Vectors in Knots

In this chapter, we will examine whether any knot can be constructed entirely of vectors which are right-handed. We will first construct the unknot such that every vector in the unknot is right-handed and we will use a method that will generalize so that given any knot, $K$, we can construct $K$ so that every vector is right-handed. We will then examine an alternative method for constructing $K$ which will give us an upper bound for the number of vectors required to make $K$ so that every vector is right-handed. We will end this chapter by giving an improvement on this bound.

We must first show that the handedness of a vector does not depend on the orientation of the knot. Knowing this allows us to put an orientation on the knot and compute the handedness of the vector without a loss of generality.

Lemma 1. The handedness of a vector is independent of the orientation on the knot.

Proof. Given three consecutive vectors in the stick representation of a knot, $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$, the triple product of $A$, $B$, and $C$ is

$$((A \times B) \bullet C) = ((a_1, a_2, a_3) \times (b_1, b_2, b_3)) \bullet (c_1, c_2, c_3)$$
$$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \bullet (c_1, c_2, c_3)$$
$$= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3.$$

Now suppose that we place the opposite orientation on the knot. Then the vectors from above would be pointed in the opposite direction and we would be left with $A' = (-a_1, -a_2, -a_3)$, $B' = (-b_1, -b_2, -b_3)$, and $C' = (-c_1, -c_2, -c_3)$. Furthermore, the order of the vectors would be $C'$, $B'$, and then $A'$. Now the triple product of $C'$, $B'$, and $A'$ is
\[(C' \times B') \cdot A' = ((-c_1, -c_2, -c_3) \times (-b_1, -b_2, -b_3)) \cdot (-a_1, -a_2, -a_3)
= (c_2b_3 - c_3b_2, c_3b_1 - c_1b_3, c_1b_2 - c_2b_1) \cdot (-a_1, -a_2, -a_3)
= -c_2b_3a_1 + c_3b_2a_1 - c_3b_1a_2 + c_1b_3a_2 - c_1b_2a_3 + c_2b_1a_3
= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3.\]

which is exactly the triple product of $A$, $B$, and $C$. Therefore the handedness of a vector does not depend on the orientation of the knot.

\[\square\]

### 2.1 A Method for Constructing Knots Entirely of Right-Handed Vectors

To construct a knot consisting of vectors which are all right-handed, we will create triplets of vectors, called **units**, which can be attached to one another. Each unit consists of three vectors. We will construct the units so that the middle vector of each unit is right-handed. We will attach each unit so that the vectors in each unit adjacent to the other unit are both right-handed. Each unit will be rotationally symmetric to the starting unit, $V_1$, so that we can rotate the frame of reference and attach any unit to the unit which was most recently attached, treating the most recently attached unit as $V_1$. This is why we will compute the triple products for the vectors in each unit, $V_i$, as it is attached to $V_i$.

Once we have each of the $V_i$ units, we will demonstrate how to construct the unknot and explain how we can construct any knot by this method of construction.

Unit $V_i$ consists of vectors $S_{i,k}$ where the first number in the subscript represents the unit to which the vector belongs while the second number in the subscript represents the order of the vector in the unit: first, second, or third. The sum of the vectors in $V_i$ is a vector which is notated $\sum V_i$. In constructing the units, we will include a constant $n$ which will be positive until otherwise stated.
The unit $V_1$ consists of $S_{1,1}$, $S_{1,2}$, and $S_{1,3}$ where

- $S_{1,1} = (n, 0, 0)$
- $S_{1,2} = (0, 0, -n)$
- $S_{1,3} = (-n, 2n, n)$

Since these are consecutive then $S_{1,1}$ and $S_{1,2}$ meet at $(n, 0, 0)$, $S_{1,2}$ and $S_{1,3}$ meet at $(n, 0, -n)$, and $S_{1,3}$ ends at $(0, 2n, 0)$. The vector $\sum V_1$ has length $2n$. Figure 2.1 shows unit $V_1$ where the positive $x_1$ axis is coming out of the page (depicted down and to the left), the positive $x_2$ axis is to the right, and the positive $x_3$ axis is up. The three vectors $S_{1,1}$, $S_{1,2}$, and $S_{1,3}$ are black and the vector $\sum V_1$ is gray.

![Figure 2.1: The $V_1$ unit](image-url)
The unit $V_2$ consists of $S_{2,1}$, $S_{2,2}$, and $S_{2,3}$ where

- $S_{2,1} = (n, 0, 0)$
- $S_{2,2} = (0, 0, -n)$
- $S_{2,3} = (-n, 2n, n)$

Attaching $V_2$ to $V_1$, we get that $S_{1,3}$ and $S_{2,1}$ attach at $(0, 2n, 0)$, $S_{2,1}$ and $S_{2,2}$ attach at $(n, 2n, 0)$, $S_{2,2}$ and $S_{2,3}$ attach at $(n, 2n, -n)$, and $S_{2,3}$ ends at $(0, 4n, 0)$. The vector $\sum V_2$ has length $2n$. Figure 2.2 shows $V_2$ attached to $V_1$ with $V_2$ slightly thicker than $V_1$ to distinguish between the units.

![Figure 2.2: The $V_2$ unit attached to the $V_1$ unit](image_url)

The handedness of $S_{1,2}$ is given by

\[
(S_{1,1} \times S_{1,2}) \cdot S_{1,3} = ((n, 0, 0) \times (0, 0, -n)) \cdot (-n, 2n, n) \\
= (0, n^2, 0) \cdot (-n, 2n, n) \\
= 0 + 2n^3 + 0 > 0.
\]

Therefore $S_{1,2}$ is right-handed. The handedness of $S_{1,3}$ is given by
\[(S_{1,2} \times S_{1,3}) \cdot S_{2,1} = ((0, 0, -n) \times (-n, 2n, n)) \cdot (n, 0, 0) \]
\[= (2n^2, n^2, 0) \cdot (n, 0, 0) \]
\[= n^3 + 0 + 0 > 0.\]
Therefore \(S_{1,3}\) is right-handed. The handedness of \(S_{2,1}\) is given by
\[(S_{1,3} \times S_{2,1}) \cdot S_{2,2} = ((-n, 2n, n) \times (n, 0, 0)) \cdot (0, 0, -n) \]
\[= (0, n^2, -2n^2) \cdot (0, 0, -n) \]
\[= 0 + 0 + 2n^3 > 0.\]
Therefore \(S_{2,1}\) is right-handed. The triple product of the three vectors with \(S_{2,2}\) in the middle is identical to the triple product of the three vectors with \(S_{1,2}\) in the middle, and so \(S_{2,2}\) is also right-handed.

The next unit, \(V_3\), attaches to the first unit at a 45 degree angle in the positive \(x_3\) direction compared to the angle that \(V_2\) attaches to \(V_1\). The unit \(V_3\) consists of:

- \(S_{3,1} = (n, 0, 0)\)
- \(S_{3,2} = (0, n\sqrt{2}, -\frac{n\sqrt{2}}{2})\)
- \(S_{3,3} = (-n, 0, \frac{3n\sqrt{2}}{2})\)

The vector \(\sum V_3 = (0, n\sqrt{2}, n\sqrt{2})\), has length \(2n\), and attaches to \(\sum V_1\) at a 45 degree angle in the positive \(x_3\) direction relative to how \(\sum V_2\) attaches to \(\sum V_1\). Figure 2.3 shows \(V_3\) attached to \(V_1\).

Attaching \(V_3\) to \(V_1\), the vectors \(S_{1,2}\) and \(S_{1,3}\) have the same handedness as when \(V_2\) is attached to \(V_1\) since \(S_{3,1} = S_{2,1}\). We must check the handedness of \(S_{3,1}\) and \(S_{3,2}\).

The handedness of \(S_{3,1}\) is given by
\[(S_{1,3} \times S_{3,1}) \cdot S_{3,2} = ((-n, 2n, n) \times (n, 0, 0)) \cdot (0, n\sqrt{2}, -\frac{n\sqrt{2}}{2}) \]
\[= (0, n^2, -2n^2) \cdot (0, n\sqrt{2}, -\frac{n\sqrt{2}}{2}) \]
\[= 0 + n^3\sqrt{2} + n^3\sqrt{2} > 0.\]
Therefore \(S_{3,1}\) is right-handed. The handedness of \(S_{3,2}\) is given by
Figure 2.3: The $V_3$ unit attached to the $V_1$ unit

\[(S_{3,1} \times S_{3,2}) \cdot S_{3,3} = \left( (n, 0, 0) \times (0, n\sqrt{2}, -\frac{n\sqrt{2}}{2}) \right) \cdot (-n, 0, n\sqrt{2}) \]
\[= (0, \frac{n^2\sqrt{2}}{2}, n^2\sqrt{2}) \cdot (-n, 0, n\sqrt{2}) \]
\[= 0 + 0 + 2n^3 > 0.\]

Therefore $S_{3,2}$ is right-handed.

A second way to check the handedness of $S_{3,2}$ is to note that we can get $V_3$ by rotating $V_2$. Since rotation preserves the value of triple products, $S_{3,2}$ is right-handed because $S_{2,2}$ is right-handed. In the rest of this section it suffices to state that $S_{k,2}$ is right-handed because all our $V_k$ are rotations of $V_2$.

The next unit, $V_4$, attaches to the first unit at a 45 degree angle in the negative $x_3$ direction compared to the angle that $V_2$ attaches to $V_1$. The unit $V_4$ consists of:

- $S_{4,1} = (n, 0, 0)$
- $S_{4,2} = (0, -\frac{n\sqrt{2}}{2}, -n\sqrt{2})$
- $S_{4,3} = (-n, \frac{3n\sqrt{2}}{2}, 0)$

The vector $\sum V_4 = (0, n\sqrt{2}, -n\sqrt{2})$, has length $2n$, and attaches to $\sum V_1$ at a
45 degree angle in the negative $x_3$ direction relative to how $\Sigma V_2$ attaches to $\Sigma V_1$. Figure 2.4 shows $V_4$ attached to $V_1$.

![Figure 2.4: The $V_4$ unit attached to the $V_1$ unit](image)

The vector $S_{1,3}$ is right-handed because $S_{4,1} = S_{2,1}$. The unit $V_4$ is $V_2$ rotated 45 degrees in the negative $x_3$ direction, therefore $S_{4,2}$ is right-handed. The only vectors we must check is $S_{4,1}$.

The handedness of $S_{4,1}$ is given by

$$(S_{1,3} \times S_{4,1}) \cdot S_{4,2} = ((-n, 2n, n) \times (n, 0, 0)) \cdot (0, \frac{-n\sqrt{2}}{2}, \frac{-n\sqrt{2}}{2})$$

$$= (0, n^2, -2n^2) \cdot (0, \frac{-n\sqrt{2}}{2}, \frac{-n\sqrt{2}}{2})$$

$$= 0 - \frac{n^3\sqrt{2}}{2} + n^3\sqrt{2}$$

$$= \frac{n^3\sqrt{2}}{2} > 0$$

Therefore $S_{4,1}$ is right-handed. It ought to be noted that this is the first time we have come to a trio of vectors where one item in the triple product, before summing the items together, is less than zero. This is why $S_{1,3}$ needs to be $(-n, 2n, n)$ rather than $(-n, n, n)$. If we were to insert $(-n, n, n)$ in the above equation, we get that the triple product of the three vectors is equal to zero which tells us that all three vectors lie in the same plane. By lengthening $S_{1,3}$ then $S_{4,1}$ becomes right-handed.
The next unit, $V_5$, attaches to the first unit at a 45 degree angle in the positive $x_1$ direction compared to the angle that $V_2$ attaches to $V_1$. The unit $V_5$ consists of:

- $S_{5,1} = \left(\frac{n\sqrt{2}}{2}, -\frac{n\sqrt{2}}{2}, 0\right)$
- $S_{5,2} = (0, 0, -n)$
- $S_{5,3} = \left(\frac{n\sqrt{2}}{2}, \frac{3n\sqrt{2}}{2}, n\right)$

The vector $\sum V_5 = (n\sqrt{2}, n\sqrt{2}, 0)$, has length $2n$, and attaches to $\sum V_1$ at a 45 degree angle in the positive $x_1$ direction relative to how $\sum V_2$ attaches to $\sum V_1$. Figure 2.5 shows $V_5$ attached to $V_1$.

![Figure 2.5: The $V_5$ unit attached to the $V_1$ unit](image)

Because $V_5$ is equal to $V_2$ when rotated in the positive $x_1$ direction, then $S_{5,2}$ is right-handed. We must check $S_{1,3}$ and $S_{5,1}$.

The handedness of $S_{1,3}$ is given by
\[(S_{1,2} \times S_{1,3}) \cdot S_{5,1} = ((0, 0, -n) \times (-n, 2n, n)) \cdot (\frac{n\sqrt{2}}{2}, -\frac{n\sqrt{2}}{2}, 0)\]
\[= (2n^2, n^2, 0) \cdot (\frac{n\sqrt{2}}{2}, -\frac{n\sqrt{2}}{2}, 0)\]
\[= n^3\sqrt{2} - \frac{n^3\sqrt{2}}{2} + 0\]
\[= \frac{n^3\sqrt{2}}{2} > 0.\]

Therefore \(S_{1,3}\) is right-handed. The handedness of \(S_{5,1}\) is given by

\[(S_{1,3} \times S_{5,1}) \cdot S_{5,2} = \left((-n, 2n, n) \times (\frac{n\sqrt{2}}{2}, -\frac{n\sqrt{2}}{2}, 0)\right) \cdot (0, 0, -n)\]
\[= (\frac{n^2\sqrt{2}}{2} - \frac{n^2\sqrt{2}}{2}) \cdot (0, 0, -n)\]
\[= 0 + 0 + \frac{n^3\sqrt{2}}{2} > 0.\]

Therefore \(S_{5,1}\) is right-handed.

The final unit, \(V_6\), attaches to the first unit at a 45 degree angle in the negative \(x_1\) direction compared to the angle that \(V_2\) attaches to \(V_1\). The unit \(V_6\) consists of:

- \(S_{6,1} = (\frac{n\sqrt{2}}{2}, \frac{n\sqrt{2}}{2}, 0)\)
- \(S_{6,2} = (0, 0, -n)\)
- \(S_{6,3} = (-\frac{3n\sqrt{2}}{2}, -\frac{n\sqrt{2}}{2}, n)\)

The vector \(\sum V_6 = (-n\sqrt{2}, n\sqrt{2}, 0)\), has length \(2n\), and attaches to \(\sum V_1\) at a 45 degree angle in the negative \(x_1\) direction relative to how \(\sum V_2\) attaches to \(\sum V_1\). Figure 2.6 shows \(V_6\) attached to \(V_1\).

Because \(V_6\) is equal to \(V_1\) when rotated in the negative \(x_1\) direction, then \(S_{6,2}\) is right-handed. We must check \(S_{1,3}\) and \(S_{6,1}\).

The handedness of \(S_{1,3}\) is given by

\[(S_{1,2} \times S_{1,3}) \cdot S_{6,1} = ((0, 0, -n) \times (-n, 2n, n)) \cdot (\frac{n\sqrt{2}}{2}, \frac{n\sqrt{2}}{2}, 0)\]
\[= (2n^2, n^2, 0) \cdot (\frac{n\sqrt{2}}{2}, \frac{n\sqrt{2}}{2}, 0)\]
\[= n^3\sqrt{2} + \frac{n^3\sqrt{2}}{2} + 0\]
\[= \frac{3n^3\sqrt{2}}{2} > 0.\]

Therefore \(S_{1,3}\) is right-handed. The handedness of \(S_{6,1}\) is given by
Figure 2.6: The $V_6$ unit attached to the $V_1$ unit

\[
(S_{1,3} \times S_{6,1}) \cdot S_{6,2} = \left( (-n, 2n, n) \times \left( \frac{n\sqrt{2}}{2}, \frac{n\sqrt{2}}{2}, 0 \right) \right) \cdot (0, 0, -n) \\
= \left( \frac{-n^2\sqrt{2}}{2}, \frac{n^2\sqrt{2}}{2}, -\frac{3n^2\sqrt{2}}{2} \right) \cdot (0, 0, -n) \\
= 0 + 0 + \frac{3n^3\sqrt{2}}{2} > 0.
\]

Therefore $S_{6,1}$ is right-handed.

In the constructing process, the origin will move via translation and then frame of reference will change via rotation. Translation and rotation both preserve the value of triple products, so vectors that were right-handed before the translation and rotation will continue to be right-handed.

Let us suppose we are attaching unit $V_n$ to unit $V_m$. Then $S_{m,1}$ begins at $(0, 0, 0)$ and ends at $(n, 0, 0)$. Unit $V_m$ in the new frame of reference is now identical to a $V_1$ unit. This is possible because all the $V_i$ are identical up to rotation. We now attach unit $V_n$ to $V_m$ as if we were attaching it to $V_1$, so the beginning of $S_{n,1}$ is placed at the end of $S_{m,3}$ at the same angle that $S_{n,1}$ would be attached to $S_{1,1}$. 
2.2 Implementing the Construction

We will now construct an unknot using this process of attaching units.

To construct an unknot, begin by placing $V_1$ and attaching $V_3$ to $V_1$. After translating the origin and rotating the frame of reference, attach a second $V_3$ to the first $V_3$. Continue attaching $V_3$ sets until we have seven $V_3$ sets and one $V_1$ set. The seven $\sum V_3$ and the one $\sum V_1$ are an octagon. The eight points of the octagon are the points of connection between any units. If we place the origin back at the start of $V_1$, with the corresponding frame of reference on that unit, the eight points of the octagon are the following:

- $(0, 0, 0)$
- $(0, 2n, 0)$
- $(0, 2n + n\sqrt{2}, n\sqrt{2})$
- $(0, 2n + n\sqrt{2}, 2n + n\sqrt{2})$
- $(0, 2n, 2n + 2n\sqrt{2})$
- $(0, 0, 2n + 2n\sqrt{2})$
- $(0, 2n + 2n\sqrt{2})$
- $(0, -n\sqrt{2}, 2n + n\sqrt{2})$
- $(0, -n\sqrt{2}, n\sqrt{2})$

Figure 2.7 is the octagon with the vectors in each $V_i$ in black and the vectors $\sum V_i$ in gray.

A second way to construct an unknot is start with $V_1$ and attach a $V_2$ to the $V_1$. Next attach a copy of $V_3$ to the $V_2$, followed by a $V_2$ to the $V_3$. Continue alternating
Figure 2.7: The unknot constructed of one $V_1$ units and seven $V_3$ units

$V_2$ and $V_3$ units until we arrive back at the $V_1$. The sum of the units are again an octagon where $\sum V_1$ and $\sum V_2$ are a side and each $\sum V_3$ and $\sum V_2$ pair are a side.

A third way to construct an unknot is to place two $V_2$ sets following the first $V_1$ and every $V_3$. The sum of the units this time are equal to an octagon where each side is a $\sum V_1$ or a $\sum V_3$ followed by two $\sum V_2$ vectors.

This illustrates one helpful process when constructing any knot; we can always change all of the previous units into that unit followed by several $V_2$ units. This process allows us to construct very tight turns. In other words, one concern with this way of constructing is that it cannot give us a knot whose stick representation has a very sharp angle between two of the sticks. If the turn is very tight, we can decrease the size of the units with respect to the overall size of the knot we are trying to replicate, and four units in a row can give us a turn that can be as small as necessary
with respect to the size of the overall knot.

The other concern when constructing knots in this way is that two units may interact with each other and accidentally change the knot itself. This can be avoided by simply replacing the unit in question by several iterations of $V_2$ units, each of which are smaller relative to the overall size of the knot. As we increase the number of $V_2$ units, the section of the knot in question gets begins to better approximate a straight line segment. Therefore, there is no worry of two units interacting with each other in undesirable ways. Therefore, we can construct any knot by the above process and have every vector in that knot be right-handed.

2.3 An Upper Bound on the Number of Vectors Required

The above construction is not a least upper bound on the number of vectors required to make a knot entirely out of vectors which are right-handed. Now we will assume the knot is in the projection where it is constructed out of the fewest number of vectors and alter this projection by adding vectors so that every vector is right-handed.

Any trio of vectors in a stick representation of a knot can be represented so that the first vector is equal to $L_1 = (0, 0, 1)$, the second vector is equal to $L_2 = (0, 1, 0)$, and the third vector is equal to $L_3 = (-1, 0, 0)$ or $L_3 = (1, 0, 0)$, where the sign of the third vector depends on the handedness of the second vector.

This is because of that fact that if a trio is non-planar, then the three vectors form a basis for the vector space because they are linearly independent. The vectors $L_1$, $L_2$, and $L_3$ or $R_3$ also form a basis. Therefore, via a linear transformation we can move between the bases. Furthermore, linear transformations which do not change the sign of the triple product of the basis do not change the sign of the triple product of any three vectors in the space, so the linear transformation taking any trio of vectors
where the middle vector is right-handed in a knot to $L_1$, $L_2$, and $R_3$ would retain the handedness for all vectors in the stick representation because $L_2$ is right-handed. If the middle vector is left-handed, then there is a linear transformation which takes it to $L_1$, $L_2$, and $L_3$.

Without loss of generality, we can assume that every vector in a stick representation of a knot is either right or left-handed. If any trio of vectors are planar then, because knots are transverse, we can push one of the vertices of the three vectors out of the plane by an arbitrarily small distance. We can choose a distance small enough so that the change is an ambient isotopy, as well as keep the sign of the triple product of all other vectors the same, so as to not inadvertently cause another trio of vectors to become planar.

Suppose there is a trio of vectors in which the middle vector is left-handed. Change the trio of vectors to $L_1$, $L_2$, and $L_3$ via a linear transformation which preserves handedness.

To change the handedness of the vector(s) between $L_1$ and $L_3$, break $L_2$ in half by creating a vertex at the point $(0, \frac{1}{2}, 1)$. Next move the vertex an arbitrarily small distance, $\varepsilon$ in the positive $x_1$ direction and the negative $x_3$ direction. The resulting set of vectors is:

- $L_1 = (0, 0, 1)$
- $L_a = (\varepsilon, \frac{1}{2}, -\varepsilon)$
- $L_b = (-\varepsilon, \frac{1}{2}, \varepsilon)$
- $L_3 = (1, 0, 0)$

where $\varepsilon > 0$ is arbitrarily small. Figure 2.8 shows all the vectors involved where $L_2$ is gray, and $L_a$ and $L_b$ are separated by a gray dot. The value of $\varepsilon$ correlates to the
bracket to the left of the $x_3$ axis; the bigger $\varepsilon$ is, the bigger the distance between the gray dot and the gray line.

Figure 2.8: Replacing $L_2$ with $L_a$ and $L_b$

Let us examine the handedness of $L_a$ and $L_b$. The handedness of $L_a$ is given by

\[
(L_1 \times L_a) \cdot L_b = ((0, 0, 1) \times (\varepsilon, \frac{1}{2}, -\varepsilon)) \cdot (-\varepsilon, \frac{1}{2}, \varepsilon) \\
= (-\frac{1}{2}, \varepsilon, 0) \cdot (-\varepsilon, \frac{1}{2}, \varepsilon) \\
= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0 \\
= \varepsilon > 0.
\]

Therefore $L_a$ is right-handed. The handedness of $L_b$ is given by

\[
(L_a \times L_b) \cdot L_3 = ((\varepsilon, \frac{1}{2}, -\varepsilon) \times (-\varepsilon, \frac{1}{2}, -\varepsilon)) \cdot (1, 0, 0) \\
= (\varepsilon, 0, \varepsilon) \cdot (1, 0, 0) \\
= \varepsilon + 0 + 0 > 0.
\]

Therefore $L_b$ is right-handed.

Because $\varepsilon$ is arbitrarily small, the handedness of $L_1$ and $L_3$ need not be affected. To understand this, we consider the set of all possible signs for the triple product. Given two vectors, the possible direction of the third vector may be represented by a sphere. The set of directions which would cause the three vectors to lie in a plane
are represented by a great circle on the sphere. Once we delete that great circle from the sphere, we are left with two hemispheres which are the open sets. One of these hemispheres represents the set of all vectors that would make the second vector right-handed and the other hemisphere represents the set of all vectors that would make the second vector left-handed. Because these sets are open, a small change in the direction of one vector will still be inside the open set, and therefore the second vector will continue to have the same handedness. Therefore, if $\varepsilon$ is small enough, we will not change the handedness of $L_1$ or $L_3$.

We can repeat this process for every vector which is left-handed. If $k$ is the number of vectors in the stick representation of the knot, and every vector was left-handed, then it would require $2k$ vectors to construct the knot out of entirely right-handed vectors. Since this is the worst case scenario, then using the above algorithm we can construct the a stick representation of that knot out of at most $2k$ vectors where every vector is right-handed.

### 2.4 A Better Upper Bound

There is one more adjustment we can make to give a better lower bound on the number of sticks required to make a projection where every vector is right-handed. Suppose two adjacent vectors are both left-handed, and let them become $L_2$ and $L_3$ from above via a linear transformation as before. We will now replace $L_2$ by two vectors, both of which are right-handed, which change the handedness of $L_3$. 
Replace $L_1$, $L_2$, and $L_3$ by the following vectors.

- $L_1 = (0, 0, 1)$
- $L_a = (\varepsilon, 1 + \varphi, -\varepsilon)$
- $L_b = (-\varepsilon, -\varphi, \varepsilon)$
- $L_3 = (1, 0, 0)$

where $1 > \varphi > \varepsilon > 0$ and $\varphi$ is arbitrarily small. Figure 2.9 shows all the vectors involved where $L_2$ is gray, and $L_a$ and $L_b$ are separated by a gray dot. The value of $\varphi$ correlates to the top bracket whereas $\varepsilon$ correlates to the right bracket.

Let us examine the handedness of $L_a$, $L_b$, and $L_3$. The handedness of $L_a$ is given by

\[
(L_1 \times L_a) \cdot L_b = ((0, 0, 1) \times (\varepsilon, 1 + \varphi, -\varepsilon)) \cdot (-\varepsilon, -\varphi, \varepsilon)
\]

\[
= (-1 - \varepsilon, \varepsilon, 0) \cdot (-\varepsilon, -\varphi, \varepsilon)
\]

\[
= (\varepsilon + \varepsilon^2) - \varepsilon \varphi + 0
\]

\[
= \varepsilon(1 + \varepsilon - \varphi) > 0.
\]
The value at the end is positive because $1 > \varphi$. Therefore $L_a$ is right-handed. The handedness of $L_b$ is given by

\[
(L_a \times L_b) \cdot L_3 = ((\varepsilon, 1 + \varphi, -\varepsilon) \times (-\varepsilon, -\varphi, \varepsilon)) \cdot (1, 0, 0) \\
= (\varepsilon, 0, \varepsilon) \cdot (1, 0, 0) \\
= \varepsilon + 0 + 0 > 0.
\]

Therefore $L_b$ is right-handed.

To show that $L_3$ now is right-handed, let us label the vector following $L_3$ to be $L_4 = (a, b, c)$. The handedness of $L_3$ in the original configuration is given by

\[
(L_2 \times L_3) \cdot L_4 = ((0, 1, 0) \times (1, 0, 0)) \cdot (a, b, c) \\
= (0, 0, -1) \cdot (a, b, c) \\
= 0 + 0 + -c = -c.
\]

Since $L_3$ is left-handed then $-c < 0$ and $c$ is positive. Let us now examine the handedness of $L_3$ after we replace $L_2$ by $L_a \cup L_b$. The handedness of $L_3$ in the new configuration is given by

\[
(L_b \times L_3 \cdot L_4 = ((-\varepsilon, -\varphi, \varepsilon) \times (1, 0, 0)) \cdot (a, b, c) \\
= (0, \varepsilon, \varphi) \cdot (a, b, c) \\
= 0 + \varepsilon a + \varphi c.
\]

Because $c$ is positive, we can make $\varepsilon a + \varphi c$ positive by forcing $\frac{\varepsilon}{\varphi} > -\frac{c}{a}$ when choosing $\varphi$ and $\varepsilon$. This is an attainable proportion because $\varphi > \varepsilon$ while $a$ and $c$ are constants. Therefore $L_3$ is right-handed and we have a new lower bound on the number of vectors needed to create a knot entirely of right-handed vectors.

**Theorem 2.1.** If $k$ is the number of vectors in the stick representation of the knot, $K$, and $k$ is even, then $K$ can be constructed of $\frac{3k}{2}$ vectors so that every vector is right-handed. If $k$ is odd, then $K$ can be constructed of $\frac{3(k+1)}{2}$ vectors so that every vector is right-handed.

All of the above constructions which create a stick representation entirely out of right-handed vectors may be done to construct a stick representation entirely out of
left-handed vectors. For the first construction, it suffices to make $n$ negative. All of the triple products that were calculated are negative if $n < 0$. In the latter sections where lower bounds were acquired, it suffices to multiply every vector used by $-1$. The reason for this is that a trio of vectors whose middle vector is left-handed is the mirror image of a trio whose middle vector is right-handed. Therefore if we reflect all of what we have done across any plane, then every right-handed vector becomes left-handed.
Chapter 3: Higher Dimensional Convex Brunnian Links

This chapter deals with Brunnian links that exist in $S^n$ with $n > 3$. A Brunnian link is three or more knots which together form a nontrivial link, but if we were to delete any one of the knots, the remaining knots form an unlink. Figure 3.1 shows two Brunnian links called the Borromean rings. Of the two Brunnian links, only the link on the right may be convex as they appear in the figure. The figure on the right consists of two 1-spheres and one 1-dimensional ellipse. These are the only convex Brunnian links in $\mathbb{R}^3$ of three, four, or five components. Proof of this fact for three and four components was given by Howards [5] and proof of this fact for five components was given by Davis [1].

![Brunnian links](image)

Figure 3.1: Brunnian links of three components in $\mathbb{R}^3$

Based on the fact the Borromean rings is the only convex Brunnian link in $\mathbb{R}^3$ up to ambient isotopy, here is a possible example of a Brunnian link consisting of the
spheres $K_1$ and $K_2$, and the ellipsoid $K_3$, which exist in $\mathbb{R}^n$:

- $K_1 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_s^2 + x_{s+1}^2 + \cdots + x_{s+t}^2 = 4, x_{s+t+1} = x_{s+t+2} = \cdots = x_n = 0\}, (0 \leq s \leq n - 2, 1 \leq t \leq n - s - 1)$.

- $K_2 = \{(x_1, x_2, \ldots, x_n) : x_{s+1}^2 + x_{s+2}^2 + \cdots + x_n^2 = 9, x_1 = x_2 = \cdots = x_s = 0\}, (0 \leq s \leq n - 2)$.

- $K_3 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 + \frac{x_{i+1}^2}{16} + \frac{x_{i+2}^2}{16} + \cdots + \frac{x_n^2}{16} = 1, x_{s+1} = x_{s+2} = \cdots = x_{s+t} = 0\}, (0 \leq s \leq n - 2, 1 \leq t \leq n - s - 1)$.

If $n = 3$, $s = 1$, and $t = 1$, then we get the case as in Figure 3.1, in $\mathbb{R}^3$.

The dimensions of $K_1$, $K_2$, and $K_3$ are $n - (s + t) + s = n - t$, $n - s$, and $s + t$ respectively. Therefore the sum of the dimensions of the three spheres is $2n$.

### 3.1 An Infinite Family of Convex Brunnian Links

In this section we prove the existence of an infinite family of convex Brunnian links.

From the previous section, if we let $s = i$ and $t = n - i - 1$, then we get the following form

- $K_1 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = 4, x_n = 0\}$.

- $K_2 = \{(x_1, x_2, \ldots, x_n) : x_{i+1}^2 + x_{i+2}^2 + \cdots + x_n^2 = 9, x_1 = x_2 = \cdots = x_i = 0\}$.

- $K_3 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 + \frac{x_{i+1}^2}{16} + \cdots + x_n^2 = 1, x_{i+1} = x_{i+2} = \cdots = x_{n-1} = 0\}$.

for $i < n$. Therefore our link, $L$, consists of $K_1$, $K_2$, and $K_3$, which are an $(n - 2)$-sphere, an $(n - i - 1)$-sphere, and an $i$-sphere respectively.

We intend to prove the following theorem in this section concerning the above three knots.
Theorem 3.1. \( L = K_1 \cup K_2 \cup K_3 \) is a convex Brunnian link.

Before we prove this theorem, we will need some tools. The rest of this section develops the necessary tools and lemmas for the proof of this theorem.

Theorem 3.2. \( S^n - S^{n-k} \) deformation retracts to \( S^{k-1} \).

To prove Theorem 3.2, we will demonstrate the situation in lower dimensions and then generalize for a proof of the Theorem. Let us consider \( S^1 - S^0 \). We will use polar coordinates, which will generalize to \( n \)-spherical coordinates, so let us look at the polar representation of \( S^1 \) in \( \mathbb{R}^2 \):

\[
S^1 = \left\{ \begin{array}{l}
x_1 = \sin(\theta_1) \\
x_2 = \cos(\theta_1)
\end{array} \right. \quad -\pi \leq \theta_1 \leq \pi
\]

Now suppose that we delete an \( S^0 \) from this \( S^1 \). Since \( S^0 \) is just two points, we may select to delete the points represented by \( \theta = \frac{\pi}{2} \) and \( \theta = -\frac{\pi}{2} \). Now let us consider the right side of the circle, \( -\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} \), and the left side of the circle, \( \frac{\pi}{2} < \theta_1 < \frac{3\pi}{2} \).

There is a deformation retraction which takes that right side to a point:

\[
R_t(\theta_1) = \left\{ (x_1, x_2) : x_1 = \sin(\theta_1(1 - t)), x_2 = \cos(\theta_1)(1 - t) \right\}
\]
\[
L_t(\theta_1) = \left\{ (x_1, x_2) : x_1 = \sin(\theta_1(1 - t) + \pi(t)), x_2 = \cos(\theta_1)(1 - t) + \pi(t) \right\}
\]

where \( R_t(\theta_1) \) is a deformation retract defined for \( -\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} \), or the right side of \( S^1 \) and \( L_t(\theta_1) \) is a deformation retract defined for \( \frac{\pi}{2} < \theta_1 < \frac{3\pi}{2} \), or the left side of \( S^1 \).

Now we need to examine the cases when \( t = 0 \) and \( t = 1 \) to make sure our homotopy goes to and from what we desire. At \( t = 0 \), we have

\[
R_0(\theta_1) = \left\{ (x_1, x_2) : x_1 = \sin(\theta_1), x_2 = \cos(\theta_1) \right\} - \frac{\pi}{2} < \theta_1 < \frac{\pi}{2}
\]
\[
L_0(\theta_1) = \left\{ (x_1, x_2) : x_1 = \sin(\theta_1), x_2 = \cos(\theta_1) \right\} \frac{\pi}{2} < \theta_1 < \frac{3\pi}{2}
\]

So \( \text{Im}(R_0) \cup \text{Im}(L_0) = S^1 - S^0 \) where \( S^0 \) is the point at the top and point at the bottom of the circle. At \( t = 1 \), we have
\[ R_1(\theta_1) = \{(x_1, x_2) : x_1 = \sin(0) = 0, x_2 = \cos(0) = 1\} \]
\[ L_1(\theta_1) = \{(x_1, x_2) : x_1 = \sin(\pi) = 0, x_2 = \cos(\pi) = -1\} \]

So \( \text{Im}(R_1) \cup \text{Im}(L_1) = S^0 \) where this \( S^0 \) is the points at the left and right of the circle. Therefore, for the \( S^1 - S^0 \) case, this is indeed deformation retractible to \( S^0-(1-1) \) or \( S^0 \) by the homotopy we have just given. Figure 3.2 shows this deformation retraction.

![Figure 3.2: The deformation retract from \( S^1 - S^0 \) to \( S^0 \)](image)

Let us now consider \( S^2 \) and subtract \( S^0 \). We can use the following spherical coordinates on \( S^2 \) in \( \mathbb{R}^3 \):

\[
S^2 = \begin{cases} 
  x_1 = \sin(\theta_1) \sin(\theta_2) \\
  x_2 = \cos(\theta_1) \sin(\theta_2) \\
  x_3 = \cos(\theta_2) 
\end{cases} \quad \begin{array}{c}
-\pi < \theta_1 \leq \pi \\
0 \leq \theta_2 \leq \pi
\end{array}
\]

There may be some concern that spherical coordinates on \( S^2 \) are not well defined at \( \theta_1 = \pi, \theta_2 = 0, \) and \( \theta_2 = \pi, \) but for \( \theta_1 \) we can easily conceive of a second parametrization which covers the great arc which is in question, so instead of overdoing the number of equations given, we will use this parametrization to represent both sets of equations. Furthermore, the concern with \( \theta_2 \) will be eliminated when we delete \( S^0 \) because the points in question are the points deleted from \( S^2 \). Let us first examine \( S^2 - S^0 \). The only difference in the above equation will be that we will
restrict the domain of $\theta_2$ to be open on the interval $(0, \pi)$, so $0 < \theta_2 < \pi$. Now we will give the following deformation retract:

$$H_t(\theta_1, \theta_2) = \begin{cases} 
    x_1 = \sin(\theta_1) \sin(\theta_2(1-t) + \frac{\pi}{2}(t)) & -\pi < \theta_1 \leq \pi \\
    x_2 = \cos(\theta_1) \sin(\theta_2(1-t) + \frac{\pi}{2}(t)) & 0 < \theta_2 < \pi \\
    x_3 = \cos(\theta_2(1-t) + \frac{\pi}{2}(t)) 
\end{cases}$$

Now let us examine the image of $H_0$ and the image of $H_1$ and make sure that the are equal to $S^2 - S^0$ and $S^1$ respectively, as predicted.

$$H_0(\theta_1, \theta_2) = \begin{cases} 
    x_1 = \sin(\theta_1) \sin(\theta_2) & -\pi < \theta_1 \leq \pi \\
    x_2 = \cos(\theta_1) \sin(\theta_2) & 0 < \theta_2 < \pi \\
    x_3 = \cos(\theta_2) 
\end{cases}$$

Therefore the image of $H_0$ is $S^2 - S^0$. Next,

$$H_1(\theta_1, \theta_2) = \begin{cases} 
    x_1 = \sin(\theta_1) \sin(\frac{\pi}{2}) = \sin(\theta_1) & -\pi < \theta_1 \leq \pi \\
    x_2 = \cos(\theta_1) \sin(\frac{\pi}{2}) = \cos(\theta_1) & 0 < \theta_2 < \pi \\
    x_3 = \cos(\frac{\pi}{2}) = 0 
\end{cases}$$

Therefore the image of $H_1$ is $S^1$. So we have just given a homotopy which takes $S^2 - S^0$ to $S^1$, so $S^2 - S^0$ deformation retracts to $S^1$. Figure 3.3 shows this deformation retraction.

![Figure 3.3: The deformation retract from $S^2 - S^0$ to $S^1$](image)

Next, we shall use the same parametrization to show that $S^2 - S^1$ is deformation retractible to $S^0$. Since $S^1$ is only one dimension below $S^2$, then subtracting $S^1$ from
$S^2$ will break $S^2$ into two parts. There will be two homotopies, $T_t$ and $B_t$, for the top and bottom hemispheres.

$$T_t(\theta_1, \theta_2) = \begin{cases} x_1 = \sin(\theta_1) \sin(\theta_2(1-t)) & -\pi < \theta_1 \leq \pi \\ x_2 = \cos(\theta_1) \sin(\theta_2(1-t)) & 0 \leq \theta_2 < \frac{\pi}{2} \\ x_3 = \cos(\theta_2(1-t)) \end{cases}$$

When $t = 0$, then we get the upper hemisphere of $S^2$:

$$T_t(\theta_1, \theta_2) = \begin{cases} x_1 = \sin(\theta_1) \sin(\theta_2) & -\pi < \theta_1 \leq \pi \\ x_2 = \cos(\theta_1) \sin(\theta_2) & 0 \leq \theta_2 < \frac{\pi}{2} \\ x_3 = \cos(\theta_2) \end{cases}$$

since $\theta_2$ is on the interval $[0, \frac{\pi}{2})$. Then, when $t = 1$, we get a single point:

$$T_t(\theta_1, \theta_2) = \begin{cases} x_1 = \sin(\theta_1) \sin(0) = 0 \\ x_2 = \cos(\theta_1) \sin(0) = 0 \\ x_3 = \cos(0) = 1 \end{cases}$$

This point is at $(0, 0, 1)$, or one of the two points of $S^0$. The respective homotopy for the bottom half is not quite symmetric.

$$B_t(\theta_1, \theta_2) = \begin{cases} x_1 = \sin(\theta_1) \sin(\theta_2(1-t) + \pi(t)) & -\pi < \theta_1 \leq \pi \\ x_2 = \cos(\theta_1) \sin(\theta_2(1-t) + \pi(t)) & \frac{\pi}{2} < \theta_2 \leq \pi \\ x_3 = \cos(\theta_2(1-t) + \pi(t)) \end{cases}$$

We can see how the image of $B_0$ is the bottom hemisphere of $S^2$ while the image of $B_1$ is $(0, 0, -1)$ or the other point of $S^0$. Therefore $S^2 - S^1$ is deformation retractible to $S^0$. Figure 3.4 shows this deformation retraction.

So we have shown that $S^1 - S^0$ is deformation retractible to $S^0$, $S^2 - S^0$ is deformation retractible to $S^1$, and that $S^2 - S^1$ is deformation retractible to $S^0$. Before going to the generic proof, let us examine one more specific homotopy which shows how $S^3 - S^1$ is deformation retractible to $S^1$. Understanding this one should give us a still better feel for how the generic case ought to be written.

First, the spherical coordinates of $S^3$ sitting in $\mathbb{R}^4$ are as follows:
Figure 3.4: The deformation retract from $S^2 - S^1$ to $S^0$

$$S^3 = \begin{cases} 
  x_1 = \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) & -\pi < \theta_1 \leq \pi \\
  x_2 = \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) & 0 \leq \theta_2 < \pi \\
  x_3 = \cos(\theta_2) \sin(\theta_3) & 0 \leq \theta_3 \leq \pi \\
  x_4 = \cos(\theta_3) & 
\end{cases}$$

Now, let us consider a homotopy which takes $S^3 - S^1$ to $S^1$. As always, we will check the image of $H_0$ and the image of $H_1$ to make sure that we begin and end with $S^3 - S^1$ and $S^1$ respectively. We should also think carefully about how this homotopy operates, and make sure that the deformation retraction is continuous. The homotopy is as follows:

$$H_t(\theta_1, \theta_2, \theta_3) = \begin{cases} 
  x_1 = \sin(\theta_1) \sin(\theta_2(1 - t) + \frac{\pi}{2}(t)) \sin(\theta_3(1 - t) + \frac{\pi}{2}(t)) & -\pi < \theta_1 \leq \pi \\
  x_2 = \cos(\theta_1) \sin(\theta_2(1 - t) + \frac{\pi}{2}(t)) \sin(\theta_3(1 - t) + \frac{\pi}{2}(t)) & 0 < \theta_2 < \pi \\
  x_3 = \cos(\theta_2(1 - t) + \frac{\pi}{2}(t)) \sin(\theta_3(1 - t) + \frac{\pi}{2}(t)) & 0 < \theta_3 < \pi \\
  x_4 = \cos(\theta_3(1 - t) + \frac{\pi}{2}(t)) & 
\end{cases}$$

So notice that for $i = 1$ and $i = 2$, we have replaced $\theta_i$ in our original equation with a weighted average between $\theta_i$ and $\frac{\pi}{2}$, i.e. $\theta_i(1 - t) + \frac{\pi}{2}(t)$. This makes it so that we nearly get our parametrization when $t = 0$. We do not quite achieve our parameterization because the domains of $\theta_2$ and $\theta_3$ no longer include 0 and $\pi$. Therefore, we get the following equation when $t = 0$. 

We ought to confirm that this is actually $S^3 - S^1$. The deleted part is composed of four parts: when $\theta_2 = 0$, $\theta_3 = 0$, $\theta_2 = \pi$ and $\theta_3 = \pi$. Call these four parts $a_1$, $a_2$, $a_3$, and $a_4$ respectively. Define these four parts:

\[
H_0(\theta_1, \theta_2, \theta_3) = \begin{cases} 
  x_1 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) & -\pi < \theta_1 \leq \pi \\
  x_2 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) & 0 < \theta_2 < \pi \\
  x_3 &= \cos(\theta_2) \sin(\theta_3) & 0 < \theta_3 < \pi \\
  x_4 &= \cos(\theta_3) 
\end{cases}
\]

The parts $a_1$ and $a_3$ are great arcs which go from one pole to another since $a_1$ spans from $(0, 0, 0, 1)$ to $(0, 0, 0, -1)$ and $a_3$ spans from $(0, 0, 0, -1)$ to $(0, 0, 0, 1)$. However, these span on opposite sides of $S^3$ since both use the variable $\theta_3$ to span, which itself spans the interval $[0, \pi]$, but $a_1(\pi) = (0, 0, 1, 0)$ while $a_3(\pi) = (0, 0, -1, 0)$ when $\theta_3 = \frac{\pi}{2}$. The other two parts, $a_2$ and $a_4$ are simply points, so all of $S^1$ is equal to $a_1 \cup a_3$. Therefore the image of $H_0$ is $S^3 - S^1$. Now let us consider $H_1$. 

\[
a_1(\theta) = \begin{cases} 
  x_1 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_2 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_3 &= \cos(\theta_2) \sin(\theta_3) = \sin(\theta_3) & 0 \leq \theta_3 \leq \pi \\
  x_4 &= \cos(\theta_3) 
\end{cases}
\]

\[
a_2 = \begin{cases} 
  x_1 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_2 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_3 &= \cos(\theta_2) \sin(\theta_3) = 0 \\
  x_4 &= \cos(0) = 1 
\end{cases}
\]

\[
a_3(\theta) = \begin{cases} 
  x_1 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_2 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_3 &= \cos(\theta_2) \sin(\theta_3) = -\sin(\theta_3) & 0 \leq \theta_3 \leq \pi \\
  x_4 &= \cos(\theta_3) 
\end{cases}
\]

\[
a_4 = \begin{cases} 
  x_1 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_2 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) = 0 \\
  x_3 &= \cos(\theta_2) \sin(\theta_3) = 0 \\
  x_4 &= \cos(\pi) = -1 
\end{cases}
\]
\( H_t(\theta_1, \theta_2, \theta_3) = \begin{cases} 
 x_1 &= \sin(\theta_1) \sin(\frac{\pi}{2}) \sin(\frac{\pi}{2}) = \sin(\theta_1) \\
 x_2 &= \cos(\theta_1) \sin(\frac{\pi}{2}) \sin(\frac{\pi}{2}) = \cos(\theta_1) \\
 x_3 &= \cos(\frac{\pi}{2}) \sin(\frac{\pi}{2}) = 0 \\
 x_4 &= \cos(\frac{\pi}{2}) = 0 
\end{cases} \quad -\pi < \theta_1 \leq \pi
\)

It is much easier to see that the image of \( H_1 \) is \( S^1 \) since it is the usual parametrization of \( S^1 \) in \( \mathbb{R}^4 \) using only one variable.

What we want to take away from all of this is that for \( S^n - S^{n-k}, k \neq 1 \), replace some of the \( \theta_i \), for \( i \neq 1 \), by the weighted average \( \theta_i(1-t) + \frac{\pi}{2}(t) \). If \( k = 1 \), the dimension of the sphere we are deleting from is only one greater than the dimension of the sphere we are deleting, and we will consider this as an alternative case for the time being requiring a different formula. Now we can begin our proof of Theorem 3.2.

**Proof of Theorem 3.2.** Here is the generalized formula for a sphere \( S^n \) sitting in \( \mathbb{R}^{n+1} \).

\[
S^n = \begin{cases} 
 x_1 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \ldots \sin(\theta_n) \\
 x_2 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \ldots \sin(\theta_n) \\
 x_3 &= \cos(\theta_2) \sin(\theta_3) \ldots \sin(\theta_n) \\
 x_4 &= \cos(\theta_3) \ldots \sin(\theta_n) \\
 \vdots & \vdots \\
 x_n &= \cos(\theta_{n-1}) \sin(\theta_n) \\
 x_{n+1} &= \cos(\theta_n) 
\end{cases} \quad -\pi < \theta_1 \leq \pi, \quad 0 \leq \theta_2 \leq \pi, \quad 0 \leq \theta_3 \leq \pi, \ldots, \quad 0 \leq \theta_n \leq \pi
\]

This is well known to be the sphere \( S^n \) sitting in \( \mathbb{R}^{n+1} \). Now here is the homotopy for \( k \neq 1 \) which should take \( S^n - S^{n-k} \) to \( S^{k-1} \).
\[ H_t = \begin{cases} 
  x_1 &= \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{k-1}) \sin(\theta_k (1 - t) + \frac{\pi}{2}(t)) \cdots \sin(\theta_n (1 - t) + \frac{\pi}{2}(t)) \\
  x_2 &= \cos(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{k-1}) \sin(\theta_k (1 - t) + \frac{\pi}{2}(t)) \cdots \sin(\theta_n (1 - t) + \frac{\pi}{2}(t)) \\
  x_3 &= \cos(\theta_2) \sin(\theta_3) \cdots \sin(\theta_{k-1}) \sin(\theta_k (1 - t) + \frac{\pi}{2}(t)) \cdots \sin(\theta_n (1 - t) + \frac{\pi}{2}(t)) \\
  x_4 &= \cos(\theta_3) \cdots \sin(\theta_{k-1}) \sin(\theta_k (1 - t) + \frac{\pi}{2}(t)) \cdots \sin(\theta_n (1 - t) + \frac{\pi}{2}(t)) \\
  \vdots & \vdots \\
  x_n &= \cos(\theta_{n-1} (1 - t) + \frac{\pi}{2}(t)) \sin(\theta_n (1 - t) + \frac{\pi}{2}(t)) \\
  x_{n+1} &= \cos(\theta_n (1 - t) + \frac{\pi}{2}(t)) \end{cases} \]

\[-\pi < \theta_1 \leq \pi \]
\[ 0 \leq \theta_2 \leq \pi \]
\[ 0 \leq \theta_3 \leq \pi \]
\[ \vdots \]
\[ 0 \leq \theta_{k-1} \leq \pi \]
\[ 0 < \theta_k < \pi \]
\[ \vdots \]
\[ 0 < \theta_n < \pi \]

There are two changes from the parametrization to the homotopy given above. The first is that for \( i \geq k \), \( \theta_i \) is replaced by the weighted average \( \theta_i (1 - t) + \frac{\pi}{2}(t) \). The second is that the domain for those \( \theta_i \) changes from being inclusive to being exclusive at the endpoints 0 and \( \pi \). Notice that, as before, \( \theta_i (1 - t) + \frac{\pi}{2}(t) \) is equal to \( \theta_i \) when \( t = 0 \). Therefore at \( H_0 \) is the parametrization of \( S^n \) with the exception of the change in domain for \( \theta_i, i \geq k \). We want to show that \( \text{Im}(H_0) = S^n - S^{n-k} \). The deleted part is composed of \( 2(n - k + 1) \) parts. For every \( i \) for \( k \leq i \leq n \), \( \theta_i \) could be 0 or \( \pi \). When \( \theta_k = 0 \) and \( \theta_k = \pi \), then the formula for a sphere becomes equal to a sphere defined on \( n - k + 1 \) variables as follows:
\[ S^n - \text{Im}(H_0) = \begin{cases} 
  x_1 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \ldots \sin(0) \ldots \sin(\theta_n) = 0 \\
  x_2 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \ldots \sin(0) \ldots \sin(\theta_n) = 0 \\
  x_3 &= \cos(\theta_2) \sin(\theta_3) \ldots \sin(0) \ldots \sin(\theta_n) = 0 \\
  x_4 &= \cos(\theta_3) \ldots \sin(\theta_n) \\
  \vdots \\
  x_{k-1} &= \cos(\theta_{k-2}) \sin(\theta_{k-1}) \sin(0) \ldots \sin(\theta_n) = 0 \\
  x_k &= \cos(\theta_{k-1}) \sin(0) \sin(\theta_{k+1}) \ldots \sin(\theta_n) = 0 \\
  x_{k+1} &= \cos(0) \sin(\theta_{k+1}) \ldots \sin(\theta_n) \\
  x_{k+2} &= \cos(\theta_{k+1}) \ldots \sin(\theta_n) \\
  \vdots \\
  x_n &= \cos(\theta_{n-1}) \sin(\theta_n) \\
  x_{n+1} &= \cos(\theta_n) \\
 \end{cases} \]

Therefore this is the parametrization of \( S^{n-k} \). If we consider any \( \theta_i \) for \( i \geq k \), we would find the parametrization of a sphere with fewer dimensions, which is actually a subset of the above \( S^{n-k} \).

Let us turn our attention to \( H_1 \) and make sure that the image of \( H_1 \) is \( S^{k-1} \).
\[ H_1(\theta_1, \ldots, \theta_n) = \begin{cases} 
 x_1 = \sin(\theta_1) \sin(\theta_2) \ldots \sin(\theta_{k-1}) \sin(\frac{\pi}{2}) \ldots \sin(\frac{\pi}{2}) \\
 x_2 = \cos(\theta_1) \sin(\theta_2) \ldots \sin(\theta_{k-1}) \sin(\frac{\pi}{2}) \ldots \sin(\frac{\pi}{2}) \\
 x_3 = \cos(\theta_2) \sin(\theta_3) \ldots \sin(\theta_{k-1}) \sin(\frac{\pi}{2}) \ldots \sin(\frac{\pi}{2}) \\
 x_4 = \cos(\theta_3) \ldots \sin(\theta_{k-1}) \sin(\frac{\pi}{2}) \ldots \sin(\frac{\pi}{2}) \\
 \vdots \\
 x_k = \cos(\theta_{k-1}) \sin(\frac{\pi}{2}) \\
x_{k+1} = \cos(\frac{\pi}{2}) \\
x_{k+2} = \cos(\frac{\pi}{2}) \\
 \vdots \\
x_n = \cos(\frac{\pi}{2}) \\
x_{n+1} = 0 \\
x_{n+2} = 0 \\
\end{cases} \]

Notice that the last \( n - k + 1 \) dimensions of our figure are equal to 0 in \( \mathbb{R}^{n+1} \) because \( \cos(\frac{\pi}{2}) = 0 \) is in the last \( n - k + 1 \) dimensions. Therefore, the figure exists in \( \mathbb{R}^k \subseteq \mathbb{R}^{n+1} \) and therefore is equal to \( S^{k-1} \). Also notice that \( \theta_i \) for \( 1 \leq i \leq k-1 \) is correctly defined since \( -\pi < \theta_1 \leq \pi \) and \( 0 \leq \theta_2 \leq \pi \) for all \( i \neq 1 \).

Since there exists a homotopy, \( H_t \) from above, taking \( S^n - S^{n-k} \) to \( S^{k-1} \) such that the homotopy is smooth and does not intersect itself, then \( S^n - S^{n-k} \) deformation retracts to \( S^{k-1} \).

\[ \Box \]

**Lemma 2.** Every proper sublink of \( L = K_1 \cup K_2 \cup K_3 \) is an unlink.
Proof. The knot $K_2$ bounds a ball $B_2$ in the complement of $K_3$, $K_3$ bounds a ball $B_3$ in the complement of $K_1$, and $K_1$ bounds a ball $B_1$ in the complement of $K_2$, where $B_1$, $B_2$, and $B_3$ are defined as follows:

- $B_1 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 4, x_n = 0\}$.
- $B_2 = \{(x_1, x_2, \ldots, x_n) : x_{i+1}^2 + x_{i+2}^2 + \cdots + x_n^2 \leq 9, x_1 = x_2 = \cdots = x_i = 0\}$.
- $B_3 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 + \frac{x_i^2}{16} \leq 1, x_{i+1} = x_{i+2} = \cdots = x_{n-1} = 0\}$.

To show $K_1$ bounds ball $B_1$ in the complement of $K_2$, we just need that $B_1 \cap K_2$ is empty. Suppose $B_1 \cap K_2$ is non-empty. Then there is some point $x \in B_1 \cap K_2$. Since $x \in K_2$, then $x_{i+1}^2 + x_{i+2}^2 + \cdots + x_n^2 = 9$. Since $x \in B_1$, then $x_n = 0$, so $x_{i+1}^2 + x_{i+2}^2 + \cdots + x_{n-1}^2 = 9$. However, since $x \in B_1$, then $x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 4$, and so $x_{i+1}^2 + x_{i+2}^2 + \cdots + x_{n-1}^2 \leq 4$, so therefore $9 \leq 4$. This is a contradiction, so it is not possible for a point to be in $B_1 \cap K_2$, and so $B_1 \cap K_2$ is empty. Similar explanations give us that $B_2 \cap K_3$ is empty and $B_3 \cap K_1$ is empty. Therefore every proper sublink of $L$ is an unlink.

Figure 3.5 demonstrates $B_1 \cup K_2 = \emptyset$ in $\mathbb{R}^3$. The outer black $S^1$ is $K_2$ while the inner gray disk is $B_1$.

Lemma 3. $K_3 - (B_1 \cap K_3)$ is not connected.

Proof. We have

- $B_1 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 4, x_n = 0\}$.
- $K_3 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 + \frac{x_i^2}{16} = 1, x_{i+1} = x_{i+2} = \cdots = x_{n-1} = 0\}$.

So therefore,

$B_1 \cap K_3 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 = 1, x_{i+1} = x_{i+2} = \cdots = x_n = 0\}$.
Figure 3.5: $K_2$ and $B_1$

Let us consider two points in $K_3 - (B_1 \cap K_3)$, $p_0 = (0, 0, \ldots, 4)$ and $p_1 = (0, 0, \ldots, -4)$. Any continuous path that connects $p_0$ and $p_1$ must contain a point where $x_n = 0$ since $x_n = 4$ in $p_0$ and $x_n = -4$ in $p_1$. This is because on any continuous path, each coordinate changes continuously, so the path cannot “jump” from $x_n = 1$ to $x_n = 2$, for example. But if $x_n = 0$ then $x_1^2 + x_2^2 + \cdots + x_i^2 = 1$ and this is exactly $(B_1 \cap K_3)$, which is deleted from the space we are considering. Therefore, by the Intermediate Value Theorem, there is no path that connects $p_0$ and $p_1$, and so $K_3 - (B_1 \cap K_3)$ is disconnected and consists of at least two pieces. \[\square\]

Figure 3.6 demonstrates Lemma 3 in $\mathbb{R}^3$. The horizontal gray disk is $B_1$ and the vertical black $S^1$ is $K_3$. Notice that $K_3$ is broken into an upper and a lower piece by $B_1$.

Let $K_3' = K_a \cup K_b$ where $K_a$ and $K_b$ are as follows:

- $K_a = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 + \frac{x_n^2}{16} = 1, x_{i+1} = x_{i+2} = \cdots = x_{n-1} = 0, x_n > 0\}$
Figure 3.6: $B_1$ and $K_3$

- $K_b = \{(x_1, x_2, \ldots x_n) : x_1^2 + x_2^2 + \cdots + x_s^2 \leq 1, x_{i+1} = x_{i+2} = \cdots = 0, \}

Figure 3.7 shows $K'_3$ when $n = 3$.

**Definition 1.** A **generalized Hopf link** is any link of two components isotopic to the following link in $\mathbb{R}^n$:

- $G_1 = \{(x_1, x_2, \ldots x_n) : x_1^2 + x_2^2 + \cdots + x_s^2 = 1, x_{s+1} = x_{s+2} = \cdots = x_n = 0, n > s \geq 0\}$
- $G_2 = \{(x_1, x_2, \ldots x_n) : (x_1 - 1)^2 + x_{s+1}^2 + x_{s+2}^2 + \cdots + x_n^2 = 1, x_2 = x_3 = \cdots = x_s = 0, n > s \geq 0\}$

The knots $G_1$ and $G_2$ share only one dimension which is not equal to zero, $x_1$, and so they do not intersect at any points. Furthermore, $G_1$ is an $(s - 1)$-sphere, and $G_2$ is an $(n - s)$-sphere, so the sum of their dimensions is one less than the ambient space $\mathbb{R}^n$ and they do not necessarily intersect. When $s = 0$, then $G_1$ and $G_2$ are both $S^1$ and sit in $\mathbb{R}^3$ forming the Hopf link in the traditional sense. Figure 3.8 shows the Hopf link in the traditional sense.
Lemma 4. $K'_3 \cup K_2$ is a generalized Hopf link.

Proof. We prove Lemma 4 by showing there is an ambient isotopy which takes $K'_3 \cup K_2$ to the generalized Hopf link above. The begin this isotopy we multiply $K'_3 \cup K_2$ by $\frac{1}{3}$. This scales $K'_3 \cup K_2$ down so that $K_2 = G_1$. Now for all the points in $K_2$, $x_{i+1}^2 + x_{i+2}^2 + \cdots + x_n^2 = 1$, and there is a point on $K_2$ at $(0, 0, \ldots, 1)$. If we consider all rays emanating from that point, the only ray which intersects both $K'_3$ and $K_2$ is the ray that goes exactly in the negative $x_3$ direction. However, all the rays that pass through $K'_3$ also pass through $G_2$. Therefore, we have an isotopy:

$$F : S^{n-s} \times I \rightarrow \mathbb{R}^n$$

where $f_t = (t - 1)K'_3 + (t)G_2$. In other words, given a point $x \in K'_3$, there is a ray $r_x$ which starts at $(0, 0, \ldots, 1)$ and passes through $x$. We can slide $x$ along the ray $r_x$ in the direction of the corresponding point where $r_x$ and $G_2$ intersect, and in every case the point will not pass through $G_1 = K_2$. The ray which intersects both $K'_3$ and $K_2$ intersects $K'_3$ and $G_2$ at the same point which is the origin. Therefore, the isotopy
Figure 3.8: The Hopf link in $\mathbb{R}^3$

which takes $K'_3$ to $G_2$ does not pass through $G_1 = K_2$, and $K'_3 \cup K_2$ is a generalized Hopf link.

Lemma 5. A generalized Hopf link is not an unlink.

Proof. Assume that there is a generalized Hopf link, $G$, which is a split link. Without loss of generality, let $G_1 = S^{s-1}$ and $G_2 = S^{n-s}$ be the components of the generalized Hopf link in $\mathbb{R}^n$ so that $G = G_1 \cup G_2$. There is a homeomorphism between $S^n$ and $\mathbb{R}^n$ with a point added at infinity given by stereographic projection. Therefore $G_1$ and $G_2$ not linked in $\mathbb{R}^n$ implies that $G_1$ and $G_2$ are not linked in the $S^n$ achieved by stereographic projection on the ambient space. If $G_1$ and $G_2$ are not linked, then $G_1$ bounds an $s$-ball, $B_{G_1}$, in the complement of $G_2$ in $S^n$. Since $S^n - G_2 = S^n - S^{n-s}$ and by Theorem 3.2, $S^n - S^{n-s}$ deformation retracts onto $S^{s-1}$, then $S^n - G_2$ deformation retracts onto $S^{s-1}$.

We should observe that if $s$ is equal to the $k$ we used in the proof of Theorem 3.2, then $G_1 = \text{Im}(H_1)$ and $G_2 = S^n - \text{Im}(H_0)$. This is because if we exchange the $x_1$ and $x_s$ coordinates of the generalized Hopf link definition, then $G_1$ is the sphere
defined on the first $s$ coordinates just as $\text{Im}(H_1)$ is the sphere defined on the first $k$ coordinates, and $G_2$ becomes the sphere defined on the last $n-s+1$ coordinates just as $S^n - \text{Im}(H_0)$ is the sphere defined on the last $n-s+1$ coordinates.

Now $G_1 = S^{s-1}$ maps into $S^{s-1}$ by inclusion. Our assumption implies that $G_1$ bounds an $s$-ball in $S^{s-1}$. Therefore $G_1$ is homotopically trivially and is homotopic to a single point. This would imply that $\pi_{s-1}(G_1) = 0$, but $\pi_{s-1}(S^{s-1}) = \mathbb{Z}$ because in general $\pi_n(S^n) = \mathbb{Z}$ [3]. This is a contradiction, therefore $G$ must be linked. Since all generalized Hopf links are isotopic to $G$, then all generalized Hopf links are linked. □

Let us return to $L = K_1 \cup K_2 \cup K_3$. If $L$ is an unlink, then $K_2$ bounds a ball, $B'_2$, disjoint from $K_1 \cup K_3$. This is equivalent to saying that $K_2$ bounds a ball, $B'_2$, disjoint from $K_3$ in the complement of $K_1$ in $S^n$, i.e. $S^n - K_1$. $K_2$ has dimension $n-i-1$, therefore $B'_2$ has dimension $n-i$. Our $B_1$ from earlier has dimension $n-1$. We can ask that $B_1$ and $B'_2$ intersect transversally. Furthermore, because $(n-i)+(n-1) \geq n$ then $B_1$ and $B'_2$ intersect in manifolds. Since $B_1$ and $B'_2$ are compact, and they intersect transversally, then there must be a finite number of intersections.

**Definition 2.** Let $U$ be a subset of $\mathbb{R}^n$ and $B$ be an $(n-1)$-ball contained in $\mathbb{R}^n$ with $x_i = 0$. Then the **reflection** $R$ of $U$ **across** $B$ is defined as the map

$$R : U \rightarrow U'$$

where $R(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, -x_i, \ldots, x_n)$.

Our reflections will always be across the ball $B_1$ where $x_n = 0$, so $i = n$ and $R(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, -x_n)$.

**Lemma 6.** If $U$ is a subset of $B'_2$, then the reflection of $U$ across $B_1$ does not intersect $K_3$ by the map $R((x_1, x_2, \ldots, x_n)) = (x_1, x_2, \ldots, x_{n-1}, -x_n)$ for any point $(x_1, x_2, \ldots, x_{n-1}, x_n) \in U$. 
Proof. Let $U'$ be the reflection of $U$ across $B_1$. Suppose that $U'$ intersects $K_3$ non-trivially. Then consider any point $p = (a_1, \ldots, a_n)$ in $K_3 \cap U'$. Since $p \in K_3$, then $(a_1, \ldots, -a_n) \in K_3$, but $(a_1, \ldots, -a_n) \in U$ and $U$ does not intersect $K_3$ since $U \subset B_2'$ which does not intersect $K_3$. Since this is a contraction $U'$ must not intersect $K_3$ and the reflection of any subset across $B_1$ does not intersect $K_3$. \hfill \square

Because $B_1$ and $B_2'$ intersect transversally, each intersection is a manifold whose dimension is one less than $B_2'$. This follows since codimension is additive for transversal intersections. $B_1$ is a ball of dimension $n - 1$ while $B_2'$ is a ball of dimension $n - i$. The codimension of $B_1$ is 1 while the codimension of $B_2'$ is $i$. Therefore the codimension of $B_1 \cap B_2'$ is $i + 1$ and the dimension of $B_1 \cap B_2'$ is $n - (i + 1)$ or one less than the dimension of $B_2'$.

The manifolds which are the intersections of $B_1$ and $B_2'$ break $B_2'$ into disjoint subsets. Manifolds which do this are called separating manifolds.

**Definition 3.** If $X$ is a connected manifold and $Y$ is a connected space, where $X$ is embedded in $Y$, then $X$ is a separating manifold of $Y$ if $Y - X$ is not connected.

The following Lemma shows that every connected manifold $B_1 \cap B_2'$ is a separating manifold.

**Lemma 7.** Let $B$ be an $n$-ball. If $A \subset B$ is an orientable $(n - 1)$-manifold, then $A$ must be a separating manifold.

**Proof.** Given $B$, an $n$-ball and $A \subset B$, an orientable $(n - 1)$-manifold, suppose $A$ is not a separating manifold. Consider an $S^1$ sphere, $C \subset B$ such that $C$ intersects $A$ transversally, and let $C \cap A$ be a single point. Consider a disk $D \in B$ which bounds $C$ and which intersects $A$ transversally. $D$ must exist in $B$ because $B$ is simply connected. If we look at $A \cap D$, the only intersections resulting from an intersecting transverse manifold and a disk are $S^1$ spheres disjoint from the boundary of the disk.
(because the intersection is transverse), and arcs across the disk which intersect the boundary in two distinct points. However, $C$ intersects $A$ in exactly one point, which cannot happen. Since this is a contradiction $A$ must be a separating manifold of $B$.

**Definition 4.** Given $B$, a manifold with boundary, and $\{A_n\}$, a set of connected manifolds such that $\bigcap A_n \subset B$, let each $A_i$ being a separating manifold of $B$. For every $A_i \subset \{A_n\}$, $B - A_i$ is disjoint and therefore breaks into two distinct subsets, $B_0$ and $B_1$. If $B_k$, $i \in \{0, 1\}$, does not contain any other $A_j$ and does not contain any part of the boundary of $B$, then $A_i$ is **innermost** on $B$ and that $B_k$ is called the **inside subset** of $A_i$ on $B$.

Figure 3.9 on the left shows the intersection of a 2-ball with a 2-ball in $\mathbb{R}^3$, while the right shows the intersection of a 3-ball with a 3-ball in $\mathbb{R}^4$. The darkest manifolds in both pictures are the innermost manifolds.

![Figure 3.9: Examples of Innermost in $\mathbb{R}^3$ and $\mathbb{R}^4$](image)

**Lemma 8.** If $K_2$ bounds a ball $B'_2$ which does not intersect $K_1 \cup K_3$, then $K_2$ bounds a ball $B''_2$ which does not intersect $K_1 \cup K_3 \cup B_1$. 
Proof of Theorem 3.1. Because none of the manifolds of $B_1 \cap B'_2$ intersect the boundary of either $B_1$ or $B'_2$ and each of the connected manifolds are separating manifolds, then there is an intersection which is innermost on $B'_2$. Call this innermost intersection $A_1$. Let $U_1$ be the inside subset of $A_1$ on $B'_2$. Reflect $U_1$ across $B_1$. Now take an $\epsilon$ neighborhood around $U_1$ on $B'_2$, named $\epsilon(U_1)$. Let this neighborhood be small enough that $\epsilon(U_1)$ does not intersect $B_1$ except at $A_1$. This is possible because $B_1$ and $B'_2$ intersect transversally.

Since $A_1$ is in $B_1 \cap B'_2$, then the reflection of $A_1$ across $B_1$ is still in $B_1$. Because $A_1$ is innermost on $B'_2$, then $A_1$ is the only part of $\epsilon(U_1)$ which intersects $B_1$. The open set $\epsilon(U_1)$ can be moved away from $B_1$ so that $\epsilon(U_1) \cap B_1 = \emptyset$. The open set $\epsilon(U_1)$ can be moved a small enough distance that it does not intersect $K_3$ since $K_3$ appears orthogonal to $B_1$ when we look very closely. It is not necessary that $B'_2$ still be embedded after this pulling movement.

Figure 3.10 shows the process described in $\mathbb{R}^3$, from left to right, described where $K_1$ is the upper $S^1$ and $K'_2$ is the lower $S^1$.

Since $A_1$ no longer intersects $B_1$, then let $A_2$ be a connected intersection, innermost on $B'_2$, and let $U_2$ be the ball that $A_2$ bounds on $B'_2$. Repeat exactly what was done with $A_1$ and $U_1$ so that $U_2$ no longer intersects $B_1$. Because $B_1$ and $B'_2$ are both compact and intersect transversally, then there are a finite number of connected
manifolds in $B_1 \cap B'_2$. By the end of the process, $B'_2$ no longer intersects $B_1$. Therefore we have a ball, $B'_2$, which is bounded by $K_2$, exists in the complement of $K_1$, and does not intersect $K_3$.

We now have enough tools to prove the theorem.

Proof. Because $B_1$, $B_2$, and $B_3$ are convex, then $K_1$, $K_2$, and $K_3$ are convex.

Suppose that $L$ is not a Brunnian link. By Lemma 2, we know that every proper subset of $L$ is an unlink. Therefore, for $L$ not to be Brunnian, it must be the case that $L$ is the unlink. If this is true, then each of the knots bounds a ball disjoint from the other two. It is sufficient to prove that only one of the knots cannot bound a ball disjoint from the others, because that shows $L$ is not an unlink. Therefore, without loss of generality, assume it is $K_2$ which bounds a ball disjoint from $K_1 \cup K_3$.

By Lemma 8, we know that if $K_2$ bounds an embedded ball disjoint from $K_1 \cup K_3$, then $K_2$ bounds a ball, $B'_2$, not necessarily embedded, disjoint from $K_1 \cup K_3 \cup B_1$. The sphere $K'_3$ is a subset of $K_1 \cup K_3 \cup B_1$, therefore $B'_2$ must be disjoint from $K'_3$. According to Lemma 4, $K'_3 \cup K_2$ is a generalized Hopf link. But that would imply that $K_2$ cannot bound a ball disjoint from $K'_3$ since they are linked. Therefore it must be that $K_2$ cannot bound a ball disjoint from $K_1 \cup K_3$. Therefore $L$ is a Brunnian link.

Since $L$ is convex and a Brunnian link, then $L$ is a convex Brunnian link.

3.2 A Special Case of the Brunnian Links

We have now proven our theorem, but a very different proof technique exists for a subset of the links. Here we show that if one of the components of the link is homeomorphic to a circle, we can use the fundamental group instead for our proof. Allow $K_3$ to be a (elliptical) circle, but symmetry dictates that the proof holds if any
of the other components had been a circle. If we set $i = 1$ in the above construction, then we get the following link $L = K_1 \cup K_2 \cup K_3$:

- $K_1 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \ldots x_{n-1}^2 = 4, x_n = 0\}$.
- $K_2 = \{(x_1, x_2, \ldots, x_n) : x_2^2 + x_3^2 + \ldots x_n^2 = 9, x_1 = 0\}$.
- $K_3 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + \frac{x_2^2}{16} = 1, x_2 = x_3 = \ldots x_{n-1} = 0\}$.

**Theorem 3.3.** $L$ is a convex Brunnian link.

**Proof.** We want to show that the loop $K_3$ does not bound a disk disjoint from $K_1 \cup K_2$. We do so by showing it is nontrivial in the fundamental group of $\mathbb{R}^n - (K_1 \cup K_2)$. We first prove the following Lemma.

**Lemma 9.** $\pi_1(\mathbb{R}^n - (K_1 \cup K_2))$ is the free group on two generators.

**Proof.** Since $K_1 \cup K_2$ by themselves in $\mathbb{R}^n$ is an unlink, there is an $(n-1)$-dimensional hyperplane that splits $\mathbb{R}^n$ into two pieces, one containing $K_1$ and the other containing $K_2$, showing that the fundamental group is the free product of the fundamental group of $\mathbb{R}^n - K_1$ and $\mathbb{R}^n - K_2$. These are symmetrical spaces and each has fundamental group the free group on one generator.

One way to see this follows. These spaces have the same fundamental group as if we take the one point compactification of $\mathbb{R}^n$ yielding $S^n$ minus an unknotted sphere $S^{n-1}$. We could then find a conformal map back to $\mathbb{R}^n$ taking the sphere to the subset of $\mathbb{R}^n$, $P = \{(x_1, x_2, \ldots, x_n) : x_1 = x_2 = 0\}$. We can find a generator for this fundamental group of the form $\alpha = x_1^2 + x_2^2 = 1, x_3 = x_4 = \ldots x_n = 0$. This is clear because the pair $(\mathbb{R}^n - P, \alpha)$ deformation retracts to the $(x_1, x_2$ plane$- (0, 0), \alpha')$ Where $\alpha' = x_1^2 + x_2^2 = 1$. Clearly the unit circle is a generator for the fundamental group of the plane minus the origin. The deformation retract is of the form $F((x_1, x_2, \ldots, x_n), t) = (x_1, x_2, (1 - t)x_3, (1 - t)x_4, \ldots (1 - t)x_n)$. \qed
Now to conclude the proof of Theorem 3.3, we observe that in the case of $L$ we have generators $\alpha(t) = (3 \cos 2\pi(\frac{3}{4} + t), 0, 0, \ldots, 3 + 3 \sin 2\pi(\frac{3}{4} + t)$ and $\beta(t) = (2 - 2 \cos 2\pi t, 0, 0, \ldots, 2 \sin 2\pi t)$ with the origin as the base point. Let $\gamma(t) = (t, 0, 0, 0, \ldots, 0)$.

Now $\gamma K_3 \gamma^{-1}$ is homotopic to $\alpha \beta^{-1} \alpha^{-1} \beta$. This is a commutator of the two generators of the fundamental group, which is the free group on two generators. Therefore this is not trivial which proves $K_3$ is nontrivial, completing the proof of Theorem 3.3.

\[ \square \]

### 3.3 Proof of the Symmetry of Our Knots

In this section, we will provide a homotopy which shows that $K_1$, $K_2$, and $K_3$ in the are interchangeable when their dimensions are the same. This shows that if we have three dimensions, $a$, $b$, and $c$, then suspending $K_1$, $K_2$, and $K_3$ in any order to get the dimensions $a$, $b$, and $c$ give rise to the same link $L$. We will give an example of this in $\mathbb{R}^3$ and then generalize it to $\mathbb{R}^n$.

Let us begin with $K_1$, $K_2$, and $K_3$ in $\mathbb{R}^3$ as follows:

- $K_1 = \{(x_1, x_2, x_3) : (x_2)^2 + (x_3)^2 = 9, x_1 = 0\}$
- $K_2 = \{(x_1, x_2, x_3) : (x_1)^2 + (x_3)^2 = 4, x_2 = 0\}$
- $K_3 = \{(x_1, x_2, x_3) : (x_1)^2 + \frac{(x_2)^2}{16} = 1, x_3 = 0\}$

We are trying to permute the $K_i$, so define three $K'_i$ as follows:

- $K'_1 = \{(x_1, x_2, x_3) : (x_2)^2 + (x_3)^2 = 4, x_2 = 0\}$
- $K'_2 = \{(x_1, x_2, x_3) : (x_3)^2 + \frac{(x_1)^2}{16} = 1, x_3 = 0\}$
- $K'_3 = \{(x_1, x_2, x_3) : (x_1)^2 + (x_2)^2 = 9, x_1 = 0\}$

Let $H_{i,t}(s)$ be the homotopy moving knot $K_i$ over time $t \in [0, 1]$ where $s$ is any
point satisfying the equation for \( K_i \). In other words

\[
H_{i,t} : K_i \times I \to K_i
\]

Then define the following three homotopies:

- \( H_{1,t}(s) = ((1 - t)K_1 + (t)K'_1) \)
- \( H_{2,t}(s) = ((1 - t)K_2 + (t)K'_2) \)
- \( H_{3,t}(s) = ((1 - t)K_3 + (t)K'_3) \)

An explicit parametrization of this homotopy is as follows:

- \( H_{1,t}(x_1, x_2, x_3) = (0, (1 - t)x_2 + (t)\frac{2}{3}x_2, (1 - t)x_3 + (t)\frac{2}{3}x_3) \) for \((x_1, x_2, x_3) \in K_1 \)
- \( H_{2,t}(x_1, x_2, x_3) = ((1 - t)x_1 + (t)2x_1, 0, (1 - t)x_3 + (t)\frac{1}{2}x_3) \) for \((x_1, x_2, x_3) \in K_2 \)
- \( H_{3,t}(x_1, x_2, x_3) = ((1 - t)x_1 + (t)3x_1, (1 - t)x_2 + (t)\frac{3}{4}x_2, 0) \) for \((x_1, x_2, x_3) \in K_3 \)

Therefore, we end with the knots \( K'_1, K'_2, \) and \( K'_3 \).

By the symmetry of \( \mathbb{R}^3 \), we may switch any of the dimensions in proving that two knots are symmetric. In the above situation, \( K_1 \) is identical to \( K'_3 \) where we use \( x_1 \) in place of \( x_2 \), \( x_2 \) in place of \( x_3 \), and \( x_3 \) in place of \( x_1 \). By the same switch of dimensions, it is also the case that \( K_2 \) is identical to \( K'_1 \) and \( K_3 \) is identical to \( K'_2 \). Therefore, in \( \mathbb{R}^3, K_1, K_2, \) and \( K_3 \) are homotopic to one another and are interchangeable.

### 3.4 Open Questions

Theorem 3.1 in section 3.1 guarantees that convex Brunnian links exist in all dimensions. This result only covers cases for when \( t = 1 \) in the three knots shown below:
• $K_1 = \{(x_1, x_2, \ldots, x_n) : x_{s+1}^2 + x_{s+2}^2 + \cdots + x_n^2 = 9, x_1 = x_2 = \cdots = x_s = 0\}, (0\leq s \leq n - 2)$.

• $K_2 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_s^2 + x_{s+t+1}^2 + x_{s+t+2}^2 + \cdots + x_n^2 = 4, x_{s+1} = x_{s+2} = \cdots = x_{s+t} = 0\}, (0 \leq s \leq n - 2, 1 \leq t \leq n - s - 1)$.

• $K_3 = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_s^2 + \frac{x_{s+1}^2}{16} + \frac{x_{s+2}^2}{16} + \cdots + \frac{x_{s+t+1}^2}{16} = 1, x_{s+t+1} = x_{s+t+2} = \cdots = x_n = 0\}, (0 \leq s \leq n - 2, 1 \leq t \leq n - s - 1)$.

We believe that $L = K_1 \cup K_2 \cup K_3$ forms a family convex Brunnian links so it remains to be shown whether this is true for any case other than when $t = 1$.

Another open question is whether any other types convex Brunnian links exist in $\mathbb{R}^n$ for a given $n$. We know that the only convex Brunnian links in $\mathbb{R}^3$ of three, four, or five components are the Borromean rings. Proof of this fact for three and four components was given by Howards [5] and proof of this fact for five components was given by Davis [1].
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