# VARIATIONAL METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

By

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# Table of Contents

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# Abstract

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The objective of this work is to explain some basic aspects of variational methods for solving a class of nonlinear partial differential equations. First, relevant mathematical background of functional analysis and variational calculus is explained. One of the main results discussed is the existence theorem for minimizers of functionals for the case of fixed boundary problems. This theorem assumes coercivity and lower semicontinuity conditions of the functional and the energy functional is defined over subsets of Sobolev spaces. After that, the technical concepts associated to this theorem are implemented to study some specific free boundary problems.

# Chapter 1: Introduction

## 1.1 Basic Idea

Many interesting problems in the natural sciences are described by partial differential equations (PDE). For example, we would like to make better predictions for the behavior of physical or biological systems, or, as an important first step, to determine their stationary patterns. The case of linear problems has been investigated in great detail by mathematicians using well-established methods. We are much more interested in the case of nonlinear problems, which often appear in applications.

First, we consider the case of problems with fixed boundary. Suppose we are interested in solving some nonlinear PDE. Let $A[u] = 0$ be a PDE where $A[\cdot]$ is a nonlinear partial differential operator and $u$ is the unknown function. We know that there does not exist a general method for solving this nonlinear PDE. However it is possible that this PDE can be obtained by minimizing an associated energy functional. Quite often, finding the minimum of this functional is easier than solving the nonlinear PDE $A[\cdot] = 0$ directly. Furthermore an important class of nonlinear PDEs related to physical problems can be obtained from an appropriate variational problem.

Variational analysis is a framework for minimizing functionals whose domain is a space of functions. Since this space of functions is usually infinite dimensional we need functional analysis to describe it formally. There are many different classes of function spaces and it is an art to select a convenient class of function space for a specific problem. One of most important issues that we need to resolve before using any analytical or numerical solving method is to determine the existence of solutions.

There exists a useful method to establish existence of minimizer of functionals. Suppose we want to minimize a functional $I[w]$ where $w$ belongs to a some class $\mathcal{C}$ of real-valued functions compatible with some boundary condition. Basically the method we will consider works as follows: First take a minimizing sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{C}$ that satisfy

$$\lim_{n\to\infty} I[u_n] = \inf_{u\in\mathcal{C}} I[u],$$

and then show that some subsequence of $\{u_n\}$ converges to a minimizer $u \in \mathcal{C}$. To do this we need three key conditions:

(a) Sequential compactness to guarantee that a minimizing sequence contains a convergent subsequence. This typically require a suitable selection of a weak topology,

(b) Closedness so that the limit of such subsequence belongs to $\mathcal{C}$, and

(c) Lower semicontinuity to ensure that the limit of this minimizing sequence is indeed a minimizer for $I[w]$.

Another case is the so called free boundary problems, which have been investigated recently with great interest. In free boundary problems we have not only an unknown function but also the domain of this function is itself unknown. Similarly to the previous case, a convenient method to solve this problem is by finding a minimizing sequence of an appropriate functional. However, in this case the functionals are usually discontinuous and thus the connection of the variational problem to the PDE is much more complicated, requiring weak convergence methods and geometric measure theory.

The plan of this dissertation is to work out the mathematical background required to deal with these classes of problems. First we discuss formally some main results of topology and functional analysis, and then we study the properties of Sobolev spaces. Then we formulate the variational problems over the space of smooth functions, and after that explain in detail the theorem of existence and uniqueness of the minimizer of functionals over Sobolev spaces with Dirichlet boundary conditions on fixed domains. Imposing additional growing conditions on the energy functional it is shown that the minimizer represents the weak solution of the nonlinear PDEs associated to the variational problem. Finally, we discuss briefly the minimizing problem for the case of free boundary conditions.

# Chapter 2: Preliminaries

## 2.1 Topology

**Definition 2.1.** A **topology** $\tau$ on a set $X$ is a collection of subsets of $X$ satisfying:

- $X, \emptyset \in \tau$.
- $\tau$ is closed under finite intersections.
- $\tau$ is closed under arbitrary unions.

A set $X$ equipped with a topology $\tau$ is called a **topological space**, and it is denoted by $(\tau, X)$ or simply $X$ if the specific topology is clear. A member of $\tau$ is called an **open** set. The **complement** of a set $A \subset X$ is denoted by $A^c$. The complement of an open set is a **closed** set.

**Definition 2.2.** Let $X$ be a topological space. A **neighborhood** of a point $x \in X$ is an open set containing $x$.

**Definition 2.3.** A topological space $X$ is called Hausdorff if for any pair of points $x, y \in X$ with $x \neq y$ there exist neighborhoods $U$ of $x$ and $V$ of $y$ such that $U \cap V = \emptyset$.

**Definition 2.4.** A non trivial set can have many different topologies. Suppose that we have two topologies $\tau$ and $\tau'$ on a set $X$. If $\tau' \subset \tau$, the we say that $\tau'$ is **weaker** than $\tau$.

**Definition 2.5.** A subfamily $\mathcal{B}$ of $\tau$ is a **base** for $\tau$ if for every $x \in X$ and every open set $U$ containing $x$ there is a set $V \in \mathcal{B}$ such that $x \in V \subset U$.

**Definition 2.6.** A collection $\mathcal{B}_x$ of neighborhoods of a point $x \in X$ is called **local base** at $x$ if every neighborhood of $x$ contains a member of $\mathcal{B}_x$.

**Definition 2.7.** A topological space is **first countable** if there exists a countable local base at each point. A topological space is called **second countable** if it has a countable base.

*Remark* 2.8. If $\mathcal{B}$. is a base for $\tau$, then each $U \in \tau$ is a union of members of $\mathcal{B}$.

**Definition 2.9.** A subfamily $\mathcal{S}$ of $\tau$ is a **subbase** of $\tau$ if the collection of all finite intersections of members of $\mathcal{S}$ is a base for $\tau$.

**Definition 2.10.** If $Y$ is a subset of a topological space $(X, \tau)$, then the **topology induced by** $\tau$ on $Y$ is defined by $Y_\tau = \{Y \cap V \mid V \in \tau\}$.

## 2.2 Closure

**Definition 2.11.** Let $X$ be a topological space. The **interior** of $A \subset X$, denoted by $A^\circ$, is the union of all open subsets of $A$. The **closure** of $A$, denoted $\overline{A}$ is the intersection of all closed supersets of $A$.

Using this definitions we can verify the following results: The interior is the largest open set included in $A$; The closure is the smallest closed set including $A$; If $A \subset B \subset X$, then $A^\circ \subset B^\circ$ and $\overline{A} \subset \overline{B}$. A set $A \subset X$ is open if and only if $A = A^\circ$, and $B$ is closed if and only if $B = \overline{B}$. For any set $A \subset X$, $\overline{(\overline{A})} = \overline{A}$ and $(A^\circ)^\circ = A^\circ$ and $(A^\circ)^c = \overline{(A^c)}$. Also, $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$ and $(A \cap B)^\circ = A^\circ \cap B^\circ$.

**Definition 2.12.** A point $x \in X$ is **limit point** of a set $A \subset X$ if any neighborhood $V$ of $x$ satisfies $A \cap (V - \{x\}) \neq \emptyset$. A point $x \in X$ is **point of closure** of a set $B \subset X$ if for any neighborhood $U$ of $x$ we have $B \cap U \neq \emptyset$.

Given $A \subset X$, the definition of limit point implies the following result: $\overline{A} = A \cup \{\text{limit points of } A\}$.

*Remark* 2.13.

**Definition 2.14.** Given a set $A \subset X$. The **boundary** of $A$ is defined by $\partial A := \overline{A} \cap \overline{A^c}$

This definition implies that if $x \in \partial A$, then every neighborhood $V$ of $x$ satisfy $V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset$. We have the following results: If $A$ is an open subset of $X$, then $\overline{A} = A \cup \partial A$ and $A \cap \partial A = \emptyset$;. A set is closed if and only if it contains all its limit points.

## 2.3 Compactness

**Definition 2.15.** Let $U \subset X$. A collection $\mathcal{O}$ of open subsets of $X$ is said to be an **open cover** of $U$ if the union of the elements of $\mathcal{O}$ contains $U$.

**Definition 2.16.** Let $U \subset X$. Then $U$ is **compact** if only if every open cover of $U$ contains a finite subcover.

Some properties of compact sets are the following: Finite sets are compact. Finite unions of compact sets are compact. Closed subsets of compact sets are compact. Compact subsets of Hausdorff spaces are closed.

**Definition 2.17.** $D \subset X$ is **dense** in $X$ if $\overline{D} = X$.

This definition implies that a set $D$ is dense if and only if every nonempty open set of $X$ contains a point in $D$. This means that any point in $X$ can be approximated by points in $D$.

**Definition 2.18.** A topological space $X$ is **separable** if it has a countable dense subset.

**Theorem 2.19.** *If a topological space is second countable, then it is separable.*

## 2.4 Sequences

**Definition 2.20.** A **sequence** in $X$ is a function from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ into $X$.

**Definition 2.21.** A sequence $\{x_n\}$ in a topological space $X$ converges to a point $x \in X$ it for each neighborhood $V$ of $x$ there is an $N \in \mathbb{N}$ such that $x_m \in V$ for all $m \geq N$. We say that $x$ is the **limit** of the sequence.

We use the short notation for sequences $\{u_k\}$ representing $\{u_k\}_{k=1}^{\infty}$, and for limits $x_k \to x$ as $k \to \infty$ representing $\lim_{k\to\infty} x_k = x$.

**Definition 2.22.** Let $\{u_k\}$ be a sequence in $X$, and $\{k_j\}$ be a strictly increasing sequence of natural numbers. We say that the sequence $\{u_{k_j}\}$ is a **subsequence** of $\{u_k\}$.

**Theorem 2.23.** *A topological space is Hausdorff if and only if every sequence converges to at most one point.*

If a sequence $\{x_n\}$ in a set $A \subset X$ converges to $x$, then $x \in \overline{A}$. However, the converse is not true. There can be points in the closure of a set but no sequences in the set $A$ converging to that point.

**Definition 2.24.** $U \subset X$ is **sequentially compact** if any sequence in $U$ has a convergent subsequence in $U$.

**Definition 2.25.** Let $U, V \subset X$. We say that $V$ is **compactly contained** in $U$ if $\overline{V}$ is compact and $\overline{V} \subset U$, which is denoted by $V \Subset U$.

**Definition 2.26.** We say that a set $U \subset X$ is **precompact** if its closure is compact.

**Definition 2.27.** If $u$ is a real-valued function defined on $U$, then the **support** of $u$ is defined by $\text{supp}(u) = \overline{\{x \in U \; : \; u(x) \neq 0\}}$.

## 2.5 Continuous Functions

**Definition 2.28.** Let $X, Y$ be topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(U)$ is open in $X$ for each open set $U$ in $Y$. We say that $f$ is **continuous at the point** $x$ if $f^{-1}(V)$ is a neighborhood of $x$ whenever $V$ is a neighborhood of $f(x)$.

**Definition 2.29.** If $X$ and $Y$ are Hausdorff spaces and $f : X \to Y$ is a function, then $f$ is **sequentially continuous** at $x$ if $\lim_{n\to\infty} f(x_n) = f(x)$ for every sequence $\{x_n\}$ in $X$ that satisfies $\lim_{n\to\infty} x_n = x$.

**Theorem 2.30.** *(a) If $f : X \to Y$ is continuous at $x$, then $f$ is sequentially continuous at $x$.*
*(b) If $f : X \to Y$ is sequentially continuous at $x$, and if $X$ has a countable local base at $x$ (in particular, if $X$ is metrizable), then $f$ is continuous at $x$.*

*Proof.* (a) Suppose $x_n \to x$ in $X$, $V$ is a neighborhood of $f(x)$ in $Y$, and $U = f^{-1}(V)$. Since $f$ is continuous, $U$ is a neighborhood of $x$, and therefore there exists an $N$ such that $x_n \in U$ for all $n \geq N$. Then $f(x_n) \in V$ for all $n \geq N$. Thus $f(x_n) \to f(x)$ as $n \to \infty$.
(b) Fix $x \in X$, let $\{U_n\}$ be a countable local base for the topology of $X$ at $x$, and assume that $f$ is not continuous at $x$. Then there is a neighborhood $V$ of $f(x)$ in $Y$ such that $f^{-1}(V)$ is not a neighborhood of $x$. Without lost generality we can consider the countable local base $\{U_n\}$ to be nested, so there is a sequence $x_n$, such that $x_n \in U_n$, $x_n \to x$ as $n \to \infty$, and $x_n \notin f^{-1}(V)$. Thus $f(x_n) \notin V$, so that $f$ is not sequentially continuous. □

This theorem shows that sequential continuity is a weaker condition than continuity, but they are equivalent for metric spaces.

**Theorem 2.31.** *Every continuous function between topological spaces maps compact sets to compact sets.*

**Corollary 2.32.** *A continuous real-valued function defined on a compact set achieves its maximum and minimum values.*

For a proof of this important corollary see [18] Theorems 4.14, 4.19.

## 2.6 Lower Semicontinuous Functions

**Definition 2.33.** Let $X$ be a topological space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$. The function $f$ is **lower semicontinuous** at $u$ if

$$\forall c \in \mathbb{R}, \quad \{u \in X \mid f(u) \leq c\} \text{ is closed in } X.$$

If $f$ is lower semicontinuous at every point $u$ of $X$, then we say that $f$ is lower semicontinuous on $X$.

**Lemma 2.34.** $f : X \to \mathbb{R} \cup \{\infty\}$ *is lower semicontinuous if and only if, for all* $c \in \mathbb{R}$ $f^{-1}((c, \infty]) = \{x \in X \mid f(x) > c\}$ *is open in* $X$.

**Definition 2.35.** Let $X$ be a topological space. A function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is **sequentially lower semicontinuous** at $u \in X$ if for any sequence $\{u_n\} \subset X$ converging to $u$

$$f(u) \leq \liminf_{n \to \infty} f(u_n).$$

If $f$ is sequentially lower semicontinuous at every point $u$ of $X$, then we say that $f$ is sequentially lower semicontinuous on $X$.

**Lemma 2.36.** *Let $X$ be a first countable topological space. An extended real valued function* $f : X \to \mathbb{R} \cup \{\infty\}$ *is lower semicontinuous if and only if it is sequentially lower semicontinuous.*

This theorem shows in particular that for metric spaces lower semicontinuity is equivalent to sequentially lower semicontinuity. We will often deal with weak topologies of normed spaces. These spaces are neither first countable nor metrizable. However, it will be enough for our purposes to use only the sequential characterization of lower semicontinuity.

*Remark* 2.37. Every extended real-valued continuous function $f : X \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous.

**Lemma 2.38.** *Let $X$ be a topological space, and let $f : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous, and $c > 0$ . Then $cf$ is also lower semicontinuous.*

*Proof.* Let $x_k \to x$ be a convergent sequence in $X$. Then $\liminf_{k\to\infty} f(x_k) \geq f(x)$. Multiply this inequality by $c$ and use the fact that $c \liminf_{k\to\infty} f(x_k) = \liminf_{k\to\infty}[cf(x_k)]$ for each $k$. We conclude that $\liminf_{k\to\infty}[cf(x_k)] \geq cf(x)$. □

**Lemma 2.39.** *Let $X$ be a topological space, and let $f, g : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous. Then $\max\{f, g\}$, $\min\{f, g\}$, and $f+g$ are also lower semicontinuous.*

*Proof.* Let $x_k \to x$ be a convergent sequence in $X$. Since $\max\{f, g\} \geq f, g$ and by monotonicity of $\liminf$ we have

$$\begin{aligned} \liminf_{k\to\infty}[\max\{f, g\}(x_k)] &\geq \liminf_{k\to\infty} f(x) \text{ and} \\ \liminf_{k\to\infty}[\max\{f, g\}(x_k)] &\geq \liminf_{k\to\infty} g(x). \end{aligned}$$

Then $\liminf_{k\to\infty}[\max\{f, g\}(x_k)] \geq \liminf_{k\to\infty}[\max\{f + g\}(x)]$.

On the other hand

$$\begin{aligned} \liminf_{k\to\infty}[(f+g)(x_k)] &= \liminf_{k\to\infty}[f(x_k) + g(x_k)] \\ &\geq \liminf_{k\to\infty} f(x_k) + \liminf_{k\to\infty} g(x_k) \text{ (superadditivity of} \liminf) \\ &\geq f(x) + g(x) = (f+g)(x). \end{aligned}$$

□

**Proposition 2.40.** *Let $U$ be a open subset of a topological space $X$. Then the characteristic function $\chi_U : X \to \mathbb{R}$ is lower semicontinuous.*

**Lemma 2.41.** *A lower semicontinuous function on a compact set $K$ assumes its infimum there.*

# Chapter 3: Functional Analysis

## 3.1 Metric Spaces

**Definition 3.1.** A **metric space** is a set $M$ equipped with a distance function $d : M \times M \to [0, \infty)$ which satisfies the following properties:

$$\begin{array}{l} (M_1)\ d(x,y) = 0 \ \text{ if and only if } \ x = y, \\ (M_2)\ \forall x, y \in M \ \ d(x,y) = d(y,x), \\ (M_3)\ \forall x, y, z \in M \ \ d(x,z) \leq d(x,y) + d(y,z). \end{array} \tag{3.1}$$

The inequality in $(M_3)$ is the triangle inequality.

*Remark* 3.2. We will use the following notation and definitions: $\mathbb{R}^n$ is the real n-dimensional **Euclidian space**. A nonempty, bounded, connected open subset $U$ of $\mathbb{R}^n$ is called a **domain**. A typical point in $\mathbb{R}^n$ is $x = (x_1, x_2, \ldots, x_n)$. The i-th standard coordinate vector is $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

The Euclidean space $\mathbb{R}^n$ is a metric space with a distance defined by $d(x,y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}$ where $x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n)$.

*Remark* 3.3. The function space $C(K) = \{f : K \to \mathbb{R} \mid f \text{ is continuous on } K\}$, where $K$ is a compact subset of $\mathbb{R}^n$, is a metric space with a distance is defined by $d(f,g) := \max_{x \in K} |f(x) - g(x)|$.

**Proposition 3.4.** *If $x, y, z \in M$, then $|d(x,z) - d(y,z)| \leq d(x,y)$.*

**Definition 3.5.** Let $M$ be a metric space and $x \in M$. The **open ball** in $M$ with center $x$ and radius $\epsilon > 0$ is defined by $B(x, \epsilon) := \{z \in X \ : \ d(z,x) < \epsilon\}$.

**Definition 3.6.** The topology generated by the collection of all open balls of a metric space $X$ is called the **metric topology** of $X$.

*Remark* 3.7. Given a subset $E$ of a metric space $X$, $E$ has empty interior in $X$ if and only if its complement, $X \setminus E$, is dense in $X$.

*Remark* 3.8. If $E$ is an open subset of a metric space $X$, for each point $x \in E$, there exist an open ball centered at $x$ whose closure is contained in $E$.

**Definition 3.9.** Let $X$ be a metric space. A sequence $\{u_k\}_{k=1}^{\infty} \subset X$ **converges** to $u \in X$ if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $k \geq N \ \rightarrow \ d(u_k, u) < \epsilon$. We write $u_k \rightarrow u$ meaning $\lim_{k\to\infty} u_k = u$ which is equivalent to $\lim_{k\to\infty} d(u_k, u) = 0$.

**Definition 3.10.** Let $X$ be a metric space. A sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is called a **Cauchy** sequence if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $k, l \geq N \ \rightarrow \ d(u_k, u_l) < \epsilon$.

**Definition 3.11.** We say that a metric space $X$ is **complete** if every Cauchy sequence in $X$ converges to a point in $x$.

**Proposition 3.12.** *Let $E$ a subset of a complete metric space $X$. Then the metric space $E$ is complete if and only if $E$ is a closed subspace of $X$.*

**Definition 3.13.** Let $E$ be a nonempty subset of a metric space $(X, d)$. We define the diameter of $E$ by

$$\operatorname{diam} E = \sup\{d(x, y) \mid x, y \in E\}$$

**Definition 3.14.** Let $X$ be a metric space. We say that $E \subset X$ is **bounded** if it has a finite diameter.

**Proposition 3.15.** *For metric spaces, compactness and sequential compactness are equivalent.*

For a proof, see [17] Theorem 16, Chapter 9.

**Theorem 3.16** (Heine-Borel)**.** *$U \subset \mathbb{R}^n$ is compact if only if it is closed and bounded.*

**Theorem 3.17** (Bolzano-Weirstrass)**.** *Every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

*Remark* 3.18. A compact metric space is separable, and every subspace of a separable metric space is separable.

**Definition 3.19.** We say that a countable collection of sets $\{E_n\}_{n=1}^{\infty}$ is **nested** if $E_{n+1} \subset E_n$ for all $n \in \mathbb{N}$.

**Definition 3.20.** A nested sequence $\{E_n\}_{n=1}^{\infty}$ of nonempty subsets of $X$ is called a **contracting** sequence if

$$\lim_{n\to\infty} \operatorname{diam}(E_n) = 0.$$

**Theorem 3.21** (The Cantor Intersection Theorem)**.** *Let $X$ be a metric space. Then $X$ is complete if and only if whenever $\{\mathcal{C}_n\}_{n=1}^{\infty}$ is a contracting sequence of nonempty closed subsets of $X$, there is a point $x \in X$ for which $\bigcap_{n=1}^{\infty} \mathcal{C}_n = \{x\}$.*

**Definition 3.22.** A subset $E$ of a metric space $X$ is called **nowhere dense** if its closure $\overline{E}$ has empty interior.

**Definition 3.23.** A subset $E$ of a metric space $X$ is said to be of the **first category** if $E$ is the union of a countable collection of nowhere dense subsets of $X$. A set that is not of the first category is said to be of the **second category**.

**Theorem 3.24** (Baire Category Theorem)**.** *Let $X$ be a complete metric space. Let $\{\mathcal{O}_n\}_{n=1}^{\infty}$ be a countable collection of open dense subsets of $X$. Then the intersection $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is also dense.*

**Corollary 3.25.** *Let $X$ be a complete metric space. Let $\{\mathcal{C}_n\}_{n=1}^{\infty}$ be a countable collection of closed with empty interior subsets of $X$. Then the union $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ also has empty interior.*

**Corollary 3.26.** *Let $X$ be a complete metric space and $\{\mathcal{C}_n\}_{n=1}^{\infty}$ be a countable collection of closed subsets of $X$. If $X = \bigcup_{n=1}^{\infty} \mathcal{C}_n$, then at least one of the $\mathcal{C}_n$'s has nonempty interior.*

The Baire Category Theorem also implies the following results:

*Remark* 3.27. An open subset of a complete metric space is of the second category.

*Remark* 3.28. In a complete metric space, the union of a countable collection of nowhere dense sets has empty interior.

**Theorem 3.29.** *Let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space $X$ that is pointwise bounded in the sense that for each $x \in X$, there is a constant $M_x$ for which*

$$|f(x)| \leq M_x \text{ for all } f \in \mathcal{F}.$$

*Then there is a nonempty open subset $\mathcal{O}$ of $X$ on which $\mathcal{F}$ is uniformly bounded in the sense that there is a constant $M$ for which*

$$|f| \leq M \text{ on } \mathcal{O} \text{ for all } f \in \mathcal{F}. \tag{3.2}$$

*Proof.* Define the collection of sets $E_n = \{x \in X \mid |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}$ where $n = 1, 2, \ldots$. $E_n$ is closed because each function $f \in \mathcal{F}$ is continuous, $E_n = \bigcap_{f \in \mathcal{F}} f^{-1}((-\infty, n])$ and because the intersection of any collection of closed sets is closed. Since $\mathcal{F}$ is pointwise bounded, for each $x \in X$ there is an index $n$ such that $|f(x)| \leq n$ for all $f \in \mathcal{F}$. This means that $x \in E_n$. Thus $X = \bigcup_{n=1}^{\infty} E_n$. Since $X$ is a complete metric space we can apply the previous corollary to conclude that there exists at least one $n$ for which $E_n$ contains an open ball $B(x, r)$. Therefore (3.2) holds for $\mathcal{O} = B(x, r)$ and $M = n$. □

## 3.2 Normed Linear Spaces

**Definition 3.30.** A real **linear space** $X$ is an Abelian group under addition $(+)$, together with a scalar multiplication from $\mathbb{R} \times X$ into $X$ such that for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$,

$\alpha(x+y) = \alpha x + \alpha y,\ (\alpha\beta)x = \alpha(\beta x),\ (\alpha+\beta)x = \alpha x + \beta x$, and $1x = x$.

**Definition 3.31.** A real linear space $X$ is called a **normed linear space** if there exists a mapping $\|\cdot\| : X \to [0, \infty)$ such that:

$$\begin{array}{l} (N_1)\ \|x\| = 0 \text{ if and only if } x = 0, \\ (N_2)\ \forall x \in X, \forall \alpha \in \mathbb{R}\ \|\alpha x\| = |\alpha|\, \|x\|, \\ (N_3)\ \forall x, y \in X\ \|x+y\| \leq \|x\| + \|y\|. \end{array} \tag{3.3}$$

Inequality $(N_3)$ is the triangle inequality.

*Remark* 3.32. Every normed space $M$ is a metric space, in which the distance function $d : M \times M \to \mathbb{R}$ satisfies $d(x, y) = \|x - y\|$.

**Definition 3.33.** A **Banach space** is a normed linear space that is complete with respect to the metric induced by the norm.

*Remark* 3.34. The Euclidian space $\mathbb{R}^n$ is a Banach space.

### 3.2.1 Bounded Linear Operators

Let $X$ and $Y$ be real Banach spaces.

**Definition 3.35.** A map $A : X \to Y$ is a **linear operator** if $\forall u, v \in X,\ \forall \lambda, \mu \in \mathbb{R}$ we have

$$A[\lambda u + \mu v] = \lambda A u + \mu A v.$$

**Definition 3.36.** A linear operator $A : X \to Y$ is **bounded** if

$$\|A\| := \sup\{\|Au\|_Y \mid \|u\|_X \leq 1\} < \infty.$$

This quantity is called the **operator norm** of $A$.

*Remark* 3.37. Every bounded linear operator is continuous and every continuous linear operator is bounded.

**Proposition 3.38.** *Let $X$ and $Y$ be normed linear spaces. Then the collection of all bounded linear operators from $X$ to $Y$, $\mathcal{L}(X,Y)$, is a normed linear space.*

**Proposition 3.39.** *Let $X$ and $Y$ be normed linear spaces. If $Y$ is a Banach space, then so is $\mathcal{L}(X,Y)$.*

### 3.2.2 The Hahn-Banach Theorem

**Definition 3.40.** A **functional** on a linear space $X$ is a real-valued function $T : X \to \mathbb{R}$. A **linear functional** is a functional $T$ such that for all $x, y \in X$, and for all $\alpha, \beta \in \mathbb{R}$ we have

$$T(\alpha \cdot x + \beta \cdot y) = \alpha \cdot T(x) + \beta \cdot T(y).$$

**Definition 3.41.** A functional $p : X \to [0, \infty)$ on a linear space is said to be **positively homogeneous** if

$$p(\lambda x) = \lambda p(x) \text{ for all } x \in X,\ \lambda > 0.$$

and it is **subadditive** if

$$p(x + y) \leq p(x) + p(y) \text{ for all } x, y \in X.$$

*Remark* 3.42. Any norm on a linear space is both subadditive and positively homogeneous.

**Theorem 3.43** (The Hahn-Banach Theorem)**.** *Let $p$ be a positively homogeneous, subadditive functional on a linear space $X$ and let $Y$ be a subspace of $X$ on which there is defined a linear functional $\psi$ for which*

$$\psi \leq p \text{ on } Y.$$

*Then $\psi$ may be extended to a linear functional on all of $X$ for which $\psi \leq p$ on $X$.*

Let $X$ be a normed linear space. Recall that a linear functional $T$ on $X$ is bounded if there is an $M \geq 0$ such that for all $x \in X$ we have

$$|T(x)| \leq M \cdot \|x\|.$$

The infimum of all such $M$ is the norm of $T$ and it is denoted by $\|T\|_*$.
We have the following results

$$\forall x, y \in X, \ |T(x) - T(y)| \leq \|T\|_* \|x - y\|.$$

and

$$\|T\|_* = \sup\{T(x) \mid x \in X, \|x\| \leq 1\}.$$

**Proposition 3.44.** *Let $X$ be a normed linear space. Then the collection of all bounded linear functionals on $X$ is a linear space on which $\|\cdot\|_*$ is a norm. This normed space is called the **dual space** of $X$, and this space is denoted by $X^*$.*

**Theorem 3.45.** *Let $X_0$ be a linear subspace of a normed linear space $X$. Then each bounded linear functional $\psi$ on $X_0$ has an extension to a bounded linear functional on all of $X$ that has the same norm as $\psi$. In particular, for each $x \in X$, there is a $\psi \in X^*$ for which*

$$\psi(x) = \|x\| \text{ and } \|\psi\| = 1.$$

**Theorem 3.46** (The Uniform Boundedness Theorem)**.** *Let $X$ be a Banach Space and $Y$ a normed linear space, and let $\mathcal{F}$ be a subset of $\mathcal{L}(X, Y)$. Suppose the family $\mathcal{F}$ is pointwise bounded in the sense that for each $x$ in $X$ there is a constant $M_x$ for which*

$$\|T(x)\| \leq M_x \text{ for all } T \in \mathcal{F}.$$

*Then the family $\mathcal{F}$ is uniformly bounded in the sense that there is a constant $M \geq 0$ for which $\|T\| \leq M$ for all $T$ in $\mathcal{F}$.*

*Proof.* For each $T \in \mathcal{F}$, the real-valued function $f_T : X \to \mathbb{R}$ defined by $f_T(x) = \|Tx\|$ is a real-valued continuous function on $X$. Since this family of continuous functions is pointwise bounded on $X$ and the metric space $X$ is complete, by Theorem (3.29), there is an open ball $B(x_0, r)$ in $X$ and a constant $C \geq 0$ for which

$$\|T(x)\| \leq C \text{ for all } x \in B(x_0, r) \text{ and } T \in \mathcal{F}.$$

Thus, for each $T \in \mathcal{F}$,

$$\|T(x)\| = \|T((x + x_0) - x_0)\| \leq \|T(x + x_0)\| + \|T(x_0)\| \leq C + M_{x_0} \text{ for all } x \in B(0, r).$$

Therefore, setting $M = \frac{C+M_{x_0}}{r}$, and using the linearity of $T$ we have $\|T\| \leq M$ for all $T$ in $\mathcal{F}$. □

**Theorem 3.47** (Banach–Steinhaus Theorem)**.** *Let $X$ be a Banach space, $Y$ a normed linear space, and $T_n : X \to Y$ a sequence of continuous linear operators. Suppose that, for each $x \in X$, $\lim_{n\to\infty} T_n(x)$ exists in $Y$.*
*(i) Then the sequence of operators $T_n$ is uniformly bounded.*
*(ii) The operator $T : X \to Y$ defined by*

$$T(x) = \lim_{n\to\infty} T_n(x) \textit{ for all } x \in X.$$

*is linear, continuous, and*

$$\|T\| \leq \liminf \|T_n\|.$$

*Proof.* $T$ is linear because the pointwise limit of linear operators is linear. By the Uniform Boundedness Principle we observe that the sequence $\{T_n\}$ is uniformly bounded. Thus $\liminf \|T_n\|$ is finite. By continuity of the norm we have $\lim_{n\to\infty} \|T_n(x)\| = \|T(x)\|$. Since $\|T_n(x)\| \leq \|T_n\| \cdot \|x\|$ for all $n$, we have $\|T(x)\| \leq \liminf \|T_n\| \cdot \|x\|$. Therefore $T$ is bounded and $\|T\| \leq \liminf \|T_n\|$. □

**Definition 3.48.** Let $Y$ be a Banach space, and let $X$ be a nonempty subset of $Y$.

(a) $X$ is **closed** in $Y$ if $\{x_k\}_{k=1}^{\infty} \subset X$ and $\lim_{k\to\infty} x_k = y$ in $Y$ implies that $y \in X$.

(b) $X$ is **bounded** if $\sup_{x\in X} \|x\| < \infty$.

(c) $X$ is **precompact** in $Y$ if from every sequence $\{x_k\}_{k=1}^{\infty} \subset X$ it is possible to select a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ which converges in $Y$.

(d) $X$ is **compact** if $X$ is precompact and closed.

**Definition 3.49.** We say that the Banach space $X$ is **embedded** in the Banach space $Y$ if

(a) $X$ is a vector subspace of $Y$, and

(b) There exists a constant $C > 0$ such that

$$\|u\|_Y \leq C\|u\|_X \quad \forall u \in X.$$

**Definition 3.50.** Let $X$ and $Y$ be Banach spaces, $X \subset Y$. We say that $X$ is **compactly embedded** in $Y$, written $X \subset\subset Y$ if

(a) $X$ is embedded in $Y$, and

(b) each bounded sequence in $X$ is precompact in $Y$.

**Definition 3.51.** Given $u \in X$, $u^* \in X^*$ we write $\langle u^*, u\rangle := u^*(u)$. The symbol $\langle\,,\rangle$ represents the **dual pairing** of $X^*$ and $X$. We define $\|u^*\|_* := \sup\{|\langle u^*, u\rangle| \mid \|u\| \leq 1\}$, which is the same as the operator norm defined previously for general linear operators.

# Chapter 4: Weak Topology

## 4.1 Weak Topology

If $\mathcal{F}$ is a collection of real-valued functions on a set $X$, the $\mathcal{F}$-weak topology on $X$ is the weakest topology on $X$ for which each function in $\mathcal{F}$ is continuous. We observe that a sequence $\{x_n\}$ in $X$ converges to $x \in X$ with respect to the $\mathcal{F}$-weak topology if and only if

$$\lim_{n\to\infty} f(x_n) = f(x) \text{ for all } f \in \mathcal{F}. \tag{4.1}$$

**Definition 4.1.** Let $X$ be a normed linear space. The weak topology induced on $X$ by the dual space $X^*$ is called the **weak topology** on $X$.

For a normed linear space we frequently call the topology induced by the norm the **strong topology**.

If $X$ is a normed linear space and $u \in X$ we define the linear functional $J(u) : X^* \to \mathbb{R}$ by

$$J(u)[u^*] = u^*(u) \text{ for all } u^* \in X^*.$$

**Definition 4.2.** Let $X$ be a normed linear space. The linear operator $J : X \to (X^*)^*$ defined by

$$J(u)[u^*] = u^*(u) \text{ for all } u \in X,\ u^* \in X^*$$

is called the **natural embedding** of $X$ into $(X^*)^*$. The space $X$ is said to be **reflexive** if $J(X) = (X^*)^*$.

We can check that the evaluation functional $J(u)$ is linear and is bounded on $X^*$ with $\|J(u)\| \leq \|u\|$. Also, the operator $J : X \to (X^*)^*$ is injective and linear and therefore $J(X)$ is a linear subspace of $(X^*)^*$.

**Theorem 4.3.** *Let $X$ be a normed linear space. Then the natural embedding $J : X \to X^{**}$ is an isometry.*

*Proof.* By definition of the norm on the dual space, for all $u \in X$ we have

$$|\psi(u)| \leq \|\psi\| \cdot \|u\| \text{ for all } \psi \in X^*.$$

Thus

$$|J(u)(\psi)| \leq \|u\| \cdot \|\psi\| \text{ for all } \psi \in X^*.$$

Therefore $J(u)$ is bounded and $\|J(u)\| \leq \|u\|$. On the other hand, according to Theorem (3.45), there is a $\psi \in X^*$ for which $\|\psi\| = 1$ and $J(u)(\psi) = \|u\|$. Therefore $\|u\| \leq \|J(u)\|$. We conclude that $\|J(u)\| = \|u\|$, i.e. $J$ is an isometry. □

**Definition 4.4.** Let $X$ be a normed linear space. The weak topology induced on $X^*$ induced by $J(X) \subset (X^*)^*$ is called the **weak-* topology** on $X^*$.

For a normed linear space $X$ we have the following inclusions:
(i) weak topology on $X \subset$ strong topology on $X$,
(ii) weak-* topology on $X^* \subset$ weak topology on $X^* \subset$ strong topology on $X^*$.

## 4.2 Weak Convergence

Let $X$ be a Banach space.

**Definition 4.5.** We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ converges weakly to $u \in X$, written $u_k \rightharpoonup u$, if $\langle u^*, u_k \rangle \to \langle u^*, u \rangle$ for each bounded linear functional $u^* \in X^*$.

**Proposition 4.6.** *If $u_k \to u$, then $u_k \rightharpoonup u$.*

*Proof.* Suppose that $u_k \to u$. Given any $u^* \in X^*$ and using the fact that $u^*$ is continuous we have $u^*(u_k) \to u^*(u)$. This means that for any $u^* \in X^*$ $\langle u^*, u_k \rangle \to \langle u^*, u \rangle$. Thus by definition of weak convergence $u_k \rightharpoonup u$. □

**Theorem 4.7.** *Let $X$ be a normed linear space. Then every weakly convergent sequence in $X$ is bounded in norm. And, if $u_n \rightharpoonup u$, then $\|u\| \leq \liminf_{k\to\infty} \|u_n\|$.*

*Proof.* Let $\{u_n\} \rightharpoonup u$ in $X$. Then $J(u_n) : X^* \to \mathbb{R}$ is a sequence of functionals that converges pointwise to $J(u) : X^* \to \mathbb{R}$ because each of the functionals $J(u_n)$ is continuous with respect to the weak topology generated by $X^*$. The Uniform Boundedness Theorem implies that $\{J(u_n)\}$ is a bounded sequence of linear functionals on $X^*$. Then since the natural embedding $J$ is an isometry, the sequence $\{u_n\}$ is bounded. Now we show the second result. From Theorem 3.45 there exists a functional $\psi \in X^*$ for which $\|\psi\| = 1$ and $\psi(u) = \|u\|$. Then

$$|\psi(u_n)| \leq \|\psi\| \cdot \|u_n\| = \|u_n\| \text{ for all } n.$$

Moreover, because $u_n \rightharpoonup u$, $|\{\psi(u_n)\}|$ converges to $|\psi(u)| = \|u\|$. Therefore using the fact that $\psi$ is continuous we have

$$\|u\| = \lim_{n\to\infty} |\psi(u_n)| \leq \liminf \|u_n\|.$$

□

## 4.3 Hilbert Spaces

Let $H$ be a real linear space.

**Definition 4.8.** A mapping $(\ ,\ ) : H \times H \to \mathbb{R}$ is called an **inner product** if

(1) $(u, u) \geq 0\ \forall u \in H$,
(2) $(u, u) = 0$ if and only if $u = 0$,
(3) $(u, v) = (v, u)\ \forall u, v \in H$,
(4) $(u, \alpha v + \beta w) = \alpha(u, v) + \beta(u, w)\ \ \forall u, v, w \in H,\ \alpha, \beta \in \mathbb{R}$.

A linear space $H$ together with an inner product on $H$ is called an **inner product space**.

**Notation 4.9.** If ( , ) is an inner product, the associated norm is

$$\|u\| := (u, u)^{1/2}.$$

The Cauchy-Schwarz inequality states

$$|(u, v)| \leq \|u\| \, \|v\| \; \forall u, v \in H.$$

**Definition 4.10.** A **Hilbert space** is an inner product space which is a Banach space with respect to the associated norm.

**Definition 4.11.** The elements $u, v \in H$ are **orthogonal** if $(u, v) = 0$. A countable collection $\{w_k\}_{k=1}^{\infty} \subset H$ is called **orthonormal** if

$$(w_k, w_l) = 0 \; \forall k \neq l, \text{ and } \|w_k\| = 1 \; \forall k.$$

**Definition 4.12.** Given an orthonormal collection $\{w_k\}_{k=1}^{\infty} \subset H$, we say that this collection is an orthonormal **basis** if for any $u \in H$ we can write

$$u = \sum_{k=1}^{\infty} (u, w_k) w_k$$

with the series converging in $H$; that is, every $u \in H$ can be represented in the inner product norm by its orthogonal expansion with respect to $\{w_k\}_{k=1}^{\infty}$.

*Remark* 4.13. If $\{w_k\}_{k=1}^{\infty}$ is an orthonormal basis, then $\forall u \in H, \; \|u\|^2 = \sum_{k=1}^{\infty} (u, w_k)^2$. This is called **Parseval's identity**.

## 4.4 Weak Compactness Theorem

**Theorem 4.14** (The Tychonoff Product Theorem)**.** *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a collection of compact spaces indexed by a set $\Lambda$. Then the Cartesian product $\Pi_{\lambda \in \Lambda} X_\lambda$, with the product topology, also is compact.*

For a proof of this theorem see p. 245 in [17].

**Theorem 4.15** (The Banach-Alaoglu Theorem). *If $X$ is a normed linear space and $B^* = \{f \in X^* \mid \|f\| \leq 1\}$ is the closed unit ball in $X^*$, then $B^*$ is a compact space in the weak* topology.*

**Theorem 4.16** (Weak Compactness Theorem). *Let $X$ be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is bounded. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that $u_{k_j} \rightharpoonup u$.*

**Theorem 4.17** (Mazur's Theorem). *A convex, closed subset of a Banach space $X$ is weakly closed.*

# Chapter 5: Function Spaces

### 5.0.1 Real-valued Functions

In this section we are going to introduce the framework of some important function spaces. We consider real-valued functions defined over a specific domain $U$ of the Euclidean space $\mathbb{R}^n$. For our purposes we always assume that the target space $\mathbb{R}$ has the usual 1-dimensional Euclidean metric topology, in particular this is a Banach space. Also for the domain space $\mathbb{R}^n$ we always consider the usual $n$-dimensional Euclidean metric topology, which makes $\mathbb{R}^n$ into a Banach space.

Let $U \subset \mathbb{R}^n$, consider a function $u : U \to \mathbb{R}$, and let $x_0 \in U$. From our definition of continuity in topological spaces we find that $u$ is continuous at $x_0$ if $\forall \epsilon > 0 \; \exists \delta > 0$ such that $x \in U$ and $|x - x_0| < \delta \; \to \; |u(x) - u(x_0)| < \epsilon$, and $u$ is continuous on $U$ if it is continuous at every point of $U$.

**Definition 5.1.** Let $x \in U \subset \mathbb{R}^n$, and consider a function $u : U \to \mathbb{R}$. The **partial derivative** of $u$ at x in the direction $e_i$ is defined by $\frac{\partial u}{\partial x_i} = \lim_{t \to 0} \frac{u(x+te_i) - u(x)}{t}$ provided that the limit exists.

**Notation 5.2.** $u_{x_i} = \frac{\partial u}{\partial x_i}$, $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $u_{x_i x_j x_k} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}$, etc.

**Notation 5.3.** A vector of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where each $\alpha_i$ is a nonnegative integer, is called a multiindex of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.
Given a multiindex $\alpha$, define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u$$

If $k$ is a nonnegative integer,

$$D^k u(x) := \{D^\alpha u(x) \mid |\alpha| = k\},$$

is the set of all partial derivatives of $u$ at $x$ of order $k$. Assigning some ordering to the various partial derivatives, we can consider $D^k u(x)$ as a point in $\mathbf{R}^{n^k}$. Note that

$$|D^k u| = \left( \sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{1/2}.$$

Special Cases:

If $k = 1$, we can arrange the elements of $Du$ into a vector:

$D^1 u \equiv Du := (u_{x_1}, \ldots, u_{x_n})$=gradient vector.

If $k = 2$, we arrange the elements of $D^2 u$ in a matrix:

$$D^2 u = \begin{pmatrix} u_{x_1 x_1} & \cdots & u_{x_1 x_n} \\ & \ddots & \\ u_{x_n x_1} & \cdots & u_{x_n x_n} \end{pmatrix}.$$

$D^2 u$ is called the Hessian matrix.

The Laplacian of $u$ is $\triangle u = \sum_{i=1}^n u_{x_i x_i} = tr(D^2 u)$.

### 5.0.2 Sequences of Real-Valued Functions

Recall that a sequence $\{x_k\}_{k=1}^\infty \subset \mathbb{R}^n$ converges to $x \in \mathbb{R}^n$ if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $k \geq N \ \rightarrow\ |x_k - x| < \epsilon$. We write $x_k \to x$ meaning $\lim_{k\to\infty} x_k = x$.

**Definition 5.4.** A **sequence of functions** on $U$ is a sequence $\{u_n\}$ in the space of functions from $U$ to $\mathbb{R}$.

**Definition 5.5.** Let $U \subset \mathbb{R}^n$, and $\{u_n\}$ be a sequence of continuous functions from $U$ to $\mathbb{R}$. We say that $\{u_n\}$ is **convergent pointwise** on $U$ if there exists a function $u : U \to \mathbb{R}$ such that $\forall x \in U\ \forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N \ \rightarrow\ |u_n(x) - u(x)| < \epsilon$.

**Definition 5.6.** Let $U \subset \mathbb{R}^n$, and let $\{u_n\}$ be a sequence of continuous functions from $U$ to $\mathbb{R}$. We say that the sequence $\{u_n\}$ is **pointwise bounded** if for each $x \in U$, the sequence of real numbers $\{u_n(x)\}$ is bounded.

**Definition 5.7.** Let $U \subset \mathbb{R}^n$ be compact, and let $\{u_n\}$ be a sequence of continuous functions from $U$ to $\mathbb{R}$. We say that the sequence $\{u_n\}$ is **uniformly bounded** on $U$ if $\exists M > 0$ such that, $\forall n \in \mathbb{N},\ \max_{x \in U} |u_n(x)| < M$.

**Definition 5.8.** Let $U \subset \mathbb{R}^n$, and let $\{u_n\}$ be a sequence of continuous functions from $U$ to $\mathbb{R}$. We say that the sequence $\{u_n\}$ is **equicontinuous** on $U$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in U\ \forall n \in \mathbb{N},\ \ |x - y| < \delta\ \rightarrow\ |u_n(x) - u_n(y)| < \epsilon$.

**Definition 5.9.** Let $U \subset \mathbb{R}^n$, and let $\{u_n\}$ be a sequence of continuous functions from $U$ to $\mathbb{R}$. We say that $\{u_n\}$ is a **uniformly convergent** sequence if there exists a function $u : U \rightarrow \mathbb{R}$ such that
$\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall x \in U\ \forall n > N\ \rightarrow\ |u_n(x) - u(x)| < \epsilon$.

**Definition 5.10.** Let $U \subset \mathbb{R}^n$, and let $\{u_n\}$ be a sequence of continuous functions from $U$ to $\mathbb{R}$. We say that $\{u_n\}$ is a **uniformly Cauchy** sequence if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall x \in U\ \forall m, n > N\ \rightarrow\ |u_m(x) - u_n(x)| < \epsilon$.

*Remark* 5.11. If $U \subset \mathbb{R}^n$ is compact, and $\{u_n\}$ is a uniformly Cauchy sequence of continuous functions, then $\{u_n\}$ is a uniformly convergent sequence of continuous functions and the limit function is also continuous. See theorems 7.8, 7.12 in [18].

## 5.1 Arzela-Ascoli Theorem

The following theorem will be very important in the theory of Sobolev spaces, which is discussed in the next chapter.

**Theorem 5.12.** *Every uniformly bounded, equicontinuous sequence of real-valued functions on a compact subset of* $\mathbb{R}^n$ *has a uniformly convergent subsequence.*

*Proof.* Let $A$ be a compact domain in $\mathbb{R}^n$, and let $(f_n)$ be a uniformly bounded, equicontinuous sequence of functions from $A$ to $\mathbb{R}$.

Let $B$ be a countable dense subset of $A$, which exists because $\mathbb{R}^n$ is separable, and let $(b_n)$ be an enumeration of $B$.

Since $f_n$ is uniformly bounded on $A$, the sequence of real numbers $(f_n(x))$ is bounded for any $x \in A$. So the sequence of real numbers $(f_1(b_1), f_2(b_1), f_3(b_1), \ldots)$ is bounded.

By the Bolzano-Weirstrass Theorem [18] this sequence has a convergent subsequence

$$(f_{1,1}(b_1), f_{1,2}(b_1), f_{1,3}(b_1), \ldots)$$

By hypothesis the subsequence $(f_{1,1}(x), f_{1,2}(x), f_{1,3}(x), \ldots)$ of $(f_n(x))$ is bounded for any $x \in A$, so $(f_{1,1}(b_2), f_{1,2}(b_2), f_{1,3}(b_2), \ldots)$ is bounded.

By the Bolzano-Weirstrass Theorem this last subsequence has a convergent subsequence

$$(f_{2,1}(b_2), f_{2,2}(b_2), f_{2,3}(b_2), \ldots)$$

By hypothesis the subsequence $(f_{2,1}(x), f_{2,2}(x), f_{2,3}(x), \ldots)$ of $(f_{1,m}(x))$ is bounded for any $x \in A$, so $(f_{2,1}(b_3), f_{2,2}(b_3), f_{2,3}(b_3), \ldots)$ is bounded.

Again by the Bolzano-Weirstrass Theorem this last subsequence has a convergent subsequence

$$(f_{3,1}(b_3), f_{3,2}(b_3), f_{3,3}(b_3), \ldots)$$

Repeating the procedure explained above we generate the following collection of subsequences of $\sigma_0(x) := (f_n(x))$

$$\begin{aligned} \sigma_1(x) &= (f_{1,1}(x), f_{1,2}(x), f_{1,3}(x), \ldots) \\ \sigma_2(x) &= (f_{2,1}(x), f_{2,2}(x), f_{2,3}(x), \ldots) \\ \sigma_3(x) &= (f_{3,1}(x), f_{3,2}(x), f_{3,3}(x), \ldots) \\ \vdots &= \vdots \\ \sigma_n(x) &= (f_{n,1}(x), f_{n,2}(x), f_{n,3}(x), \ldots) \\ \vdots &= \ddots \end{aligned} \tag{5.1}$$

This collection of subsequences has the following properties:

1. $\sigma_n$ is a subsequence of $\sigma_{n-1}$, $\forall n \in \mathbb{N}$.

2. $\sigma_n(b_n) = (f_{n,m}(b_n))$ converges for $n$ fixed and $m \to \infty$.

3. The order of the functions on each sequence is preserved. This means that if $j < k$ and both $f_{n,j}$ and $f_{n,k}$ appear in $\sigma_n$, then these functions appear in $\sigma_{n-1}$ with $f_{n,j} = f_{n-1,i}$, $f_{n,k} = f_{n-1,l}$, and $i < l$.

Now we select the diagonal subsequence $\sigma$ defined by

$$\sigma(x) := (f_{1,1}(x), f_{2,2}(x), f_{3,3}(x), \ldots)$$

Property 3 implies that if we move upward from the $n$-th row to the $n-1$-th row in the collection of functions in (5.1), then the functions can only move to the right and never to the left.

Properties 1 and 3 imply that $(f_{n,n}(x), f_{n+1,n+1}(x), f_{n+2,n+2}(x), \ldots)$ is a subsequence of $\sigma_n(x)$ for each $n \in \mathbb{N}$ because the elements of the diagonal subsequence come from functions in $\sigma_n(x)$ after the first $n-1$ functions. Specifically, by property 3, for each $k \geq n$, $f_{k,k}(x) = f_{n,m}(x)$ for some $m \geq n$. Using mathematical induction we can conclude that $\sigma_{m \geq n}$ is a subsequence of $(f_n)$.

By Property 2 we know that $\sigma_1(x)$ is convergent on $b_1$. By properties 1 and 2 $\sigma_2(x)$ is convergent on the points $\{b_1, b_2\}$. By properties 1 and 2 $\sigma_3(x)$ is convergent on the points $\{b_1, b_2, b_3\}$. Then by mathematical induction we can show that $\sigma_n(x)$ is convergent on $\{b_m\}$ for all $m \leq n$.

Therefore, properties 1, 2, and 3 together imply that

$$\sigma \quad \text{is a subsequence of} \quad (f_n) \quad \text{convergent on} \quad B \subset A. \tag{5.2}$$

Next we will use the fact that $(f_n)$ is equicontinuous to show that the subsequence $\sigma$ converges uniformly on $A$.

The fact that $\sigma$ is a subsequence of the original equicontinuous sequence $(f_n)$ implies that $\sigma$ itself is also equicontinuous. Formally this means that
$\forall \epsilon > 0 \; \exists \delta > 0$ such that

$$(\forall x, y \in A) \; (\forall n \in \mathbb{N}) \;\; |x - y| < \delta \;\rightarrow\; |f_{n,n}(x) - f_{n,n}(y)| < \epsilon/3. \tag{5.3}$$

The number $\delta$ depends on $\epsilon$, but it is independent of the choice of $x, y \in A$, and $n$.

For the $\delta$ selected above, for each term of the enumeration $b_m \in B, \; m \in \mathbb{N}$ we associate an open ball $B_\delta(b_m) \subset \mathbb{R}^n$ with radius $\delta$ and center $b_m$. The infinite collection of open balls $\mathcal{C} := \{B_\delta(b_m) \mid m \in \mathbb{N}\}$ is an open cover of $A$ because $B$ is dense in $A$.

Since $A$ is compact, $\mathcal{C}$ has a finite subcover, so there are finitely many points $b_1, b_2, \ldots b_{r_\delta} \in B$ for some $r_\delta \in \mathbb{N}$ such that we can write $A \subset \cup_{j=1}^{r_\delta} B_\delta(b_j)$.
By the previous conclusion (5.2), $\sigma(x)$ is a convergent sequence for each point $b_j$ with

$1 \leq j \leq r_\delta$. This fact implies that $\sigma(x)$ is a Cauchy sequence at each one of these points.
Then $(\exists k_{r_\delta,j} \in \mathbb{N})$ such that

$$\forall m, n > k_{r_\delta,j} \;\rightarrow\; |f_{m,m}(b_j) - f_{n,n}(b_j)| < \epsilon/3. \tag{5.4}$$

To avoid the dependence on $j$ we set $K_\delta = \max_{1\leq j\leq r_\delta}\{k_{r_\delta,j}\}$.

Thus, we have found that $\forall\delta > 0\; \exists K_\delta \in \mathbb{N}$ such that $\forall m, n > K_\delta, \forall x \in A$

$$|f_{m,m}(x) - f_{n,n}(x)| \;\leq\; |f_{m,m}(x) - f_{m,m}(b_j)| + |f_{m,m}(b_j) - f_{n,n}(b_j)| + |f_{n,n}(b_j) - f_{n,n}(x)|,$$

for some $j$ with $1 \leq j \leq r_\sigma$, by the triangle inequality.
Then, using the inequality (5.3) twice, and inequality (5.4) once we get

$$|f_{m,m}(x) - f_{n,n}(x)| \;<\; \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

Hence we have found that the sequence $\sigma$, which is a subsequence of $(f_n)$, is a uniformly Cauchy sequence on A. Then using the remark (5.11), we conclude that $\sigma$ converges uniformly on $A$. □

## 5.2 $C^k(U)$ Spaces

In this section are introduced the basic classes of real-valued continuously differentiable functions. These classes of functions are defined on open subsets of the Euclidean metric space $\mathbb{R}^n$. After introducing some metric structure in these classes of functions, we can use them to make useful approximations in the class of Sobolev spaces.

**Definition 5.13.** Let $U$ be an open subset of $\mathbb{R}^n$.
$C(U) \equiv C^0(U) := \{u : U \rightarrow \mathbb{R}^n \mid u \text{ is continuous}\}$ is the linear space of continuous real-valued functions on $U$.

*Remark* 5.14. $C(U)$ is a linear space under pointwise addition and scalar multiplication meaning that for any $f, g \in C(U)$, $U \in \mathbb{R}$, and $x \in U$ we have $(f+g)(x) = f(x) + g(x)$, $(\alpha f)(x) = \alpha f(x)$.

**Definition 5.15.** $C(\overline{U}) := \{u \in C(U) \mid u \text{ is uniformly continuous on } U\}$.

*Remark* 5.16. Recall that a real-valued function $u$ defined on the bounded open set $U \subset \mathbb{R}^n$ is uniformly continuous if and only if it can be extended to a real-valued continuous function $\tilde{u}$ defined on $\overline{U}$.

**Definition 5.17.** $C^k(U) := \{u : U \to \mathbb{R}; \mid u \text{ is } k\text{-times continuously differentiable}\}$ is the linear space of $k$-times continuously differentiable real valued functions on $U$.

**Definition 5.18.** $C^k(\overline{U}) := \{u \in C^k(U) \mid D^\alpha u \text{ is uniformly continuous on } U, \text{for all } |\alpha| \leq k\}$.

**Definition 5.19.** $C^\infty(U) := \{u : U \to \mathbb{R} \mid u \text{ is infinitely differentiable}\} = \bigcap_{k=0}^\infty C^k(U)$ is the linear space of infinitely continuously differentiable real-valued functions on $U$. We say that a function $u \in C^\infty$ is **smooth**. $C^\infty(\overline{U}) := \bigcap_{k=0}^\infty C^k(\overline{U})$.

**Definition 5.20.** $C_c^\infty(U) := \{u : U \to \mathbb{R} \mid u \in C^\infty(U) \text{ and } \operatorname{supp}(u) \Subset U\}$ is the linear space of smooth real valued functions on $U$ with compact support. Similarly defined are the linear spaces $C_c(U)$, $C_c^k(U)$, $C_c(\overline{U})$, $C_c^k(\overline{U})$, and $C_c^\infty(\overline{U})$.

## 5.3 Measure Theory

### 5.3.1 Lebesgue Measure

**Definition 5.21.** A collection $\mathcal{M}$ of subsets of $\mathbb{R}^n$ is called a $\sigma$-algebra if

(a) $\emptyset, \mathbb{R}^n \in \mathcal{M}$,

(b) $A \in \mathcal{M}$ implies $A^c \in \mathcal{M}$,

(c) if $\{A_k\}_{k=1}^\infty \subset \mathcal{M}$, then $\bigcup_{k=1}^\infty A_k$, and $\bigcap_{k=1}^\infty A_k$ are also elements of $\mathcal{M}$.

**Theorem 5.22** (Existence of Lebesgue measure and Lebesgue measurable sets)**.** *There exists a maximal $\sigma$-algebra $\mathcal{M}$ and a mapping*

$$|\cdot| : \mathcal{M} \to [0, +\infty]$$

*with the following properties:*

*(a) Every open subset of $\mathbb{R}^n$, and thus every closed subset of $\mathbb{R}^n$, belongs to $\mathcal{M}$.*

*(b) If $B$ is any ball in $\mathbb{R}^n$, then $|B|$ equals the n-dimensional volume of $B$.*

*(c) If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}$ and the sets $\{A_k\}_{k=1}^{\infty}$ are pairwise disjoint, then*

$$|\bigcup_{k=1}^{\infty} A_k| = \sum_{k=1}^{\infty} |A_k|.$$

**Notation 5.23.** The sets in $\mathcal{M}$ are called **Lebesgue measurable sets** and $|\cdot|$ is the n-dimensional **Lebesgue measure**.

**Notation 5.24.** If some property holds everywhere on $\mathbb{R}^n$, except for on a measurable set with Lebesgue measure zero, we say the property holds almost everywhere, abbreviated "a.e.".

**Definition 5.25.** Let $f : \mathbb{R}^n \to \mathbb{R}$. We say $f$ is a **measurable function** if for each open subset $U \subset \mathbb{R}$,

$$f^{-1}(U) \in \mathcal{M}.$$

*Remark* 5.26. In particular, if $f$ is a continuous function, then $f$ is measurable. The sum and product of two measurable functions are measurable. In addition if $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions, then $\limsup f_k$ and $\liminf f_k$ ar also measurable functions.

**Theorem 5.27** (Egoroff's Theorem)**.** *Let $\{f_k\}_{k=1}^{\infty}$, and $f$ be measurable functions, and suppose that $f_k \to f$ pointwise a.e. on $A$, where $A \subset \mathbb{R}^n$ is measurable, and $|A| < \infty$. Then for each $\epsilon > 0$ there exists a measurable subset $E \subset A$ such that* *(a)* $|A - E| \leq \epsilon$, *and*

*(b)* $f_k \to f$ *uniformly on* $E$.

**Theorem 5.28** (Monotone Convergence Theorem)**.** *Assume the functions* $\{f_k\}_{k=1}^{\infty}$ *are nonnegative and measurable, with* $f_1 \leq f_2 \leq \cdots f_k \leq f_{k+1} \leq \cdots$.
*Then*

$$\int_{\mathbb{R}^n} \lim_{k\to\infty} f_k \, dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k \, dx.$$

## 5.4 $L^p$ Spaces

Let $U$ be a measurable subset of $\mathbb{R}^n$.

**Definition 5.29.** Let $f : U \to \mathbb{R}$ be measurable. We define the **essential supremum** by $\text{ess sup}_U f := \inf\{C > 0 \mid f(x) \leq C \text{ for almost all } x \in U\}$.

**Definition 5.30.** $L^p(U) := \{u : U \to \mathbb{R} \mid u \text{ is measurable and } \|u\|_{L^p(U)} < \infty\}$ where $\|u\|_{L^p(U)} = (\int_U |u|^p dx)^{1/p}$ and $1 \leq p < \infty$.

**Definition 5.31.** $L^\infty(U) := \{u : U \to \mathbb{R} \mid u \text{ is measurable and } \|u\|_{L^\infty(U)} < \infty\}$ where $\|u\|_{L^\infty(U)} = \text{ess sup}_U |u|$.

**Definition 5.32.** $L^p_{loc}(U) = \{u : U \to \mathbb{R} \mid u \in L^p(V) \text{ for all } V \subset\subset U\}$.

**Lemma 5.33** (Young's Inequality)**.** *Let* $a, b > 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

**Theorem 5.34** (Holder's Inequality)**.** *Let* $p, q \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $u \in L^p(U)$, $v \in L^q(U)$. *Then* $uv \in L^1(U)$ *and* $\|uv\|_1 \leq \|u\|_p \cdot \|v\|_q$ *or explicitly*

$$\int_U |u(x)v(x)|dx \leq (\int_U |u(x)|^p dx)^{\frac{1}{p}} (\int_U |v(x)|^q dx)^{\frac{1}{q}}.$$

**Corollary 5.35** (Minkowski's Inequality)**.** *Let* $1 \leq p \leq \infty$, $u, v \in L^p(U)$. *Then* $u + v \in L^p(U)$ *and* $\|u + v\|_{L^p(U)} \leq \|u\|_{L^p(U)} + \|v\|_{L^p(U)}$.

**Theorem 5.36** (Interpolation Inequality for $L^p$-norms)**.** *Assume* $1 \leq s \leq r \leq t \leq \infty$, $1/r = \theta/s + (1-\theta)/t$ *for* $0 < \theta < 1$, *and* $u \in L^s(U) \cap L^t(U)$. *Then* $u \in L^r(U)$, *and* $\|u\|_{L^r(U)} \leq \|u\|^{\theta}_{L^s(U)} \|u\|^{1-\theta}_{L^t(U)}$.

**Proposition 5.37.** *The function space* $L^p(U)$ *with* $1 \leq p \leq \infty$ *is a Banach space provided we identify two functions that agree a.e. on* $U$.

**Proposition 5.38.** *The dual of* $L^p(U)$ *with* $1 \leq p < \infty$ *is* $(L^p(U))^* = L^q(U)$ *where* $1/p + 1/q = 1$, $1 < q \leq \infty$.

We have the following results: (a) $(L^\infty(U))^* \neq L^1(U)$, (b) $L^p(U)$ is separable if $1 \leq p < \infty$, and (c) $L^p(U)$ is reflexive if $1 < p < \infty$.

**Theorem 5.39.** *Let* $U \subset \mathbb{R}^n$ *be a bounded domain, and* $1 \leq p \leq q \leq \infty$. *Then* $L^q(U) \subset L^p(U)$. *Furthermore*

$$\|u\|_{L^p(U)} \leq C\|u\|_{L^q(U)}, \quad \textit{for all}\, u \in L^q(U),$$

*where* $C = |U|^{\frac{1}{p}-\frac{1}{q}}$ *if* $q < \infty$ *and* $C = |U|^{\frac{1}{p}}$ *if* $q = \infty$.

**Theorem 5.40.** *Let* $U \subset \mathbb{R}^n$ *be a bounded domain. Then uniform convergence implies convergence in* $L^p(U)$ *for any* $1 \leq p < \infty$.

*Proof.* Let $\{u_k\} \subset L^p(U)$ be a sequence of $L^p$ functions that converges uniformly to a function $u$ defined on $U$. Then for any $\epsilon > 0$ there exists $k' \in \mathbb{N}$ such that

$$|u_k(x) - u(x)| < \epsilon \quad \text{for all } k \geq k' \text{ and for all } x \in U.$$

Now

$$\begin{aligned} (|u|)^p &\leq (|u_k - u| + |u_k|)^p \ \text{(Triangle Inequality)} \\ &\leq (2\max\{|u_k - u|, |u_k|\})^p \\ &\leq 2^p(|u_k - u|^p + |u_k|^p) \end{aligned}$$

Then we integrate

$$\begin{aligned}\int_U |u|^p dx &\le 2^p \int_U |u_{k'} - u|^p dx + 2^p \int_U |u_{k'}|^p dx \\ &\le 2^p \epsilon^p |U| + 2^p \|u_{k'}\|^p_{L^p(U)} < \infty.\end{aligned}$$

Thus $u \in L^p(U)$.

Then use uniform convergence inequality raised to the $p$ power, integrate, and finally raise to $\frac{1}{p}$ power with $k > k'$

$$\begin{aligned}(\int_U |u_k(x) - u(x)|^p dx)^{\frac{1}{p}} &< (\int_U \epsilon^p dx)^{\frac{1}{p}}, \\ \|u_k(x) - u(x)\|_{L^p(U)} &< \epsilon |U|^{\frac{1}{p}}.\end{aligned}$$

□

**Theorem 5.41.** *Let $U$ be an open subset of $\mathbb{R}^n$, and $1 \le p < \infty$, and $q$ the conjugate of $p$. Suppose that $u_k \rightharpoonup u$ weakly in $L^p(U)$ and $v_k \to v$ strongly in $L^q(U)$. Then*

$$\lim_{k \to \infty} \int_U u_k v_k \, dx = \int_U uv \, dx.$$

*Proof.* For each $k$,

$$\int_U u_k v_k dx - \int_U uv dx = \int_U (v_k - v) u_k dx + \int_U v u_k dx - \int_U uv dx.$$

Since every weakly convergent is strongly bounded there exists a constant $C > 0$ such that $\|u_k\|_{L^p(U)} \le C$ for all $k$.

$$\begin{aligned}|\int_U u_k v_k dx - \int_U uv dx| &\le |\int_U (v_k - v) u_k dx| + |\int_U v u_k dx - \int_U uv dx| \text{ ( Triangle Ineq.)} \\ &\le C\|v_k - v\|_{L^q(U)} + |\int_U v u_k dx - \int_U uv dx|. \text{ (Holder's Ineq.)}.\end{aligned}$$

Finally use the fact that $\lim_{k\to\infty} \|v_k - v\|_{L^q(U)} = 0$ and $\lim_{k\to\infty} \int_U v u_k dx = \int_U uv dx$.

□

**Theorem 5.42.** *If $u_k \to u$ in $L^p(U)$ then a subsequence $\{u_{n_k}\} \subset u_k$ converges a.e. to $u$.*

## 5.5 Convolution and Smoothing

This procedure will be used to construct smooth approximations to given functions.

**Notation 5.43.** Let $U \subset \mathbb{R}^n$ be open, and $\epsilon > 0$. We define

$$U_\epsilon = \{x \in U \mid dist(x, \partial U) > \epsilon\}$$

**Definition 5.44.** Define the **standard mollifier** $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} C\exp(\frac{1}{|x|^2-1}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

such that $\text{supp}(\eta) \subset B(0,1)$ and $C > 0$ yields $\int_{\mathbb{R}^n} \eta dx = 1$.
For each $\epsilon > 0$, we define $\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$.
The functions $\eta_\epsilon$ are in $C^\infty$ and satisfy $\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1, \;\; \text{supp}(\eta_\epsilon) \subset B(0,\epsilon)$.

**Definition 5.45.** If $f \in L^1_{loc}(U)$ we define its mollification as $f^\epsilon := \eta_\epsilon * f$. That is, $f^\epsilon(x) = \int_U \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta(z)f(x-z)dz, \; \forall x \in U_\epsilon$. The last equality is obtained by using the change of variable $x - y = z$ and the fact that the support of $\eta_\epsilon$ is contained in the ball $B(0,\epsilon)$.

**Theorem 5.46** (Properties of mollifiers)**.**

*(a)* $f^\epsilon \in C^\infty(U_\epsilon)$,
*(b)* $f^\epsilon \to f$ *a.e. as* $\epsilon \to 0$,
*(c) If* $f \in C(U)$, *then* $f^\epsilon \to f$ *uniformly on compact subsets of* $U$.
*(d) If* $1 \leq p \leq \infty$ *and* $f \in L^p_{loc}(U)$, *then* $f^\epsilon \to f$ *in* $L^p_{loc}(U)$.

These properties are used to show the existence of a partition of unity

**Theorem 5.47.** *Let* $U \subset \mathbb{R}^n$ *be a domain covered by a family* $\{V_i\}_{i\in I}$ *of open sets i.e.* $U \subset \bigcup_{i\in I} V_i$. *Then there exists a* ***partition of unity*** *i.e a family* $\{\zeta_j\}_{j\in J} \subset C^\infty_c(\mathbb{R}^n)$ *such that*

*(a) For each* $j \in J$, *supp*$(\zeta_j) \subset U_i$ *for some* $i \in I$,

*(b)* $0 \leq \zeta_j(x) \leq 1$ *for all* $j \in J$, *for all* $x \in \mathbb{R}^n$,

*(c)* $\sum_{j\in J} \zeta_j(x) = 1$ *for all* $x \in U$*,*

*(d) For each compact set* $K \subset U$*, there are only finitely many* $\zeta_j$ *that are not identically zero on* $K$*.*

# Chapter 6: Sobolev Spaces

## 6.1 The Sobolev Function Space $W^{k,p}(U)$

**Definition 6.1.** Let $u, v \in L^1_{loc}(U)$. For a multiindex $\alpha$, we say that $v$ is the $\alpha$-**weak derivative** of $u$ on $U$, written $D^\alpha u = v$, if $\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx$ for all **test functions** $\phi \in C_c^\infty(U)$.

**Definition 6.2.** Suppose $p \in \mathbb{R}$ such that $1 \leq p \leq \infty$, and $k \in \mathbb{Z}$ such that $k \geq 0$. The **Sobolev space** $W^{k,p}$ is defined as
$W^{k,p}(U) = \{u \in L^p(U) \mid D^\alpha u$ exists in the weak sense and is in $L^p(U)\ \forall |\alpha| \leq k\}$, with the norm $\|u\|_{W^{k,p}(U)} = (\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p)^{1/p}$ if $1 \leq p < \infty$ and $\|u\|_{W^{k,\infty}(U)} = \sum_{|\alpha|\leq k} \text{ess sup}_U |D^\alpha u|$.

We identify functions in $W^{k,p}(U)$ which agree a.e.

**Definition 6.3.** $W^{k,p}_{loc}(U) = \{u \mid u \in W^{k,p}(V)$ for all $V \subset\subset U\}$.

**Definition 6.4.** $W^{k,p}_0(U)$ is the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. This means that $u \in W^{k,p}_0(U)$ if and only if there exists a sequence $\{u_m\} \subset C_c^\infty(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.

*Remark* 6.5. If $k = 0$ we have $W^{0,p}(U) = L^p(U)$. If $p = 2$ and $0 \leq k$ we write $H^k(U) = W^{k,2}(U)$. $H^k$ is a Hilbert space with a inner product given by $(u, v)_{H^k} = \sum_{|\alpha|\leq k} \int_U (D^\alpha u(x) \cdot D^\alpha v(x) dx$. In particular $H^0(U) = L^2(U)$.

**Theorem 6.6** (Properties of weak derivatives)**.** *Let $u, v \in W^{k,p}(U)$, $\||\alpha| \leq k$. Then*

1. *$D^\alpha u \in W^{k-|\alpha|}(U)$ and $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}$ for all $\alpha, \beta$ with $|\alpha|+|\beta| \leq k$.*

*2. For all $a, b \in \mathbb{R}$, $au + bv \in W^{k,p}(U)$ and $D^\alpha(au + bv) = aD^\alpha u + bD^\alpha v$*

*3. If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and $D^\alpha(\zeta u) \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta \zeta D^{\alpha-\beta} u$.*

**Proposition 6.7.** *For all $k \geq 0$ and for all $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space which is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.*

**Definition 6.8.** Let $U \subset \mathbb{R}^n$ be bounded domain and let $k \in \mathbb{N}$. We say that the boundary $\partial U$ is $C^k$ if for each point $x_0 \in \partial U$ there exist $r > 0$ and a $C^k$ function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n = f(x_1, x_2, \cdots, x_{n-1})\}.$$

**Theorem 6.9** (Global approximation by functions smooth up to the boundary)**.** *Assume $U$ is bounded and $\partial U$ is $C^1$. Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$ and for any $k \geq 1$. Then there exist functions $u_m \in C^\infty(\overline{U})$ such that*

$$u_m \to u \text{ in } W^{k,p}(U). \tag{6.1}$$

**Theorem 6.10** (Extension Theorem)**.** *Assume $U$ is bounded and $\partial U$ is $C^1$. Select a bounded open set $V$ such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n), \tag{6.2}$$

*and a constant $C$ depending only on $p$, $U$ and $V$ such that for each $u \in W^{1,p}(U)$ :*

*(a) $Eu = u$ a.e. in $U$,*

*(b) $Eu$ has support within $V$,*

*(c) there exist a constant $C > 0$ that depend only on $p$, $U$, and $V$ such that*
$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$.

**Definition 6.11.** We call $Eu$ an **extension** of $u$ to $\mathbb{R}^n$.

**Theorem 6.12** (Trace Theorem). *Let $U \subset \mathbb{R}^n$ be a bounded domain, and $\partial U$ $C^1$. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U), \tag{6.3}$$

*and a constant $C$ depending only on $p$ and $U$ such that*

*(a) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$,*

*(b) for each $u \in W^{1,p}(U)$, $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$.*

**Definition 6.13.** We call $Tu$ the **trace** of $u$ on $\partial U$.

**Theorem 6.14** (Trace-zero functions in $W^{1,p}$). *Let $U \subset \mathbb{R}^n$ be a bounded domain, and $\partial U$ is $C^1$, and let $u \in W^{1,p}(U)$. Then*

$$u \in W_0^{1,p}(U) \text{ if and only if } \quad Tu = 0 \text{ on } \partial U. \tag{6.4}$$

## 6.2 Sobolev Inequalities

**Definition 6.15.** If $1 \leq p < n$, the Sobolev conjugate of $p$ is $p^* := \frac{np}{n-p}$.

**Theorem 6.16** (Gagliardo-Nirenberg-Sobolev inequality). *Assume $1 \leq p < n$. There exists a constant $C = C(p, n)$ such that for all $u \in C_c^1(\mathbb{R}^n)$*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}. \tag{6.5}$$

**Theorem 6.17** (Estimates for $W^{1,p}$, $1 \leq p < n$). *Let $U\mathbb{R}^n$ be a bounded domain, and suppose $\partial U$ is $C^1$. Assume $1 \leq p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, and there exists a constant $C = C(p, n, U)$ with estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}. \tag{6.6}$$

**Theorem 6.18** (Estimates for $W_0^{1,p}$, $1 \leq p < n$). *Let $U \subset \mathbb{R}^n$ be a bounded domain. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate with a constant $C = C(p,\, q,\, n, U)$*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)} \tag{6.7}$$

*for each* $q \in [1, p^*]$.

*In particular, for all* $1 \leq p \leq \infty$,

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}. \tag{6.8}$$

*This estimate is called* **Poincare's inequality**.

## 6.3 Rellich-Kondrachov Compactness Theorem

The Gagliardo-Nirenberg-Sobolev Inequality (6.16) states the embedding of $W^{1,p}(U)$ into $L^{p^*}(U)$ for $1 \leq p < n$. Now we will prove another result that shows that $W^{1,p}(U)$ is compactly embedded in $L^q(U)$ for $1 \leq q < p^*$.

**Theorem 6.19** (Rellich-Kondrachov Compactness Theorem)**.** *Let* $U$ *be a bounded, open subset of* $\mathbb{R}^n$*, such that* $\partial U$ *is* $C^1$*, and* $1 \leq p < n$*. Then*

$$W^{1,p}(U) \subset\subset L^q(U), \ \forall q \text{ such that } 1 \leq q < p^*.$$

*Proof.*

1. Theorem (6.17) implies that $W^{1,p}(U) \subset L^q(U)$ with $\|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}$ for all $1 \leq q < p^*$. This means that $W^{1,p}(U)$ is embedded in $L^q(U)$. We therefore only need to show that the embedding is compact, i.e. if $\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in $W^{1,p}(U)$, then there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ which converges in $L^q(U)$.

2. By the extension theorem (6.10), and without lost of generality we can assume that all the functions $\{u_m\}_{m=1}^{\infty}$ have compact support in some bounded open set $V \subset \mathbb{R}^n$. We may also assume that

$$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty. \tag{6.9}$$

3. Let $\varepsilon > 0$ and consider the smoothed functions

$$u_m^{\varepsilon} := \eta_{\varepsilon} * u_m \ \forall m \in \mathbb{N}$$

Here, $\eta_\varepsilon$ is the standard mollifier. We may also assume that for $\varepsilon$ sufficiently small the functions $\{u_m^\varepsilon\}_{m=1}^\infty$ all have support in $V$ because $\mathrm{supp}(u_m^\varepsilon) \subset \mathrm{supp}(u_m) + \mathrm{supp}(\eta_\varepsilon)$.

4. First, we make the following claim:

$$u_m^\varepsilon \to u_m \text{ in } L^q(U) \text{ as } \varepsilon \to 0, \text{ uniformly in } m. \tag{6.10}$$

To prove this claim, note that if $u_m$ is smooth, then

$$\begin{aligned} u_m^\varepsilon(x) - u_m(x) &= \int_{B(0,1)} \eta(y)(u_m(x-\varepsilon y) - u_m(x))dy, \\ &\quad \text{(using the first fundamental theorem of calculus)} \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt}(u_m(x-\varepsilon ty)dtdy, \\ &\quad \text{(using the chain rule)} \\ &= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\varepsilon ty)\cdot y\; dtdy. \end{aligned}$$

Now using the property of Lebesgue integrals of bounded measurable functions $f$ on bounded domains, $|\int_V f| \le \int_V |f|$.

$$\begin{aligned} \int_V |u_m^\varepsilon(x) - u_m(x)|dx &\le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x-\varepsilon ty)|dxdtdy, \\ &\quad \text{(setting } z = x - \varepsilon ty \text{ and integrating over } y, z \\ &\le \varepsilon \int_V |Du_m(z)|dz. \end{aligned}$$

We have found that

$$\begin{aligned} \|u_m^\varepsilon - u_m\|_{L^1(V)} &\le \varepsilon \|Du_m\|_{L^1(V)} \\ &\le \varepsilon C \|Du_m\|_{L^p(V)} \text{ (because } V \text{ is bounded).} \end{aligned}$$

For $u_m \in W^{1,p}(V)$ the integral $\int_V |Du_m(z)|dz$ is finite for all $m$.
By (6.9) we have

$$u_m^\varepsilon \to u_m \text{ in } L^1(V), \text{ uniformly in } m. \tag{6.11}$$

Then, since $1 \leq q < p^*$, we see using the interpolation inequality for $L^p$-norms (5.36) that

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \|u_m^\varepsilon - u_m\|_{L^1(V)}^{\theta} \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta},$$

where $\theta$ is chosen so that $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$, $0 < \theta < 1$.
Now inequality (6.9) and the Gagliardo-Nirenberg-Sobolev Inequality (6.16) imply that

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq C\|u_m^\varepsilon - u_m\|_{L^1(V)}^{\theta},$$

hence result (6.11) implies (6.10).

5. Next we make the following second claim:

For each fixed $\varepsilon > 0$, the sequence $\{u_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded and equicontinuous. (6.12)

In fact, if $x \in \mathbb{R}^n$, then

$$\begin{aligned} |u_m^\varepsilon(x)| &\leq \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u_m(y)|dy, \\ &\leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \\ &\quad \text{(because } \|u_m\|_{L^1(V)} \text{ is bounded, whenever } V \text{ is bounded)} \\ &\leq \frac{C}{\varepsilon^n} < \infty, \end{aligned}$$

for all $m \in \mathbb{N}$. This result means that the sequence $\{u_n^\varepsilon\}$ is uniformly bounded In the same way

$$\begin{aligned} |Du_m^\varepsilon(x)| &\leq \int_{B(x,\varepsilon)} |D\eta_\varepsilon(x-y)||u_m(y)|dy, \\ &\leq \|D\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\varepsilon^{n+1}} < \infty, \end{aligned}$$

for all $m \in \mathbb{N}$. This result means that the sequence $\{u_n^\varepsilon\}$ is equicontinuous. These two estimates imply (6.12).

6. Now fix $\delta > 0$. We can show that there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(v)} \leq \delta. \tag{6.13}$$

7. To show this, we use the claim (6.10) to select $\varepsilon$ such that

$$\forall m \in \mathbb{N},\ \|u_m^{\varepsilon} - u_m\|_{L^q(V)} \leq \delta/2. \tag{6.14}$$

8. Since the functions $\{u_m\}_{m=1}^{\infty}$ and $\{u_m^{\varepsilon}\}_{m=1}^{\infty}$ have support in a fixed bounded set $V \subset \mathbb{R}^n$, we can use the claim (6.12) and the Arzela-Ascoli theorem, (5.12), to obtain a subsequence $\{u_{m_j}^{\varepsilon}\}_{j=1}^{\infty} \subset \{u_m^{\varepsilon}\}_{m=1}^{\infty}$ which converges uniformly on $V$. In particular this subsequence is a Cauchy sequence therefore

$$\limsup_{j,k\to\infty} \|u_{m_j}^{\varepsilon} - u_{m_k}^{\varepsilon}\|_{L^q(V)} = 0. \tag{6.15}$$

Now results (6.14), (6.15), together the triangle inequality imply that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta.$$

so we have proved (6.13).

9. We use (6.13) with $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$. Then use a standard diagonal argument to extract a subsequence $\{u_{m_j}\}_{j=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ satisfying

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(v)} = 0.$$

□

# Chapter 7: Variational Analysis in $C^\infty$

## 7.1 Variational Problem

Suppose that a real-valued function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is given. By definition, $f$ has a local minimum at $x_0 \in [a, b]$ if and only if there exists a neighborhood $U(x_0)$ of $x_0$ such that $f(x) \geq f(x_0)$ for all $x \in U(x_0)$, where $x \neq x_0$.

If $f$ has a local minimum at $x_0$, assuming that $x_0$ is an interior point of $[a, b]$, and the derivative $f'(x_0)$ exists, then $f'(x_0) = 0$.

We will generalize this extremal problem from real calculus to variational calculus. Given a functional $F : D \subset X \to \mathbb{R}$ where $X$ is a Banach space. If $F$ has a local minimum at $u_0$, and $u_0 \in \overset{\circ}{D}$, then we have an operator equation $F'(u_0) = 0$. The generalized derivative $F'(x_0)$ is a Frechet derivative.

A variational problem is a generalization of extremal problems in real calculus where the domain $D$, also called the admissible set, is a space of functions. Since spaces of functions are infinite dimensional we need the framework of functional analysis to characterize the problem.

The functional $F$ is given by an integral involving the unknown function and its first derivative, and it is not necessarily linear. The generalized derivative is determine by calculating the first variation of the functional, and if the function is sufficiently smooth yields a second-order PDE called the Euler-Lagrange Equation of the variational problem.

In this chapter, we discuss the simple case where the domain of the functional is the space of smooth functions that satisfying a Dirichlet boundary condition on a fixed boundary. The technique to calculate the first variation is explained explicitly.

## 7.2 Euler-Lagrange Equation

**Definition 7.1.** Let $U \subset \mathbb{R}^n$ be a nonempty, connected, open set. We call $U$ a **domain**.

**Definition 7.2.** Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial U$, and let $L : \mathbb{R}^n \times \mathbb{R} \times \overline{U} \to \mathbb{R}$ be a smooth function. We call $L$ a **Lagrangian**.

**Definition 7.3.** Given a fixed smooth function $g : \overline{U} \to \mathbb{R}$, we define the **set of admissible functions**

$$\mathcal{A} = \{w \in C^\infty(\overline{U}) \mid w|_{\partial U} = g|_{\partial U}\}$$

This set of admissible functions corresponds to a Dirichlet boundary condition.

**Notation 7.4.** $L = L(p, z, x) = L(p_1, \ldots, p_n, z, x_1, \ldots, x_n)$ for $p \in \mathbb{R}^n,\ z \in \mathbb{R}$, and $x \in U$. Here, if $w$ is any function in $\mathcal{A}$, $p$ is the variable that represents the gradient vector $Dw = (w_{x_1}, \ldots, w_{x_n})$, $z$ is the variable that represents $w(x)$. We also we use the following short notations for the derivatives of $L$ $D_pL = (L_{p_1}, \ldots, L_{p_n}),\ D_zL = L_z,\ D_xL = (L_{x_1}, \ldots, L_{x_n})$.

**Definition 7.5.** The functional $I[\cdot]$ is defined by

$$I[w] = \int_U L(Dw(x), w(x), x)dx \tag{7.1}$$

for $w \in \mathcal{A}$.

Assume that some particular smooth function $u$ is a minimizer of $I[\cdot]$ and satisfies the boundary condition $u|_{\partial U} = g|_{\partial U}$. We will show that $u$ is a solution of a specific nonlinear PDE. Take any smooth function $v \in C_c^\infty(U)$ and define the function $i : \mathbb{R} \to \mathbb{R}$ by

$$i(\tau) := I[u + \tau v]. \tag{7.2}$$

Since $v$ has compact support in $U$ we have $v|_{\partial U} = 0$. Then $(u+\tau v)|_{\partial U} = u|_{\partial U}+\tau v|_{\partial U} = u|_{\partial U} = g$, so that $u + \tau v$ is admissible for any $\tau$. Therefore, since $u$ is a minimizer of $I[\cdot]$, $i(\cdot)$ has a minimum at $\tau = 0$. Therefore $\tau = 0$ is a critical point of this one-variable function, i.e. the necessary condition

$$i'(0) = 0, \tag{7.3}$$

must be satisfied. Writing out $i(\tau)$ explicitly we have

$$i(\tau) = \int_U L(Du + \tau Dv, u + \tau v, x)dx. \tag{7.4}$$

Calculating the derivative $i'(\tau)$, using the fact that $\tau$ is a parameter inside the integral and applying the chain rule, we obtain

$$i'(\tau) = \int_U [\sum_{k=1}^{n} L_{p_k}(Du + \tau Dv, u + \tau v, x)v_{x_k} + L_z(Du + \tau Dv, u + \tau v, x)v]dx. \tag{7.5}$$

Setting $\tau = 0$ and substituting into (7.3), we get

$$i'(0) = \int_U [\sum_{k=1}^{n} L_{p_k}(Du, u, x)v_{x_k} + L_z(Du, u, x)v]dx = 0. \tag{7.6}$$

Now, integrating by parts on the first term and using the fact that $v$ has compact support within $U$, we get

$$\int_U [-\sum_{k=1}^{n}(L_{p_k}(Du, u, x))_{x_k} + L_z(Du, u, x)]vdx = 0. \tag{7.7}$$

Since this equality holds for arbitrary test functions $v$, we conclude that $u$ satisfies the following second-order, possibly nonlinear PDE

$$-\sum_{k=1}^{n}(L_{p_k}(Du, u, x))_{x_k} + L_z(Du, u, x) = 0. \tag{7.8}$$

This is the Euler-Lagrange equation associated with the functional $I[\cdot]$ defined in (8.1). Any smooth minimizer of $I[\cdot]$ is a solution of the Euler-Lagrange equation (7.8), and thus we can try to find a solution of equation (7.8) by searching for minimizers of

(8.1). But there may be other, non-minimizing solutions to the PDE and also the minimizer may not be smooth. It is important to point out that the Euler-Lagrange Equation is a possibly quasilinear second order PDE in divergence form.

One important example related with problems in physics is the Poisson equation. Consider $L(p, z, x) = \frac{1}{2}|p|^2 - zf(x)$. Then $L_{p_i} = p_i$ with $i = 1, \ldots, n$, $L_z = -f(x)$. The functional is

$$I[w] := \int_U [\frac{1}{2}|Dw|^2 - wf]dx,$$

and the associated Euler-Lagrange equation is $\triangle u = f$ in $U$.

# Chapter 8: Variational Analysis in $W^{1,q}$

Let $U \subset \mathbb{R}^n$ be a bounded domain such that $\partial U$ is $C^1$, let the Lagrangian function $L$ be smooth, and let the admissible set of functions $\mathcal{A}$ be a subset of some appropriate Sobolev space. Take also the same kind of functional discussed in previous chapter

$$I[w] = \int_U L(Dw(x), w(x), x)dx \tag{8.1}$$

for $w \in \mathcal{A}$.

As we anticipated in the introduction in order to have a existence proof for minimizers we will need three key requirements: compactness, closedness, and lower semicontinuity. When these requirements are satisfied we have available powerful weak convergence methods to guarantee existence of minimizers.

We have some reasons why we want to define the functional over a Sobolev space. First, we know that Sobolev spaces are more general than the space of smooth functions because it includes discontinuous functions. Second, Sobolev spaces are reflexive Banach spaces so that we can use the corresponding weak sequential compactness theorem which is a very important property to prove existence of minimizers.

We will need additional conditions, other than smoothness, on the Lagrangian $L$. We will identify some specific conditions on the Lagrangian $L$ that ensures that the functional $I[\cdot]$ defined over a Sobolev spaces indeed has a minimizer.

## 8.1 Coercivity

We know that in general arbitrary smooth functionals do not necessarily attain a global minimum or even a local one. This problem becomes clear if we think of smooth functions such as $f(x) = e^{-x^2}$. The problem happens in finite dimensions only when the smooth function has horizontal asymptotes. For this reason we need to control from below the growth of the functional $I[w]$ for large functions $w$. One way to do this is to require that $I[w] \to \infty$ as $|w| \to \infty$. We will assume that for any $q \in \mathbb{R}$ with $1 < q < \infty$ there exist constants $\alpha > 0,\ \beta \geq 0$ such that

$$L(p, z, x) \geq \alpha |p|^q - \beta, \tag{8.2}$$

for all $p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in U$. This is called a **coercivity condition on** $L$. Then integrating over $U$ and using monotonicity property of integral we get

$$I[w] \geq \alpha \|Dw\|^q_{L^q(U)} - \gamma \tag{8.3}$$

for $\gamma := \beta|U|$. Thus $I[w] \to \infty$ as $\|Dw\|_{L^q} \to \infty$.
We say that (8.3) is a **coercivity condition on** $I[w]$.

The crucial fact that $U$ is bounded, implies that $\gamma$ is finite i.e $0 \leq \gamma < \infty$. Then inequality (8.3) implies that $I[w]$ is bounded below i.e $-\infty < -\gamma \leq I[w]$. On the other hand we know, by Poincare's Theorem, that the $L^q$-norm of $Dw$ controls the $L^q$-norm of $w$. These two norms contributes to the norm of $W^{1,q}(U)$.

These results suggests that we can define the functional $I[w]$ for functions $w \in W^{1,q}(U)$ and not only for smooth functions as we did in the previous chapter. Furthermore we can also require a Dirichlet boundary condition $w|_{\partial U} = g|_{\partial U}$ that must be interpreted now in the trace sense.

The class of admissible functions is defined by

$$\mathcal{A} := \{w \in W^{1,q}(U) : w|_{\partial U} = g|_{\partial U} \text{ in the trace sense}\}.$$

## 8.2 Lower Semicontinuity

By the coercivity condition, $I(w)$ is bounded below, so we can set

$$m := \inf_{w \in \mathcal{A}} I[w]. \tag{8.4}$$

Using the definition of infimum we can select a sequence of functions $u_k \in \mathcal{A}$ so that

$$I[u_k] \to m \text{ as } k \to \infty. \tag{8.5}$$

We call $\{u_k\}_{k=1}^{\infty}$ a *minimizing sequence.* We want to show that some subsequence of $\{u_k\}_{k=1}^{\infty}$ converges to an actual minimizer $u \in \mathcal{A}$. To do this it is necessary to use a kind of sequential compactness of the admissible set $\mathcal{A}$. This condition is not possible using only norm convergence because the space $W^{1,q}(U)$ is infinite dimensional. Recall that in infinite dimensional spaces the unit closed ball is not necessarily compact.

The inequality (8.3) of the coercivity condition will only allows us to conclude that the minimizing sequence is contained in a bounded subset of $W^{1,q}(U)$. This does not imply that there exists any subsequence of $\{u_k\}_{k=1}^{\infty}$ which actually converges in the $W^{1,q}(U)$ norm topology.

However, we can recover enough sequential compactness if we consider the weak topology. We know that when $1 < q < \infty$, the space $L^q(U)$ is a reflexive Banach space. Recall that the weak compactness theorem (4.16) states that if a sequence in a reflexive Banach space is bounded, then there exists a subsequence that converges weakly to some limiting element in the given Banach space. Using this theorem for our sequence $\{u_k\}_{k=1}^{\infty}$ and our Banach space $W^{1,q}(U)$ which is reflexive, we conclude that there exist $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u \in W^{1,q}(U)$ such that

$$\begin{aligned} u_{k_j} &\rightharpoonup u \text{ weakly in } L^q(U) \\ Du_{k_j} &\rightharpoonup Du \text{ weakly in } L^q(U;\mathbb{R}^n). \end{aligned} \tag{8.6}$$

We abbreviate this result by saying

$$u_{k_j} \rightharpoonup u \text{ weakly in } W^{1,q}(U). \tag{8.7}$$

Now we observe that we have another potential problem because in almost interesting applications the functional $I[\cdot]$ is not continuous with respect to weak convergence.

This means that from (8.7) and (8.5) we can not conclude that

$$I[u] = \lim_{j\to\infty} I[u_{k_j}], \tag{8.8}$$

meaning that $u$ is a really a minimizer. The reason is that in general $Du_{k_j} \rightharpoonup Du$ does not imply $Du_{k_j} \to Du$ in norm. However, is not necessary to require equality in (8.8). It is enough to know only that

$$I[u] \leq \liminf_{j\to\infty} I[u_{k_j}]. \tag{8.9}$$

From this inequality and (8.5) we can deduce $I[u] \leq m$. On the other hand (8.4) implies $I[u] \geq m$. Thus $I[w] = m$. This means that, if the lower semicontinuity condition (8.9) holds $u$ will indeed be a minimizer.

**Definition 8.1.** We say that a functional $I[\cdot]$ is sequentially weakly lower semicontinuous on $W^{1,q}(U)$, if

$$u_k \rightharpoonup u \text{ weakly in } W^{1,q}(U) \text{ implies that } I[u] \leq \liminf_{j\to\infty} I[u_k]. \tag{8.10}$$

The next objective is to determine reasonable conditions on the Lagrangian $L$ that ensure that $I[\cdot]$ is weakly sequentially lower semicontinuous.

## 8.3 Convexity

To determine an absolute minimum of $I[w]$ we need a global condition analogous to the positivity of the Hessian used in extremal problems of finite dimensional real analysis. This argument suggest that it is reasonable to require that $L$ be convex in its first argument $p$.

**Definition 8.2.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **convex** if $f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$, $\lambda, \mu > 0$ such that $\lambda + \mu = 1$.

**Theorem 8.3** (Supporting hyperplanes). *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Then for each $x \in \mathbb{R}^n$ there exists $r \in \mathbb{R}^n$ such that*

$$f(y) \geq f(x) + r \cdot (y - x) \quad \forall\, y \in \mathbb{R}^n. \tag{8.11}$$

*Remark* 8.4. (a) The mapping $y \mapsto f(x) + r \cdot (y - x)$ determines the supporting hyperplane to $f$ at $x$. The inequality above says that the graph of $f$ lies above each supporting hyperplane. If $f$ is differentiable at $x$, $r$ is unique and $r = Df(x)$.
(b) If $f$ is $C^2$, then $f$ is convex if and only if $D^2 f \geq 0$.

The following theorem proves that a convexity condition on $L$ implies weak lower semicontinuity which is a crucial result that will be used in the proof of the existence of a minimizer in the next section.

## 8.4 Weak Sequential Lower Semicontinuity Theorem

**Theorem 8.5** (Weak Sequential Lower Semicontinuity Theorem). *Assume that $L$ is smooth and bounded below, and in addition the mapping $p \mapsto L(p, z, x)$ is convex, $\forall z \in \mathbb{R}$, $\forall x \in U$. Then $I[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(U)$.*

*Proof.*

1. Take any sequence $\{u_k\}_{k=1}^{\infty}$ such that $u_k \rightharpoonup u$ weakly in $W^{1,q}(U)$, and let $l := \liminf_{k\to\infty} I[u_k]$. We want to show that $I[u] \leq l$. Note that the sequence $\{u_k\}$ is not necessarily convergent strongly in $W^{1,q}(U)$ and it is not necessarily a minimizing sequence.

2. From the functional analysis result (4.7) we know that any weakly convergent sequence is bounded in norm. Thus, we have

$$\sup_k \|u_k\|_{W^{1,q}(U)} < \infty. \tag{8.12}$$

From the definition of lim inf we know that $l$ is the smallest cluster point of the real sequence $\{I[u_k]\}$. We also know that if a sequence converges, then its limit and its liminf coincide. Therefore, using a subsequence of $u_k$ if necessary, we can suppose

$$l = \lim_{j\to\infty} I[u_k]. \tag{8.13}$$

3. We know from the Rellich-Kondrachov compactness theorem (6.19) that from any bounded sequence in $W^{1,q}(U)$ we can extract a subsequence $\{u_{k_j}\}$ that converges strongly on $L^q(U)$. For convenience we will use the notation $\{u_k\}$ for this subsequence. So we have

$$u_k \to u, \text{ strongly in } L^q(U).$$

4. Using the Riesz-Fischer Theorem (see [17] page 148) we can, if necessary, extract another subsequence such that

$$u_k \to u \text{ a.e. in } U. \tag{8.14}$$

5. Fix $\epsilon > 0$. Then using the previous result and Egoroff's Theorem (5.27) we obtain that there exists a smaller measurable set $E_\epsilon \subset U$ with

$$|U - E_\epsilon| \le \epsilon. \tag{8.15}$$

so that

$$u_k \to u \text{ uniformly on } E_\epsilon. \tag{8.16}$$

To restrict temporarily the limit function and its derivative to be in $L^\infty(U)$ it is convenient to define another subset of $U$

$$F_\epsilon := \{x \in U : |u(x)| + |Du(x)| \le \frac{1}{\epsilon}\}. \tag{8.17}$$

Observe that

$$|U - F_\epsilon| \to 0 \text{ as } \epsilon \to 0, \tag{8.18}$$

because as $\epsilon \to 0$ the restrictions on the on $u$ and $Du$ are removed. This is a basic property of measurable functions, see for example [17].

To combine the restriction of uniform convergence and to be inside $L^1$ at the same time we set

$$G_\epsilon := E_\epsilon \cap F_\epsilon. \tag{8.19}$$

From relations (8.15), (8.18), we observe that $|U - G_\epsilon| \to 0$ as $\epsilon \to 0$.

6. Let $\epsilon > 0$ be fixed. By the coercivity condition (8.2), we know that $L$ is bounded below by $-\beta$ . Without lost of generality we can consider $L \geq 0$. Otherwise set $\tilde{L} = L + \beta$. So

$$\begin{aligned} I[u_k] &= \int_U L(Du_k, u_k, x)dx \geq \int_{G_\epsilon} L(Du_k, u_k, x)dx \\ &\geq \int_{G_\epsilon} L(Du, u_k, x)dx + \int_{G_\epsilon} D_pL(Du, u_k, x) \cdot (Du_k - Du)dx. \end{aligned} \tag{8.20}$$

The first inequality follows from the monotonicity of Lebesgue integrals.

The second inequality follows from the convexity of $L$ in its first argument using the smoothness of $L$ and the supporting hyperplane theorem (8.3).

7. From results (8.16), (8.17), and (8.19), and using the fact that for uniformly convergent sequences the limit of the sequence of integrals is equal to the integral of its limit, we see that

$$\lim_{k\to\infty} \int_{G_\epsilon} L(Du, u_k, x)dx = \int_{G_\epsilon} L(Du, u, x)dx. \tag{8.21}$$

The result (8.17) is required to maintain the integrand on the left-hand side bounded, thus integrable, and to make $L$ uniformly continuous in its arguments.

8. $Du_k \rightharpoonup Du$ weakly in $L^q(U; \mathbb{R}^n)$ because $u_k \rightharpoonup u$ weakly in $W^{1,q}(U)$. The expression $D_pL(Du, u_k, x) \to D_pL(Du, u, x)$ converges uniformly on $G_\epsilon$ because

$L$ is smooth and $D_pL$ is uniformly continuous on bounded functions. Then by result (5.4) the expression converges strongly in $L^p(G_\epsilon)$ for any $1 \leq p < \infty$; in particular it converges in the dual of $L^q((G_\epsilon))$. Thus from (5.41)

$$\lim_{k\to\infty} \int_{G_\epsilon} D_pL(Du, u_k, x) \cdot (Du_k - Du)dx = 0. \tag{8.22}$$

Substituting results (8.21), and (8.22) into (8.20) we obtain

$$l = \lim_{k\to\infty} I[u_k] \geq \int_{G_\epsilon} L(Du, u, x)dx. \tag{8.23}$$

9. Finally letting $\epsilon \to 0$ and using the monotone convergence theorem (5.28) we conclude that

$$l \geq \int_U L(Du, u, x)dx = I[u].$$

□

Observe that this result is independent of the specific admissible set $\mathcal{A}$ used i.e. the boundary condition of the problem.

## 8.5 Existence of minimizers Theorem

This following theorem assumes the same hypothesis on the domain $U$ as well as on the lagrangian $L$, smoothness, and coercivity conditions of $L$, and the same definition of functional $I[\cdot]$ over the same admissible set $\mathcal{A}$.

**Theorem 8.6** (Existence of Minimizer Theorem)**.** *Assume that $L$ satisfies the coercivity inequality (8.2) and is convex in the variable $p$. Suppose also that the admissible set $\mathcal{A}$ is nonempty. Then there exists at least one function $u \in \mathcal{A}$ solving*

$$I[u] = \min_{w\in\mathcal{A}} I[w].$$

*Proof.* 1. We know that the functional $I[\cdot]$ is bounded below by the coercivity condition. Since $\mathcal{A}$ is not empty by hypothesis, the set $\{I[w] \mid w \in \mathcal{A}\}$ is not empty and bounded below. Thus the infimum of this set exists, and we can set $m := \inf_{w\in\mathcal{A}} I[w]$, with $-\infty < m \leq +\infty$. If $m = +\infty$ we are done, we do not need to prove nothing. So we assume that $m$ is finite. Meaning $-\infty < m < \infty$.

2. Select a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset W^{1,q}(U)$, which by definition satisfies

$$I[u_k] \to m \text{ as } k \to \infty. \tag{8.24}$$

Observe that we only say that the real sequence $\{I[u_k]\}$ converges to a limit $m$ in $\mathbb{R}$, and the minimizing sequence $\{u_k\}$ does not yet necessarily converge either strongly or weakly in $W^{1,q}(U)$.

3. Without lost of generality, we can take the constant $\beta$ in $L$ be zero, since otherwise we can make the same shifting $\bar{L} := L + \beta$ used before. Thus using the coercivity condition, we have $L \geq \alpha|p|^q$ with $\alpha > 0$, so

$$I[w] \geq \alpha \int_U |Dw|^q \, dx. \tag{8.25}$$

Since $m$ is finite we conclude from (8.24) and (8.25) that

$$\sup_k \|Du_k\|_{L^q(U)} < \infty. \tag{8.26}$$

because the sequence of real numbers $\{I[u_k]\}$ is convergent, thus bounded, and $I[u_k]$ controls the norm $\|Du_k\|_{L^q(U)}$ for each $k$.

4. Fix any function $w \in \mathcal{A}$. Since $u_k$ and $w$ are both equal to $g$ on $\partial U$ in the trace sense, we have $u_k - w \in W_0^{1,q}(U)$. Therefore using the triangle inequality, (8.25), the fact that $w \in \mathcal{A} \subset W^{1,q}(U)$, the fact that $U$ is a bounded set in $\mathbb{R}^n$, and the Poincare's inequality (6.8) we have

$$\begin{aligned} \|u_k\|_{L^q(U)} &\leq \|u_k - w\|_{L^q(U)} + \|w\|_{L^q(U)} \\ &\leq C_1\|Du_k - Dw\|_{L^q(U)} + C_2 \leq C_3, \end{aligned}$$

where $C_1, C_2, C_3$ are real positive constants. Hence $\sup_k \|u_k\|_{L^q(U)} < \infty$. This estimate together with the finiteness (8.25) imply $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{1,q}(U)$.

5. Now we recall the important fact that $W^{1,q}(U)$ is a reflexive Banach space. Applying the weak compactness theorem for $W^{1,q}(U)$, there exist a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u \in W^{1,q}(U)$ such that

$$u_{k_j} \rightharpoonup u \text{ weakly in } W^{1,q}(U). \tag{8.27}$$

6. Now the next important step to do is to check that $u \in \mathcal{A}$. This is the closedness condition in our proof. To see this, note that for $w \in \mathcal{A}$ as above, $u_k - w \in W_0^{1,q}(U)$.

   Now $W_0^{1,q}(U)$ is a linear subspace of $W^{1,q}(U)$. Every linear subspace is convex. Since $W_0^{1,q}(U)$ is defined as a closure it is closed. Then, by Mazur's theorem (4.17), $W_0^{1,q}(U)$ is weakly closed. Hence $u - w \in W_0^{1,q}(U)$. Consequently the trace of $u$ on $\partial U$ is $g$. The Dirichlet boundary condition is satisfied in the trace sense. so $u \in \mathcal{A}$.

7. In the previous weak lower semicontinuity theorem (8.5), we proved that under the given conditions,

$$I[u] \leq \liminf_{j\to\infty} I[u_{k_j}] = m.$$

   But since the infimum $m$ is realized, we conclude that

$$I[u] = m = \min_{w\in\mathcal{A}} I[w]. \tag{8.28}$$

□

## 8.6 Weak Solutions

Consider the boundary-value problem for the Euler-Lagrange PDE associated the functional $L$

$$
\begin{aligned}
-\sum_{i=1}^{n}(L_{p_i}(Du,u,x))_{x_i} + L_z(Du,u,x) &= 0 \text{ in } U, && (8.29)\\
u &= g \text{ on } \partial U. && (8.30)
\end{aligned}
$$

If we multiply (8.29) by a test function $v \in C_c^\infty(U)$, and integrate by parts, we get

$$\int_U [\sum_{i=1}^{n} L_{p_i}(Du,u,x)v_{x_i} + L_z(Du,u,x)v]dx = 0. \tag{8.31}$$

Now we assume that the following growth conditions on $L$ and its derivatives hold.

$$
\begin{aligned}
|L(p,z,x)| &\leq C(|p|^q + |z|^q + 1), && (8.32)\\
|D_pL(p,z,x)| &\leq C(|p|^{q-1} + |z|^{q-1} + 1), && (8.33)\\
|D_zL(p,z,x)| &\leq C(|p|^{q-1} + |z|^{q-1} + 1) && (8.34)
\end{aligned}
$$

for some constant $C$ and all $p \in \mathbb{R}$, $x \in U$.

Under these growth conditions we can verify that the equation (8.31) is valid for a minimizer $u$ and any $v \in W_0^{1,q}(U)$.

**Definition 8.7.** We say that $u \in \mathcal{A}$ is a **weak solution** of the boundary-value problem (8.29) for the Euler Lagrange equation if

$$\int_U [\sum_{i=1}^{n} L_{p_i}(Du,u,x)v_{x_i} + L_z(Du,u,x)v]dx = 0. \tag{8.35}$$

for all $v \in W_0^{1,q}(U)$.

**Theorem 8.8** (Solution of the Euler-Lagrange equation)**.** *Assume $L$ verifies the growth conditions (8.32,8.33,8.34), and $u$ satisfies*

$$I[u] = \min_{w\in\mathcal{A}} I[w]. \tag{8.36}$$

*Then $u$ is a weak solution of (8.29).*

*Proof.* Take $v \in W_0^{1,q}$ and set

$$i(\tau) := I[u + \tau v] \text{ for all } \tau \in \mathbb{R}. \tag{8.37}$$

Using continuity of $i(\tau)$, the growing conditions (8.32,8.33,8.34), the Young's inequality, and the Dominated Convergence Theorem we can verify that

$$i'(0) = \lim_{i \to 0} \frac{i(\tau) - i(0)}{\tau} = \int_U [\sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v] dx = 0. \tag{8.38}$$

thus $u$ is a weak solution. □

*Remark* 8.9. In general the Euler-Lagrange equation (8.31) is a only a necessary condition for minimizers. However, if the joint map $(p, z) \mapsto L(p, z, x)$ is convex for each $x$, then each weak solution is in fact a minimizer.

# Chapter 9: Free Boundary Problems

## 9.1 Existence of Minimizers for problems with Free Boundaries

Let $U \subset \mathbb{R}^n$ be a domain. Consider the the following functional with a variable domain of integration

$$J[u] = \int_{U\cap\{u>0\}} f(x, u(x), \nabla u(x))dx.$$

We want to minimize $J(u) \to \min$, with the conditions: $u = 0$ on $\Gamma := U \cap \partial\{u > 0\}$, $u = 0$ on $U \setminus \{u > 0\}$. The set $\Gamma$ is called the **free boundary**.

If we assume that the Lagrangian function $f = f(x, z, p)$ is smooth and, the free boundary $\Gamma$ is smooth, we can calculate the first variation using a Taylor's expansion around the absolute minimum $u$ of $J$. This is done for example in the book Calculus of Variations by Gelfand and Fomin. The Euler-Lagrange equation is

$$\nabla \cdot f_p - f_z = 0 \text{ in } N := U \cap \{u > 0\}, \tag{9.1}$$

and the free boundary conditions

$$u = 0, \;\; f_p \cdot \nabla u - f = 0 \text{ in } \Gamma. \tag{9.2}$$

Two global conditions for an absolute minimum are

$$f \text{ convex in } p, \qquad f(x, z, 0) \geq 0$$

Consider the Lagrangian $f(x, z, p) := |p|^2 + Q^2$, where $Q = Q(x)$ is a function. Using (9.1), and (9.2) we obtain

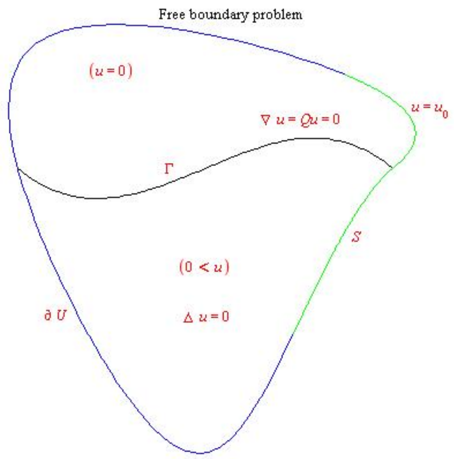
$$\triangle u = 0 \text{ in } N, \qquad u = 0, \; |\nabla u| = Q \text{ in } \Gamma. \tag{9.3}$$

If we do not impose any restriction on the class of sets $\{u > 0\}$ there is an existence proof done in reference [3]. Here we describe the basic steps of a bounded version of this proof.

Let $U \subset \mathbb{R}^n$ be a bounded domain, $\partial U$ is $C^1$. $S \subset \partial U$ is open and has $n-1$ dimensional Hausdorff measure $|S| > 0$. The Dirichlet boundary condition on $S$ is determined by a function $w \in H^1(U)$ such that $w \geq 0$. Consider the functional

$$J[u] := \int_U (|\nabla u|^2 + \chi_{\{u>0\}} Q^2) dx$$

in the admissible set $K := \{u \in H^1(U) \mid u = w \text{ on } S\}$ so $K$ is a closed convex subspace of $H^1(U)$. Observe that the functional $J[u]$ is not differentiable as a function of $u$. For this reason we need some technicality to determine, in appropriate sense, its first variation and boundary conditions on the fixed boundary $\partial U$ and on the free boundary $\Gamma$.




**Theorem 9.1** (Existence Theorem for a Free Boundary Problem)**.** *There exists an absolute minimum $u \in K$ of the functional $J$.*

*Proof.* We assume that $0 < Q < Q_{max} < \infty$, and the domain $U$ is a bounded. Since $0 \leq J[u]$ for all $u \in H^1(U)$ we can set the number $m := \inf\{J[u] \mid u \in K\}$. If $m = +\infty$ we are done. So we assume that $m$ is finite and nonnegative $0 \leq m < \infty$. Then there exists a minimizing sequence $\{u_n\} \subset K$ that satisfies $J[u_k] \to m$.

We can rewrite the functional as $J[u] = J_1[u] + J_2[u]$ where $J_1[u] := \int_U |\nabla u|^2 dx$, $J_2[u] := \int_U \chi_{\{u>0\}} Q^2 dx$. The first term $J_1[\cdot]$ satisfies coercivity because its integrand $|p|^2$ is coercive. The second term $J_2[\cdot]$ is bounded because $0 \leq \chi_{\{u>0\}} Q^2 < \infty$ implies $0 \leq \int_U \chi_{\{u>0\}} Q^2 dx < \infty$. Adding a bounded functional to a coercive functional the coercivity is preserved. Thus $J[\cdot]$ satisfies coercivity.

Using the same argument in previous existence theorem (8.6), we can have a sequence $\{u\}$ such that $u_k \rightharpoonup u$ weakly in $K$.

We need to show that $J[\cdot]$ is weakly lower semicontinuous. According with the weak lower semicontinuity theorem (8.5), the first term $J_1[\cdot]$ is is weak lower semicontinuous because its integrand $|p|^2$ is smooth, convex in $p$ and bounded below.

We will show that the second term $J_2[\cdot]$ is also weakly lower semicontinuous. Using the similar steps at the beginning of the proof the weak lower semicontinuity theorem (8.5) we have $u_k \to u$ pointwise a.e. in $U$. Now, for any $x \in U$, $\chi_{\{u>0\}}(x) > 0$ implies $u(x) > 0$. So $u_k(x) > 0$ for large enough $k$. Then fixing such large enough $k$ we have for almost all $x \in U$

$$\chi_{\{u>0\}}(x) \leq \chi_{\{u_k>0\}}(x)$$

because $\chi_{\{u_k>0\}}(x) = 1$ for $k$ sufficiently large for almost all $x \in U$ such that $\chi_{\{u_k>0\}}(x) = 1$. Hence using monotonicity of lim inf

$$\chi_{\{u>0\}}(x) \quad \leq \quad \liminf_{k\to\infty} \chi_{\{u_k>0\}}(x)$$

Multiplying both sides of the last inequality by $Q^2(x)$ and integrating over $U$, we have

$$\begin{aligned} \int_U Q^2(x)\chi_{\{u>0\}}(x)dx &\leq \int_U \liminf_{k\to\infty} [Q^2 \chi_{\{u_k>0\}}(x)]dx \ \ \text{(monotonicity)} \\ &\leq \liminf_{k\to\infty} \int_U \chi_{\{u_k>0\}}(x)dx \ \ \text{(Fatou's lemma).} \end{aligned}$$

Finally we use the fact that the sum of two weakly lower semicontinuous functionals is also a weakly lower semicontinuous functional by (2.39). Thus there exists at least one $u \in K$ such that $J[u] = \min\{J[v] \mid v \in K$. □

Consider the following example of solution of free boundary problem in two dimensions

$$U = \{(x, y) \in \mathbb{R}^2 \mid |x| < 2 \text{ and } |y| < 1\}$$

and $S = \{(x, y) \in \partial U \mid |x| = 2\}$, $w = 1$, $Q = 1$. The absolute minimum solution is $u(x) = \max\{0, |x| - 1\}$. In this case we have $U = U_1 \cup U_2 \cup U_3$ where $U_1 = \{(x, y) \in U \mid -2 < x < -1\}$, $U_2 = \{(x, y) \in U \mid |x| < 1\}$, $U_3 = \{(x, y) \in U \mid 1 < x < 2$. $u(x) = 0$ in $R_2$, and $u(x)$ is harmonic in $R_1 \cup R_3$. The Dirichlet data is $u(x)|_S = 1$. The free-boundary is $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \{(x, y) \in U \mid x = -1\}$ and $\Gamma_2 = \{(x, y) \in \mathbb{R}^2 \mid x = 1\}$. The free-boundary conditions are $u(x)|_S = 0$ and $\partial_{-\nu} u = 1$.

## 9.2 Applications

In this section we are going describe some applications problems that are related with free-boundary problems. The free-boundary conditions we have considered appears, for example, in fluid problems such as jets and cavities. To simplify the description we consider the case of incompressible, irrotational, and inviscid fluids. In this case the velocity of the fluid is given by a $\nabla\phi$ where $\phi$ is the velocity potential. In two dimensions the harmonic conjugate $\psi$ is called the stream function. $\phi$ and $\psi$ are related by the Cauchy-Riemann equations $\phi_x = \psi_y$, $\phi_y = -\psi_x$. The stream lines are determined by $\psi = const$.

The velocity potential $\phi$ satisfy the Bernoulli's law $|\nabla\phi|^2 + 2p = \text{const}$ where $p$ is the pressure. The free-boundary in this case is the interface between the fluid and the air. On the free boundary, the pressure is constant and we have that $|\nabla\phi|$ is constant. The stream function $\psi$ satisfies

$$\triangle\psi = 0 \qquad \text{in the fluid,} \tag{9.4}$$

$$\psi = 0, \ \frac{\partial\psi}{\partial\nu} = c \qquad \text{on the free boundary,} \tag{9.5}$$

where $c$ is a constant. Using conformal mappings the solution of the two-dimensional jet problem has been established in the literature.

In three dimensions we can not use conformal methods to solve free boundary problems. However in the reference [4] they considered axially symmetric jet problem and used the variational approach to establish existence of solutions. This work was based on the general existence theorem by Alt and Caffarelli [3].

# Chapter 10: Conclusion

We have considered a class of boundary value problems that can be reformulated as a variational problem. This kind of problems includes many important problems in science and technology. We only used the Lagrangian formulation of variational problems. This formulation is suitable for elliptic problems or steady state of evolution problems. The specific method to prove existence of minimizers is the direct method. One advantage of this method is that we can deal with nonlinear problems. The class of nonlinear problems are very interesting because it can capture new features that are not captured by linear models.

The source of nonlinearity we have been motivated by in the present work comes from the fact that in many cases the domain of the problem is a priori unknown, so that we have free-boundaries. It turns out that the direct method for existence theorems can also be used successfully for some specific free boundary problems. Two key concepts are weak sequential compactness and lower semicontinuity in Sobolev spaces. The goal of the present work is to provide a basic exploration of the mathematical tools required to begin to study applications problems related to free-boundary problems. As possible next step it would be interesting to study more advanced aspects of stationary problems, after that evolution problems for example free-moving problems.

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# Vita

Carlos Tello was born in Lima, Peru. He did his undergraduate study in Peru, obtaining a BS degree in Physics. In 2001 he completed a PhD degree in Physics in Brazil. Next he moved to USA and began working as an adjunct faculty teaching Physics and Mathematics in New Jersey and North Carolina for 5 years. Recently he finished his graduate study at Wake Forest University for a MA degree in Mathematics. He also has worked at the Institutional Research Office of this institution. He has interest in continue his career in applied research and development sector requiring good mathematical physics background.