RESONANCE PROBLEMS OF THE FUČÍK SPECTRUM USING VARIATIONAL METHODS

BY

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Abstract

Quinn A. Morris

The Fučík spectrum of a linear operator, L, is defined to be the set,

$$\Sigma = \left\{ (a,b) \in \mathbb{R}^2 : \text{ there is a non-trivial solution to } Lu = au^+ - bu^- \right\}.$$

The Fučík spectrum is important since it reflects parameter values for which the existence of solutions to the equation $Lu = au^+ - bu^- + g(u)$ may change. In this thesis, we examine the Fučík spectrum of both matrix and differential operators. A variational characterization due to Castro and a standard saddle point theorem are used to determine existence of solutions in non-resonance and resonance cases, with the development of generalized orthogonality conditions for existence of solutions in the resonance cases. Our results improve upon existing theorems due to Marguiles and Marguiles in the matrix case, and Bliss, Buerger, and Rumbos in the differential case.

Chapter 1: Introduction

Given some linear operator, L, consider the equation

$$Lu = au^+ - bu^- + g(u), \tag{1.1}$$

where, if $\mathbf{u}$ is a vector, then $\mathbf{u}^+ = \begin{bmatrix} u_1^+ \\ \vdots \\ u_n^+ \end{bmatrix}$, $\mathbf{u}^- = \begin{bmatrix} u_1^- \\ \vdots \\ u_n^- \end{bmatrix}$ and $u_i^\pm = \max\{0, \pm u_i\}$ and if u is a function, $u^\pm(t) = \max\{0, \pm u(t)\}$.

The object of interest in this thesis is the Fučík spectrum of L, denoted Σ, where

$$\Sigma = \left\{(a,b) \in \mathbb{R}^2 : \text{ there is a non-trivial solution to } Lu = au^+ - bu^-\right\}. \tag{1.2}$$

The Fučík spectrum is of interest since it is known to identify parameter values where the solvability of equations of the form $Lu = au^+ - bu^- + g(u)$ may change. Motivation for studying such problems often depends on the choice of operator. When L is a second derivative operator, such equations model asymmetric oscillating systems. An example is a suspension bridge, which experiences a spring-like restoring force in both directions due to the stiffness of the road bed and a tension force from the suspension cables when it moves below equilibrium (see [4], [6]). If L is a matrix, such problems generally arise in the numerical study of differential equations, as in [7].

Success has been found proving existence theorems for problems of this type using degree-theoretic arguments, as in [4] and [7]. More recently, theorems due to Castro and Drábek and Robinson (see [2] and [3], respectively) have used a variational approach to solve such problems. Using the method of Castro in [2], we wish to find a variational characterization of the Fučík spectrum for both matrix and differential operators, and then use the variational characterization to find conditions for existence of solutions to (1.1), in both resonance and non-resonance cases (i.e., $(a, b) \in \Sigma$ and $(a, b) \notin \Sigma$, respectively). In the resonance case, we establish a generalized orthogonality condition, a type of Fredholm Alternative, for existence of a solution.

By treating the matrix and ODE cases in parallel, we are able to highlight the improvements that we have made to the existing literature (see [7] and [1]) and clearly highly where the finite and infinite dimensional cases differ. Not surprisingly, the most prominent differences arise the in compactness arguments.

The idea of the variational method is that in some cases, if it is difficult to find solutions to a particular equation, say $f(x) = 0$, directly, it may be easier to identify an appropriate functional, call it F, such that $F'(x) = f(x)$, and look for critical points of F. As a more concrete example, consider that, if we wished to solve $x^3 + x + 1 = 0$, we could examine the function $F(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2 + x$ and look for critical points. While this may not make it easier to solve the equation analytically, it does make it easier to prove the existence of solutions.

In order to determine the existence of a solution, there are two main areas which must be examined. First, we must look at the geometry of the appropriate functional. In particular, we will show that the chosen functional is concave and anticoercive on certain subsets of the domain. We will then show that the functional is bounded below, or possibly coercive, on a complementary subset of the domain.

Definition 1. *A functional, $F : D \to \mathbb{R}$, is concave on a subset $U \subset D$ if, for all $x, y \in U$, $\langle \nabla F(y) - \nabla F(x), y - x \rangle \leq 0$. If $\langle \nabla F(y) - \nabla F(x), y - x \rangle < 0$, the F is called strictly concave.*

Definition 2. *A functional, $F : D \to \mathbb{R}$, is anticoercive on a subset $U \subset D$ if, given any sequence $\{x_k\}_{k=1}^{\infty} \subset U$ such that $\|x_k\|_D \to \infty$, $F(x_k) \to -\infty$.*

Thinking of these two properties geometrically, the concavity property will give us that any line drawn between two points in the image of F will lie below the actual functional values along that line. The anticoercive property tells us that as one goes out to ∞ in any direction, the values of the functional go towards $-\infty$. Once we have established that our functional is concave and anticoercive on some subset, then we will show that the functional obtains an absolute maximum on that subset. If we then take the minimum over all such subsets, we hope to find a critical value. Such an approach is often called a minimax characterization of the critical value.

In addition to the geometry, however, we must also establish a second property, which is known as the Palais-Smale compactness condition.

Definition 3 (PS)**.** *Let H be a Hilbert space. A functional $F \in C^1(H, \mathbb{R})$ satisfies the Palais-Smale compactness condition if each sequence $\{u_k\}_{k=1}^{\infty}$, such that $\{F(u_k)\}_{k=1}^{\infty}$ is bounded and $\nabla F(u_k) \to 0$ in H, is precompact in H.*

In a finite number of dimensions, this condition is automatically satisfied if the given functional is such that the inverse image of any compact set is itself compact. In infinite dimension, the condition is more obviously necessary due to the more complex notion of compactness. Establishing (PS) for cases where $(a, b) \in \Sigma$ often requires the use of a generalized orthogonality condition. Specifically, a Landesman-Lazer condition is often used. There are a great variety of Landesman-Lazer type conditions. In finite dimension, we will use a Landesman-Lazer condition of the form,

Definition 4 (LLM)**.** *Let $(a, b) \in \Sigma$. For a bounded gradient field $g : \mathbb{R}^n \to \mathbb{R}^n$, if every sequence, $\{\mathbf{x_k}\}_{k=1}^{\infty} \subset \mathbb{R}^n$, such that $\mathbf{x_k} \to \mathbf{x}$, a Fučík eigenvector associated with (a, b), is such that*

$$\lim_{k\to\infty} \langle g(\mathbf{x_k}), \mathbf{x} \rangle > 0,$$

then the Landesman-Lazer condition is satisfied.

In the infinite dimensional case, we will use a Landesman-Lazer condition of the form,

Definition 5 (LLD). *Let $(a, b) \in \Sigma$ and $g : \mathbb{R} \to \mathbb{R}$ be a bounded, continuous function with $G(u) := \int_0^u g(t)dt$. If every sequence $\{u_k\}_{k=1}^{\infty} \subset H$ such that $u_k \to \Psi$, a Fučík eigenfunction associated with (a, b), is such that*

$$\left[G^+ \int_{\Psi > 0} \Psi \, dt + G^- \int_{\Psi < 0} \Psi \, dt \right] \neq 0,$$

where

$$G^+ = \lim_{u \to +\infty} \frac{G(u)}{u} \text{ and } G^- = \lim_{u \to -\infty} \frac{G(u)}{u},$$

then the Landesman-Lazer condition is satisfied.

Establishing such a compactness condition is necessary in order to apply a standard Saddle Point Theorem. The theorem stated below is taken as a consequence of Theorems 2.8 & 2.9 in [11].

Theorem 1.1. *Let H be a Hilbert space, let $X \subset H$, and let $E : H \to \mathbb{R}$ be a C^1 functional. Let $B_R := \{x \in X : \|x\| \leq R\}$, let $\gamma_0 : \partial B_R \to H$ be a continuous function, let $\Gamma := \{\gamma : B_R \to H : \gamma \in C(B_R, H), \gamma|_{\partial B_R} \equiv \gamma_0\}$, and let $c := \inf_{\gamma \in \Gamma} \sup_{x \in B_R} E(\gamma(x))$. If $\sup_{x \in \partial B_R} E(\gamma_0(x)) < c$ and if E satisfies (PS), then c is a critical value of E.*

1.1 Preliminaries

Before beginning, we will define some basic notation which we will use throughout the following chapters.

In the case where L is a differential operator, we will make heavy use of two very important function spaces, the Lebesgue space, $L^2[0, 2\pi]$, and the Sobolev space,

$W^{1,2}[0,2\pi]$. Define

$$L^2[0,2\pi] := \{f : [0,2\pi] \to \mathbb{R} : f \text{ is measurable, and } \int_0^{2\pi} f^2\,dt < \infty\}.$$

In other words, $L^2[0,2\pi]$ is the space of square integrable function. The norm associated with the space is

$$\|f\|_{L^2} := \left(\int_0^{2\pi} f^2\,dt\right)^{\frac{1}{2}}.$$

Using the definition of $L^2[0,2\pi]$, we can now define $W^{1,2}[0,2\pi]$. Let

$$W^{1,2}[0,2\pi] := \{f \in L^2[0,2\pi] : f \text{ is absolutely continuous, and } f' \in L^2[0,2\pi]\},$$

with associated norm,

$$\|f\|_H = \left(\int_0^{2\pi} f^2\,dt + \int_0^{2\pi} (f')^2\,dt\right)^{\frac{1}{2}}.$$

We will henceforth refer to the space simply as H and the norm as $\|\cdot\|_H$, consistent with a number of other texts. A rather important property of H, which we will make use of quite often, is the fact that H has a compact embedding into both L^2 and C. This standard theorem taken from [5] can be found in many functional analysis textbooks.

Theorem 1.2. *There is a $c > 0$ such that $|u(x) - u(y)| \leq c|x-y|^{\frac{1}{2}}$ $\forall x, y \in [0,2\pi]$ and $\forall u \in H$.*

An application of the Arzela-Ascoli theorem will then give that the inclusion map, $H \to C[0,2\pi]$ is a compact operator. It follows that $H \to L^2[0,2\pi]$ is also compact. A full proof of the Sobolev Embedding theorem can be found in [5].

In the matrix case, we will be interested in vectors $\mathbf{u} \in \mathbb{R}^n$. We will decompose our space $\mathbb{R}^n = X \oplus Y$, where X and Y are defined as the span of eigenvectors of a matrix,

A. It will often be helpful to decompose individual vectors, $\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{u} = \mathbf{x}+\mathbf{y}$ with $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Note, however, that $\mathbf{x}$ and $\mathbf{y}$ are assumed to have to same dimension as that of the full space. Occasionally, it will also be helpful to consider a sort of directional derivative, which we will denote $\nabla_X F(\mathbf{u}) \cdot (\mathbf{x} + \mathbf{y}) = \nabla F(\mathbf{u}) \cdot \mathbf{x}$. Similar notation will be used for a derivative in the Y direction.

Finally, we will use the same notation for the differential equations case, except our space will be H, and the decomposition $H = X \oplus Y$ is used, assuming that X is finite dimensional.

Chapter 2: Matrix Case

In this first chapter, we consider the case $L = A$, where A is an $n \times n$ symmetric matrix and we let $\mathbf{u} \in \mathbb{R}^n$.

We must first identify an appropriate functional. In general, we wish to find a functional J such that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\nabla J(\mathbf{u}) \cdot \mathbf{v} = \left\langle A\mathbf{u} - a\mathbf{u}^+ + b\mathbf{u}^-, \mathbf{v} \right\rangle$$

Lemma 1. *Let $J : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$J(\mathbf{u}) = \frac{1}{2} \langle A\mathbf{u}, \mathbf{u} \rangle - \frac{a}{2} \left\langle \mathbf{u}^+, \mathbf{u}^+ \right\rangle - \frac{b}{2} \left\langle \mathbf{u}^-, \mathbf{u}^- \right\rangle$$

Then $\nabla J(\mathbf{u}) \cdot \mathbf{v} = \langle A\mathbf{u} - a\mathbf{u}^+ + b\mathbf{u}^-, \mathbf{v} \rangle$.

Proof. Let $f(x) = (x^+)^2$. Then

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Clearly, f is continuous on all of $\mathbb{R}$ and f is differentiable for any $x \neq 0$. Moreoever, at zero, we see that,

$$\begin{aligned} \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^-} \frac{0}{x} \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^-} \frac{x^2}{x} \\ &= \lim_{x \to 0^-} x \\ &= 0. \end{aligned}$$

So,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 0,$$

and hence f is a continuously differentiable function, with

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Written in another form, $f'(x) = 2x^+$.

Now, let $h(x) = (x^-)^2$. Then

$$h(x) = \begin{cases} 0 & x \geq 0 \\ x^2 & x < 0 \end{cases}$$

Similarly, $h(x) = (x^-)^2$ is continuously differentiable, with $h'(x) = -2x^-$. The negative sign comes from an application of the chain rule.

If we now consider the quantity $\langle \mathbf{u}^{\pm}, \mathbf{u}^{\pm} \rangle$ we note that

$$\left\langle \mathbf{u}^{\pm}, \mathbf{u}^{\pm} \right\rangle = \left(u_1^{\pm}\right)^2 + \left(u_2^{\pm}\right)^2 + \cdots + \left(u_n^{\pm}\right)^2,$$

is continuously differentiable in each variable, so $\nabla \left(\langle \mathbf{u}^{\pm}, \mathbf{u}^{\pm} \rangle \right) = \pm 2\mathbf{u}^{\pm}$.

As for the one remaining term, note that

$$\lim_{\|\mathbf{u}-\mathbf{u_0}\| \to 0} \left| \frac{\langle A\mathbf{u}, \mathbf{u} \rangle - \langle A\mathbf{u_0}, \mathbf{u_0} \rangle - 2 \langle A\mathbf{u_0}, \mathbf{u} - \mathbf{u_0} \rangle}{\|\mathbf{u} - \mathbf{u_0}\|} \right|$$

$$= \lim_{\|\mathbf{u}-\mathbf{u_0}\| \to 0} \left| \frac{\langle A\mathbf{u}, \mathbf{u} \rangle - \langle A\mathbf{u_0}, \mathbf{u_0} \rangle - \langle A\mathbf{u_0}, \mathbf{u} \rangle - \langle A\mathbf{u_0}, \mathbf{u} \rangle + \langle A\mathbf{u_0}, \mathbf{u_0} \rangle + \langle A\mathbf{u_0}, \mathbf{u_0} \rangle}{\|\mathbf{u} - \mathbf{u_0}\|} \right|$$

$$= \lim_{\|\mathbf{u}-\mathbf{u_0}\| \to 0} \left| \frac{\langle A(\mathbf{u} - \mathbf{u_0}), \mathbf{u} - \mathbf{u_0} \rangle - \langle A\mathbf{u_0}, \mathbf{u} - \mathbf{u_0} \rangle}{\|\mathbf{u} - \mathbf{u_0}\|} \right|$$

$$= \lim_{\|\mathbf{u}-\mathbf{u_0}\| \to 0} \left| \frac{\langle A\mathbf{u} - \mathbf{u_0}, \mathbf{u} - \mathbf{u_0} \rangle}{\|\mathbf{u} - \mathbf{u_0}\|} \right|$$

$$= 0.$$

So, $\langle A\mathbf{u}, \mathbf{u}\rangle$ is differentiable, with $\nabla(\langle A\mathbf{u}, \mathbf{u}\rangle) = 2A\mathbf{u}$.

Therefore, we may conclude that $\nabla J(\mathbf{u}) \cdot \mathbf{v} = \langle A\mathbf{u} - a\mathbf{u}^+ + b\mathbf{u}^-, \mathbf{v}\rangle$. $\square$

We now examine the geometry of the functional J on certain subsets of $\mathbb{R}^n$. We begin by showing a lemma, which will be of use in establishing properties of the functional. It should be noted here that, while the lemma may seem arbitrary and might not be the most logical way to proceed upon first seeing this type of problem, after repeated estimations of the same type, it was determine that this single inequality would help establish many of the desired properties.

Lemma 2. *Let A be a real-valued, $n \times n$ symmetric matrix with eigenvalues, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, and corresponding eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Given a such that $\lambda_k < a < \lambda_{k+1}$ for some k and $b > a$, define $X := span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and $Y := span\{\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n\}$. Let $J : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$J(\mathbf{u}) = \frac{1}{2}\langle A\mathbf{u}, \mathbf{u}\rangle - \frac{a}{2}\left\langle \mathbf{u}^+, \mathbf{u}^+\right\rangle - \frac{b}{2}\left\langle \mathbf{u}^-, \mathbf{u}^-\right\rangle.$$

Let $s = b - a > 0$. Then there is an $\epsilon > 0$ such that, for all $\mathbf{x_1}, \mathbf{x_2} \in X$ and $\mathbf{y_1}, \mathbf{y_2} \in Y$,

$$\begin{aligned}\langle \nabla J(\mathbf{x_2} + \mathbf{y_2}) - &\ \nabla J(\mathbf{x_1} + \mathbf{y_1}), \mathbf{x_2} - \mathbf{x_1}\rangle \\ &\leq -\epsilon\|\mathbf{x_2} - \mathbf{x_1}\|^2 + s(\|\mathbf{x_2} - \mathbf{x_1}\| + \|\mathbf{y_2} - \mathbf{y_1}\|)\|\mathbf{y_2} - \mathbf{y_1}\|.\end{aligned}$$

Proof. Let

$$D = \langle \nabla J(\mathbf{x_2} + \mathbf{y_2}) - \nabla J(\mathbf{x_1} + \mathbf{y_1}), \mathbf{x_2} - \mathbf{x_1}\rangle.$$

This gives

$$\begin{aligned}
D = \Big\langle A(\mathbf{x_2}+\mathbf{y_2}) - a(\mathbf{x_2}+\mathbf{y_2})^+ + b(\mathbf{x_2}+\mathbf{y_2})^-, \mathbf{x_2}-\mathbf{x_1}\Big\rangle \\
- \Big\langle A(\mathbf{x_1}+\mathbf{y_1}) - a(\mathbf{x_1}+\mathbf{y_1})^+ + b(\mathbf{x_1}+\mathbf{y_1})^-, \mathbf{x_2}-\mathbf{x_1}\Big\rangle \\
= \Big\langle A(\mathbf{x_2}+\mathbf{y_2}) - a(\mathbf{x_2}+\mathbf{y_2}) + s(\mathbf{x_2}+\mathbf{y_2})^-, \mathbf{x_2}-\mathbf{x_1}\Big\rangle \qquad (\text{since } \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-) \\
- \Big\langle A(\mathbf{x_1}+\mathbf{y_1}) - a(\mathbf{x_1}+\mathbf{y_1}) + s(\mathbf{x_1}+\mathbf{y_1})^-, \mathbf{x_2}-\mathbf{x_1}\Big\rangle \\
= \langle A(\mathbf{x_2}-\mathbf{x_1}) - a(\mathbf{x_2}-\mathbf{x_1}), \mathbf{x_2}-\mathbf{x_1}\rangle + \langle A(\mathbf{y_2}-\mathbf{y_1}) - a(\mathbf{y_2}-\mathbf{y_1}), \mathbf{x_2}-\mathbf{x_1}\rangle \\
+ s\Big\langle (\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-, \mathbf{x_2}-\mathbf{x_1}\Big\rangle \\
= \langle A(\mathbf{x_2}-\mathbf{x_1}), \mathbf{x_2}-\mathbf{x_1}\rangle - a\langle \mathbf{x_2}-\mathbf{x_1}, \mathbf{x_2}-\mathbf{x_1}\rangle \qquad (\text{since } X \perp Y) \\
+ s\Big\langle (\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-, \mathbf{x_2}-\mathbf{x_1}\Big\rangle .
\end{aligned}$$

If we then make the substitution $\mathbf{x_2}-\mathbf{x_1} = (\mathbf{x_2}+\mathbf{y_2}) - (\mathbf{x_1}+\mathbf{y_1}) - (\mathbf{y_2}-\mathbf{y_1})$, and we note that

$$\begin{aligned}
\Big\langle (\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-,\ \mathbf{x_2}-\mathbf{x_1}\Big\rangle \\
= \Big\langle (\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-, (\mathbf{x_2}+\mathbf{y_2}) - (\mathbf{x_1}+\mathbf{y_1})\Big\rangle \\
- \Big\langle (\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-, \mathbf{y_2}-\mathbf{y_1}\Big\rangle \\
\leq - \Big\langle (\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-, \mathbf{y_2}-\mathbf{y_1}\Big\rangle ,
\end{aligned}$$

since $f(\mathbf{x}) = \mathbf{x}^-$ is a monotone decreasing function. By application of the Cauchy-Schwarz and triangle inequalities, coupled with the fact that $\|\mathbf{v}^- - \mathbf{w}^-\| \leq \|\mathbf{v}-\mathbf{w}\|$ we get

$$
\begin{aligned}
D \leq\ & \langle A(\mathbf{x_2}-\mathbf{x_1}), \mathbf{x_2}-\mathbf{x_1}\rangle - a\langle \mathbf{x_2}-\mathbf{x_1}, \mathbf{x_2}-\mathbf{x_1}\rangle \\
& \qquad - s\left\langle (\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-, \mathbf{y_2}-\mathbf{y_1}\right\rangle \\
\leq\ & \langle A(\mathbf{x_2}-\mathbf{x_1}), \mathbf{x_2}-\mathbf{x_1}\rangle - a\langle \mathbf{x_2}-\mathbf{x_1}, \mathbf{x_2}-\mathbf{x_1}\rangle \\
& \qquad + s\|(\mathbf{x_2}+\mathbf{y_2})^- - (\mathbf{x_1}+\mathbf{y_1})^-\|\|\mathbf{y_2}-\mathbf{y_1}\| \qquad (2.1) \\
\leq\ & \langle A(\mathbf{x_2}-\mathbf{x_1}), \mathbf{x_2}-\mathbf{x_1}\rangle - a\langle \mathbf{x_2}-\mathbf{x_1}, \mathbf{x_2}-\mathbf{x_1}\rangle \\
& \qquad + s\|(\mathbf{x_2}+\mathbf{y_2}) - (\mathbf{x_1}+\mathbf{y_1})\|\|\mathbf{y_2}-\mathbf{y_1}\| \\
\leq\ & \langle A(\mathbf{x_2}-\mathbf{x_1}), \mathbf{x_2}-\mathbf{x_1}\rangle - a\langle \mathbf{x_2}-\mathbf{x_1}, \mathbf{x_2}-\mathbf{x_1}\rangle \\
& \qquad + s\left[\|\mathbf{x_2}-\mathbf{x_1}\| + \|\mathbf{y_2}-\mathbf{y_1}\|\right]\|\mathbf{y_2}-\mathbf{y_1}\|
\end{aligned}
$$

Since $a > \lambda_k$, choose ϵ such that $a = (1+\epsilon)\lambda_k$. Then

$$
\begin{aligned}
\langle A(\mathbf{x_2}-\mathbf{x_1}), \mathbf{x_2}-\mathbf{x_1}\rangle - a\langle \mathbf{x_2}-\mathbf{x_1}, \mathbf{x_2}-\mathbf{x_1}\rangle &\leq \lambda_k\|\mathbf{x_2}-\mathbf{x_1}\|^2 - (1+\epsilon)\lambda_k\|\mathbf{x_2}-\mathbf{x_1}\|^2 \\
&\leq -\epsilon\|\mathbf{x_2}-\mathbf{x_1}\|^2 \qquad (2.2)
\end{aligned}
$$

since λ_k is the largest eigenvalue acting on elements of X, and hence $\langle A\mathbf{x}, \mathbf{x}\rangle \leq \lambda_k \langle \mathbf{x}, \mathbf{x}\rangle$ for all $\mathbf{x} \in X$.

Combining (2.1) and (2.2), we get the desired inequality,

$$
D \leq -\epsilon\|\mathbf{x_2}-\mathbf{x_1}\|^2 + s\left[\|(\mathbf{x_2}-\mathbf{x_1})\| + \|(\mathbf{y_2}-\mathbf{y_1})\|\right]\|\mathbf{y_2}-\mathbf{y_1}\| \qquad (2.3)
$$

□

Now that we have established this important inequality, we proceed to prove the properties of the functionals J and $\tilde{J}$.

Theorem 2.1. *Under the hypotheses of Lemma 2,*

1. *For fixed $\mathbf{y} \in Y$, J is convex and anticoercive on the set $\mathbf{y} + X$ and achieves a unique maximum.*

2. There exists a continuous function, $r : Y \to X$ such that,

(a) $\tilde{J}(\mathbf{y}) := J(r(\mathbf{y}) + \mathbf{y}) = \max\{J(\mathbf{y} + \mathbf{x}) : \mathbf{x} \in X\}$*,*

(b) $\tilde{J} \in C^1(Y, \mathbb{R})$*, and*

(c) given $t \geq 0$*,* $r(t\mathbf{y}) = tr(\mathbf{y})$ *and* $\tilde{J}(t\mathbf{y}) = t^2 \tilde{J}(\mathbf{y})$ $\forall \mathbf{y} \in Y$*.*

Proof. Since X and Y are complementary subspaces of $\mathbb{R}^n$, then for any vector $\mathbf{u} \in \mathbb{R}^n$, we can write $\mathbf{u} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Now, we consider J restricted to the set $\mathbf{y} + X = \{\mathbf{y} + \mathbf{x} : \mathbf{x} \in X\}$, where $\mathbf{y}$ is some fixed element of Y, and examine D, a difference quotient for ∇J.

With $\mathbf{y} \in Y$ fixed and $\mathbf{x_1}, \mathbf{x_2} \in X$, note that (2.3) simplifies to

$$\langle \nabla J(\mathbf{x_2} + \mathbf{y}) - \nabla J(\mathbf{x_1} + \mathbf{y}), \mathbf{x_2} - \mathbf{x_1} \rangle \leq -\epsilon \|\mathbf{x_2} - \mathbf{x_1}\|^2, \tag{2.4}$$

which shows that J is strictly concave on $\mathbf{y} + X$.

Examining the value of J on $\mathbf{y} + X$, we note that, since X and Y are orthogonal and X and Y are invariant with respect to the matrix A (i.e., $A\mathbf{x} \in X$ $\forall \mathbf{x} \in X$ and $A\mathbf{y} \in Y$ $\forall \mathbf{y} \in Y$),

$$\begin{aligned}
2J(\mathbf{x} + \mathbf{y}) &= \langle A(\mathbf{x} + \mathbf{y}), \mathbf{x} + \mathbf{y} \rangle - a \left\langle (\mathbf{x} + \mathbf{y})^+, (\mathbf{x} + \mathbf{y})^+ \right\rangle \\
&\qquad\qquad - b \left\langle (\mathbf{x} + \mathbf{y})^-, (\mathbf{x} + \mathbf{y})^- \right\rangle \\
&= \langle A(\mathbf{x} + \mathbf{y}), \mathbf{x} + \mathbf{y} \rangle - a \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\
&\qquad\qquad - s \left\langle (\mathbf{x} + \mathbf{y})^-, (\mathbf{x} + \mathbf{y})^- \right\rangle \\
&= \langle A\mathbf{x}, \mathbf{x} \rangle + \langle A\mathbf{y}, \mathbf{y} \rangle - a \langle \mathbf{x}, (\mathbf{x}) \rangle \\
&\qquad\qquad - a \langle \mathbf{y}, \mathbf{y} \rangle - s \left\langle (\mathbf{x} + \mathbf{y})^-, (\mathbf{x} + \mathbf{y})^- \right\rangle \\
&\leq \langle A\mathbf{x}, \mathbf{x} \rangle - a \langle \mathbf{x}, \mathbf{x} \rangle + \langle A\mathbf{y}, \mathbf{y} \rangle - a \langle \mathbf{y}, \mathbf{y} \rangle \\
&\leq \lambda_k \|\mathbf{x}\|^2 - a\|\mathbf{x}\|^2 + C \\
&= (\lambda_k - a)\|\mathbf{x}\|^2 + C,
\end{aligned}$$

for some $C \geq 0$, where C comes from inner products dealing only with $\mathbf{y} \in Y$, which is fixed. Hence, since $\lambda_k < a$, J is bounded above and anticoercive on $\mathbf{y} + X$. Now, let $M = \sup_X J(\mathbf{x} + \mathbf{y}) < \infty$. Since $J(\mathbf{x} + \mathbf{y})$ is anticoercive, then there exists $R > 0$ such that $M = \sup_{\|\mathbf{x}\| \leq R} J(\mathbf{x} + \mathbf{y})$. J is a continuous functional, and so it achieves a maximum on the compact set $\|\mathbf{x}\| \leq R$. The maximum is unique since if $\mathbf{x_1}, \mathbf{x_2} \in X$ are both maxima, then $0 = \langle \nabla J(\mathbf{x_1} + \mathbf{y}) - \nabla J(\mathbf{x_2} + \mathbf{y}), \mathbf{x_1} - \mathbf{x_2} \rangle < 0$, a contradiction. Hence the first claim of the theorem holds.

Define a function, $r : Y \to X$, such that $r(\mathbf{y}) \in X$ is the unique element of X such that J is maximized on $\mathbf{y} + X$. We wish to show that $r(\mathbf{y})$ is a continuous function. We first note that

$$\langle \nabla J(r(\mathbf{y_2}) + \mathbf{y_2}) - \nabla J(r(\mathbf{y_1}) + \mathbf{y_1}), r(\mathbf{y_2}) - r(\mathbf{y_1}) \rangle = 0,$$

since $r(\mathbf{y_1})$ and $r(\mathbf{y_2})$ are both in X and are maxima on the sets $\mathbf{y_1} + X$ and $\mathbf{y_2} + X$, respectively. Substituting this fact into (2.3) and making the substitutions $\mathbf{x_1} = r(\mathbf{y_1})$ and $\mathbf{x_2} = r(\mathbf{y_2})$, we conclude that

$$\frac{\epsilon}{s} \|r(\mathbf{y_2}) - r(\mathbf{y_1})\|^2 - \|r(\mathbf{y_2}) - r(\mathbf{y_1})\| \|\mathbf{y_2} - \mathbf{y_1}\| \leq \|\mathbf{y_2} - \mathbf{y_1}\|^2,$$

and hence,

$$\|r(\mathbf{y_2}) - r(\mathbf{y_1})\| \leq \left(\frac{s + \sqrt{s^2 + 4\epsilon s}}{2\epsilon} \right) \|\mathbf{y_2} - \mathbf{y_1}\|, \tag{2.5}$$

so $r(\mathbf{y})$ is a continuous function.

Now, let $\tilde{J} : Y \to Y$ be defined such that $\tilde{J}(\mathbf{y}) := J(r(\mathbf{y}) + \mathbf{y})$. Note that

$$\begin{aligned}
\tilde{J}(\mathbf{y_2}) - \tilde{J}(\mathbf{y_1}) &= J(r(\mathbf{y_2}) + \mathbf{y_2}) - J(r(\mathbf{y_1}) + \mathbf{y_1}) \\
&= (J(r(\mathbf{y_2}) + \mathbf{y_2}) - J(r(\mathbf{y_2}) + \mathbf{y_1})) + (J(r(\mathbf{y_2}) + \mathbf{y_1}) \\
&\qquad - J(r(\mathbf{y_1}) + \mathbf{y_1})) \\
&\leq (J(r(\mathbf{y_2}) + \mathbf{y_2}) - J(r(\mathbf{y_2}) + \mathbf{y_1})),
\end{aligned}$$

since $J(r(\mathbf{y_1})+\mathbf{y_1})$ is the maximum at $\mathbf{y_1}$. Then

$$\begin{aligned}
\tilde{J}(\mathbf{y_2})-\tilde{J}(\mathbf{y_1}) &\leq \nabla J(r(\mathbf{y_2})+\mathbf{y_1})\cdot(\mathbf{y_2}-\mathbf{y_1})+o(\|\mathbf{y_2}-\mathbf{y_1}\|) \qquad (J\in C^1(\mathbb{R}^n,\mathbb{R}))\\
&= (\nabla J(r(\mathbf{y_1})+\mathbf{y_1})+\nabla J(r(\mathbf{y_2})+\mathbf{y_1})\\
&\qquad -\nabla J(r(\mathbf{y_1})+\mathbf{y_1}))\cdot(\mathbf{y_2}-\mathbf{y_1})+o(\|\mathbf{y_2}-\mathbf{y_1}\|)\\
&= (\nabla J(r(\mathbf{y_1})+\mathbf{y_1}))\cdot(\mathbf{y_2}-\mathbf{y_1})+(\nabla J(r(\mathbf{y_2})+\mathbf{y_1})\\
&\qquad -\nabla J(r(\mathbf{y_1})+\mathbf{y_1}))\cdot(\mathbf{y_2}-\mathbf{y_1})+o(\|\mathbf{y_2}-\mathbf{y_1}\|)\\
&= \nabla J(r(\mathbf{y_1})+\mathbf{y_1})\cdot(\mathbf{y_2}-\mathbf{y_1})+o(\|\mathbf{y_2}-\mathbf{y_1}\|), \qquad (2.6)
\end{aligned}$$

since

$$\begin{aligned}
\lim_{\|\mathbf{y_2}-\mathbf{y_1}\|\to 0}&\left|\frac{(\nabla J(r(\mathbf{y_2})+\mathbf{y_1})-\nabla J(r(\mathbf{y_1})+\mathbf{y_1}))\cdot(\mathbf{y_2}-\mathbf{y_1})}{\|\mathbf{y_2}-\mathbf{y_1}\|}\right|\\
&\leq \lim_{\|\mathbf{y_2}-\mathbf{y_1}\|\to 0}\nabla J(r(\mathbf{y_2})+\mathbf{y_1})-\nabla J(r(\mathbf{y_1})+\mathbf{y_1})\\
&= 0,
\end{aligned}$$

by the continuity of both ∇J and r. If, instead of adding and subtracting $J(r(\mathbf{y_2})+\mathbf{y_1})$ in the first step of the above inequality, we had added and subtracted $J(r(\mathbf{y_1})+\mathbf{y_2})$, we would have concluded that

$$\tilde{J}(\mathbf{y_2})-\tilde{J}(\mathbf{y_1}) \geq \nabla J(r(\mathbf{y_1})+\mathbf{y_1})\cdot(\mathbf{y_2}-\mathbf{y_1})+o(\|\mathbf{y_2}-\mathbf{y_1}\|).$$

Combining these two results, we conclude that

$$\tilde{J}(\mathbf{y_2})-\tilde{J}(\mathbf{y_1}) = \nabla J(r(\mathbf{y_1})+\mathbf{y_1})\cdot(\mathbf{y_2}-\mathbf{y_1})+o(\|\mathbf{y_2}-\mathbf{y_1}\|),$$

and therefore $\tilde{J}\in C^1(Y,\mathbb{R})$ and $\nabla\tilde{J}(\mathbf{y})=\nabla_Y J(r(\mathbf{y})+\mathbf{y})$.

Finally, note that if $t\in\mathbb{R}$ is given, then for $t\geq 0$,

$$\begin{aligned}
J(t\mathbf{u}) &= \frac{1}{2}(\langle A(t\mathbf{u}),t\mathbf{u}\rangle - a\left\langle (t\mathbf{u})^+,(t\mathbf{u})^+\right\rangle + b\left\langle (t\mathbf{u})^-,(t\mathbf{u})^-\right\rangle)\\
&= \frac{1}{2}(t^2\langle A(\mathbf{u}),\mathbf{u}\rangle - t^2a\left\langle (\mathbf{u})^+,(\mathbf{u})^+\right\rangle + t^2b\left\langle (\mathbf{u})^-,(\mathbf{u})^-\right\rangle)\\
&= t^2J(\mathbf{u}), \qquad (2.7)
\end{aligned}$$

since positive constants can be factored out of $(\cdot)^+$ and $(\cdot)^-$.

Given (2.7), consider

$$J(\mathbf{x}+t\mathbf{y}) = J\left(t\left((\frac{\mathbf{x}}{t}+\mathbf{y}\right)\right)$$
$$= t^2 J\left(\frac{\mathbf{x}}{t}+\mathbf{y}\right) \tag{2.8}$$

Since $J(\mathbf{x}+t\mathbf{y})$ is maximized at $\mathbf{x} = r(t\mathbf{y})$ and $t^2J(\frac{\mathbf{x}}{t}+\mathbf{y})$ will be maximized at $\mathbf{x} = tr(\mathbf{y})$, then $r(t\mathbf{y}) = tr(\mathbf{y})$. Finally, we combine these results to see that,

$$\begin{aligned}\tilde{J}(t\mathbf{y}) &= J(r(t\mathbf{y})+t\mathbf{y})\\ &= J(tr(\mathbf{y})+t\mathbf{y})\\ &= t^2J(r(\mathbf{y})+\mathbf{y})\\ &= t^2\tilde{J}(\mathbf{y}).\end{aligned}$$

□

Having established the appropriate geometry of the function, we may now show a variational characterization of the Fučík spectrum. This characterization exhibits that the characterization of the Fučík spectrum given by Castro in [2] for a more general class of operators is applicable to our case.

We begin with several lemmas which relate critical points of $\tilde{J}\Big|_{\|\mathbf{y}\|=1}$ to critical points of the unrestricted $\tilde{J}$, and critical points of $\tilde{J}$ to critical points of J.

Lemma 3. *Given* $\mathbf{y_0} \in Y$ *with* $\|\mathbf{y_0}\| = 1$, $\nabla\, \tilde{J}(\mathbf{y_0})\Big|_{\|\mathbf{y}\|=1} = \mathbf{0}$ *and* $\tilde{J}(\mathbf{y_0}) = \mathbf{0}$ *if and only if* $\nabla\tilde{J}(\mathbf{y_0}) = 0$.

Proof. Let $\mathbf{y_0} \in Y$ such that $\|\mathbf{y_0}\| = 1$ and let $V := \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{y_0}\rangle = 0\}$. Then every $\mathbf{y} \in Y$ may be represented as $\mathbf{y} = t\mathbf{y_0} + \mathbf{v}$ for some $t \in \mathbb{R}$ and $\mathbf{v} \in V$. Note that the curve $\gamma(t) = \frac{t\mathbf{y_0}+\mathbf{v}}{\|t\mathbf{y_0}+\mathbf{v}\|}$, with $\mathbf{y_0} \in Y$ and $\mathbf{v} \in V$ both fixed, is a smooth curve on the set $\{\mathbf{y} \in Y : \|\mathbf{y}\| = 1\}$ with $\gamma'(0) = \mathbf{v}$. First note that

$$\begin{aligned} \frac{d}{dt}\left(\tilde{J}(\gamma(t))\right)\Big|_{t=0} &= \nabla J(\gamma(0)) \cdot \gamma'(0) \\ &= \nabla J(\mathbf{y_0}) \cdot \mathbf{v} \end{aligned}$$

Similarly, note that

$$\begin{aligned} \frac{d}{dt}\left(\tilde{J}(t\mathbf{y_0})\right)\Big|_{t=1} &= \frac{d}{dt}\left(t^2\tilde{J}(\mathbf{y_0})\right)\Big|_{t=1} \\ &= 2\tilde{J}(\mathbf{y_0}) \end{aligned}$$

If we now represent $\mathbf{y} \in Y$ as $\mathbf{y} = t\mathbf{y_0} + \mathbf{v}$, then we see that $\nabla\tilde{J}(\mathbf{y_0}) \cdot (t\mathbf{y_0} + \mathbf{v}) = 2t\tilde{J}(\mathbf{y_0}) + \nabla\tilde{J}(\mathbf{y_0}) \cdot \mathbf{v}$. If $\tilde{J}(\mathbf{y_0}) = 0$ and $\nabla\tilde{J}(\mathbf{y_0}) \cdot \mathbf{v} = 0 \ \forall\mathbf{v} \in V$, then clearly $\nabla\tilde{J}(\mathbf{y_0}) \cdot \mathbf{y} = 0 \ \forall\mathbf{y} \in Y$. If $\nabla\tilde{J}(\mathbf{y_0}) \cdot \mathbf{y} = 0 \ \forall\mathbf{y} \in Y$, then clearly it must be true for $\mathbf{y} = \mathbf{y_0}$. In that case, we see that $\mathbf{v} = \mathbf{0}$, so $0 = \nabla\tilde{J}(\mathbf{y_0}) \cdot \mathbf{y_0} = 2t\tilde{J}(\mathbf{y_0}) \ \forall t > 0$. Hence $\tilde{J}(\mathbf{y_0}) = 0$, and therefore we conclude that $\nabla\tilde{J}(\mathbf{y_0}) = \mathbf{0}$.

□

Lemma 4. $\nabla\tilde{J}(\mathbf{y_0}) \cdot \mathbf{y} = 0 \ \forall\mathbf{y} \in Y$ *if and only if* $\nabla J(r(\mathbf{y_0}) + \mathbf{y_0}) \cdot \mathbf{u} = 0 \ \forall\mathbf{u} \in \mathbb{R}^n$.

Proof. Assume $\nabla\tilde{J}(\mathbf{y_0}) \cdot \mathbf{y} = 0 \ \forall\mathbf{y} \in Y$. Taking the gradient of J, we see that

$$\nabla J(r(\mathbf{y_0}) + \mathbf{y_0}) \cdot \mathbf{u} = \nabla_X J(r(\mathbf{y_0}) + \mathbf{y_0}) \cdot \mathbf{x} + \nabla_Y J(r(\mathbf{y_0}) + \mathbf{y_0}) \cdot \mathbf{y} \tag{2.9}$$

Since J achieves a maximum with respect to X, then $\nabla_X J(r(\mathbf{y_0}) + \mathbf{y_0}) \cdot \mathbf{x} = 0$. Substituting this in to (2.9), we see that

$$\begin{aligned} \nabla J(r(\mathbf{y_0}) + \mathbf{y_0}) \cdot \mathbf{u} &= \nabla_Y J(r(\mathbf{y_0}) + \mathbf{y_0})\mathbf{y} \\ &= \nabla\tilde{J}(\mathbf{y}) \end{aligned}$$

The other direction of the proof follows from (2.9) and the fact that critical points in the X-direction are unique from Theorem 2.1.

Therefore, the lemma holds. □

Now, using these lemmas, we may provide a variational characterization of the Fučík spectrum. For the statement and proof of this theorem, we adopt the notation $\tilde{J}_b$, to highlight the fact that the functional $\tilde{J}$ takes b as a parameter. There is no difference between $\tilde{J}$ and $\tilde{J}_b$, but given the importance of the reliance of $\tilde{J}$ on b to this proof, we have adopted a different notation. Furthermore, this notation is consistent with that used by Castro in [2]

Theorem 2.2. *Let $\lambda_k < a < \lambda_{k+1}$ and define*

$$b(a) := \sup\{b \geq a : \inf_{\|y\|=1} \tilde{J}_b(y) > 0\}.$$

Then,

1. *$(a, b(a)) \in \Sigma$, as defined in (1.2), or $b(a) = \infty$.*
2. *if $a < b < b(a)$, then $(a, b) \notin \Sigma$, and*
3. *$b(a) \geq \lambda_{k+1}$.*

Proof. Assume $b(a) < \infty$ and let $h(b) := \inf_{\|\mathbf{y}\|=1} \tilde{J}_b(\mathbf{y})$. It is clear from the definition of $\tilde{J}_b$ that $h(b)$ is nonincreasing and continuous in b. Furthermore, it is the case that $h(b(a)) = 0$, since if $h(b(a)) = \epsilon > 0$, then we can choose some $b_1 > b(a)$ so that $h(b_1) = \frac{\epsilon}{2}$, contradicting the definition of $b(a)$. We can similarly rule out $h(b(a)) < 0$. If $h(b(a)) = 0$, then for some $\mathbf{y_0} \in Y$ with $\|\mathbf{y_0}\|$, $\tilde{J}_b(\mathbf{y_0} = 0$ is the minimum value and is achieved at $\mathbf{y_0}$. This $\mathbf{y_0}$ is guaranteed to exists due to the compactness of the set $\{\mathbf{y} \in Y : \|y\| = 1\}$ and the continuity of $\tilde{J}_b$. So, $\tilde{J}_b\Big|_{\|\mathbf{y}\|=1}$ has a critical point with critical value is zero. By Lemma 3 this corresponds to a critical point for the unrestricted $\tilde{J}_b$ functional. Since critical points of $\tilde{J}_b$ correspond to critical points of J_b by Lemma 4 and the critical point is nontrivial since $\|\mathbf{y}\| = 1$, then $(a, b(a)) \in \Sigma$. If $b < b(a)$, then $h(b) \neq 0$. Since $h(b) \neq 0$, then the critical points of $\tilde{J}_b\Big|_{\|\mathbf{y}\|=1}$ do not correspond to critical points of the unrestricted functional $\tilde{J}_b$. So, (a, b) is not a critical point of J_b and hence $(a, b) \notin \Sigma$. Finally, we show that $b(a) \geq \lambda_{k+1}$ by assuming to the contrary that $a < b(a) < \lambda_{k+1}$. Let $\kappa = \lambda_{k+1} - a > 0$. Then $\forall \mathbf{y} \in Y$,

$$\begin{aligned}
\tilde{J}_b(\mathbf{y}) &= J_b(r(\mathbf{y}) + \mathbf{y}) \\
&\geq J_b(\mathbf{y}) \\
&= \frac{1}{2}\langle A\mathbf{y}, \mathbf{y}\rangle - \frac{1}{2}a\left\langle \mathbf{y}^+, \mathbf{y}^+\right\rangle - \frac{1}{2}b\left\langle \mathbf{y}^-, \mathbf{y}^-\right\rangle \\
&\geq \frac{1}{2}\lambda_{k+1}\|\mathbf{y}\|^2 - \frac{1}{2}(\lambda_{k+1} - \kappa)\|\mathbf{y}\|^2 \qquad (\lambda_{k+1} \text{ is the smallest eigenvalue on } Y.) \\
&= \frac{1}{2}\kappa\|\mathbf{y}\|^2 \\
&> 0.
\end{aligned}$$

Hence, $\inf_{\|\mathbf{y}\|=1} \tilde{J}_b(\mathbf{y}) > 0$, a contradiction to our definition of $b(a)$, so $b(a) \geq \lambda_{k+1}$. $\square$

Now, using the variational approach, we wish to identify a functional, $E(\mathbf{u})$, so that $\nabla E(\mathbf{u}) = 0$ is exactly (1.1). Let

$$E(\mathbf{u}) = J(\mathbf{u}) - G(\mathbf{u}), \text{ where } \nabla G(\mathbf{u}) = g(\mathbf{u}).$$

We easily see that, in light of Lemma 1, this is the appropriate functional.

Consider the functional $E(\mathbf{u})$ restricted to the subspace X. If we assume that $g(\mathbf{u})$ is bounded, then, due to (2.4), we can conclude that

$$E(\mathbf{x}) \leq -\epsilon\|\mathbf{x}\|^2 + M\|\mathbf{x}\|, \tag{2.10}$$

and therefore $E(\mathbf{x})$ is anticoercive on X.

Now consider $E(\mathbf{x})$ restricted to the set $\mathcal{Y} := \{r(\mathbf{y}) + \mathbf{y} : y \in Y\}$. Note first that

$$\begin{aligned}
\tilde{J}(\mathbf{y}) &= \tilde{J}\left(\|\mathbf{y}\|\frac{\mathbf{y}}{\|\mathbf{y}\|}\right) \\
&= \|\mathbf{y}\|^2\tilde{J}(\hat{\mathbf{y}}), \qquad (2.11)
\end{aligned}$$

and so if $\inf_{\|\mathbf{y}\|=1} \tilde{J}(\mathbf{y}) \geq \epsilon$, we conclude that $\tilde{J}(\mathbf{y}) \geq \epsilon \|\mathbf{y}\|^2$.

Now, recalling that $r(\mathbf{y})$ satisfies (2.5), we can rearrange (2.5) to show that $\|r(\mathbf{y})\| \leq M' \|\mathbf{y}\|$, for some $M' > 0$.

Combining these results, we see that

$$\begin{aligned} E(r(\mathbf{y}) + \mathbf{y}) &\geq \epsilon \|\mathbf{y}\|^2 - M \|r(\mathbf{y}) + \mathbf{y}\| \\ &\geq \epsilon \|\mathbf{y}\|^2 - M \left(\|r(\mathbf{y})\| + \|\mathbf{y}\| \right) \\ &\geq \epsilon \|\mathbf{y}\|^2 - M(M' + 1)\|\mathbf{y}\|. \end{aligned} \tag{2.12}$$

if $\inf_{\|\mathbf{y}\|=1} \tilde{J}_b(\mathbf{y}) \geq \epsilon$. It follows that there exists some R sufficiently large such that,

$$\sup_{\|\mathbf{x}\|=R} E(\mathbf{x}) < \inf_{\mathbf{y} \in \mathcal{Y}} E(\mathbf{y}).$$

On the issue of existence of solutions, we need also to establish an appropriate compactness condition for our functional. In particular, we wish to show that the functional satisfies (PS), which depends on whether our parameter values are in the Fučík spectrum. When $(a, b) \in \Sigma$, we show that (LLM) is sufficient to establish compactness.

Theorem 2.3. *If $(a, b) \notin \Sigma$ or if both $(a, b) \in \Sigma$ and (LLM) is satisfied then the functional $E : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$E(\mathbf{u}) = \frac{1}{2} \langle A\mathbf{u}, \mathbf{u} \rangle - \frac{a}{2} \left\langle \mathbf{u}^+, \mathbf{u}^+ \right\rangle - \frac{b}{2} \left\langle \mathbf{u}^-, \mathbf{u}^- \right\rangle - \langle G(\mathbf{u}), \mathbf{u} \rangle$$

satisfies (PS).

Proof. We wish to show that, given $\{\mathbf{x_k}\}_{k=1}^{\infty}$ such that $\{E(\mathbf{x_k})\}_{k=1}^{\infty}$ is bounded and $\nabla E(\mathbf{x_k}) \to 0$, $\{\mathbf{x_k}\}_{k=1}^{\infty}$ has a convergent subsequence. Since we are working in $\mathbb{R}^n$, it suffices to show that $\{\mathbf{x_k}\}_{k=1}^{\infty}$ is bounded.

Assume to the contrary that $\{x_k\}_{k=1}^{\infty}$ is not a bounded sequence. Then without loss of generality, $\|\mathbf{x_k}\| \to \infty$ so let $\widehat{\mathbf{x}}_\mathbf{k} = \frac{\mathbf{x_k}}{\|\mathbf{x_k}\|}$. Then there exists a subsequence, $\{\widehat{\mathbf{x}}_\mathbf{k}\}_{k=1}^{\infty}$ and an $\widehat{\mathbf{x}}$ such that $\widehat{\mathbf{x}}_\mathbf{k} \to \hat{\mathbf{x}}$. By the continuity of the norm, $\|\hat{\mathbf{x}}\| = 1$. Dividing the functional equation through by $\|\mathbf{x_k}\|$, we see that

$$\frac{\nabla E(\mathbf{x_k})}{\|\mathbf{x_k}\|} = A\hat{\mathbf{x}}_\mathbf{k} - a\hat{\mathbf{x}}_\mathbf{k}^+ + b\hat{\mathbf{x}}_\mathbf{k}^- - \frac{g(\mathbf{x_k})}{\|\mathbf{x_k}\|}.$$

Letting $k \to \infty$, we get that

$$0 = A\hat{\mathbf{x}} - a\hat{\mathbf{x}}^+ + b\hat{\mathbf{x}}^- \tag{2.13}$$

a contradiction if $(a, b) \notin \Sigma$. If $(a, b) \in \Sigma$, we note that $\hat{\mathbf{x}}$ is an eigenvector, and then we consider the equation

$$\langle \nabla E(\hat{\mathbf{x}}_\mathbf{k}), \hat{\mathbf{x}} \rangle = \langle A\hat{\mathbf{x}}_\mathbf{k}, \hat{\mathbf{x}} \rangle - a \left\langle \hat{\mathbf{x}}_\mathbf{k}^+, \hat{\mathbf{x}} \right\rangle + b \left\langle \hat{\mathbf{x}}_\mathbf{k}^-, \hat{\mathbf{x}} \right\rangle - \langle g(\hat{\mathbf{x}}_\mathbf{k}), \hat{\mathbf{x}} \rangle$$

Since A is symmetric and $\hat{\mathbf{x}}$ satisfies (2.13), we may rewrite the equation as

$$\langle \nabla E(\hat{\mathbf{x}}_\mathbf{k}), \hat{\mathbf{x}} \rangle = a \left(\left\langle \hat{\mathbf{x}}_\mathbf{k}, \hat{\mathbf{x}}^+ \right\rangle - \left\langle \hat{\mathbf{x}}_\mathbf{k}^+, \hat{\mathbf{x}} \right\rangle \right) - b \left(\left\langle \hat{\mathbf{x}}_\mathbf{k}, \hat{\mathbf{x}}^- \right\rangle - \left\langle \hat{\mathbf{x}}_\mathbf{k}^-, \hat{\mathbf{x}} \right\rangle \right) - \langle g(\hat{\mathbf{x}}_\mathbf{k}), \hat{\mathbf{x}} \rangle$$

Taking a limit of each side as $k \to \infty$ and using the continuity of $(\cdot)^{\pm}$, we conclude that

$$\lim_{k \to \infty} \langle g(\hat{\mathbf{x}}_\mathbf{k}), \hat{\mathbf{x}} \rangle = 0,$$

which is a contradiction of (LLM). Therefore, $\{\mathbf{x_k}\}_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^n$ which implies that it has a converging subsequence. So $\{\mathbf{x_k}\}_{k=1}^{\infty}$ satisfies (PS). □

Now, given properties of the functional E, our variational characterization, and the Palais-Smale compactness condition, now we need only show one last lemma before applying the saddle point theorem.

Lemma 5. *Let*

$$\Gamma := \left\{ \gamma : B_R(0) \subseteq X \to H : \gamma \mid_{\partial B_{R(0)}} (\mathbf{x}) = \mathbf{x}, \gamma \in C \right\}.$$

Then,

$$\inf_{\gamma \in \Gamma} \sup_{\mathbf{x} \in B_R} E(\gamma(\mathbf{x})) > \sup_{\mathbf{x} \in \partial B_R} E(\mathbf{x}).$$

Proof. Let $\gamma : \overline{B_R(0)} \subseteq X \to H$ be a continuous function such that $\gamma(\partial B_R(0)) = \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} = \mathbf{0}, \|\mathbf{x}\| = R\}$. Let $\gamma(\mathbf{x}) = \gamma_X(\mathbf{x}) + \gamma_Y(\mathbf{x})$ where $\gamma_X(\mathbf{x}) \in X$ and $\gamma_Y(\mathbf{x}) \in Y$. In order to show that $\gamma(\overline{B_R(0)}) \cap \mathcal{Y} \neq \varnothing$, we wish to find $\mathbf{x} \in X$ so that $\gamma_X(\mathbf{x}) = r(\gamma_Y(\mathbf{x}))$. Let $F(\mathbf{x}) = \gamma_X(\mathbf{x}) - r(\gamma_Y(\mathbf{x}))$. Now, let $h(\mathbf{x}, t) = tF(\mathbf{x}) + (1-t)\mathbf{x}$. Note first that if $x \in \partial B_R(0)$, then $F(\mathbf{x}) = \mathbf{x} \neq 0$, so $h(\mathbf{x}, t) = t\mathbf{x} + (1-t)\mathbf{x} = 1$ for $x \in \partial B_R(0)$. Then $\deg(F, \overline{B_R(0)}, 0) = \deg(I, \overline{B_R(0)}, 0) = 1$, and hence,

$$\begin{aligned} \inf_{\gamma \in \Gamma} \sup_{\mathbf{x} \in B_R} E(\gamma(\mathbf{x})) &\geq \inf_{\mathbf{y} \in \mathcal{Y}} E(\mathbf{y}) \\ &> \sup_{\mathbf{x} \in \partial B_R} E(\mathbf{x}) \end{aligned}$$

□

Theorem 2.4. *Under the hypotheses of Theorem 2.3, there exists a solution to* (1.1)*, where $g(\mathbf{u})$ is a bounded gradient vector field.*

Proof. Recall that the functional E satisfies (PS) due to Theorem 2.3 and that

$$\inf_{\gamma \in \Gamma} \sup_{\mathbf{x} \in B_R} E(\gamma(\mathbf{x})) > \sup_{\mathbf{x} \in \partial B_R} E(\mathbf{x}),$$

due to Lemma 5. Hence by the saddle point theorem, if

$$\Gamma := \left\{ \gamma : B_R(0) \subseteq X \to H : \gamma \mid_{\partial B_{R(0)}} (\mathbf{x}) = (\mathbf{x}, 0), \gamma \in C \right\}$$

and

$$d := \inf_{\gamma \in \Gamma} \sup_{\mathbf{x} \in X} E(\gamma(\mathbf{x})),$$

then d is a critical value. Hence, for some $\mathbf{u_0} \in \mathbb{R}^n$, $\nabla E(\mathbf{u_0}) \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{R}^n$. Hence, $\nabla E(\mathbf{u_0}) \equiv 0$, so $\mathbf{u_0} \in \mathbb{R}^n$ is a solution to $A\mathbf{u} - a\mathbf{u}^+ - b\mathbf{u}^- - g(\mathbf{u}) = 0$. $\square$

Chapter 3: ODE Case

In the differential equations case, the problem of interest is the boundary value problem,

$$\begin{cases} -u'' = au^+ - bu^- + g(u) \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi) \end{cases} \tag{3.1}$$

where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$.

The method of proving existence of solutions follows just as it did in the previous chapter. It is worth noting that much of the geometry of the functional is exactly as it was before, with only a few exceptions which will be noted. Where the computations are nearly identical, the reader will be referred to the computations in the previous chapter, and only the final result will be given. The major difference in the ODE case is proving the Palais-Smale condition, which turns out to be substantially more difficult. One might expect this, given that we are now trying to prove a compactness condition for an infinite dimensional space.

Lemma 6. *Let $J : H \to \mathbb{R}$ be given by*

$$J(u) = \frac{1}{2}\int_0^{2\pi} (u')^2\, dt - \frac{a}{2}\int_0^{2\pi} (u^+)^2\, dt - \frac{b}{2}\int_0^{2\pi} (u^-)^2\, dt.$$

Then

$$\nabla J(\mathbf{u}) \cdot \mathbf{v} = \int_0^{2\pi} u'v'\, dt - a\int_0^{2\pi} u^+ v\, dt + b\int_0^{2\pi} u^- v\, dt.$$

This lemma is justified by a more general theorem.

Theorem 3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^1(\mathbb{R})$ function. Consider $F : H \to \mathbb{R}$ given by $F(u) = \int_0^{2\pi} g(u)\, dt$. Then F is $C^1(\mathbb{R})$ with $\nabla F \cdot v = \int_0^{2\pi} g'(u)v\, dt$.*

Proof. Note first that

$$\left|F(u) - F(u_0) - \int_0^{2\pi} g'(u_0)(u-u_0)\,dt\right| = \left|\int_0^{2\pi} g(u) - g(u_0) - g'(u_0)(u-u_0)\,dt\right|$$
$$= \left|\int_0^{2\pi} g'(\tilde{u})(u-u_0) - g'(u_0)(u-u_0)\,dt\right|,$$

where $\tilde{u}(x)$ is the function guaranteed by the Mean Value Theorem such that $\tilde{u}(x)$ is between $u(x)$ and $u_0(x)$ and $g(u) - g(u_0) = g'(\tilde{u}(x))(u-u_0)$. Then

$$\left|F(u) - F(u_0) - \int_0^{2\pi} g'(u_0)(u-u_0)\,dt\right| \le \int_0^{2\pi} |g'(\tilde{u}) - g'(u_0)|\,|u-u_0|\ dt$$
$$\le \|g'(\tilde{u}) - g'(u_0)\|_{L^2}\|u-u_0\|_{L^2}$$

Therefore,

$$\lim_{\|u-u_0\|_H \to 0} \frac{\left|F(u) - F(u_0) - \int_0^{2\pi} g'(u)v\,dt\right|}{\|u-u_0\|_H} \le \lim_{\|u-u_0\|_H \to 0} \frac{\|g'(\tilde{u}) - g'(u_0)\|_{L^2}\|u-u_0\|_{L^2}}{\|u-u_0\|_{L^2}}$$
$$\le \lim_{\|u-u_0\|_H \to 0} \|g'(\tilde{u}) - g'(u_0)\|_{L^2}$$
$$= 0,$$

since $\tilde{u}(x)$ is between $u(x)$ and $u_0(x)$ and $\tilde{u} \to u_0$ uniformly in $C[0, 2\pi]$. Hence, F is differentiable with $\nabla F(u) \cdot v = \int_0^{2\pi} g'(u)v\,dt$. □

This theorem allows us to reduce the problem of finding a derivative of our functional to a single-variable problem. Therefore, recall that if $f(x) = \frac{1}{2}(x^{\pm})^2$, then $f'(x) = \pm x^{\pm}$. Therefore J is differentiable with derivative

$$\nabla J(u) \cdot c = \int_0^{2\pi} u'v'\,dt - a\int_0^{2\pi} u^+ v\,dt + b\int_0^{2\pi} u^- v\,dt$$

As before, we wish to establish an inequality which will make the proofs of several parts of the theorem quite simple.

Lemma 7. *Choose ϵ such that $a = (1+\epsilon)\lambda_k$. Let $\delta = \min\left\{\frac{\epsilon}{2}\lambda_k, \frac{\epsilon}{2}\right\}$ and let $D = \langle \nabla J(x_2+y_2) - \nabla J(x_1+y_1), x_2 - x_1\rangle$. Then,*

$$D \leq -\delta\|x_2 - x_1\|_H^2 + s(\|x_2 - x_1\|_{L^2} + \|y_2 - y_1\|_{L^2})\|y_2 - y_1\|_{L^2}$$

Proof. We proceed just as in the proof of Lemma 2. The same calculations as before will show that

$$D \leq \|x_2' - x_1'\|_{L^2}^2 - a\|x_2 - x_1\|_{L^2} + s(\|x_2 - x_1\|_{L^2} + \|y_2 - y_1\|_{L^2})\|y_2 - y_1\|_{L^2}$$

Now we choose ϵ such that $a = (1+\epsilon)\lambda_k$. If we then examine the first piece of the right-hand side, we find that

$$\begin{aligned}
\|x_2' - x_1'\|_{L^2}^2 &= \int_0^{2\pi} (x_2' - x_1')^2\,dt && (3.2)\\
&= \left(1 + \frac{\epsilon}{2}\right)\int_0^{2\pi} (x_2' - x_1')^2\,dt - \frac{\epsilon}{2}\int_0^{2\pi} (x_2' - x_1')^2\,dt\\
&\leq \left(1 + \frac{\epsilon}{2}\right)\lambda_k\int_0^{2\pi} (x_2 - x_1)^2\,dt - \frac{\epsilon}{2}\int_0^{2\pi} (x_2' - x_1')^2\,dt && (3.3)
\end{aligned}$$

since λ_k is the largest eigenvalue on X. Now, making the substitution $a = (1+\epsilon)\lambda_k$, we note that

$$\begin{aligned}
\|x_2' - x_1'\|_{L^2}^2 - a\|x_2 - x_1\|_{L^2} &\leq \left(1 + \frac{\epsilon}{2}\right)\lambda_k\int_0^{2\pi} (x_2 - x_1)^2\,dt - \frac{\epsilon}{2}\int_0^{2\pi} (x_2' - x_1')^2\,dt\\
&\qquad - (1+\epsilon)\lambda_k\int_0^{2\pi} (x_2 - x_1)^2\,dt\\
&= -\frac{\epsilon}{2}\lambda_k\int_0^{2\pi} (x_2 - x_1)^2\,dt - \frac{\epsilon}{2}\int_0^{2\pi} (x_2' - x_1')^2\,dt\\
&\leq -\delta\left(\int_0^{2\pi} (x_2 - x_1)^2\,dt + \int_0^{2\pi} (x_2' - x_1')^2\,dt\right)\\
&= -\delta\|x_2 - x_1\|_H.
\end{aligned}$$

Substituting this back into the original inequality, we conclude that

$$D \leq -\delta\|x_2 - x_1\|_H^2 + s(\|x_2 - x_1\|_{L^2} + \|y_2 - y_1\|_{L^2})\|y_2 - y_1\|_{L^2}$$

as claimed. $\square$

Now we proceed as before with a theorem about the properties of a functional J. We recall that the eigenvectors of the second derivative operator are $\{sin(nx)\}$ with corresponding eigenvalues $\{n^2\}$ for n a non-negative integer.

Theorem 3.2. *Let L be the second derivative operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ and corresponding eigenfunctions $\{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$. Given $\lambda_k < a < \lambda_{k+1}$ define $X := span\{\phi_1, \phi_2, \ldots, \phi_k\}$ and $Y := span\{\phi_{k+1}, \phi_{k+2}, \ldots, \phi_n, \ldots\} = X^\perp$. Let $J : H \to \mathbb{R}$ be defined as*

$$J(u) = \frac{1}{2}\int_0^{2\pi} (u')^2\, dt - \frac{a}{2}\int_0^{2\pi} (u^+)^2\, dt - \frac{b}{2}\int_0^{2\pi} (u^-)^2\, dt$$

Then,

1. *For fixed $y \in Y$, J is concave and anticoercive on the set $y + X$ and achieves a unique maximum.*
2. *There exists a continuous function, $r : Y \to X$ such that,*
   (a) $\tilde{J}(y) = J(r(y) + y) = \max\{J(y + x) : x \in X\}$,
   (b) $\tilde{J} \in C^1(Y, \mathbb{R})$, and
   (c) given $t \geq 0$, $r(ty) = tr(y)$ and $\tilde{J}(ty) = t^2\tilde{J}(y)$ $\forall y \in Y$.

Proof. As in (2.4), we may conclude that that for fixed $y \in Y$,

$$\langle \nabla J(x_2 + y) - \nabla J(x_1 + y), x_2 - x_1\rangle_{L^2} \leq -\delta\|x_2 - x_1\|_H^2,$$

and hence we conclude that J is concave and anticoercive on the set $y + X$.

Similarly, to show that $r(y)$ is a continuous function, we use the same substitution as in (2.5) (along with the fact that $\|\cdot\|_{L^2} \leq \|\cdot\|_H$) and conclude that,

$$\left(\frac{s+\sqrt{s^2+4\epsilon s}}{2\epsilon}\right)\|y_2-y_1\|_{L^2}. \tag{3.4}$$

So $r(y)$ is continuous in L^2. Note also that if $\{y_k\}_{k=1}^{\infty}$ is a bounded sequence in H, then $\{y_k\}_{k=1}^{\infty}$ has a convergent subsequence in L^2, call it $\{y_{k_i}\}_{i=1}^{\infty}$, which by the previous inequality gives us that $\{r(y_{k_i})\}_{i=1}^{\infty}$ converges in H.

The proofs of parts 2(b) and 2(c) of the theorem follow exactly as in (2.6), (2.7), and (2.8). These two final parts conclude the proof of the theorem. □

The variational characterization exhibited in 2.2 remains almost exactly the same, with one detail needing to be checked. In the finite dimensional case, we used the compactness of the set $\{\mathbf{y} \in \mathbb{R}^n : \|y\| = 1\}$ to show that $\inf_{\|y\|=1} \tilde{J}_b(y)$ was achieved at some $\mathbf{y}$ with $\|\mathbf{y}\| = 1$. In the infinite dimensional case, we must argue that such a y exists.

Let $\{y_k\}_{k=1}^{\infty}$ be a minimizing sequence with $\|y_k\|_{L^2} = 1$ such that $\tilde{J}(y_n) \searrow \inf_{\|y\|_{L^2}=1} \tilde{J}_b(y)$. Since $\{y_k\}_{k=1}^{\infty}$ is a minimizing sequence, then $\{2\tilde{J}_b(y_k)\}_{k=1}^{\infty}$ is bounded above. Let $M > 2\tilde{J}_b(y_k)$ for all k. Then,

$$\begin{aligned} M &> 2\tilde{J}_b(y_k) \\ &\geq 2J_b(y_k) \\ &= \int_0^{2\pi} (y_k')^2 \, dt - a\int_0^{2\pi} \left(y_k{}^+\right)^2 \, dt - b\int_0^{2\pi} \left(y_k{}^-\right)^2 \, dt. \end{aligned}$$

Since $\|y_k\|_{L^2} = 1$, then the two rightmost integrals are bounded, and hence

$$\int_0^{2\pi} (y_k')^2 \, dt < \infty.$$

Therefore, $\|y_k\|_H$ is bounded for all k. Since H embeds compactly in $C[0, 2\pi]$, there must be a subsequence of $\{y_k\}_{k=1}^{\infty}$ that converges in $C[0, 2\pi]$. Also, without loss of generality, $r(y_k) \to r(y)$ in H by (3.4). We also note that, due to weak lower semicontinuity,

$$\liminf_{k\to\infty} \int_0^{2\pi} ((r(y_k) + y_k)')^2 \, dt \geq \int_0^{2\pi} ((r(y) + y)')^2 \, dt.$$

Hence,

$$\begin{aligned}
2 \inf_{\|y\|_{L^2}=1} \tilde{J}_b(y) &= \liminf_{k\to\infty} \Bigg(\int_0^{2\pi} ((r(y_k) + y_k)')^2 \, dt - a \int_0^{2\pi} \left((r(y_k) + y_k)^+\right)^2 \, dt \\
&\qquad\qquad -b \int_0^{2\pi} \left((r(y_k) + y_k)^-\right)^2 \, dt \Bigg) \\
&\geq \int_0^{2\pi} ((r(y) + y)')^2 \, dt - a \int_0^{2\pi} \left((r(y) + y)^+\right)^2 \, dt \\
&\qquad\qquad - b \int_0^{2\pi} \left((r(y) + y)^-\right)^2 \, dt \\
&= 2\tilde{J}_b(y).
\end{aligned}$$

Hence, $\inf_{\|y\|_{L^2}=1} \tilde{J}_b(y) = 2\tilde{J}_b(y)$ for some $y \in H$.

Now, the only difficulty remaining is establishing the Palais-Smale condition.

Theorem 3.3. *If $(a, b) \notin \Sigma$ or if $(a, b) \in \Sigma$ and (LLD) is satisfied, then the functional $E : H \to \mathbb{R}$ defined by*

$$E(u) = \frac{1}{2} \int_0^{2\pi} (u')^2 \, dt - \frac{a}{2} \int_0^{2\pi} (u^+)^2 \, dt - \frac{b}{2} \int_0^{2\pi} (u^-)^2 \, dt - \int_0^{2\pi} G(u) \, dt \qquad (3.5)$$

satisfies (PS).

First, suppose that $\{u_k\}_{k=1}^{\infty}$ is a sequence such that $\{E(u_k)\}_{k=1}^{\infty}$ is bounded and $\nabla E(u_k) \to 0$ in H. We wish to show that $\|u_k\|_\infty$ is bounded. Suppose to the contrary that $\|u_k\|_\infty \to \infty$. Then let $v_k = \frac{u_k}{\|u_k\|_\infty}$. Note that if we divide the energy functional through by $\|u_k\|_\infty^2$, we get

$$\frac{E(u_k)}{\|u_k\|_\infty^2} = \frac{1}{2}\int_0^{2\pi} (v_k')^2\,dt - \frac{a}{2}\int_0^{2\pi} ({v_k}^+)^2\,dt - \frac{b}{2}\int_0^{2\pi} ({v_k}^-)^2\,dt - \int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty^2}\,dt$$

Note that if we take a limit, the term $\frac{E(u_k)}{\|u_k\|_\infty^2} \to 0$ since $\{E(u_k)\}_{k=1}^\infty$ is bounded and

$$\int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty^2}\,dt \to 0$$

since $G' = g$ is a bounded function, and thus $|G(u_k)| \leq C\,|u_k|$, where $|g(u_k)| \leq C$ $\forall u_k$. Also note that $\|v_k^\pm\|_\infty \leq 1$, so $\int_0^{2\pi}(v^\pm)^2\,dt$ is likewise bounded. Therefore, we may conclude that

$$\frac{1}{2}\int_0^{2\pi} (v_k')^2\,dt < +\infty,$$

and therefore $\|v_k\|_H$ is bounded.

Thus, without loss of generality, there exists $\Psi \in H$ such that $v_k \rightharpoonup \Psi$ in H and $v_k \to \Psi$ in $L^2\,[0, 2\pi]$ and $C\,[0, 2\pi]$, by Alaoglu's theorem and a standard compact embedding theorem. We know that $\|\Psi\|_\infty = 1$ since $\|v_k\|_\infty = 1\ \forall k$, so Ψ is nontrivial. Using this convergence, we can now show that

$$0 = \lim_{k\to\infty} \frac{\nabla E(u_k)}{\|u_k\|_\infty} \cdot w \tag{3.6}$$

$$= \lim_{k\to\infty} \left[\int_0^{2\pi} v_k' w'\,dt - a\int_0^{2\pi} {v_k}^+ w\,dt + b\int_0^{2\pi} {v_k}^- w\,dt - \int_0^{2\pi} \frac{g(u_k)}{\|u_k\|_\infty} w\,dt\right] \tag{3.7}$$

$$= \int_0^{2\pi} \Psi' w'\,dt - a\int_0^{2\pi} \Psi^+ w\,dt + b\int_0^{2\pi} \Psi^- w\,dt \tag{3.8}$$

Thus, Ψ is a weak solution to the boundary value problem

$$\begin{cases} -u'' = au^+ - bu^- \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi) \end{cases} \tag{3.9}$$

and Ψ is a non-trivial Fučík eigenfunction. If $(a,b) \notin \Sigma$, then this is a contradiction and $\|u_k\|_\infty$ is bounded as claimed. If $(a,b) \in \Sigma$, the consider the quantity,

$$\frac{2E(u_k) - \nabla E(u_k)\cdot u_k}{\|u_k\|_\infty} = -2\int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty}\,dt + \int_0^{2\pi} g(u_k)\frac{u_k}{\|u_k\|_\infty}\,dt. \tag{3.10}$$

Note first that, by assumption

$$\lim_{k\to\infty} \frac{2E(u_k) - \nabla E(u_k)\cdot u_k}{\|u_k\|_\infty} = 0.$$

We can rewrite the first term on the right hand side of (3.10) so that

$$\begin{aligned}
\lim_{k\to\infty}\int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty}\,dt &= \lim_{k\to\infty}\int_0^{2\pi} \frac{G(u_k)}{u_k}\frac{u_k}{\|u_k\|_\infty}\,dt \\
&= \lim_{k\to\infty}\int_{\Psi<0} \frac{G(u_k)}{u_k}\frac{u_k}{\|u_k\|_\infty}\,dt + \int_{\Psi>0} \frac{G(u_k)}{u_k}\frac{u_k}{\|u_k\|_\infty}\,dt \\
&= G^-\int_{\Psi<0} v\,dt + G^+\int_{\Psi>0} v\,dt
\end{aligned} \tag{3.11}$$

Now, we need only to determine what the last integral in (3.10) converges to in order to reach a contradiction, which will show that $\|u_k\|_\infty$ is bounded. We begin with a lemma.

Lemma 8. *Let $E : H \to \mathbb{R}$ be defined as before, and let $\{u_k\}_{k=1}^\infty$ be a sequence such that $\{E(u_k)\}_{k=1}^\infty$ is bounded and $\nabla E(u_k) \to 0$ in H. Then $\frac{u_k}{\|u_k\|_\infty}$ has a convergent subsequence in H.*

Proof. Let

$$P(u)\cdot v = \int_0^{2\pi} u'v'\,dt + \int_0^{2\pi} uv\,dt = \langle u, v\rangle_H$$

$$S(u)\cdot v = -(a+1)\int_0^{2\pi} u^+ v\,dt + (b+1)\int_0^{2\pi} u^- v\,dt$$

$$T(u)\cdot v = -\int_0^{2\pi} g(u)v\,dt$$

so that

$$\nabla E(u) \cdot v = (P(u) + S(u) + T(u)) \cdot v.$$

First, let us consider $S(u)$. Since $\left(\frac{u_k}{\|u_k\|_\infty}\right) \xrightarrow{L^2} \Psi$, then $\left(\frac{u_k}{\|u_k\|_\infty}\right)^+ \xrightarrow{L^2} \Psi^+$ and $\left(\frac{u_k}{\|u_k\|_\infty}\right)^- \xrightarrow{L^2} \Psi^-$ by the Lebesgue Dominated Convergence Theorem. Noting that

$$\frac{S(u_k)}{\|u_k\|_\infty} \cdot v = S\left(\frac{u_k}{\|u_k\|_\infty}\right) \cdot v = -a \int_0^{2\pi} \left(\frac{u_k}{\|u_k\|_\infty}\right)^+ v\, dt + b \int_0^{2\pi} \left(\frac{u_k}{\|u_k\|_\infty}\right)^- v\, dt,$$

we conclude that $S\left(\frac{u_k}{\|u_k\|_\infty}\right) \cdot v \to S(\Psi) \cdot v \; \forall v \in H$. Since

$$\begin{aligned} \left|\left(S\left(\frac{u_k}{\|u_k\|_\infty}\right) - S(\Psi)\right) \cdot v\right| &= \left| -(a+1) \int_0^{2\pi} \left(\left(\frac{u_k}{\|u_k\|_\infty}\right)^+ - \Psi^+\right) v\, dt \right. \\ &\qquad \left. +(b+1) \int_0^{2\pi} \left(\left(\frac{u_k}{\|u_k\|_\infty}\right)^- - \Psi^-\right) v\, dt \right| \\ &\leq (a+1) \| \left(\frac{u_k}{\|u_k\|_\infty}\right)^+ - \Psi^+ \|_{L^2} \\ &\qquad + (b+1) \| \left(\frac{u_k}{\|u_k\|_\infty}\right)^- - \Psi^- \|_{L^2}, \end{aligned}$$

for $\|v\|_{L^2} \leq 1$, then $S\left(\frac{u_k}{\|u_k\|_\infty}\right) \to S(\Psi)$ in H^*.

Now, considering $T(u)$, we see that

$$T(u) \cdot v = -\int_0^{2\pi} g(u) v\, dt,$$

so $\{T(u_k)\}$ is bounded in H^* since

$$\|T(u)\|_{H^*} \leq \|g(u)\|_{L^2} \leq C.$$

So $\|\frac{T(u_k)}{\|u_k\|_\infty}\| \to 0$ as $k \to \infty$.

Finally, considering $P(u)$, we first note that $P(u) \cdot v = \langle u, v \rangle_H$. Therefore, by the Riesz Representation Theorem, there is an isomorphism, $i : H^* \to H$ such that $i \circ P(u) = u \ \forall u \in H$. So, P is an invertible linear operator with continuous inverse.

Recalling that $\nabla E(u) = P(u) + S(u) + T(u)$ and that by a hypothesis of the Palais-Smale condition, $\nabla E(u_k) \to 0$ in H^* as $k \to \infty$, we see that

$$\frac{\nabla E(u_k)}{\|u_k\|_\infty} = P\left(\frac{u_k}{\|u_k\|_\infty}\right) + S\left(\frac{u_k}{\|u_k\|_\infty}\right) + \frac{T(u_k)}{\|u_k\|_\infty}$$

can be rewritten as

$$\frac{u_k}{\|u_k\|_\infty} = P^{-1}\left(\frac{\nabla E(u_k)}{\|u_k\|_\infty} - S\left(\frac{u_k}{\|u_k\|_\infty}\right) - \frac{T(u_k)}{\|u_k\|_\infty}\right).$$

Therefore, invoking the continuity of P^{-1} and taking a limit as $k \to \infty$, we conclude that

$$\frac{u_k}{\|u_k\|_\infty} \xrightarrow{H} P^{-1}\left(0 - S(\Psi) - 0\right) = P^{-1}(-S(\Psi)) = \Psi.$$

□

Lemma 9.

$$g(u_k) \rightharpoonup G^+\chi_{\Psi>0} + G^-\chi_{\Psi<0}$$

By Alaoglu's Theorem, we know that $\{g(u_k)\}_{k=1}^{\infty}$ has a weakly convergent subsequence since $\{g(u_k)\}_{k=1}^{\infty}$ is bounded in $L^2[0, 2\pi]$. Let $g(u_k) \rightharpoonup \mathscr{g}$. Now we need only to show that

$$\mathscr{g} = G^+\chi_{\Psi>0} + G^-\chi_{\Psi<0}$$

It will be helpful to recall some standard properties of Fučík eigenfunctions, Ψ, namely that they are continuously differentiable and have a finite number of critical points. For a proof of such properties and an explicit formulation for such Ψ, see [1]. Let

$v = \chi_{[c,d]}$ be the characteristic function of some closed interval where $0 \le c < d \le 2\pi$ and $[c, d] \subset \{x : \Psi(x) > 0, \Psi'(x) > 0\}$. Then we may write

$$\begin{aligned}
\int_0^{2\pi} g(u_k)\chi_{[c,d]}\,dt &= \int_c^d g(u_k)\,dt \\
&= \int_c^d g(u_k)\left(1 - \frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\right)dt + \int_c^d g(u_k)\left(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\right)dt, \qquad (3.12)
\end{aligned}$$

where $c < e < d$ such that $\Psi'(e) = \frac{\Psi(d)-\Psi(c)}{d-c}$, as guaranteed by the Mean Value Theorem. Analyzing the second term, we find

$$\begin{aligned}
\int_c^d g(u_k)\left(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\right)dt &= \frac{1}{\Psi'(e)\|u_k\|_\infty}\int_c^d g(u_k)u_k'\,dt \\
&= \frac{1}{\Psi'(e)\|u_k\|_\infty}\left(G(u_k(d)) - G(u_k(c))\right) \\
&= \frac{1}{\Psi'(e)}\frac{G(u_k(d))}{u_k(d)}\frac{u_k(d)}{\|u_k\|_\infty} - \frac{G(u_k(c))}{u_k(c)}\frac{u_k(c)}{\|u_k\|_\infty}
\end{aligned}$$

Now, taking a limit of both sides, we see that,

$$\begin{aligned}
\lim_{k\to\infty}\int_c^d g(u_k)\left(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\right)dt &= \lim_{k\to\infty}\frac{1}{\Psi'(e)}\frac{G(u_k(d))}{u_k(d)}\frac{u_k(d)}{\|u_k\|_\infty} - \frac{G(u_k(c))}{u_k(c)}\frac{u_k(c)}{\|u_k\|_\infty} \\
&= \frac{1}{\Psi'(e)}\left(G^+\Psi(d) - G^+\Psi(c)\right) \\
&= (d-c)G^+ \\
&= \int_0^{2\pi} G^+\chi_{[c,d]}\,dt
\end{aligned}$$

Focusing now on the first term of (3.12), we note that,

$$\left(1 - \frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\right) \to \left(1 - \frac{\Psi'(x)}{\Psi'(e)}\right),$$

so

$$\int_c^d g(u_k)\left(1-\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\right)\,dt \to \int_c^d \mathcal{g}\left(1-\frac{\Psi'(x)}{\Psi'(e)}\right)\,dt.$$

Let $\epsilon > 0$ and define $M := \|\mathcal{g}\|_\infty$. The fact that M exists is a consequence of the boundedness of g. Choose c_i, d_i such that

$$\bigcup_{i=i}^{n}[c_i, d_i] = [c, d],\ |d-c| = \sum_{i=1}^{n}|d_i - c_i|,\ \text{and}\ \left|1-\frac{\Psi'(x)}{\Psi'(e_i)}\right| < \frac{\epsilon}{M}\ \forall x_1, x_2 \in [c_i, d_i].$$

Then,

$$\sum_{i=1}^{n}\left|\int_{c_i}^{d_i}\mathcal{g}\left(1-\frac{\Psi'(x)}{\Psi'(e_i)}\right)\,dt\right| \leq \sum_{i=1}^{n}\epsilon|d_i - c_i| = \epsilon(d-c)$$

Since ϵ was chosen arbitrarily, we may let $\epsilon \to 0$, and hence, by substituting back into (3.12) we find that

$$\lim_{k\to\infty}\int_0^{2\pi} g(u_k)\chi_{[c,d]}\,dt = \int_0^{2\pi} G^+\chi_{[c,d]}\,dt\ \forall[c,d] \subset \{x : \Psi(x) > 0, \Psi'(x) > 0\}\,. \quad (3.13)$$

The exact same calculations will show that, given $[c, d] \subset \{x : \Psi(x) > 0, \Psi'(x) < 0\}$, we get the same conclusion as in (3.13) . For $[c, d] \subset \{x : \Psi(x) < 0, \Psi'(x) > 0\}$ and $[c, d] \subset \{x : \Psi(x) < 0, \Psi'(x) < 0\}$, we can complete the same calculations, but will this time find that

$$\lim_{k\to\infty}\int_0^{2\pi} g(u_k)\chi_{[c,d]}\,dt = \int_0^{2\pi} G^-\chi_{[c,d]}\,dt.$$

Hence, we may recombine the integrals to see that

$$\lim_{k\to\infty}\int_0^{2\pi} g(u_k)\chi_{[c,d]}\,dt = \int_0^{2\pi}\left(G^+\chi_{\Psi>0} + G^-\chi_{\Psi<0}\right)\chi_{[c,d]}\,dt. \quad (3.14)$$

We have now shown that $g(u_k) \rightharpoonup G^+\chi_{\Psi>0} + G^-\chi_{\Psi<0}$ for all characteristic functions of closed intervals, so long as the closed intervals avoid critical points. If any intervals do include a critical point, however, we may delete some arbitrarily small neighborhood of each of the finitely many critical point so that the total change in the integral is less than some ϵ. Standard arguments will show that since all L^2 functions can be approximated by step functions (which are themselves linear combinations of characteristic functions over intervals), then the claim holds for all $v \in H$.

Combining (3.10), (3.11), and (3.14), we now find that

$$0 = -\left[G^+ \int_{\Psi>0} \Psi \, dt + G^- \int_{\Psi<0} \Psi \, dt \right],$$

a contradiction because this quantity was nonzero by assumption. Hence, $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in L^∞. By examining (3.5), we note that $\{E(u_k)\}_{k=1}^{\infty}$ is bounded by hypothesis and all the integral terms, except the one involving u_k', are bounded by virtue of $\{u_k\}_{k=1}^{\infty}$ being bounded in L^∞. Hence, $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in H. Now, as before, consider $\nabla E(u_k) = P(u_k) + S(u_k) + T(u_k)$. Since $\{u_k\}_{k=1}^{\infty}$ is bounded in H, then there exists a subsequence $u_k \overset{H}{\rightharpoonup} u$ and $u_k \overset{L^2,C}{\rightarrow} u$. Now, taking a limit, we see that

$$0 = \lim_{k\to\infty} \nabla E(u_k) = \lim_{k\to\infty} P(u_k) + S(u_k) + T(u_k) = P(u) + S(u) + T(u)$$

and since P is invertible, we may rearrange the equation to see that,

$$u_k \overset{H}{\rightharpoonup} u = P^{-1}(-S(u) - T(u)).$$

Hence $\{u_k\}_{k=1}^{\infty}$ has a subsequence which converges in H, and therefore we have satisfied (PS).

As before, we may now apply the saddle point theorem.

Theorem 3.4. *Under the hypotheses of Theorem 3.3, there exists a solution to* (3.1)*, where $g(u)$ is a bounded function.*

Proof. Recall that the functional E satisfies (PS) due to Theorem 3.3 and that

$$\inf_{\gamma \in \Gamma} \sup_{\mathbf{x} \in B_R} E(\gamma(\mathbf{x})) > \sup_{\mathbf{x} \in \partial B_R} E(\mathbf{x}),$$

due to Lemma 5. Hence by the saddle point theorem, if

$$\Gamma := \left\{ \gamma : B_R(0) \subseteq X \to H : \gamma \mid_{\partial B_R(0)} (\mathbf{x}) = \mathbf{x}, \gamma \in C \right\}$$

and

$$d := \inf_{\gamma \in \Gamma} \sup_{\mathbf{x} \in X} E(\gamma(\mathbf{x})),$$

then d is a critical value. Hence, for some $u_0 \in H$, $\nabla E(u_0) \cdot v = 0$ for all $v \in H$. Hence, $\nabla E(u_0) \equiv 0$, so $u_0 \in \mathbb{R}^n$ is a solution to $Au - au^+ - bu^- + g(u) = 0$. $\square$

Chapter 4: Conclusion and Further Research

In conclusion, our results in Theorem 2.4 improve upon the results of Margulies and Margulies in [7] by proving existence of solutions in both resonance and non-resonance case. The proof of Margulies and Marguiles was also degree-theoretic, and not variational. Our results in Theorem 3.4 improve upon recent results from Bliss, Buerger, and Rumbos in [1]. Their argument, which made very careful use of a series of estimates, was only able to show that (LLD) was sufficient for the existence of a solution if the parameters $(a, b) \in \Sigma$ were sufficiently close to the main diagonal.

There still remain several open problems in both the discrete and ODE cases. In the discrete case, determining a Landesman-Lazer type condition which puts the restriction on the function G, as in (LLD), instead of restricting the function g, remains an open problem. Also, proving existence theorems for non-symmetric matrices remains an open and challenging problem. See [9] for an example of existence theorems for persymmetric matrices.

In both cases, how to use the variational characterization for certain pieces of the Fučík spectrum remains an open problem. Castro's characterization in [2] gives a characterization for some curves, but only within a strip determined by consecutive eigenvalues. In some situations, there are points in $\mathbb{R}^2$ that are in a connected component of $\mathbb{R}^2 - \Sigma$ that is not accessible from the main diagonal. For these so-called Type II regions, our existence theorems do not apply, but extending the variational characterization in a new way might allow for those type of results.

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