Coherent Spatial Structure versus Time Chaos in a Perturbed Sine-Gordon System

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A novel interplay of coherent spatial structure and temporal chaos is reported for the ac-driven, damped sine-Gordon system with breather initial conditions and periodic boundary conditions. The competing tendencies of spatial structures to be stable and of single-particle motions to be chaotic can lead to suppression of chaos, spatio-temporal intermittency, and symmetry-breaking tendencies. Dynamic structure factors provide direct evidence for spatial period halving and renormalized-phonon generation in chaotic regimes.

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The fascinating phenomenon of chaotic behavior in deterministic nonlinear systems is attracting a great deal of interest in a wide range of disciplines. One facet of this rich topic is of particular importance in understanding the chaotic behavior of physical systems in which the spatial degrees of freedom cannot be neglected: namely the role of coherent, quasistable, spatial structures in chaotic regimes and in the transitions into (and out of) these regimes. Examples of current interest include “clumping” phenomena in turbulent plasmas, dislocations in Couette-flow experiments, and (perturbed) solitons in damped, ac-driven, sine-Gordon systems.1–3

These last investigations1–3 are relevant to a large variety of soliton-bearing condensed matter systems,4 including Josephson arrays and transmission lines,5 charge-density-wave materials, magnetic chains, etc. The governing equation of motion for these cases can be put in the dimensionless form

\[ \varphi_{tt} - \varphi_{xx} + \sin \varphi = \Gamma \sin(\omega_x t) - \epsilon \varphi_t, \]

where the terms on the right-hand side of this otherwise unperturbed sine-Gordon (SG) equation represent a spatially uniform, oscillating driving force (amplitude \( \Gamma \) and frequency \( \omega_x \)) and linear damping, respectively. It may be helpful to view Eq. (1) as appropriate to the continuum limit of a chain of torsion-coupled pendula4 immersed in a viscous medium and subjected to an oscillatory torque.

Several studies6–7 have been made of the response of a “single pendulum” [ \( \varphi_{xx} = 0 \) in Eq. (1), i.e., flat initial conditions] to these forces and many periodic → chaotic transitions have been
mapped out in the parameter space \((\Gamma, \omega, \epsilon)\). Numerical integrations\(^{6,7}\) of the full ("many-pendulum") partial differential equation (1) have revealed numerous responses depending on parameter values and choices of initial conditions.

Here we report fascinating new effects involving (i) the tendency for the system to preserve spatial structures which evolve from an initial breather profile and (ii) the competing tendency for the single pendulum to become chaotic in some \(\Gamma\) regimes.\(^{1-3}\) This competition is "relieved" in various ways leading to (i) suppression of time chaos by developing higher spatial symmetries, (ii) spatio-temporal intermittency in which time chaos is accompanied by lower spatial symmetry, or (iii) symmetry-breaking tendencies. The competition is a general phenomenon, which we illustrate below which specific boundary conditions, initial data, chain length, and parameter values.

Our numerical method for solving the continuum Eq. (1) is described in Ref. 3. Periodic boundary conditions were used with \(\epsilon = 0.2\), \(\omega = 0.6\), and \(\Gamma\) varied from 0.0 to 2.0. As initial conditions we imposed the SG breather profile:

\[
\varphi(x, 0) = 0
\]

\[
\varphi(x, 0) = 4 \tan^{-1}(\omega \varphi^{-1}(1 - \omega \varphi^2)^{1/2})
\]

\[
\times \operatorname{sech}\left\{x(1 - \omega \varphi^2)^{1/2}\right\}
\]

with \(\varphi = 0.2\). Since this starting profile is symmetric about \(x = 0\), the evolving profile remains symmetric. We have found significantly different results when this initial spatial symmetry is even slightly broken\(^8\) in parameter ranges where chaotic tendencies exist.

In addition to being able to view \(\varphi(x, t)\) and \(\varphi_t(x, t)\) vs \(x\) at any time \(t\) we have used several other diagnostic tools. These include single-particle phase-plane plots \((\varphi, \varphi_t)\) for any location \(x\) and corresponding Poincaré sections;\(^6\) local power spectra\(^1,3,6,7,9\) \(S(x, \omega)\); plots of the spatial averages \(\langle \varphi \rangle\) and \(\langle \varphi_t \rangle\) as functions of time; and spatially Fourier-transformed\(^9\) dynamic structure factors \(S_A(q, \omega)\) (with \(A = \varphi, \varphi_t, \sin\varphi, \cos\varphi\), etc.), which provide useful information on the presence and character of spatial structures.

For values of \(\Gamma \leq 0.57\), the initial breather profile (both symmetric and slightly asymmetric) smooths out spatially and the response becomes identical to the periodic entrained motion of the single pendulum.\(^{3,6,7}\) For \(\Gamma\) in the range \(0.57 \leq \Gamma \leq 0.62\), the initial breather shifts its frequency\(^10\) to lock to the driving frequency, \(\omega = 0.6\). A similar synchronization effect has been studied\(^11\) analytically for small amplitude breathers.

For still larger \(\Gamma\) values, it is convenient to consider the symmetric and slightly asymmetric initial breather cases separately. For the asymmetric case,\(^8\) as \(\Gamma\) is increased above 0.616, the "background" (cf. the single pendulum) can have a tendency\(^3,6,7\) to run through large \(\varphi\) variations, the synchronized breather increasingly distorts into a kink-antikink pattern. At \(\Gamma \approx 0.6197\) a sharp transition\(^12\) to a stable spatial period-\(\frac{1}{2}\) structure occurs [see Fig. 1(a)], somewhat similar to the Jacobi elliptic-function solutions known for the unperturbed SG system. Since these \(\Gamma\) values are in a range\(^3\) where the background can have tendencies to chaos, we see that the spatial structure inhibits this tendency, although the competition results in a change in the spatial pattern. For \(0.6196 \leq \Gamma \leq 0.685\) this same competition results in intermittent periods of evolution where the spatial structure is period 1 only and translates in \(x\) (nonuniformly in time) before it relocks in the period-\(\frac{1}{2}\) pattern. An example is shown in Fig. 1 for \(\Gamma = 0.6805\); For a while the period-\(\frac{1}{2}\) pattern is locked [Fig. 1(a)]; the pattern then changes to period 1 and translates [Fig. 1(b)] before relocking as a period-\(\frac{1}{2}\) structure [Fig. 1(c)]. The frequency and length of these intermittent bursts vary nonuniformly as \(\Gamma\) is increased in the above range, presumably.

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**FIG. 1.** Intermittent spatial period-\(\frac{1}{2}\) structure resulting from a slightly asymmetric initial breather profile; \(\Gamma = 0.6805\). \(\varphi\) vs \(x\) shown at times when the structure is (a) locked in \(x\) space executing periodic oscillation \((t = 510)\), (b) delocked translating in \(x\) space \((t = 1500)\) with period 1, and (c) relocked to an almost symmetric period-\(\frac{1}{2}\) profile \((t = 1820)\).
related to the variety\textsuperscript{3} of background flow tendencies. The intermittent delocking is accompanied by large random fluctuations in $\phi$. During the spatially locked periods the time evolution of $\phi$ is confined to very few wells and is non-chaotic. This spatio-temporal intermittency is the way the competing tendencies are accommodated by the system (i.e., the way the “frustration” is relieved). There is a quite sharp change\textsuperscript{3} of character at $\Gamma \approx 0.885$, but even for $\Gamma > 0.885$ there are intermittent bursts of spatially locked period-$\frac{1}{2}$ motion although the period-$\frac{1}{2}$ structure is now completely broken for most of the time. The power spectra show\textsuperscript{3} a very broad noisy background and Poincaré sections\textsuperscript{3} a diffuse scatter of points with little structure (suggesting an attractor of high dimension).

For the case of the strictly symmetric breather the above competition cannot be relieved by developing asymmetries in $x$ space. For $\Gamma \approx 0.9$ the motion is nonchaotic (predominantly simply periodic, with the same thresholds at $\Gamma \approx 0.57$ and 0.6197, as above), although as $\Gamma \rightarrow 0.9$, the spatial patterns become increasingly complex and there are increasingly long chaotic, spatial period-1 transients (see Fig. 2) (indicating the increasing ease with which the periodic motion can be disturbed). In this sense, for the whole range $0.57 \approx \Gamma \approx 0.9$, the presence of spatial structure dominates and totally inhibits the chaotic or large-amplitude $\phi$ variations typical of the single pendulum.\textsuperscript{3} However, for $\Gamma \gtrsim 0.9$ the converse holds and background motions dominate producing chaos and spatial period 1 until a transition occurs at $\Gamma \approx 1.41$ to a periodic response with a spatial period-$\frac{1}{2}$ structure, followed at $\Gamma \approx 1.44$ by a transition to period $\frac{1}{4}$. Figure 3 shows intermittent behavior at $\Gamma = 1.40$ where short “bursts” of periodic, spatial period-$\frac{1}{2}$ (almost period-$\frac{1}{4}$) response occur. We find evidence for the existence of a precursor to the period-$\frac{1}{4}$ structure for $\Gamma = 1.4$ in plots of the dynamic structure factors $S_n(q, \omega)$ for several wave vectors $q_n = \pm n/L$, where $L$ is the length of the system and $n$ is an integer, in the sense that for $n = 0, 4, 8, \ldots$ we observe (see Fig. 4, $A = \phi_i$) harmonics of $\omega_4 = 0.6$ most strongly.

Another feature revealed by, e.g., $S_n(q, \omega)$ is the strong component of (renormalized) linear modes (“phonons”) in the chaotic spectrum. For example, at $\Gamma = 1.4$ we observe a broad peak denoted by arrows in Fig. 4 whose frequency, $\omega_4$, increases with $q$ according to $\omega_4^2 = \omega_0^2 + c_\phi^2 q^2$, where $c_\phi \approx 1.0$ is the phonon velocity in the dimensionless units of Eq. (1) and $\omega_0^2 \approx 0.05$ is the squared frequency of long-wavelength modes and is strongly renormalized downward from its value (unity) for the linearized, unperturbed SG equa-

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2}
\caption{Response at $\Gamma = 0.89$ just below a transition ($\Gamma \approx 0.9$) to chaos, for the strictly symmetric breather. After a long chaotic transient, periodic motion sets in as seen in (a) spatial averages of $\phi$ vs $t$ and (b) the scatter of points in the phase-plane plot for $x = 0.0$ (the solid line corresponds to the periodic motion).
}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3}
\caption{Response at $\Gamma = 1.4$ just below a transition ($\Gamma \approx 1.44$) from chaotic to periodic behavior, for the symmetric-breather case. (a) The spatial average of $\phi$ shows precursor intermittent bursts of periodic response, and phase-plane plots [e.g., (b) at $x = 0.0$] reflect this with “fuzzy,” overlapping orbits. (As in Fig. 2, successive points are unconnected.)
}
\end{figure}
tion. This value of \( \omega_\infty^2 \) is rather insensitive to \( \Gamma \) in this range \( 0.90 < \Gamma < 1.42 \) of chaotic response. The downward renormalization of \( \omega_\infty \) is reminiscent of the same effect in "self-consistent-phonon" perturbation treatments of anharmonicity in thermally driven sine-Gordon systems. Note that phonons are playing a central role here characterizing "noise" in a deterministic system.

The periodic response with exact spatial period-\( \frac{1}{4} \) structures persists until \( \Gamma \approx 1.49 \) (further period halving is possible on longer systems\(^4\)), where another transition to chaotic response occurs, although we find complicated quasiperiodic motions as well above this value. Interestingly, with \textit{flat} initial conditions unusual "running" motion\(^3\) occurs in the range \( 1.745 < \Gamma < 2.005 \), but with the symmetric-breather initial condition this mode is inhibited. Nevertheless, \( \Gamma \) is sufficiently large that large-amplitude \( \psi \) variations still occur (partly through kink-antikink production\(^1\)) and result in chaotic response. In contrast, for the breather initial condition with weak asymmetry,\(^6\) the breather deforms to a low-amplitude, extended profile that can "run" in \( \psi \) just as for flat initial conditions.\(^3\)

In summary, we have studied the influence of an initial breather profile on the appearance and character of temporal chaos in the ac-driven, damped, continuum SG system. We have found a novel \textit{interplay of spatial structures and time chaos}. Spatial patterns tend to develop higher symmetry and \textit{inhibit} temporal chaos of the sort seen with flat initial conditions, but \textit{competing tendencies} of background motion versus spatial structure can lead to rich intermittency. Strict spatial symmetry imposes stringent conditions on the dynamics, but when such a symmetry is even slightly broken\(^7\) the system relieves the competition and produces chaos of a much different type than for the "single pendulum." Dynam-ic structure factors \( S(q, \omega) \) are valuable diagnostic tools from which, for example, we can deduce the presence of subtle spatial structures and provide evidence for the phenomenon of \textit{renormalized-phonon generation} in chaotic regimes (dressing these structures). Further simulation and analytic studies are underway to elucidate the above effects. Responses to \textit{kink} initial conditions\(^8\) will be described elsewhere.

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\begin{enumerate}
\item A. R. Bishop, K. Fesser, P. S. Lomdahl, and S. E. Trullinger, to be published.
\item For example, we can introduce asymmetry by very slightly asymmetrizing a few grid-point locations.
\item The power spectra \( S_\psi(q, \omega) \) and \( S_\xi(q, \omega) \) are calculated as the squared modulus of the Fourier transforms of \( A \). Data smoothing is included and care taken to avoid aliasing.
\item This effect is rather insensitive to the choice of the initial breather frequency, \( \omega_\infty \).
\item This threshold value of \( \Gamma \) and the resulting spatial pattern depend on the length of the system. Here we have used 120 grid points with 5 per spatial unit.
\end{enumerate}