

TWO-COMPONENT LINK SYMMETRIES

BY

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## Abstract

A mathematical knot is a closed curve in a three-dimensional space that does not intersect itself, while a link is two or more such curves that do not intersect each other. We consider the “intrinsic” symmetry group of a two-component link  $L$  which records directly whether  $L$  is isotopic to a link obtained by reversing the orientation of the ambient space, reversing the orientations of the components, or permuting the components of  $L$ . It is a subgroup of the Whitten group, the group of all such isotopies.

For two-component links, we catalog the 27 possible intrinsic symmetry groups, which represent the subgroups of the Whitten group up to conjugacy, as well as explain several methods for determining that a link does not possess some symmetries. We are able to provide prime, nonsplit examples for 21 of these groups. Ten of these examples are classically known from Hillman and others; the rest, joint work with my collaborators, are new to the literature.

We then look at the symmetry groups of a family of links that result from repeatedly Bing doubling a Hopf link; for three components and above, these are Brunnian links. We conjecture that every link in this family has an index two subgroup in its Whitten group. Some preliminary results towards this conjecture are discussed.

## Chapter 1: Introduction

Knot theory first arose in mathematics as a solution to a problem that does not exist. In the late 1800s Lord Kelvin speculated that atoms were knotted vortices in the ether; while this belief was entirely erroneous, it spurred mathematicians to develop and study the field now known as knot theory. A knot in mathematics is a closed non-self-intersecting curve in three dimensions; it can also be thought of as the embedding of a circle into either  $\mathbb{R}^3$  or  $S^3$ . Two examples of knots are found in Figure 1.1.

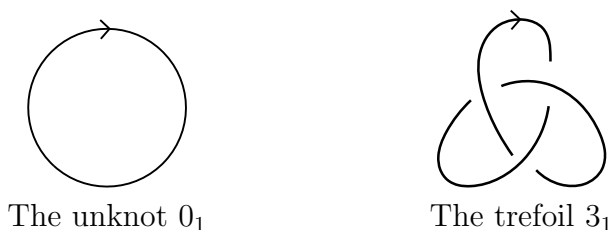


Figure 1.1: The unknot and the trefoil, two examples of knots.

One can ask if a given knot is equivalent to the mirror that knot, or to reversing the orientation of that knot, or to the knot formed by mirroring and reversing orientation at the same time. These are the “intrinsic” symmetries of a knot. These symmetries have many applications. Arguably, the main question in knot theory is that of classification; which knots are equivalent, and which are not? The intrinsic symmetry group of a knot is one natural part of this classification.

Knot symmetries also have applications beyond mathematics. In biochemistry, the chirality, that is to say the mirrorability, of proteins and other molecules have important effects. Determining and understanding the chirality of these molecules often reduces to determining and understanding the chirality of related knots, and

thus the study of intrinsic symmetry groups for knots has led to a greater insight into the effects of molecular chirality; for more information see Flapan [9].

The group of intrinsic symmetries of knots was studied extensively in the middle of the 20th century. In 1961 Fox posed a list of topology problems [10] that included many questions about symmetries of knots. One question was whether there are any noninvertible knots, knots where reversing the orientation produces a knot that is not isotopic to the original.

This question was answered in the affirmative by Trotter [19]. He was able to show that pretzel knots of the form  $P(p, q, r)$  are noninvertible if  $p$ ,  $q$ , and  $r$  are all distinct odd integers with absolute value greater than 1. Note that the examples for noninvertible knots given in Section 2.1 do not lie in this family. The symmetry group for knots is now well understood; for example, see the textbook [7]. Knots are discussed in more depth in this thesis in Section 2.1.

A link is the union of multiple knots (called the components of the link) that do not intersect each other. Thus a one-component link is a knot. Examples of links can be found in Figure 1.2. Section 2.2 gives more background on links.

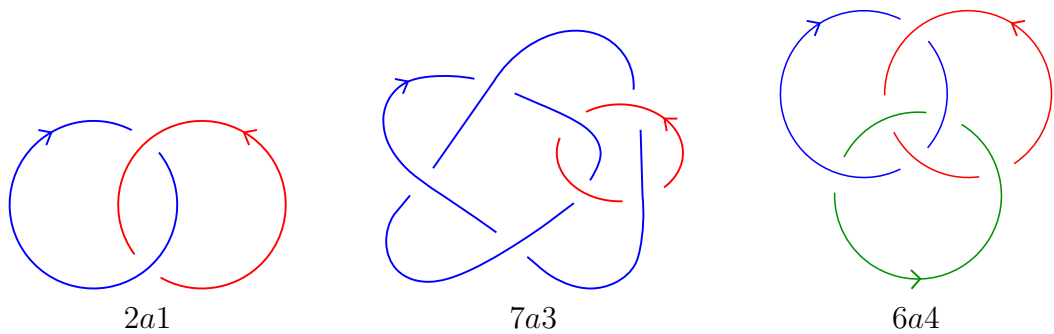


Figure 1.2: Three link examples, two with two components and one with three components.

The intrinsic symmetries of a link were first described by Whitten [20], who studied under and collaborated with Fox. This group includes the same actions of mirroring

the ambient space and reversing the orientations of the components, but for links one can also permute components.

Whitten showed that for links with  $\mu$  components, the group  $\Gamma_\mu$  of all such symmetries is isomorphic to  $\mathbb{Z}_2 \times (\mathbb{Z}_2^\mu \rtimes S_\mu)$ ; thus we refer to  $\Gamma_\mu$  as the *Whitten group*. Then for a link  $L$ , let  $\Sigma(L)$  be the group of symmetries exhibited by  $L$ ; thus  $\gamma \in \Gamma_\mu$  is an element of  $\Sigma(L)$  if  $\gamma$  acting on  $L$  is isotopic to  $L$ . Whitten was the first to raise the topic of this thesis, namely for what subgroups  $\Sigma$  of  $\Gamma_\mu$  is there a link  $L$  such that  $\Sigma(L) = \Sigma$ .

The search for example links continued with Hillman [14], who attempted to find examples in the case of  $\mu = 2$  components. He was able to find links for 8 of the 12 subgroups of  $\Gamma_2$  that do not contain symmetries that invert the individual components. He was also the first to compute the number of nonconjugate subgroups of  $\Gamma_2$ , although his computation of 28 such subgroups was corrected by our paper [3]; there are 27 nonconjugate subgroups of  $\Gamma_2$ .

A number of authors have attempted to use polynomials to determine restrictions on  $\Sigma(L)$ ; for examples see [2, 6, 11, 14]. A discussion of the restrictions one can place on  $\Sigma(L)$  using various invariants for the link  $L$  is found in Section 3.1. When first defining his polynomial, Conway [6] was able to show that if  $L$  is a prime nonsplit alternating link with an even number of components then one can rule out many symmetries from  $\Sigma(L)$ . Cerf [4] was able to define a new invariant, nullification writhe, to achieve the same result. Both of these methods are presented in Section 3.2. Cerf also used her invariant to help her make an atlas of oriented links up to nine crossings in [5].

In their paper, Berglund et al. [2] analytically classified the intrinsic symmetry groups of all links with eight or fewer crossings. This search found that only 5 of the 27 subgroups of  $\Gamma_2$  were exhibited by links with eight or fewer crossings. For all

but one of the links  $L$  they considered, standard invariants were used to rule out all symmetries that were not in the link's group of intrinsic symmetries  $\Sigma(L)$ . For one link,  $8_5^3$ , they rely on a computer search to rule out certain symmetries.

Along with Cantarella, Mastin and Parsley [3], I attempted to find examples for all 27 subgroups of  $\Gamma_2$ . We were able to find links for 21 out of the 27 groups using both computer searches and analytical approaches. We were also able to catalog the frequency at which each group appears among all 77,036 of the hyperbolic two-component links of 14 or fewer crossings using the computer program **SnapPy** [8], a user interface for **SnapPea**. The results from [3] are found in Chapter 4. Some possible ways to expand on these results can be found in Chapter 5.

## Chapter 2: Background

Simple closed curves have long been an area of interest to mathematicians. Gauss, among others, studied these objects. When embedded in a three-dimensional space, simple closed curves are known as knots.

**Definition 2.1.** A *knot* is the image of an embedding of a circle  $S^1$  into  $\mathbb{R}^3$  or a three-dimensional sphere  $S^3$ . A knot is *oriented* if one has given the knot an orientation; give the circle that is the preimage of your knot a clockwise or counterclockwise orientation and then define the orientation of your knot in relation to your orientation of  $S^1$ .

The concept of a knot can be generalized to multiple nonintersecting knots.

**Definition 2.2.** A *link with  $n$  components* is the union of  $n$  nonintersecting knots in  $\mathbb{R}^3$  or  $S^3$ .

We will assume in this thesis that all knots and links are oriented.

### 2.1 Knots

We first need to define when two knots are the same. We consider two knots equivalent if there is a way to deform one into the other without breaking any strand of the knots or passing one strand through another. The rigorous way to define such a deformation is via ambient isotopies.

**Definition 2.3.** Let  $K_1$  and  $K_2$  be oriented knots in  $\mathbb{R}^3$ . A continuous map  $F : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$  is an *ambient isotopy* from  $K_1$  to  $K_2$  if  $F(\mathbb{R}^3, 0)$  is the identity map,  $F(\mathbb{R}^3, t)$  is one-to-one for each  $t \in [0, 1]$ , and  $F(K_1, 1) = K_2$ . If there is an ambient isotopy from  $K_1$  to  $K_2$  we say the two knots  $K_1$  and  $K_2$  are *equivalent* or *isotopic*, denoted  $K_1 \approx K_2$ .



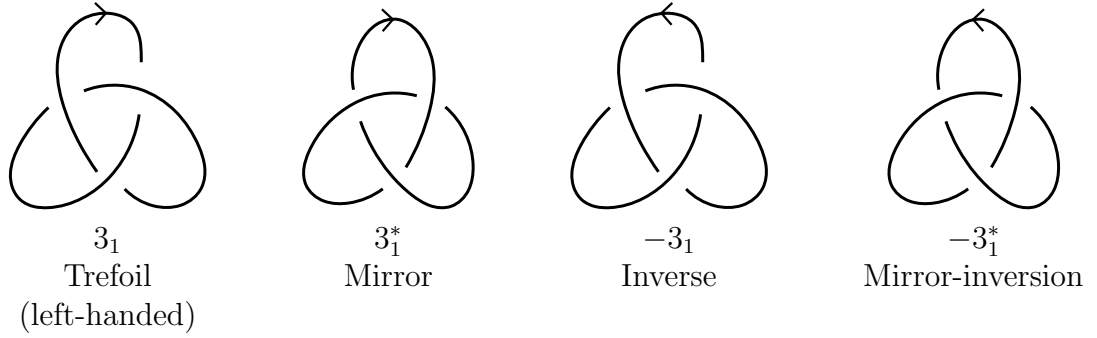


Figure 2.1: The action of  $\Gamma_1$  on the trefoil  $3_1$ . For this knot,  $3_1 \approx -3_1$  and  $3_1^* \approx -3_1^*$  but  $3_1 \not\approx 3_1^*$ .

One of the most important questions in the field of knot theory is that of classification, trying to determine when two knots are different or the same. The symmetries of a knot play an important role in this question. For a knot  $K$ , let  $-K$  be the knot with the opposite orientation. We call this the *inverse* of  $K$ . Let  $K^*$  be the result of changing the orientation of the ambient space  $\mathbb{R}^3$  or  $S^3$ , so it is like we have held a mirror up to  $K$ . We call this the *mirror* of  $K$ . Finally, we could both mirror and invert  $K$  to obtain  $-K^*$ . We call this the *mirror-inversion* of  $K$ . If we mirror twice or invert the orientation of a knot twice, we obtain the original knot, so these two along with mirror-inverting form a group  $\Gamma_1$  isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Examples of these symmetries acting on the trefoil  $3_1$  are shown in Figure 2.1.

For a knot  $K$ , we next form a group  $\Sigma(K)$  that is the stabilizer of  $K$  under the action of  $\Gamma_1$ . Thus  $\gamma \in \Sigma(K)$  if  $\gamma \in \Gamma_1$  and  $\gamma(K) \approx K$ . We call  $\Sigma(K)$  the *intrinsic symmetry group* of  $K$ . Thus the possible intrinsic symmetries for  $K$  consist of the subgroups of  $\Gamma_1$ , which is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . This group has five subgroups: the entire group, the trivial group, and the three subgroups generated by the three elements of order two. For a list of these subgroups, see Table 2.1.

A knot  $K$  with  $\Sigma(K)$  equal to all of  $\Gamma_1$  is said to have *full symmetry*. If  $\Sigma(K)$  is trivial, then  $K$  is said to have *no symmetry*. If  $\Sigma(K)$  is equal to the group generated

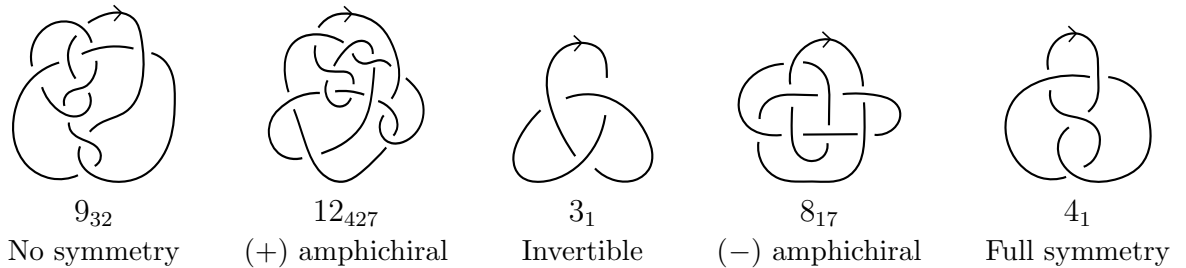


Figure 2.2: Knots which exhibit each of the 5 different types of intrinsic symmetry groups for knots, corresponding to the 5 subgroups of  $\Gamma_1$ .

by the inversion symmetry, then  $K$  is said to be *invertible*. If  $\Sigma(K)$  is equal to the group generated by the mirror symmetry, then  $K$  is said to be *positive amphichiral* or *+ amphichiral*. If  $\Sigma(K)$  is equal to the group generated by the mirror-inversion symmetry, then  $K$  is said to be *negative amphichiral* or *- amphichiral*. These five subgroups, with examples, are listed in Table 2.1 and the examples are shown in Figure 2.2.

Symmetry subgroup of $\Gamma_1$	Name	Example
$\{(1, 1)\}$	No symmetry	$9_{32}$
$\{(1, 1), (-1, 1)\}$	(+) amphichiral symmetry	$12_{427}$
$\{(1, 1), (1, -1)\}$	invertible symmetry	$3_1$
$\{(1, 1), (-1, -1)\}$	(-) amphichiral symmetry	$8_{17}$
$\Gamma_1$	full symmetry	$4_1$

Table 2.1: The five standard symmetry types for knots correspond to the five subgroups of the Whitten group  $\Gamma_1$ . This table comes from [2].

Taking two knots  $K$  and  $J$ , we can define the *composition* or *connect sum* of these two knots, a new knot denoted  $K\#J$  obtained by removing a small arc from both  $K$  and  $J$  and then connecting the four endpoints by two new arcs. This is well-defined with respect to which small arcs one chooses to delete, as one can shrink one of the knots, say  $K$ , arbitrarily small until one can move  $K$  around  $J$  to wherever one pleases. Note this also leads to the connect sum being a commutative and associative

operation, with the unknot serving as the identity, as shown in the textbook [1]. Thus the algebraic structure of knot composition is that of a commutative monoid.

A knot is called *composite* if it can be expressed as the composition of two non-trivial knots, and *prime* if it is not composite nor the unknot. The unknot is not composite, although this is not easy to prove. Moreover, every knot has a unique decomposition into prime knots. This was originally proven by Schubert; for a modern, readable proof see [18]. This leads to a theorem about composite knots and symmetry.

**Theorem 2.4.** *Let the knot  $K$  be the connected sum of  $n$  prime factor knots, so  $K = \#_{i=1}^n K_i$  where  $K_i$  is prime for all  $i$ . Then  $K$  admits symmetry  $\gamma \in \Gamma_1$  if and only if there exists a permutation  $\rho \in S_n$  so that  $K_{\rho(i)} \approx \gamma(K_i)$ .*

*Proof.* Let  $\gamma \in \Gamma_1$ . Suppose there exists a permutation  $\rho \in S_n$  so that  $K_{\rho(i)} \approx \gamma(K_i)$ . A symmetry  $\gamma$  acts on  $K$  by acting on each factor, so

$$\gamma(K) = \#_{i=1}^n \gamma(K_i) \approx \#_{i=1}^n K_{\rho(i)} \approx \#_{i=1}^n K_i = K$$

where we use the fact that the connect sum of knots is commutative to obtain the second isotopy.

Now suppose  $K$  admits symmetry  $\gamma \in \Gamma_1$ . Then  $\#_{i=1}^n K_i = K \approx \gamma(K) = \#_{i=1}^n \gamma(K_i)$ . Therefore as  $\gamma(K_i)$  is prime if  $K_i$  is prime,  $\#_{i=1}^n K_i$  and  $\#_{i=1}^n \gamma(K_i)$  are two prime decompositions of the same knot. So for all  $i$  there is a  $j$  such that  $K_i \approx \gamma(K_j)$  with no two distinct  $i$  getting assigned the same  $j$ ; this gives us our permutation  $\rho \in S_n$ . □

Let us see an example of this theorem.

**Example 2.5.** Consider  $3_1 \# 4_1$ . This knot is invertible, as  $3_1 \approx -3_1$  and  $4_1 \approx -4_1$  as Figure 2.2 indicates. However, this knot is not mirrorable. While the figure-eight

knot is mirrorable, e.g.  $4_1 \approx 4_1^*$ , the trefoil  $3_1$  is not isotopic to  $3_1^*$  or  $4_1^*$ , so there is no permutation of the factor knots that leaves the composite knot unchanged after the mirror action. Thus  $3_1\#4_1$  is invertible;  $\Sigma(3_1\#4_1) = \{(1, 1), (1, -1)\}$ .

While  $3_1 \not\approx 3_1^*$ , we know mirroring a knot  $K$  twice recovers  $K$ . Since  $3_1 \approx (3_1^*)^*$ , let us consider the square knot  $3_1\#3_1^*$ , which thus has full symmetry by Theorem 2.4. We can generalize this idea. The knot  $K\#\varepsilon K^*$  is called the  $\varepsilon$ -square of  $K$ , where  $\varepsilon$  is  $\pm 1$  depending on whether one wishes to invert or not. Then  $K\#\varepsilon K^*$  admits the  $\varepsilon$ -amphichiral symmetry, as  $\varepsilon(K\#\varepsilon K^*)^* = \varepsilon K^*\#\varepsilon(\varepsilon K^*)^* = K\#\varepsilon K^*$  as in Theorem 2.4. We can use this to classify all  $\varepsilon$ -amphichiral knots, from a proof by [15].

**Theorem 2.6.** *A knot  $K$  admits the  $\varepsilon$ -amphichiral symmetry if and only if it is the connected sum of prime  $\varepsilon$ -amphichiral knots and  $\varepsilon$ -squares of prime knots.*

*Proof.* If  $K_1$  and  $K_2$  are  $\varepsilon$ -amphichiral, so is their connect sum; this fact along with the above fact about  $\varepsilon$ -squares gives us one direction.

Now suppose  $K$  is  $\varepsilon$ -amphichiral. Due to the prime decomposition of knot,  $K$  has a decomposition  $K \approx m_1 K_1 \# \dots \# m_n K_n$ , where the knots  $K_1, \dots, K_n$  are all prime and not isotopic but possibly  $K_i^* = K_j$  and  $m_1, \dots, m_n$  are natural numbers which are the multiplicities. Then as  $K$  is  $\varepsilon$ -amphichiral, we have  $K \approx \varepsilon K^*$ , or

$$m_1 K_1 \# \dots \# m_n K_n \approx m_1 (\varepsilon K_1^*) \# \dots \# m_n (\varepsilon K_n^*)$$

Then  $m_i (\varepsilon K_i^*)$  are also all prime and distinct, so as prime decompositions are unique, for each index  $i$  there is a unique index  $j$  such that  $K_i \approx (\varepsilon K_j^*)$  and  $m_i = m_j$ . Then if  $i = j$ ,  $K_i$  is  $\varepsilon$ -amphichiral. If  $i \neq j$ , then the pair  $K_i \# K_j$  is an  $\varepsilon$ -square by definition. □

There are many systematic ways to manipulate diagrams of knots to show that two knots are isotopic. The Reidemeister moves are perhaps the most used today,

but Peter Tait had a different method. Tait, a collaborator of Lord Kelvin, is often considered the father of knot theory; he made many advances in the field as well as three seminal conjectures, which were not proven until almost a century later. A *flype* consists of twisting a certain part of a knot, a tangle, by 180 degrees. See Figure 3.6. One of Tait's conjectures was that two reduced alternating diagrams of an alternating link can be transformed to each other via a finite sequence of flypes. This result was proven over 90 years later in [16]. Flypes are used in Section 3.2 as one way to prove a theorem about alternating links.

## 2.2 Links

One way to generalize knots is to consider multiple knots in  $\mathbb{R}^3$  or  $S^3$  which do not intersect but which may be linked together. Such an object is called a link. Two links are considered equivalent if there is an ambient isotopy between them. The knots that make up a link are called the *components* of a link; a knot by itself may then be thought of as a one-component link.

One may take the connect sum of links  $L_1$  and  $L_2$  in much the same way as one can for knots, by deleting a small arc from both  $L_1$  and  $L_2$  and connecting the four endpoints by two new arcs. However there is the additional complication that choosing different knots in  $L_1$  or  $L_2$  to delete arcs from and thus join together can result non-isotopic links being formed. Despite this, one can also prove that links have a unique prime decomposition; see [12].

It is also possible to define the intrinsic symmetry group for links. However, there is an additional symmetry type; a *permutation* of a link  $L$  switches the positions of the components. A permutation of two components in a link without changing anything else is called a *pure exchange* symmetry. Together with mirroring the whole link and inverting components, these form the intrinsic symmetry group  $\Gamma_\mu$  for links with  $\mu$

components introduced by Whitten [20]; we refer to it as the *Whitten group*. Note that  $\Gamma_\mu$  is isomorphic to  $\mathbb{Z}_2 \times (\mathbb{Z}_2^\mu \rtimes S_\mu)$ . The first  $\mathbb{Z}_2$  records the orientation of the ambient space, the  $\mu$  inner  $\mathbb{Z}_2$ 's record the orientations of the knot components, and the  $S_\mu$  records the permutation of those components.

One can also consider the symmetry group  $\text{Sym}(L)$  of a link  $L$ , which is defined to be the mapping class group of the pair  $(S^3, L)$ , denoted  $\text{MCG}(S^3, L)$ . A mapping class group of a manifold is defined to be the group of isotopy classes of homeomorphisms of that manifold. Thus  $\text{MCG}(S^3, L)$  is the group of isotopy classes of homeomorphisms of  $S^3$  that fix  $L$ . One can compute this group in many different ways; see for instance [13]. The intrinsic symmetry group then becomes the image of the homomorphism

$$\pi : \text{Sym}(L) = \text{MCG}(S^3, L) \rightarrow \text{MCG}(S^3) \times \text{MCG}(L).$$

Therefore  $\Sigma(L)$  records an action on  $L$  itself, only recording the orientation of the ambient  $S^3$ , while  $\text{Sym}(L)$  records all of these symmetries as well as symmetries of  $S^3$  itself. This fact is why  $\Sigma(L)$  is called the group of intrinsic symmetries of  $L$ .

To understand why  $\Gamma_\mu$  possesses a semidirect product structure, note that the order in which we invert or exchange components matters. If we invert the knot in the first component and then exchange the first and second components we obtain a different symmetry than if we exchanged the first and second components and then inverted the knot in the first component, as which knot is in the first component has changed. We thus define the order we carry out the symmetries of  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mu, \rho)$ , where  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mu \in \mathbb{Z}_2$  and  $\rho \in S_\mu$ , as

- 1) permute components according to  $\rho$
- 2) invert  $K_i$  if  $\varepsilon_i = -1$
- 3) mirror if  $\varepsilon_0 = -1$ .

Notice that this is why we want a semidirect product instead of a direct product

with the permutation group  $S_\mu$ . Consider  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mu, \rho) * (\bar{\varepsilon}_0, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_\mu, \bar{\rho})$ , where we first undertake the symmetry  $(\bar{\varepsilon}_0, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_\mu, \bar{\rho})$  and then the symmetry  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mu, \rho)$ . We wish to have  $\bar{\varepsilon}_i$  happen after permutation  $\bar{\rho}$  but before permutation  $\rho$ , so the knot inverted by  $\bar{\varepsilon}_i$  gets carried on by  $\bar{\rho}$ , so it's perfectly fine to have  $\bar{\varepsilon}_i$  happen after  $\rho\bar{\rho}$ . On the other hand,  $\varepsilon_i$  was referring to the component that was inverted after only  $\rho$ , and now we wish to invert that component after both permutations. To see how  $\rtimes$ , the semidirect product, will describe this construction, let us first consider a definition.

**Definition 2.7.** Let  $H, K$  be groups, and let  $\phi: K \rightarrow \text{Aut}(H)$  be a homomorphism, where  $\text{Aut}(H)$  is the set of isomorphisms from  $H$  to  $H$ . Then the semidirect product with respect to  $\phi$  of  $H$  and  $K$ , written as  $H \rtimes_\phi K$ , is a group with elements  $(h, k)$  and operation  $(h_1, k_1)(h_2, k_2) = (h_1 \cdot (\phi(k_1)(h_2)), k_1 k_2)$ .

For links with  $\mu$  components, we use the homomorphism  $\phi: S_\mu \rightarrow \text{Aut}(\mathbb{Z}_2^{\mu+1})$  such that  $\phi(\rho)$  is the isomorphism  $\phi(\rho)(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mu) = (\varepsilon_0, \varepsilon_{\rho(1)}, \dots, \varepsilon_{\rho(\mu)})$ . We then obtain that if  $\gamma, \bar{\gamma} \in \mathbb{Z}_2 \times (\mathbb{Z}_2^\mu \rtimes S_\mu)$ ,

$$\begin{aligned} \gamma * \bar{\gamma} &= (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mu, \rho) * (\bar{\varepsilon}_0, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_\mu, \bar{\rho}) \\ &= ((\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\mu) \cdot (\phi(\rho)(\bar{\varepsilon}_0, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_\mu)), \rho\bar{\rho}) \\ &= (\varepsilon_0\bar{\varepsilon}_0, \varepsilon_1\bar{\varepsilon}_{\rho(1)}, \dots, \varepsilon_\mu\bar{\varepsilon}_{\rho(\mu)}, \rho\bar{\rho}). \end{aligned}$$

Notice that this give us the structure we wish to have, as  $\bar{\varepsilon}_i$  happens after  $\rho\bar{\rho}$ , while  $\varepsilon_{\rho(i)}$  now refers to the correct component for  $\varepsilon_i$  to invert.

The next question is to explore the group structure of  $\Gamma_2 = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes S_2)$ , as this thesis is mostly focused on two-component links. We will look at this question by first simplifying and looking only at the semidirect product portion,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes S_2$ , ignoring the mirror symmetry for now. Using the notation from Table 2.2 and following the definitions above we obtain the group table for  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes S_2$ , Table 2.3.

There are two elements of order 4,  $a_1$  and  $a_2$ , as these elements invert some

Notation

$m = (-1, 1, 1, e)$	$PI = (1, -1, -1, e)$
$i_1 = (1, -1, 1, e)$	$i_2 = (1, 1, -1, e)$
$\rho = (1, 1, 1, (12))$	$b = (1, -1, -1, (12))$
$a_1 = (1, -1, 1, (12))$	$a_2 = (1, 1, -1, (12))$

Table 2.2: Notation for certain elements of  $\Gamma_2$ , the group of intrinsic symmetries of two-component links.

	$i_1$	$i_2$	$\rho$	$PI$	$a_1$	$a_2$	$b$
$i_1$	$\{id\}$	$PI$	$a_1$	$i_2$	$b$	$\rho$	$a_2$
$i_2$	$PI$	$\{id\}$	$a_2$	$i_1$	$\rho$	$b$	$a_1$
$\rho$	$a_1$	$a_2$	$\{id\}$	$b$	$i_2$	$i_1$	$PI$
$PI$	$i_2$	$i_1$	$b$	$\{id\}$	$a_2$	$a_1$	$\rho$
$a_1$	$\rho$	$b$	$i_1$	$a_2$	$PI$	$\{id\}$	$i_2$
$a_2$	$b$	$\rho$	$i_2$	$a_1$	$\{id\}$	$PI$	$i_1$
$b$	$a_2$	$a_1$	$PI$	$\rho$	$i_1$	$i_2$	$\{id\}$

Table 2.3: The group table for  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes S_2$

elements and then permute, which brings the semidirect product structure into play, while the other elements are of order 2. Because of this, and because it has 8 elements,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes S_2$  is therefore isomorphic to  $D_4$ , the dihedral group with 8 elements. As the whole group  $\Gamma_2$  is formed by taking a direct product of this with another copy of  $\mathbb{Z}_2$  that represents the mirror symmetry, we find that  $\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes S_2)$  is isomorphic to  $\mathbb{Z}_2 \times D_4$

**Definition 2.8.** For a link  $L$  with  $\mu$  components and  $\gamma \in \Gamma_\mu$ , let  $L^\gamma$  be the link obtained by  $\gamma$  acting on  $L$ . We then define the *intrinsic symmetry group* of  $L$  to be  $\Sigma(L) := \{\gamma \in \Gamma_\mu \mid \text{there is an isotopy between } L \text{ and } L^\gamma \text{ that preserves component numbering and orientation}\}$ .

Thus  $\Sigma(L)$  is the stabilizer of  $L$  under the action of  $\Gamma_\mu$ , so  $\Sigma(L)$  is a subgroup of  $\Gamma_\mu$ . Thus we are interested next in the subgroups of  $\Gamma_2$ , as these will give us the possible intrinsic symmetry groups for two-component links. However, we wish for



$\Sigma(L)$  to be the same no matter how one orients and numbers the components of  $L$ . We thus need to introduce the concept of conjugate subgroups.

**Definition 2.9.** If  $H_1$  and  $H_2$  are subgroups of  $G$ , then we say that  $H_1$  is *conjugate* to  $H_2$ , denoted by  $H_1 \sim H_2$ , if there exists a  $g \in G$  such that  $H_2 = gH_1g^{-1}$

Let us look at some examples in  $\Gamma_2$ .

**Example 2.10.** First, consider  $\langle \rho \rangle = \{e, \rho\}$  and  $\langle b \rangle = \{e, b\}$  in  $\Gamma_2$ , where  $e$  is the identity symmetry. Then  $\langle rho \rangle \sim \langle b \rangle$  as  $i_1 \rho i_1^{-1} = P I \rho = b$ . See Figure 2.3.

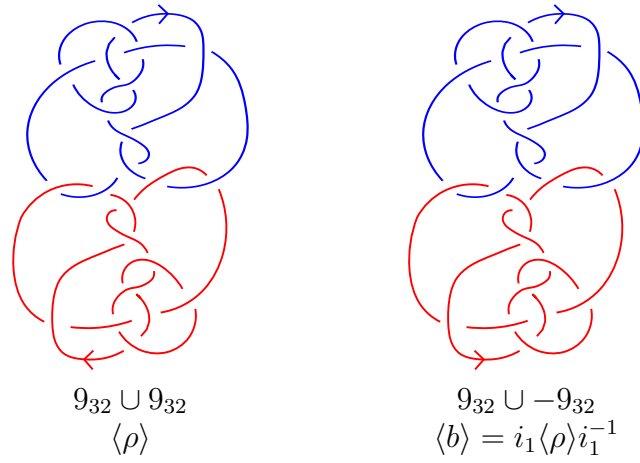


Figure 2.3: Examples of links whose intrinsic symmetry groups differ by conjugacy. The symmetry group for the link on the left is  $\langle \rho \rangle$ ; see Theorem 4.1. The symmetry group for the link on the right is  $\langle b \rangle$  for similar reasons. However, these links differ only in how one orients the bottom, red link, and thus they should have the same intrinsic symmetry group..

**Example 2.11.** Next let us examine the following:  $\langle m, i_1 \rangle = \{e, m, i_1, mi_1\}$  and  $\langle m, i_2 \rangle = \{e, m, i_2, mi_2\}$ , both subgroups of  $\Gamma_2$ . Then  $\langle m, i_1 \rangle \sim \langle m, i_2 \rangle$  as  $\rho(m)\rho^{-1} = m$ ,  $\rho(i_1)\rho^{-1} = i_2$  and  $\rho(mi_1)\rho^{-1} = mi_2$ .

**Example 2.12.** Next consider  $\langle m, \rho \rangle = \{e, m, \rho, m\rho\}$  and  $\langle m, b \rangle = \{e, m, b, mb\}$  in  $\Gamma_2$ . Then as  $i_1(m)i_1^{-1} = m$ ,  $i_1(\rho)i_1^{-1} = b$ , and  $i_1(m\rho)i_1^{-1} = mb$ , then  $\langle m, \rho \rangle \sim \langle m, b \rangle$ .

**Example 2.13.** However  $\langle m, i_1 \rangle$  and  $\langle m, \rho \rangle$  are not conjugate in  $\Gamma_2$ . For any  $\gamma \in \Gamma_2$ ,  $\gamma \langle m, i_1 \rangle \gamma^{-1} = \langle m, i_1 \rangle$  or  $\gamma \langle m, i_1 \rangle \gamma^{-1} = \langle m, i_2 \rangle$ ; that is to say  $\langle m, i_1 \rangle$  is conjugate to only one other subgroup in  $\Gamma_2$ , given in Example 2.11. Similarly  $\langle m, \rho \rangle$  is conjugate to only one other subgroup in  $\Gamma_2$ , given in Example 2.12.

Consider two links  $L = K_1 \cup K_2$  and  $\bar{L} = \bar{K}_1 \cup \bar{K}_2$  that are isotopic but have different labels; that is to say the knot in  $L$  labeled  $K_1$  is the knot labeled  $\bar{K}_2$  in  $\bar{L}$  and vice versa. Then  $\Sigma(L)$  and  $\Sigma(\bar{L})$  as subgroups of  $\Gamma_2$  are conjugate via  $\rho$  as this switches the labels. If  $L$  and  $\bar{L}$  are different only due to the orientations given to knots (where say  $K_1 \approx \bar{K}_1$  but  $K_2 \approx -\bar{K}_2$  as in Figure 2.3) then  $\Sigma(L)$  and  $\Sigma(\bar{L})$  will be conjugate via  $i_1$ , as  $i_1 \Sigma(L) i_1^{-1}$  will reverse those orientations. This extends to all  $\gamma \in \Gamma_2$  and indeed all  $\gamma \in \Gamma_\mu$ .

## 2.3 Notation

We next explain some notation for knots and links. Following [2, 3], we refer to subgroups of  $\Gamma_2$  as  $\Sigma_{k,j}$  where  $k$  is the order of the subgroup and  $j$  is an index. In this thesis, knots are referred to by Rolfsen notation,  $C_i$ , where  $C$  is the crossing number of the knot and  $i$  is an index. Links are referred to by Thistlethwaite notation and DT codes. The Thistlethwaite notation for an alternating link is  $Ca_i$  and for a non-alternating link is  $Cn_i$ , where again  $C$  is the crossing number and  $i$  is an index. The DT code, or Dowker-Thistlethwaite code, for a link diagram is a sequence of  $m$  even integers where  $m$  is the crossing number. Thus while Rolfsen notation and Thistlethwaite notation give unique designations to knots and links, a given link will have many different DT codes.

To obtain a DT code for a diagram of a knot, fix a starting point and direction and assign positive integers to crossings according to the order you arrive at crossings until you arrive back at your starting point. Note that each crossing will thus be

assigned two integers, an even integer and an odd integer. Make each even integer negative if you traveled “over” the crossing when you assigned the even integer to that crossing. Thus alternating knots will have all even integers assigned the same sign, while non-alternating knots will have both positive and negative even integers. Now write the even integers in the order obtained from their associated odd integer; this sequence of even integers is a DT code for that knot.

For a DT code for a link diagram, assign starting points to each component  $K_i$ , and when you arrive back at your starting point for  $K_i$ , jump to the starting point for  $K_{i+1}$ . When writing the sequence of even integers, use a colon in between the integers to indicate when you changed components. An example for a DT code is DT[8,10,-16:12,14,24,2,-20,-22,-4,-6,-18]. This is a 12 crossing two-component link, in which the first component has 3 crossings and the second component has 9 crossings. We can run the algorithm used to find a DT code from a link  $L$  in reverse to find a diagram for  $L$  from its DT code. In this thesis DT codes are used for links with greater than 11 crossings, as such links do not have an assigned index for Thistlethwaite notation.

The computer program **SnapPy** was used in [3] and thus in this thesis to obtain examples for subgroups of  $\Gamma_2$ . **SnapPy** uses a variant of the DT code to designate links with a high number of crossings. Let  $L$  be a  $\mu$ -component link. First, **SnapPy** uses letters instead of integers for the DT code for  $L$ , with the  $n^{th}$  letter representing the integer  $2n$ , using capitalized letters to represent negative integers. Second, the first  $\mu + 2$  letters are not part of the sequence, but rather encode other information about  $L$ . The first letter represents the crossing number, the second represents the number of components  $\mu$ , and the  $(2 + i)^{th}$  letter represents the number of crossings in the  $i^{th}$  component; the  $n^{th}$  letter represents the integer  $n$  for each of these. Notice that this gives information about where the colons should be placed in the DT code.

**Example 2.14.** One such DT code from **SnapPy** is [ebbccdaeb]. The first letter,  $e$ ,

tells us that the crossing number for this link is five while the next letter,  $b$ , tells us the number of components is two. The letters  $b$  and the  $c$  following give us that one of the components has two crossings while the other has three. Finally, the last five letters  $cdaeb$  tell us the DT code  $DT[6,8;2,10,4]$ . This represents the Whitehead link  $5a1$ .

## Chapter 3: Symmetries of Two-Component Links

The symmetries of any given link  $L$  form a subgroup of  $\Gamma_2$ . From the semidirect product structure we can see that 11 of the 15 nonidentity elements of  $\Gamma_2$  are order two. The exceptions are  $a_1 = (1, -1, 1, (12))$ ,  $a_2 = (1, 1, -1, (12))$ ,  $ma_1 = (-1, -1, 1, (12))$  and  $ma_2 = (-1, 1, -1, (12))$ ; these elements are order four. From this, we can deduce that there will be 11 subgroups of  $\Gamma_2$  of order two. Similarly, we can find 15 subgroups of order four and 7 of order eight. Combined with the trivial subgroup and all of  $\Gamma_2$ , we arrive at a total of 35 subgroups of  $\Gamma_2$ . Among these are 8 conjugate pairs. Four pairs have order 2 and four pairs have order 4. All of these conjugate pairs are obtained by relabeling or reorienting the link. There are therefore 27 subgroups of  $\Gamma_2$  that of interest; thus there are 27 possibilities for  $\Sigma(L)$ . A subgroup lattice for  $\Gamma_2$  is given in Figure 3.1.

### 3.1 Obstructions to Symmetries

We next look at ways to rule out symmetries from  $\Sigma(L)$  for given links  $L$ .

**Definition 3.1.** The *linking number* of a two-component link  $L = K_1 \cup K_2$ , denoted  $\ell k(L)$ , is found by assigning +1 or -1 to each crossing of  $K_1$  with  $K_2$  via the right-hand rule, adding together each of these values, and then dividing by two.

Linking number is a link invariant, which can be proved by noting that Reidemeister moves do not change the linking number. The action of some intrinsic symmetries  $\gamma \in \Gamma_2$  on a link  $L$  do change the sign of  $\ell k(L)$ . Let  $\gamma = (\varepsilon_0, \varepsilon_1, \varepsilon_2, p) \in \Gamma_2$  and  $L$  be a link. Then  $\ell k(L) = -\ell k(L^\gamma)$  if  $\varepsilon_0\varepsilon_1\varepsilon_2 = -1$ .

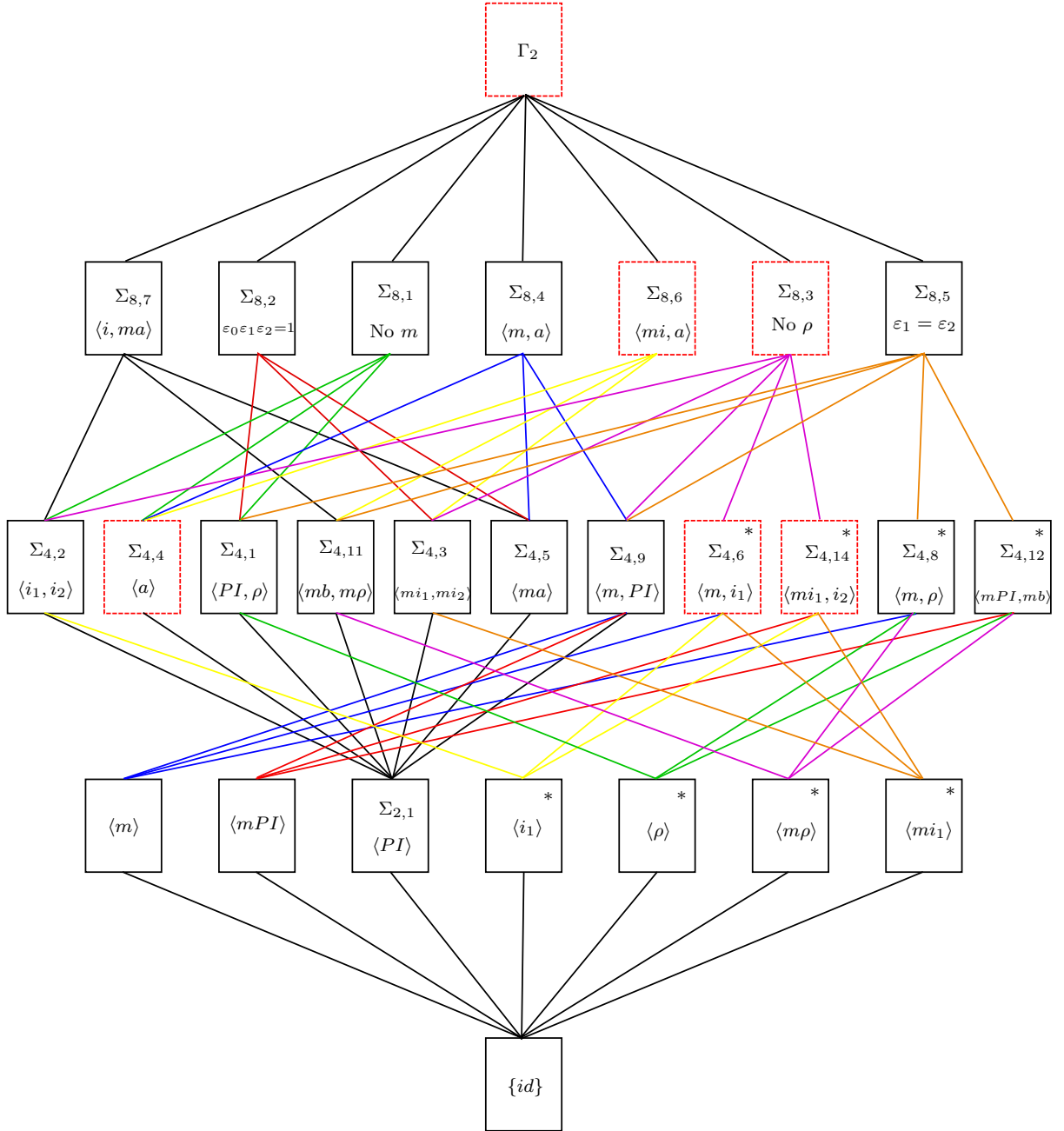


Figure 3.1: There are 27 subgroups of the Whitten group  $\Gamma_2$  up to conjugacy. Here they are shown as a subgroup lattice. Subgroups for which no example link has been found are denoted by a red dashed boarder. Subgroups that are conjugate to another subgroup that is not shown are denoted by \*. This table is from [3].

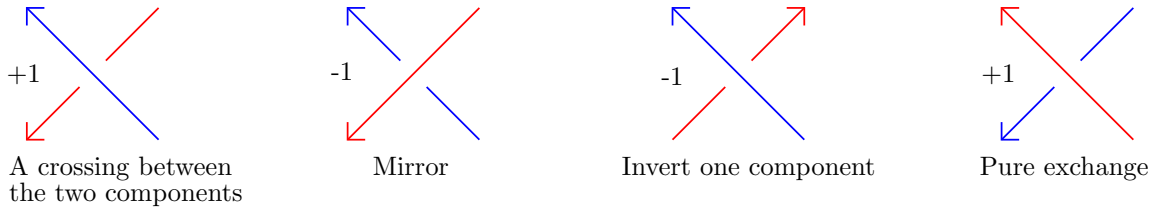


Figure 3.2: How the identity, mirror, inversion of one component and pure exchange symmetries affect linking number. Note we obtain the displayed linking number for these crossings via the right hand rule.

This is because mirroring or inverting one of the components of  $L$  changes the sign of every crossing of  $K_1$  with  $K_2$ , as shown in Figure 3.2. Thus if  $\varepsilon_i = -1$  for exactly one  $i$ ,  $1 \leq i \leq 3$ , then  $\ell k(L) = -\ell k(L^\gamma)$ . However, if  $\varepsilon_i = -1$  for exactly two  $i$ ,  $1 \leq i \leq 3$ , the sign of  $\ell k(L)$  has flipped twice and  $\ell k(L) = \ell k(L^\gamma)$ . Finally, if  $\varepsilon_i = -1$  for all  $i$ ,  $1 \leq i \leq 3$ , then the sign of  $\ell k(L)$  has flipped three times and so  $\ell k(L) = -\ell k(L^\gamma)$ . However, an exchange does not change the sign of the crossings.

Therefore, if the linking number of  $L$  is nonzero,  $\ell k(L) \neq \ell k(L^\gamma)$  if  $\varepsilon_0 \varepsilon_1 \varepsilon_2 = -1$ , so in that case  $L \not\approx L^\gamma$ . Thus all symmetries of  $L$ , that is to say all  $\gamma \in \Sigma(L)$ , have  $\varepsilon_0 \varepsilon_1 \varepsilon_2 = 1$ ; thus  $\Sigma(L)$  is a subgroup of  $\Sigma_{8,2}$ .

**Definition 3.2.** The *self-writhe* of a diagram of a link  $L = K_1 \cup K_2$ , denoted  $\text{sw}(L)$ , is found by assigning  $+1$  or  $-1$  to each crossing of  $K_1$  with itself and  $K_2$  with itself via the right-hand rule and then adding together each of these values.

Flypes preserve self-writhe, therefore as there is a finite sequence of flypes between reduced diagrams of alternating links by the Tait flyping conjecture [16], self-writhe is preserved for reduced diagrams of an alternating link. Since mirroring changes the sign of every crossing it swaps the sign of self-writhe. However inverting does not affect self-writhe as we are only concerned about intracomponent crossings. Therefore, for any element of the form  $\gamma = (-1, \varepsilon_1, \varepsilon_2, \rho)$  we have  $\text{sw}(L) = -\text{sw}(L^\gamma)$ . Thus if  $L$  is alternating and the self-writhe of  $L$  is non-zero we have  $\varepsilon_0 = 1$ ; thus  $\Sigma(L)$  is a subgroup of  $\Sigma_{8,1}$ .

If the intrinsic symmetry group of a link  $L = K_1 \cup K_2$  includes an element  $\gamma = (\varepsilon_0, \varepsilon_1, \varepsilon_2, (12))$  then knot  $K_1$  would have to take the place of  $K_2$  after permutation and the other actions contained in  $\gamma$ , and vice versa. Thus if  $\gamma_i = (\varepsilon_0, \varepsilon_i) \in \Gamma_1$  and  $K_1$  is not isotopic to  $K_2^{\gamma_1}$  or  $K_2$  is not isotopic to  $K_1^{\gamma_2}$  then  $L$  is not isotopic to  $L^\gamma$ .

We can also use the “satellite lemma” from [2, 3] to rule out exchanges between components.

**Lemma 3.3.** *Let  $L = \bigcup_{i=1}^n K_i$  be a link. Suppose that  $L(J, K_i)$  is a satellite of  $L$  constructed by replacing component  $K_i$  with a knot or link  $J$ . Then  $L$  cannot have a pure exchange symmetry  $(1, 1, \dots, 1, (ij))$  exchanging components  $K_i$  and  $K_j$  unless  $L(J, K_i)$  and  $L(J, K_j)$  are isotopic.*

*Proof.* Such a pure exchange would carry an oriented solid tube around  $K_i$  to a corresponding oriented solid tube around  $K_j$ . If we imagine  $J$  embedded in this tube, this generates an isotopy between  $L(J, K_i)$  and  $L(J, K_j)$ .  $\square$

The point of this lemma is that we can often distinguish  $L(J, K_i)$  and  $L(J, K_j)$  using classical invariants which are insensitive to the original labeling of the link.

There are also polynomial invariants that one can use to rule out symmetries from  $\Sigma(L)$  for given links  $L$ . The *Conway polynomial* for a link  $L$ , denoted  $\nabla(L)$ , was devised by Conway [6] and can be obtained from the Alexander polynomial by a change of variables. It may also be computed using the two skein relations; the first is  $\nabla(0_1) = 1$  where  $0_1$  is any version of the unknot. The second skein relation the Conway polynomial satisfies is that  $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$  where  $L_+$ ,  $L_-$  and  $L_0$  are links that differ in only a small region around a single crossing as in Figure 3.3 and  $z$  is the polynomial variable. Lemma 3.6 in this thesis uses the Conway polynomial extensively.

Another polynomial invariant is the *Jones polynomial*, denoted  $V(L)$  for a link  $L$ . This polynomial satisfies the skein relations that the Jones polynomial of the unknot is 1 and  $(t^{1/2} - t^{-1/2})V(L_0) = t^{-1}V(L_+) - tV(L_-)$ , where  $t$  is the polynomial variable



$\Sigma(K_1)$	$\Sigma(K_2)$	Same type?	$\ell k(L) = 0?$	$\Sigma(L)$ subgroup of
Full	Full	Yes	Yes	$\Gamma_2$
Full	Full	No	Yes	$\Sigma_{8,3}$ (No exchanges)
Full	Full	Yes	No	$\Sigma_{8,2}$ ( $\varepsilon_0\varepsilon_1\varepsilon_2 = 1$ )
Full	Full	No	No	$\Sigma_{4,3}$ or $\langle mi_1, mi_2 \rangle$
None	None	No	-	$\{id\}$
None	None	$K_1 \cong K_2$	-	$\langle \rho \rangle$
None	None	$K_1 \cong K_2^*$	-	$\langle m\rho \rangle$
$\langle i \rangle$	$\langle i \rangle$	$K_1 \cong K_2$	Yes	$\Sigma_{8,1}$ (No mirror)
$\langle i \rangle$	$\langle i \rangle$	No	Yes	$\Sigma_{4,2} = \langle i_1, i_2 \rangle$
$\langle i \rangle$	$\langle i \rangle$	$K_1 \cong K_2$	No	$\Sigma_{4,1} = \langle PI, \rho \rangle$
$\langle i \rangle$	$\langle i \rangle$	No	No	$\Sigma_{2,1} = \langle PI \rangle$
$\langle i \rangle$	$\langle i \rangle$	$K_1 \cong K_2^*$	Yes	$\Sigma_{8,7} = \langle i_1, ma \rangle$
$\langle i \rangle$	$\langle i \rangle$	$K_1 \cong K_2^*$	No	$\Sigma_{4,5} = \langle ma \rangle$
$\langle m \rangle$	$\langle m \rangle$	$K_1 \cong K_2$	Yes	$\Sigma_{4,8} = \langle m, \rho \rangle$
$\langle m \rangle$	$\langle m \rangle$	No	Yes	$\langle m \rangle$
$\langle m \rangle$	$\langle m \rangle$	$K_1 \cong K_2$	No	$\langle \rho \rangle$
$\langle m \rangle$	$\langle m \rangle$	No	No	$\{id\}$
$\langle mi \rangle$	$\langle mi \rangle$	$K_1 \cong K_2$	Yes	$\Sigma_{4,12} = \langle mPI, mb \rangle$
$\langle mi \rangle$	$\langle mi \rangle$	No	Yes	$\langle mPI \rangle$
$\langle mi \rangle$	$\langle mi \rangle$	$K_1$	No	$\langle \rho \rangle$
$\langle mi \rangle$	$\langle mi \rangle$	No	No	$\{id\}$
Full	None	-	Yes	$\langle i_1 \rangle$
Full	None	-	No	$\{id\}$
$\langle i \rangle$	None	-	Yes	$\langle i_1 \rangle$
$\langle i \rangle$	None	-	No	$\{id\}$
$\langle m \rangle$ or $\langle mi \rangle$	None	-	-	$\{id\}$
Full	$\langle i \rangle$	-	Yes	$\Sigma_{4,2} = \langle i_1, i_2 \rangle$
Full	$\langle i \rangle$	-	No	$\Sigma_{2,1} = \langle PI \rangle$
$\langle m \rangle$ or $\langle mi \rangle$	$\langle i \rangle$	-	Yes	$\langle i_2 \rangle$
$\langle m \rangle$ or $\langle mi \rangle$	$\langle i \rangle$	-	No	$\{id\}$
Full	$\langle m \rangle$	-	Yes	$\Sigma_{4,6} = \langle m, i_1 \rangle$
Full	$\langle m \rangle$	-	No	$\langle mi_1 \rangle$
Full	$\langle mi \rangle$	-	Yes	$\Sigma_{4,14} = \langle i_1, mi_2 \rangle$
Full	$\langle mi \rangle$	-	No	$\langle mi_2 \rangle$
$\langle mi \rangle$	$\langle m \rangle$	-	-	$\langle mi_1 \rangle$

Table 3.1: A table giving the largest group a link  $L = K_1 \cup K_2$  with various obstructions can have as its symmetry group.

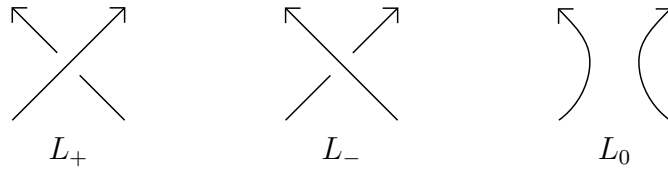


Figure 3.3: The ways a link  $L$  could differ at a crossing, used for polynomial skein relations. The link with a  $+1$  crossing is  $L_+$ , the link with a  $-1$  crossing is  $L_-$ , and the link with the crossing nullified as in Definition 3.8 is  $L_0$ .

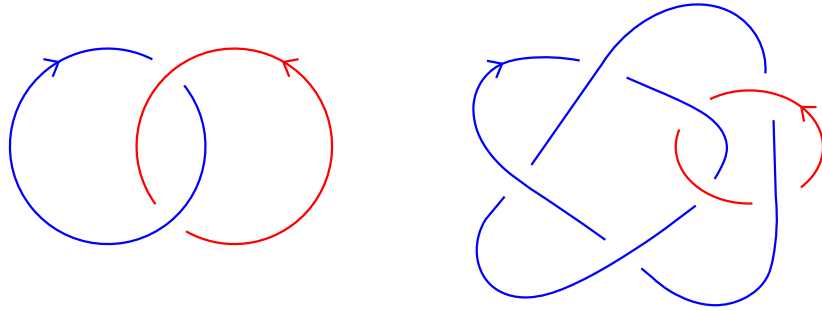


Figure 3.4: The Hopf link on the left and  $7a3$  on the right.

and once again  $L_+$ ,  $L_-$  and  $L_0$  are defined as in Figure 3.3. The Jones polynomial and the Conway polynomial can also be combined and generalized as the *HOMFLY polynomial*. The Jones, Conway, and HOMFLY polynomials are all used extensively to show that certain links do not exhibit certain symmetries.

We turn to some examples to illustrate how we use these invariants to find  $\Sigma(L)$  for two links  $L$ .

**Example 3.4.** The Hopf link has 8 of the possible 16 symmetries. It is found on the left in Figure 3.4. It can be shown that you can invert both elements at once, permute the two components, and mirror the link while inverting one component. These three symmetries generate the subgroup of  $\Gamma_2$  that this link exhibits,  $\Sigma_{8,2}$ . As the linking number is nonzero, the Hopf link cannot have any other symmetries.

**Example 3.5.** The link  $7a3$  is found on the right in Figure 3.4. It is alternating and its self-writhe is nonzero, which means that it cannot have any mirror symmetry. Moreover, its components are the unknot and the trefoil knot, so it cannot have any exchange symmetry. We have isotopies showing that that one can invert either of the

components individually or together, so its subgroup is  $\Sigma_{4,2}$ .

### 3.2 Alternating Two-Component Links

Two-component alternating links have a simpler subgroup lattice as such links cannot have certain symmetries. If  $L$  is an alternating nonsplit link then  $\Sigma(L)$  is a subgroup of  $\Sigma_{8,1}$  or  $\Sigma_{8,2}$  as we can see in two different proofs. A link is *alternating* if in a diagram of the link the crossings alternate between over and under as one travels around the link in a fixed direction. We shall first look at the Conway polynomial.

**Lemma 3.6.** *Let  $L$  be a link with  $\mu$  components. Then if  $\gamma = (-1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_\mu, p)$  where  $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_\mu$  and  $p \in S_\mu$ ,  $\nabla(L) = (-1)^{\mu+1} \nabla(L^\gamma)$ .*

*Proof.* We shall proceed by induction on the number of components. For knots, this is well known; see [7]. Let  $\gamma$  be defined as above. Let  $K$  be one of the components of  $L$ . Resolve each of  $n$  overcrossings that  $K$  has with the other components of  $L$  according to the skein relation of the Conway polynomial, as well as the corresponding  $n$  undercrossings for  $K$  in  $L^\gamma$ . As this will turn every crossing  $K$  has with  $L$  into undercrossings, we eventually make split links  $L - K$  and  $L^\gamma - K$ . We obtain

$$\nabla(L) = (-1)^{i_1} z \nabla(L_{0,1}) + (-1)^{i_2} z \nabla(L_{0,2}) + \dots + (-1)^{i_n} z \nabla(L_{0,n}) + \nabla(L - K)$$

and

$$\nabla(L^\gamma) = (-1)^{i_1+1} z \nabla(L_{0,1}^\gamma) + (-1)^{i_2+1} z \nabla(L_{0,2}^\gamma) + \dots + (-1)^{i_n+1} z \nabla(L_{0,n}^\gamma) + \nabla(L^\gamma - K)$$

where  $i_j = 0$  or  $1$  if the  $j$ th overcrossing of  $K$  in  $L$  was a  $+1$  crossing or a  $-1$  crossing, respectively.

As  $L - K$  and  $L^\gamma - K$  are split,  $\nabla(L - K) = \nabla(L^\gamma - K) = 0$ . Moreover, every crossing we resolved was an intercomponent crossing, so  $L_{0,j}$  is a link with  $\mu - 1$  components for every  $j$ ; therefore by induction hypothesis,  $\nabla(L_{0,j}) = (-1)^\mu \nabla(L_{0,j}^\gamma)$ .

We thus have

$$\begin{aligned}\nabla(L) &= (-1)^{i_1+\mu} z \nabla(L_{0,1}^\gamma) + (-1)^{i_2+\mu} z \nabla(L_{0,2}^\gamma) + \dots + (-1)^{i_n+\mu} z \nabla(L_{0,n}^\gamma) \\ &= (-1)^{\mu+1} \nabla(L^\gamma)\end{aligned}\quad \square$$

As the Conway polynomial does not vanish for nonsplit alternating links and is a link invariant, this therefore implies that nonsplit alternating links with an even number of components cannot exhibit the symmetries from Lemma 3.6.

**Theorem 3.7.** *If  $L$  is a nonsplit, alternating link with  $\mu$  components where  $\mu$  is even and  $\gamma = (-1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_\mu, p)$  where  $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_\mu$  and  $p \in S_\mu$ , then  $\gamma \notin \Sigma(L)$ .*

*Proof.* By Lemma 3.6 and the above,  $\nabla(L) \neq \nabla(L^\gamma)$  so  $\gamma \notin \Sigma(L)$ .  $\square$

Therefore, if  $L$  is a nonsplit alternating link with two components and  $\gamma = (-1, \varepsilon, \varepsilon, p)$ , then  $L$  cannot be isotopic to  $L^\gamma$ . Therefore for such links, the subgroup lattice diagram is the order ideal generated by two subgroups,  $\Sigma_{8,1}$  and  $\Sigma_{8,2}$ . Alternating examples for all of these subgroups have been found except for  $\Sigma_{4,4}$ , where no example link of any type has yet to be found.

Alternatively, we can follow the example of Cerf in [4] to prove this same result.

**Definition 3.8.** The *nullification of a crossing* of an oriented link projected in  $\mathbb{R}^2$  takes a crossing and removes it by regluing the strands that make up the crossing in a way such that the original orientation is preserved. This process is described in Figure 3.5.

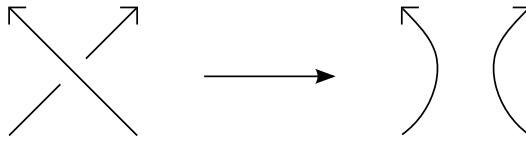


Figure 3.5: The nullification of a crossing. Notice that nullifying the mirror of the crossing on the left would result in the same nullification.

**Definition 3.9.** The *nullification of an oriented link projection* in  $\mathbb{R}^2$  consists of nullifying a series of crossings until all remaining crossings are nugatory, that is to say all crossings can be removed by twisting, while preserving the number of nonsplit link components. The set of crossings so nullified is the *nullification set*.

Note that if we start with a nonsplit link, we obtain the unknot after nullification, while if we start with a split link with  $n$  split components, we obtain the unlink with  $n$  components after nullification, as according to this definition we nullify in such a way as to preserve the number of nonsplit link components.

One may obtain the *Seifert diagram* of a link diagram by nullifying all of the crossings of the link. This results in a set of topological circles in the plane; call these *Seifert circles*. Then the *Seifert graph* of a link diagram is obtained by placing a vertex in each Seifert circle and placing an edge between vertices if the Seifert circles shared a crossing in the original link diagram. For more information, see Chapter 4 of [1]. When transposed to the Seifert graph representation of a link, nullification consists finding a spanning tree (or spanning forest for a split link) for that graph. This is due to the fact that a bridge in the Seifert graph of a link corresponds to a nugatory crossing in the projection of that link.

**Definition 3.10.** The *nullification writhe*  $w_x$  of an oriented link projection in  $\mathbb{R}^2$  is the sum of the signs of the crossings in the nullification set.

**Definition 3.11.** The *remaining writhe*  $w_y$  of an oriented link projection in  $\mathbb{R}^2$  is the sum of the signs of the crossings of the crossings not in the nullification set.

We can immediately see that for a given link  $L$ , the writhe of  $L$  is equal to the sum of the nullification writhe and the remaining writhe.

To show that nullification writhe and remaining writhe are well-defined for a given reduced alternating link projection, we shall need a lemma.

**Lemma 3.12.** *In the Seifert diagram corresponding to an alternating link projection, all connections on the same side of a Seifert circle have the same sign, and all connections on opposite sides of a Seifert circle have opposite signs.*

*Proof.* If adjacent connections on the same side of a Seifert circle had opposite signs, then the corresponding link projection would be non-alternating. Similarly, if two adjacent connections on opposite sides of a Seifert circle had the same sign, then the corresponding link projection would be non-alternating.  $\square$

**Theorem 3.13.** *The nullification writhe and the remaining writhe of a reduced alternating projection of a link are independent of the choice of the nullification set.*

*Proof.* In the Seifert graph description, the nullification set consists of all edges removed in order to obtain a spanning tree (or spanning forest if the link is split), that is to say, all edges but one between each pair of vertices linked by multiple edges, and one edge from each cycle of edges. As the number of edges removed to obtain the spanning tree (or forest) is constant no matter which edges are chosen, the cardinality of the nullification set is constant for a given link projection.

By Lemma 3.12, all multiple edges between a given pair vertices in a Seifert graph have the same sign, so removing all but one will add the same contribution to the nullification writhe. Moreover, all edges in a given cycle have the same sign. If not, you would have a vertex  $v$  in the cycle that would correspond to a Seifert circle with one neighbor in the cycle being inside  $v$  while the other neighbor would be outside  $v$ . But then the only way to close the cycle by the Jordan Curve Theorem would be for the cycle to go through  $v$  again, which is a contradiction of the definition of cycle. Therefore, removing any edge in a cycle will add the same contribution to the nullification writhe. Therefore nullification writhe is independent of the choice of the nullification set, therefore the remaining writhe is independent of the choice of the nullification set.  $\square$

We next wish to prove that nullification writhe is invariant for any reduced alternating projection of a given link. We will first need another lemma.

**Lemma 3.14.** *Given any crossing of a reduced projection of an alternating link, there exists a nullification set containing this crossing.*

*Proof.* Because the projection is reduced, there are no nugatory crossings. This implies that in the Seifert graph, there are no bridges, as bridges correspond to nugatory crossings. Therefore, we can start the spanning tree (or forest) by removing any edge, as removing any edge does not disconnect the graph by definition. This implies that in the nullification process, we can begin by nullifying any crossing.  $\square$

**Theorem 3.15.** *Any two reduced projections of an oriented alternating link have the same nullification writhe and the same remaining writhe.*

*Proof.* By the Tait Flying Conjecture proved by Menasco and Thistlethwaite [16], any two reduced projections of an alternating link are related by a finite sequence of flypes. We thus need to show that flypes preserve nullification writhe and remaining writhe.

Two projections related by a flype can be represented as in Figure 3.6, where  $R$  and  $S$  are tangles. The sign of the crossing might be positive or negative, but it is unchanged by the flying operation.

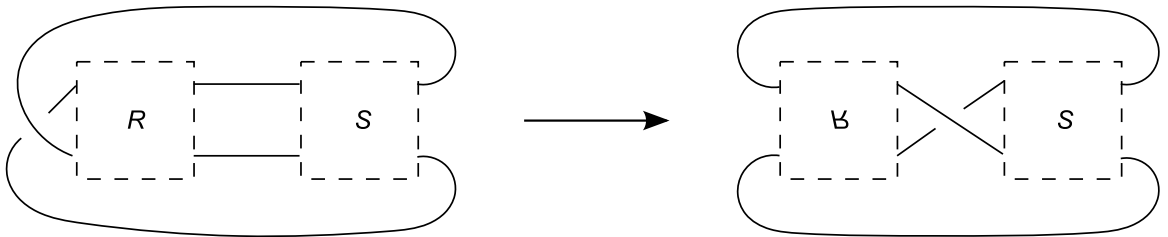


Figure 3.6: An example of a flype of a tangle  $R$ . Notice that the sign of the crossing before the flype might be positive or negative, but that the sign of the crossing is unchanged by flying.

Let us analyze the first situation, when the sign of the crossing outside of the tangles is negative. We nullify the diagram in Figure 3.6 before the flype by including the crossing outside of  $R$  or  $S$  in the nullification set, as on the left in Figure 3.7. Notice that we obtain a diagram where the unknot (or unlink) remaining after nullification is the connected sum of tangles  $R$  and  $S$  after nullification; call these  $R'$  and  $S'$ . This implies that  $R'$  and  $S'$  are both unknots (or unlinks). Then, for the diagram on the right in Figure 3.6, we choose to nullify the image of the same crossings that

we nullified in the left-hand diagram. This implies that  $R'$  and  $S'$  on the right are both unknots (or unlinks), so as the link on the right is also the connect sum of  $R'$  and  $S'$ , it is also the unknot or unlink. Therefore, this set of crossings is a nullification set, so the nullification writhe of the left and right diagrams is the same.

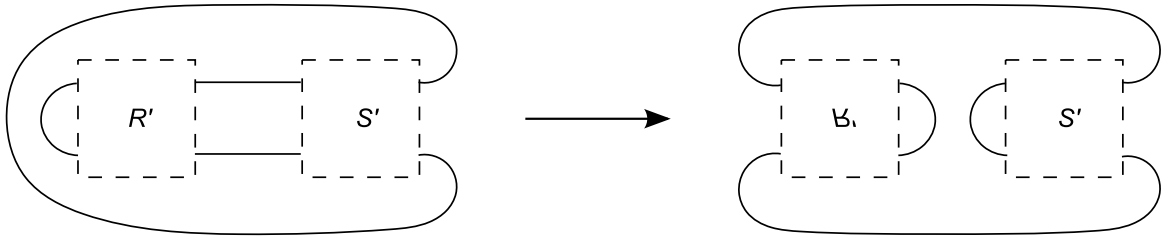


Figure 3.7: The nullification of Figure 3.6 when the crossing outside of  $R$  or  $S$  was negative.

For the second situation, where the sign of the crossing outside of the tangles is positive, we once again nullify the left-hand diagram in Figure 3.6 by including the crossing outside of  $R$  or  $S$  in the nullification set to obtain the left-hand diagram in Figure 3.8. We then nullify the image of those crossings in the right-hand diagram in Figure 3.8.

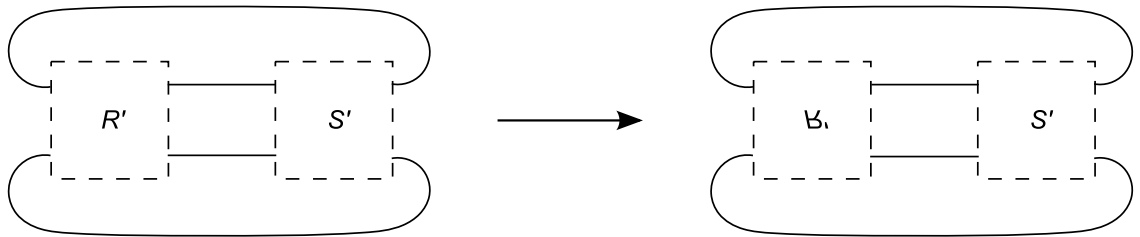


Figure 3.8: The nullification of Figure 3.6 when the crossing outside of  $R$  or  $S$  was positive.

Four subcases arise in the Seifert diagram depiction of these links after nullification, depending on how the endpoints of the tangles  $R'$  and  $S'$  are connected to each other within  $R'$  and  $S'$ . In the first two subcases, shown in Figure 3.9, we know there must remain at most one connection (direct or indirect) between both displayed Seifert circles, as otherwise there would be a cycle in the corresponding Seifert graph. Therefore, either  $R'$  or  $S'$  is not a connected tangle, so one of those two can be split



into two tangles. Then all of the diagrams below become the connected sum of three tangles. Since the links on the left in Figure 3.9 are trivial, all three of these tangles must be trivial, which implies that the links on the right in Figure 3.9 are also trivial. Therefore, the crossings nullified in the right hand diagrams form a nullification set, so the nullification writhe of the left and right diagrams in Figure 3.9 are the same.

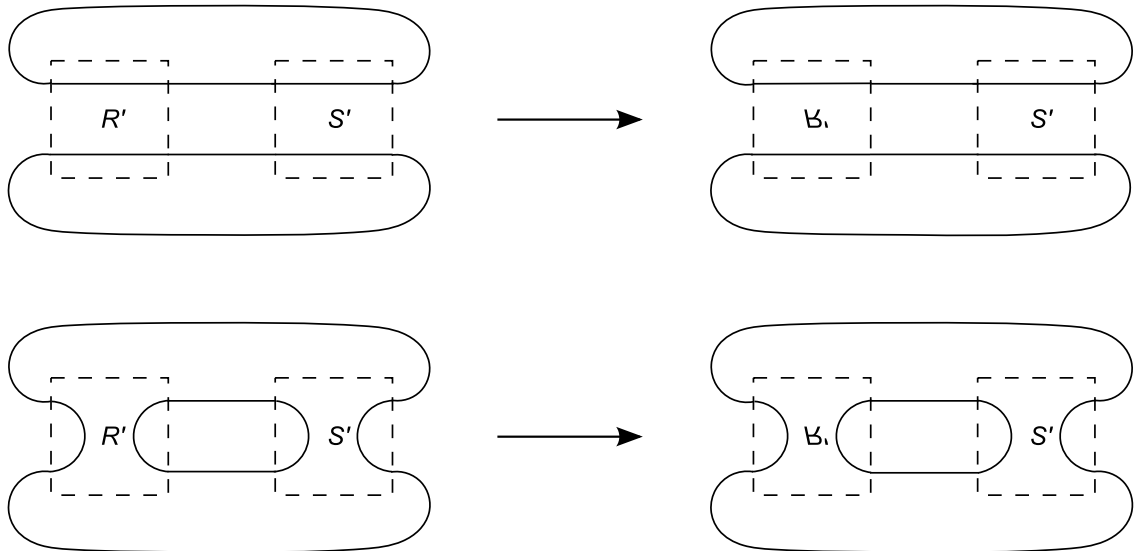


Figure 3.9: The first two subcases of Figure 3.8, when the endpoints of the tangles  $R'$  and  $S'$  are both connected within  $R'$  and  $S'$  NW to NE on the top and NW to SW on the bottom. Note the lines within  $R'$  and  $S'$  indicate these connections, but are not meant to indicate that these are all of the arcs within  $R'$  and  $S'$ .

In the second two subcases, shown in Figure 3.10, you cannot have a connection from a circle to itself, nor can you have a cycle, therefore in the top diagrams in Figure 3.10  $R'$  is not a connected tangle while in the bottom diagrams in Figure 3.10  $S'$  is not a connected tangle. Therefore in each of these subcases, we can split either  $R'$  or  $S'$  into two tangles where all of the diagrams below become the connected sum of three tangles. Since the links on the left Figure 3.10 are trivial, all three of these tangles must be trivial, which implies that the links on the right Figure 3.10 are also trivial. Therefore, the crossings nullified in the right-hand diagrams in Figure 3.10 form a nullification set, so the nullification writhe of the left and right diagrams are the same.

Therefore, in all cases, flypes preserve nullification writhe. Therefore, flypes pre-

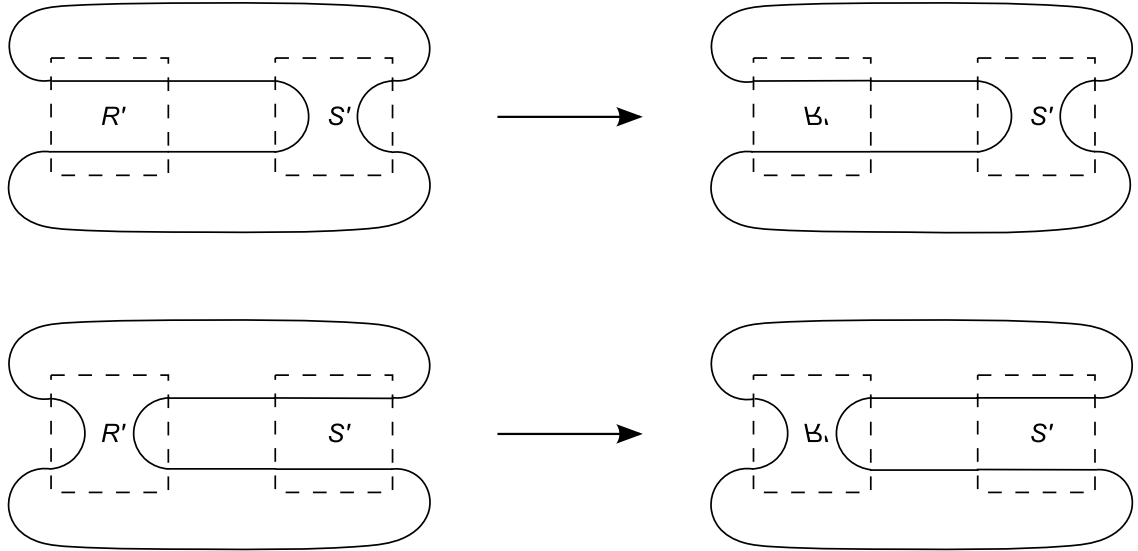


Figure 3.10: The second two subcases of Figure 3.8. On top the endpoints of the tangle  $R'$  are connected within  $R'$  NW to NE and the endpoints of the tangle  $S'$  are connected within  $S'$  NW to SE. The bottom diagrams are the opposite. Note the lines within  $R'$  and  $S'$  indicate these connections, but are not meant to indicate that these are all of the arcs within  $R'$  and  $S'$ .

serve remaining writhe, as flypes preserve writhe. Therefore, all reduced alternating projections of a link have the same nullification writhe and remaining writhe.  $\square$

Notice that this implies that for an alternating link to have certain amphichiral symmetries, it must have a nullification writhe and a remaining writhe of zero.

**Theorem 3.16.** *If  $L$  is an alternating link represented by a reduced projection then  $w_x(L) = -w_x(L^\gamma)$  where  $\gamma$  is  $m$ ,  $mPI$ ,  $m\rho$  or  $mPI\rho$ .*

*Proof.* Take a nullification set for  $L$ . As  $m, mPI, m\rho$  and  $mPI\rho$  all switch the sign of every crossing in  $L$ , they switch the sign of every crossing in the nullification set for  $L$ . Moreover, nullifying this set of crossings produces a link with only nugatory crossings. If we were to nullify these same crossings in  $L^\gamma$ , we would still be left with only nugatory crossings, as a switched sign on a nugatory crossing leaves a nugatory crossing. Therefore, this set of crossings is a nullification set for  $L^\gamma$ , so  $w_x(L) = -w_x(L^\gamma)$   $\square$

So for an alternating link to have the mirror, mirror pure invertibility, mirror exchange, or mirror pure invertibility exchange symmetries, it must have a nullification writhe of zero, as nullification writhe is preserved for reduced diagrams of alternating links. However, alternating nonsplit links with an even number of components cannot have a nullification writhe of zero.

**Theorem 3.17.** *If  $L$  is an alternating nonsplit link then  $w_x(L) \neq 0$*

*Proof.* Take a reduced alternating nonsplit diagram of link  $L$ , and begin to nullify it. Each step in the nullification process either increases or decreases the number of components of the link. If the strands of the link in the crossing nullified are part of the same component, the number of components in the link increases by one, while if the strands in the crossing are part of different components, the number of components in the link decreases by one.

Now as  $L$  is nonsplit, this implies that the unknot will be the result of the nullification process. Therefore, as we start with an even number of components and end up with one component after the nullification process, the cardinality of the nullification set must be odd. Therefore, the nullification writhe is adding up an odd number of positive and negative ones, so  $w_x(L) \neq 0$ .  $\square$

Thus we see that Cerf's nullification writhe argument also proves Theorem 3.7.

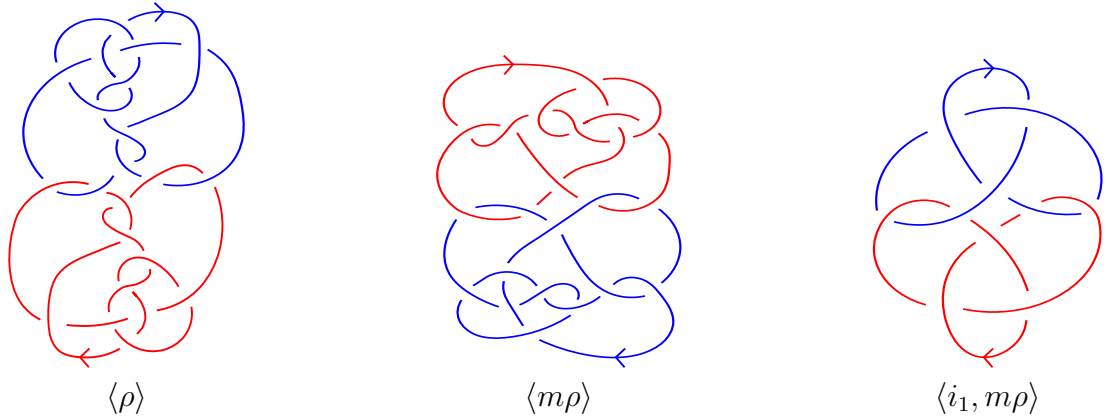


Figure 4.1: Links with Whitten groups for which no example link had been found prior to [3]. Their symmetry groups are  $\langle \rho \rangle$ ,  $\langle m\rho \rangle$  and  $\langle i_1, m\rho \rangle$ ; see Theorem 4.1.

## Chapter 4: Results

Of the 27 different two-component link symmetry types, prime nonsplit examples for 21 of those symmetry types have been found. The example links for the groups  $\{id\}$  and  $\Sigma_{8,5}$  were originally found by Hillman in his paper [14]. He listed examples for many of the 12 subgroups that do not include  $i_1$  or  $i_2$ .

The examples for the groups  $\Sigma_{8,1}$ ,  $\Sigma_{8,2}$ ,  $\Sigma_{4,1}$ ,  $\Sigma_{4,2}$ , and  $\Sigma_{2,1}$  were generally known; for example, proofs that these links exhibit the given symmetry groups are found in [2]. The examples for  $\langle \rho \rangle$ ,  $\langle m\rho \rangle$  and  $\langle i_1, m\rho \rangle$  were first found in [3], and devised by the authors. For diagrams of these links, see Figure 4.1. All other examples are first given in [3], and found using **SnapPea**.

We next show that the three links in Figure 4.1 have the claimed symmetry groups.

**Theorem 4.1.** *The three links in Figure 4.1 possess the symmetry groups  $\langle \rho \rangle$ ,  $\langle m\rho \rangle$  and  $\Sigma_{8,7}$ , respectively.*

*Proof.* The link on the far left in Figure 4.1 is comprised of two components of  $9_{32}$ , while the link in the center is comprised of knots  $9_{32}$  and  $9_{32}^*$ . The knot  $9_{32}$  has no symmetry, therefore the only possible nontrivial symmetries for these links are  $\rho$

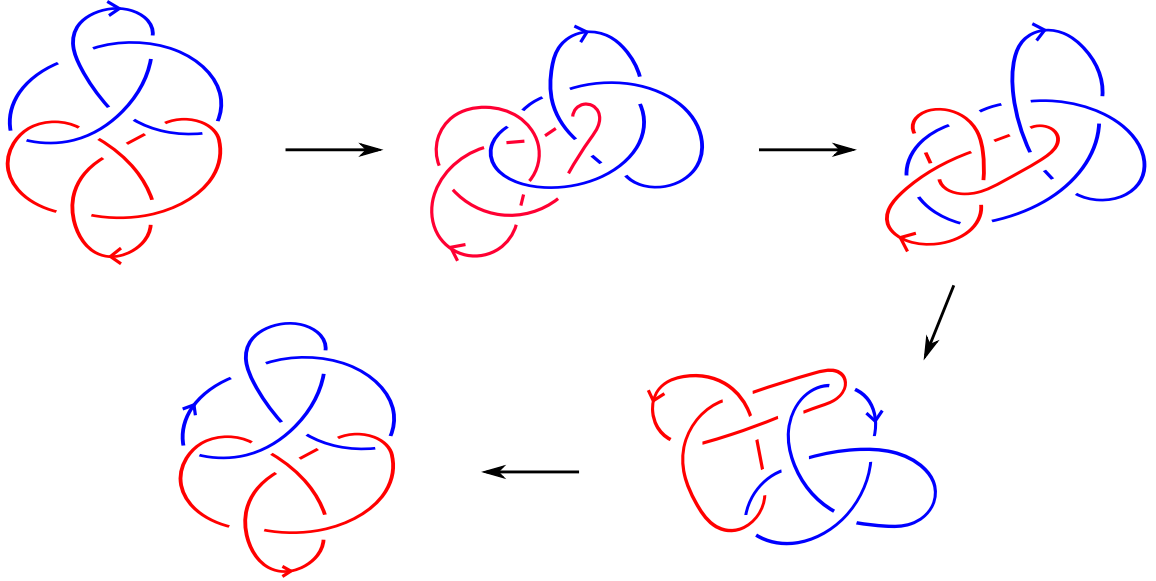


Figure 4.2: Isotopy diagram showing that  $i_2 \in \Sigma(10n59)$ . This figure first appeared in [3].

and  $m\rho$  respectively. These two links achieve said symmetries through a 180 degree rotation about a line perpendicular to the page.

The two components of the link  $10n59$  in the far right in Figure 4.1 are a left- and a right-handed trefoil ( $3_1$  and  $3_1^*$ ). Since the trefoil is not mirrorable, we cannot mirror link  $10n59$  without permuting components, and vice-versa. Therefore its symmetry group is a subgroup of  $\Sigma_{8,7} = \langle i_2, m\rho \rangle$ . The link exhibits the  $m\rho$  symmetry through a 180 degree rotation about a line perpendicular to the page; Figure 4.2 shows it also admits the symmetry  $i_2$ .  $\square$

With my coauthors in [3], we managed to find examples for 21 out of the 27 different subgroups of  $\Gamma_2$ . These results are presented in Table 4.1.

Moreover, using **SnapPy** [8], we were able to find the number of hyperbolic two-component links that exhibit each symmetry group for two-component links with 14 crossings or less. This information is found in Table 4.2.

Finally, there are 293 non-hyperbolic two-component links with 14 crossings or less in Thistlethwaite's link table. **SnapPea** is unable to calculate the symmetry groups

for these links. In [3], we were able to obtain partial information about the symmetry groups using the invariants and methods listed in Chapter 3. These methods gave us a subgroup of  $\Gamma_2$  guaranteed to contain  $\Sigma(L)$  for our 293 non-hyperbolic two-component links.

We found 9 links with  $\Sigma(L) < \Sigma_{4,1}$ , 89 links with  $\Sigma(L) < \Sigma_{4,2}$ , 194 links with  $\Sigma(L) < \Sigma_{2,1}$  and 1 link with  $\Sigma(L) < \Sigma_{8,1}$ . Note that from the subgroup lattice of  $\Gamma_2$  only this last link might have had a new symmetry group,  $\Sigma_{4,4} < \Sigma_{8,1}$ , one for which we had not already found an example link. This potential new example was the 194th 14-crossing two-component non-hyperbolic link in Thistlethwaite's table. However, we were able to realize a diagram of the link  $L$  that revealed a pure-exchange symmetry  $\rho$ , and as  $\rho \notin \Sigma_{4,4}$  we thus have  $\Sigma(L) \neq \Sigma_{4,4}$ . We therefore have the following proposition.

**Proposition 4.2.** *Among all two-component links with less than or equal to 14 crossings, we obtain precisely 21 different intrinsic symmetry groups; missing are 6 of the 27.*

*Proof.* From Table 4.2 we have that exactly 21 different intrinsic symmetry groups exhibited by hyperbolic two-component links with less than or equal to 14 crossings, while the above shows that non-hyperbolic two-component links do not exhibit any new intrinsic symmetry groups.  $\square$

Symmetry Group	Example
$\{id\}$	11a164
$\Sigma_{2,2} = \langle m \rangle$	DT[8,10,-16:12,14,24,2,-20,-22,-4,-6,-18]
$\Sigma_{2,3} = \langle mPI \rangle$	DT[8,10,-14:12,22,24,-18,-20,-4,-6,-16,2]
$\Sigma_{2,1} = \langle PI \rangle$	7a2
$\Sigma_{2,4} = \langle i_1 \rangle$	10a98
$\Sigma_{2,5} = \langle \rho \rangle$	DT[14,16,44,20,4,2,18,10,12,8,28:36,38,22,42,26,24,40,32,34,30,6]
$\Sigma_{2,6} = \langle m\rho \rangle$	DT[44,-20,-16,-22,-14,-4,-6,-10,-12,-28,-8:2,34,-18,24,42,26,40,30,32,36,38]
$\Sigma_{2,7} = \langle mi_1 \rangle$	10a81
$\Sigma_{4,2}$	7a3
$\Sigma_{4,4}$	<i>No example known</i>
$\Sigma_{4,1}$	4a1
$\Sigma_{4,11}$	DT[16,-6,-12,-20,-22,-4,28:2,14,26,-10,-8,18,24]
$\Sigma_{4,3}$	10n46, 10a56
$\Sigma_{4,5}$	DT[14,6,10,16,4,20:22,8,2,24,12,18]
$\Sigma_{4,9}$	10n36
$\Sigma_{4,6}$	<i>No example known</i>
$\Sigma_{4,7}$	<i>No example known</i>
$\Sigma_{4,8}$	DT[10,-14,-18,24:2,28,-4,-12,-20,-6,-16,26,8,22]
$\Sigma_{4,10}$	DT[10,-14,-20,24:2,28,-16,-4,-12,-6,-18,26,8,22]
$\Sigma_{8,7}$	10n59
$\Sigma_{8,2}$	2a1
$\Sigma_{8,1}$	5a1
$\Sigma_{8,4}$	DT[14,-16,22,-28,24,-18:12,-20,-10,-2,26,8,5,-6]
$\Sigma_{8,6}$	<i>No example known</i>
$\Sigma_{8,3}$	<i>No example known</i>
$\Sigma_{8,5}$	DT[10,-14,-18,22:2,24,-4,-12,-6,-16,8,10]
$\Gamma_2$	<i>No example known</i>

Table 4.1: Prime nontrivial links can currently be found that exhibit 21 out of the 27 possible symmetry groups. Links are specified using Thistlethwaite notation and DT codes, as described in Section 2.2 . This table can be found in [3]

Group	Number of Links	Group	Number of Links	Group	Number of Links
$\{id\}$	53484	$\Sigma_{4,1}$	1396	$\Sigma_{8,1}$	52
		$\Sigma_{4,2}$	2167	$\Sigma_{8,2}$	25
		$\Sigma_{4,3}$	24	$\Sigma_{8,3}$	0
$\Sigma_{2,1}$	17951	$\Sigma_{4,4}$	0	$\Sigma_{8,4}$	1
$\Sigma_{2,2}$	7	$\Sigma_{4,5}$	12	$\Sigma_{8,5}$	2
$\Sigma_{2,3}$	9	$\Sigma_{4,6}$	0	$\Sigma_{8,6}$	0
$\Sigma_{2,4}$	1336	$\Sigma_{4,7}$	0	$\Sigma_{8,7}$	8
$\Sigma_{2,5}$	418	$\Sigma_{4,8}$	2		
$\Sigma_{2,6}$	3	$\Sigma_{4,9}$	11		
$\Sigma_{2,7}$	123	$\Sigma_{4,10}$	1	$\Gamma_2$	0
		$\Sigma_{4,11}$	4		

Table 4.2: The number of links admitting each symmetry group among the 77,036 hyperbolic two-component links included in the Thistlethwaite table distributed with SnapPy. Among these links, almost two-thirds had no symmetry (trivial symmetry group) and there were no examples with full symmetry. This table is from [3].



## Chapter 5: Future Directions

The results presented in this thesis can be extended in a number of ways. There are still six subgroups of  $\Gamma_2$  for which no example links has been found. One can also start looking at  $\Gamma_\mu$  when  $\mu \geq 3$ , but trying to find examples for all subgroups of  $\Gamma_\mu$  quickly becomes untenable. Another new direction of study is to look at families of links with interesting properties.

### 5.1 Subgroups of $\Gamma_\mu$

Subgroup	Generators for subgroup
$\Sigma_{4,4}$	$\langle a \rangle$
$\Sigma_{4,14}$	$\langle mi_1, i_2 \rangle$
$\Sigma_{4,6}$	$\langle m, i_1 \rangle$
$\Sigma_{8,6}$	$\langle mi_1, a \rangle$
$\Sigma_{8,3}$	$\langle m, i_1, i_2 \rangle$
$\Gamma_2$	

Table 5.1: Two-component symmetry groups for which there is no known prime, nonsplit example link that demonstrates that group.

An obvious direction for future study would be to find examples for the six different symmetry groups for which examples of prime nonsplit links have not yet been found. A list of these is found in Table 5.1.

Of particular interest is an example of a link with full symmetry. The unlink with  $\mu$  components possesses full symmetry, as does a link comprised of two split copies of knots with full symmetry such as  $4_1$ . To date no nonsplit examples are known. A two-component link  $L$  with symmetry group  $\Sigma(L) = \Gamma_2$  must have linking number zero and be non-alternating; its two knot components must be of the same knot type and possess full symmetry.

One natural generalization of these results would be to look at the symmetry groups of links with more than two components. An exhaustive survey quickly be-

comes unwieldy, however; there are 131 and 994 subgroups up to conjugacy for three- and four-component links, respectively, as can be seen in Table 5.2.

$\mu$	$ \Gamma_\mu $	# subgroups	# subgroups (up to conjugacy)
1	4	5	5
2	16	35	27
3	96	420	131
4	768	9417	994
5	7680	270131	6382

Table 5.2: The number of subgroups of  $\Gamma_\mu$ ; each one represents a different intrinsic symmetry group possible for a  $\mu$ -component link. This table is from [2, 3].

## 5.2 Symmetries of Brunnian links

Finding family of examples that have interesting symmetry groups seems more manageable. A *Brunnian link*  $L = \bigcup_{i=1}^n K_i$  is a nonsplit link with  $n$  components where deleting any component results in the unlink with  $n - 1$  components, i.e.,  $L - K_i$  is an unlink with  $n - 1$  components for all  $i$ ,  $1 \leq i \leq n$ . The Borromean rings, the simplest example of a Brunnian link, are depicted in Figure 5.1.

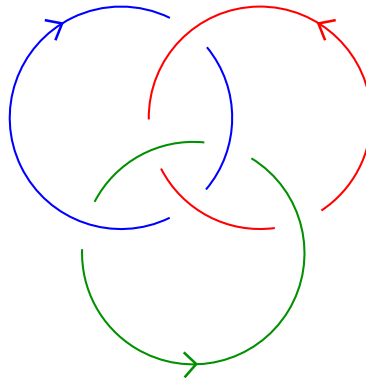


Figure 5.1: The Borromean rings,  $6a4$ .

**Definition 5.1.** The *Bing double* of a link  $L$  takes a tubular neighborhood  $V$  of one

component  $K$  of the link, and replaces  $V$  and  $K$  with the torus and the Bing link depicted in Figure 5.2.

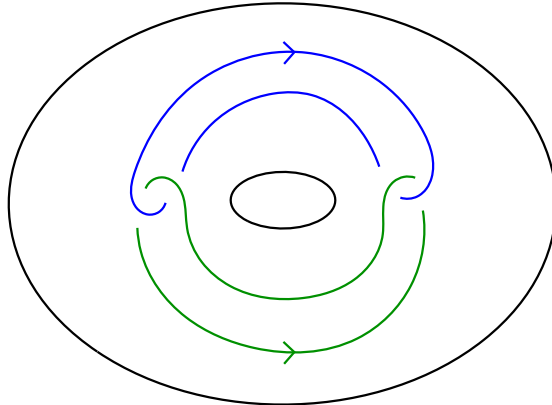


Figure 5.2: The Bing link inside of a solid torus.

Note that taking the Bing double of the Hopf link as in Figure 5.3 results in a link that is isotopic to the Borromean rings, so the two links in Figures 5.1 and 5.3 are isotopic. Moreover, both the Hopf link and the Borromean rings have index two symmetry subgroups in their respective Whitten groups  $\Gamma_2$  and  $\Gamma_3$ . For the Hopf link,  $\Sigma(2a1) = \Sigma_{8,2} = \langle mi_1, \rho \rangle$  and for the Borromean rings,  $\Sigma(6a4) = \langle (-1, 1, 1, 1, (1)), (1, 1, 1, 1, (123)), (1, -1, 1, 1, (12)) \rangle$ . This leads to the conjecture below.

Let  $B_n$  be the  $n$ -component link that results from Bing doubling the Hopf link  $n - 2$  times; let the two new Bing doubled components be the  $(n - 1)^{th}$  and  $n^{th}$  components.

**Conjecture 5.2.** *Taking the Bing double of a Hopf link  $n - 2$  times results in a Brunnian link  $B_n$  for which  $\Sigma(B_n)$  has index two in  $\Gamma_n$ .*

We anticipate proving this conjecture via the following inductive approach. We will use the existing symmetries of  $B_{n-1}$  and the Bing link, as well as new permutation symmetries, to show that the order of  $\Sigma(B_n)$  is at least  $|\Gamma_n|/2$ . We will then rule out a symmetry by using Milnor  $\bar{\mu}$ -invariants. We need seven lemmas, where five have been proven and the remaining two are conjectures.

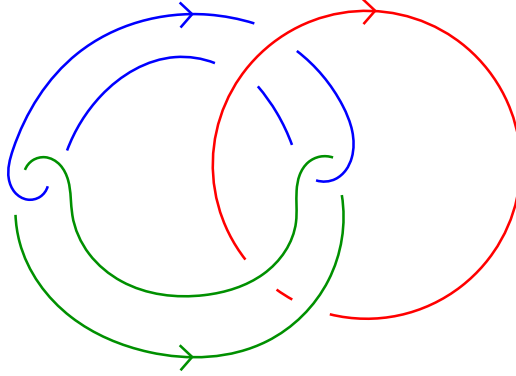


Figure 5.3: The Borromean rings as the Bing double of the Hopf link.

**Lemma 5.3.** *The Bing double of a Brunnian link is Brunnian.*

*Proof.* Suppose  $L = \bigcup_{i=1}^n K_i$  is a Brunnian link with  $n$  components, all of which are unknots. Bing double one of the components,  $K_n$ , to create  $L^* = \bigcup_{i=1}^{n-1} K_i \cup \overline{K}_n \cup \overline{K}_{n+1}$ , a link with  $n + 1$  components. Delete one of those components,  $K_i$ . We have two cases.

Case 1: If  $i < n$ , the deleted component,  $K_i$ , was part of your original link  $L$ . Then there is an isotopy that takes  $L - K_i$  to the unlink; use this isotopy on  $L^* - K_i$  using the tubular neighborhood around  $K_n$  to isotope the Bing double components  $\overline{K}_n \cup \overline{K}_{n+1}$ . This produces a split link with the only crossings in a projection being the Bing double components; as  $K_n$  was the unknot these components can therefore be split apart and  $L^* - K_i$  becomes the unlink.

Case 2: If  $i \geq n$ , the component you deleted was not part of the original link  $L$ , instead it was one of the Bing double components; without loss of generality let  $\overline{K}_n$  be the component you deleted.. You may stretch  $\overline{K}_n$  inside of the tubular neighborhood around where  $K_n$  used to be until  $\overline{K}_{n+1}$  has no crossings in a projections with any knot other than  $\overline{K}_n$ . Delete  $\overline{K}_n$ . Then as the other Bing double component  $\overline{K}_{n+1}$  has no crossings in a projection, it can be moved away from the rest of  $L^* - \overline{K}_n$ , which is now the same as  $L - K_n$ ; as  $L$  is Brunnian, there is an isotopy that takes this to the unlink, so  $L^* - \overline{K}_n$  is isotopic to the unlink.  $\square$

Thus  $B_n$  is Brunnian for  $n \geq 3$ .

**Lemma 5.4.** *For every symmetry  $\gamma_{n-1} = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-2}, 1, \rho) \in \Sigma(B_{n-1})$ , where  $\rho \in S_{n-1}$  fixes the  $(n-1)^{th}$  component, there exists a  $\gamma_n = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-2}, \varepsilon_{n-1}, \varepsilon_n, \rho) \in \Sigma(B_n)$ .*

*Proof.* We consider separate cases depending on if we have mirrored or not.

Case 1: If  $\varepsilon_0 = 1$ , then nothing we have done has affected our Bing doubled component. We thus set  $\varepsilon_{n-1} = \varepsilon_n = 1$  and see that the same ambient isotopy  $F$  that shows that  $B_{n-1} \approx B_{n-1}^{\gamma_{n-1}}$  shows that  $B_n \approx B_n^{\gamma_n}$  after we extend  $F$  to a solid tubular neighborhood of the Bing doubled  $(n-1)^{th}$  component of  $B_{n-1}$ .

Case 2: If  $\varepsilon_0 = -1$  we use a result from [15]. Consider the Bing link inside its solid torus  $V$  as in Figure 5.2. Let  $\xi = (\xi_0, \xi_1, \xi_2)$ , where  $\xi_0$  records the orientation of the core of  $V$  and  $\xi_1, \xi_2$  record the orientations of the Bing components. Then there is an orientation-reversing homeomorphism from  $V$  to  $V$  that reverses the orientation of the core of  $V$  and the Bing components according to  $\xi$  if and only if  $\xi_0 \xi_1 \xi_2 = -1$  by [15].

We thus once again extend the ambient isotopy  $F$  from  $B_{n-1}$  to  $B_{n-1}^{\gamma_{n-1}}$  to a solid tubular neighborhood  $V$  of the  $(n-1)^{th}$  of  $B_{n-1}$ . In this case, we set exactly one of  $\varepsilon_{n-1}, \varepsilon_n$  to 1 and the other to  $-1$ , and see that  $B_n \approx B_n^{\gamma_n}$ . This is because we are reversing the orientation of  $V$  and, as  $\gamma_{n-1}$  does not reverse the orientation of the  $(n-1)^{th}$  component, not reversing the orientation of the core of  $V$ . Thus by [15], we achieve an isotopy if and only if we reverse the orientation of exactly one of the Bing links.  $\square$

The following is an outline for the rest of our proof.

**Conjecture 5.5.** *For every  $\rho = (i, n-1)$  or  $(i, n) \in S_n$  where  $i \in \{1, \dots, n-2\}$ ,  $(1, 1, \dots, 1, \varepsilon_{n-1}, \varepsilon_n, \rho) \in \Sigma(B_n)$ , where  $\varepsilon_n = -\varepsilon_{n-1}$ .*

**Conjecture 5.6.** *The link  $B_n$  does not depend on the component of  $B_{n-1}$  that is Bing doubled.*

**Lemma 5.7.** *The link  $B_n$  admits the intrinsic symmetries  $(1, 1, \dots, 1, 1, -1, (n, n-1))$  and  $(1, 1, \dots, 1, -1, -1, (1))$ .*

*Proof.* These symmetries are achievable by isotopies within the torus surrounding the Bing double components and are thus not affected by the rest of the link. To achieve  $(1, 1, \dots, 1, 1, -1, (n, n-1))$ , rotate the Bing link within the torus 180 degrees. This mirrors the crossings within the torus, so one must then flip one of the components which results in exactly one of the components being inverted. To achieve  $(1, 1, \dots, 1, -1, -1, (1))$ , flip both components, which mirrors the crossings twice and inverts both components.  $\square$

Milnor defined  $\bar{\mu}$ -invariants in order to classify links up to link homotopy [17]. The following fact is a well understood consequence of our definitions.

**Lemma 5.8.** *The appropriate Milnor  $\bar{\mu}$ -invariant is  $\pm 1$  for  $B_n$ .*

**Lemma 5.9.** *If  $L$  is a  $n$ -component link and  $\bar{\mu}_{12\dots n}(L) \neq 0$ , then  $\Sigma(L) \neq \Gamma_n$ .*

*Proof.* By [17], if  $\gamma = (1, -1, 1, 1, \dots, 1, (1))$ , then  $\bar{\mu}_{12\dots n}(L) = -\bar{\mu}_{12\dots n}(L^\gamma)$ . Thus as  $\bar{\mu}_{12\dots n}(L) \neq 0$ , we have that  $\bar{\mu}_{12\dots n}(L) \neq \bar{\mu}_{12\dots n}(L^\gamma)$ , so  $L$  is not link homotopic to  $L^\gamma$ . Thus  $L$  is not ambient isotopic to  $L^\gamma$ , so  $\Sigma(L) \neq \Gamma_n$ .  $\square$

For links  $L$  with  $\mu$  components where  $\mu > 1$  there are no known links with full symmetry, where  $\Sigma(L) = \Gamma_\mu$ . Thus if we can prove this conjecture, we would have found a family of links that have the largest known number of intrinsic symmetries for each  $\mu$ .

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# Vita

## Education

Wake Forest University Winston-Salem, NC	Mathematics Thesis: “Two-Component Link Symmetries”	M.A. expected May, 2012
Pomona College Claremont, CA	Mathematical Economics Major Theatre Minor	B.A. May, 2007
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## Experience

Graduate Teaching Assistant at Wake Forest	Summer and fall, 2011 Math 109 and Math 311/611
Math Center Tutor at Wake Forest	2010-2012
Graduate Assistant Wake Forest Institute for Public Engagement	2010-2012
Independent Tutor, Portland OR High school mathematics	2008-2010
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## Presentations

Graduate Student Seminar talk on link symmetries, 2012.

Poster presented at the 2011 Fall Southeastern Sectional AMS Meeting, Special Session on Geometric Knot Theory and Its Applications.

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