OBSERVING WAVE PROPAGATION OF A FLOAT SERVE ON A VOLLEYBALL

BY

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Abstract

Katherine Novacek

There has been research that has determined that the drag forces on a non-spinning volleyball's boundary layer causes the ball to swerve on its trajectory. The purpose of this paper is to develop a numerical model for wave propagation on a volleyball, which we model as a smooth sphere, to show that the waves lead to some turbulence on the flight path of a volleyball. By applying various initial conditions to the three-dimensional wave equation, we can observe waves travelling on its surface. We first solve the wave equation with spherical coordinates using separation of variables. For our model, we study the wave equation restricted to a sphere and describe it solutions, which are known as spherical harmonics. By using the software *Matlab*, we are able to implement our equations and initial conditions to observe the wave oscillations on the volleyball.

Chapter 1: Introduction

1.1 Background

This study is concerned with analytical and numerical solutions to the three-dimensional wave equation $u_{tt} = \nabla^2 u = c^2(u_{xx} + u_{yy} + u_{zz})$ in spherical coordinates. One of the essential topics in a differential equations' course is the wave equation, which is the main focus of this paper.

Definition 1. The *wave equation*, written as

$$u_{tt} = c^2 \nabla^2 u,$$

is a second-order linear hyperbolic partial differential equation (PDE). The scalar function u models the amplitude of the wave depending on the time variable t and spatial variables $x_1, x_2, ..., x_n$. In the equation above, c is the speed of our wave and ∇^2 is the Laplacian.

Definition 2. The *Laplacian* is a second order differential operator given by the divergence of the gradient of a function in n-dimensional Euclidean space. In Cartesian coordinates, the Laplacian is $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$.

Definition 3. *Harmonic functions* are the solutions to *Laplace's equation*, which is written as $\nabla^2 u = 0$.

The wave equation arises in physics, in fields such as mathematics, acoustics, and fluid dynamics where it is used to model phenomena such as sound or light. While studying vibrating strings, Pythagoras noticed how harmonics were produced by dividing the length of the string by whole numbers. These ratios of string lengths began his interest in understanding pitches [in music]. Though Galileo and Newton did research on wave equations, it wasn't until the 1700's that Jean Le Rond d'Alembert was the first to publish solutions to the one-dimensional wave equation. d'Alembert first derived the general solution to the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ on the real line as

$$u(x,t) = f(x+ct) + g(x-ct).$$

This tells us that solutions to the one-dimensional wave equation are sums of a left-travelling function f and a right-travelling function g. We use the word 'travelling' to describe that the shape of these functions stay constant. The speed of these functions as they travel is c.

Unique solutions to the wave equation are obtained by imposing conditions regarding to the wave at certain times and places. In particular, we apply *initial conditions*, where we specify the function of its derivative at an initial time, and *boundary conditions*, where we specify what happens at the boundary of the domain for all times. The solution to the Cauchy problem for the one-dimensional wave equation on the real line with the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ is known as *d'Alembert's formula* is

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\omega)d\omega \qquad -\infty < x < \infty \text{ and } t > 0.$$

Assuming that $\phi \in C^2$ and $\psi \in C^1$, we can see that $u \in C^2$. [11].

1.2 Derivation of the Wave Equation

To derive the wave equation, we will apply Newton's law to an elastic, flexible, homogeneous string. Consider a small segment of this string.

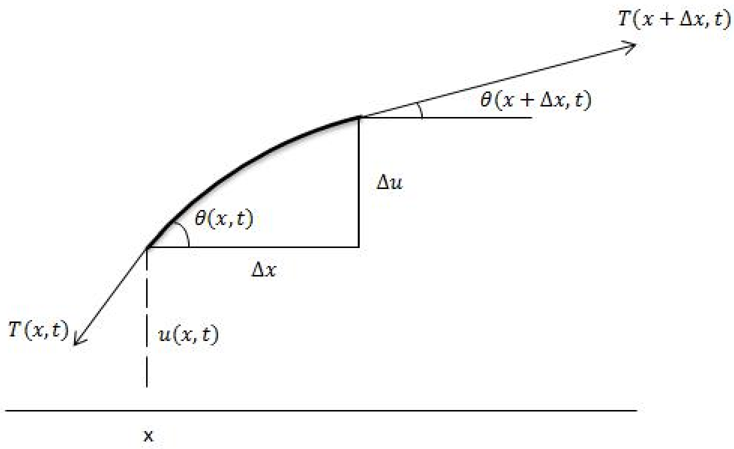


Figure 1.1: A segment of an elastic, flexible, homogeneous string.

In Figure 1.1, $u(x, t)$ is the vertical displacement of the string at position x and time t. The function $\theta(x, t)$ is the angle between the string and a horizontal line at position x and time t. Function $T(x, t)$ is the tension in the string at position x and time t. The mass density of the string is represented by ρ. This value is constant because the string is homogeneous.

We need to look at the forces that are acting on the string and use Newton's method to help us form the wave equation. There is tension pulling our segment to the right, which has a magnitude of $T(x+\Delta x, t)$ and acts at an angle of $\theta(x+\Delta x, t)$. Opposite of this, the tension pulling the string to the left has a magnitude of $T(x, t)$ and acts at an angle of $\theta(x, t)$. The other forces that act on our string are vertical forces, such as gravity. These forces will be denoted as $F(x, t)\Delta x$, which is the magnitude acting on the segment of the string. By viewing our length of string as it were a line segment, the section of string that we are considering has a mass of approximately $\rho\sqrt{\Delta x^2 + \Delta u^2}$ by using the Pythagorean Theorem.

Applying Newton's second law of motion, $\mathbf{F} = m\mathbf{a}$, we observe both the transverse and longitudinal components. Transverse corresponds to the direction that is perpendicular to our x-axis, which is vertical, while longitudinal refers to the same direction of our x-axis, which is left and right.

First, the transverse component of Newton's law states that

$$T(x+\Delta x,t)\sin(\theta(x+\Delta x,t)) - T(x,t)\sin(\theta(x,t)) + F(x,t)\Delta x$$
$$= \frac{\partial u^2}{\partial x^2}\rho\sqrt{\Delta x^2+\Delta u^2}. \qquad (1.1)$$

By dividing by Δx and taking the limit as $\Delta x \to 0$, we get

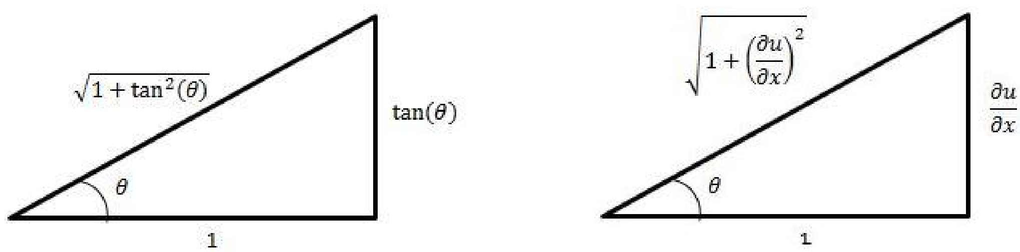
$$\lim_{\Delta x\to 0}\frac{T(x+\Delta x,t)\sin(\theta(x+\Delta x,t)) - T(x,t)\sin(\theta(x,t))}{\Delta x} + \lim_{\Delta x\to 0}\frac{F(x,t)\Delta x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\partial u^2}{\partial x^2}\frac{\rho\sqrt{\Delta x^2+\Delta u^2}}{\Delta x}$$

$$\frac{\partial}{\partial x}T(x,t)\sin(\theta(x,t)) + F(x,t) = \rho\frac{\partial u^2}{\partial x^2}\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}$$

$$\frac{\partial}{\partial x}T(x,t)\sin(\theta(x,t)) + T(x,t)\cos(\theta(x,t)) + F(x,t) = \rho\frac{\partial u^2}{\partial x^2}\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}.$$

Notice from Figure 1.1 that $\tan(\theta(x,t)) = \frac{\partial u}{\partial x}(x,t)$. By using the figure below, we can solve for and eliminate the terms with θ's.



$$\sin(\theta(x,t)) = \frac{\frac{\partial u}{\partial x}(x,t)}{\sqrt{1+(\frac{\partial u}{\partial x}(x,t))^2}} \qquad \cos(\theta(x,t)) = \frac{1}{\sqrt{1+(\frac{\partial u}{\partial x}(x,t))^2}}$$

If we plug these formulas back into (1.1), we end up with a large and messy equation. We can instead consider only small vibrations of the string to simplify (1.1). We will be defining a small vibration by $|\theta(x,t)| << 1$ for all x and t. This means that $|\tan(\theta(x,t))| << 1$ and thus $|\frac{\partial u}{\partial x}(x,t)| << 1$. Therefore

$$\sqrt{1+\left(\frac{\partial u}{\partial x}(x,t)\right)^2} \approx 1 \tag{1.2}$$

Now, substituting these formulas into (1.1) we have

$$\frac{\partial}{\partial x}T(x,t)\frac{\partial u}{\partial x}(x,t)+T(x,t)\frac{\partial u^2}{\partial x^2}(x,t)+F(x,t)=\rho\frac{\partial u^2}{\partial x^2}. \tag{1.3}$$

Now that we have simplified the transverse component of Newton's second law, we now need to look at the longitudinal component since (1.3) has two unknowns, u and T. For this derivation, let us assume that only transverse vibrations occur. This means that the longitudinal force on the string must be zero. So the longitudinal component is

$$T(x+\Delta x,t)\cos(\theta(x+\Delta x,t))-T(x,t)\cos(\theta(x,t))=0. \tag{1.4}$$

In the same method, we divide by Δx and take the limit as Δx approaches zero to obtain

$$\lim_{\Delta x\to 0}\frac{T(x+\Delta x,t)\cos(\theta(x+\Delta x,t))-T(x,t)\cos(\theta(x,t))}{\Delta x}=\lim_{\Delta x\to 0}0$$

$$\frac{\partial}{\partial x}\left[T(x,t)\cos(\theta(x,t))\right]=0.$$

From above we know that $\cos(\theta)$ is approximately one by assuming small transverse vibrations. This means that

$$\frac{\partial}{\partial x}T(x,t)\cdot\cos(\theta(x,t)) + T(x,t)\cdot(-\sin(\theta(x,t))\frac{\partial\theta}{\partial x}) = 0$$

$$\frac{\partial}{\partial x}T(x,t)\cdot 1 + T(x,t)\cdot 0 = 0$$

$$\frac{\partial}{\partial x}T(x,t) = 0$$

This tells us that it is okay to assume that our tension T is a function of t only.

The last thing that we need to note is that we are assuming that the string is homogeneous. Recall that we have stated that this means that the mass density ρ is constant. This implies that our string tension is now a constant and there are no external forces $F(x,t)$. Thus, by assuming small, transverse, vibrations (1.3) simplifies to

$$T\frac{\partial u^2}{\partial x^2}(x,t) = \rho\frac{\partial u^2}{\partial x^2}$$

$$\frac{\partial u^2}{\partial x^2}(x,t) = c^2\frac{\partial u^2}{\partial x^2} \qquad \text{where} \quad c = \sqrt{\frac{T}{\rho}}.$$

This is the *wave equation* and c is the speed of the wave. [4, 11]

1.3 Conservation of Energy

Another principle that is needed for the wave equation is to know that it upholds the *Conservation of Energy.* The principle of Conservation of Energy states that the total amount of energy in an isolated system remains constant over time. The equations that will portray the Conservation of Energy are

$$E_W = E_k + E_p \tag{1.5}$$

and

$$0 = \frac{dE_k}{\partial t} + \frac{dE_p}{\partial t} \tag{1.6}$$

where E_W is the total energy, E_k is the kinetic energy, and E_p is the potential energy. The first equation states that the total amount of energy is the sum of how much kinetic energy and potential energy are present. The second equation is stating that there is no change of the total energy. Hence, the total amount of energy stayed constant over time. We will show that the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ with periodic boundary conditions, which will arise in Chapter 2, conserves energy.

To start, we will find the equation of kinetic energy, which is the energy that an object it possesses due to its motion. The formula for kinetic energy for a single particle is well-known as $E_k = \frac{1}{2}mv^2$, where m is mass and v is velocity. To derive the equation for kinetic energy of a string, we shall focus on a small section of the string, x_1 to x_2. We assume our string has mass density ρ which has units of mass per unit length. The velocity of the wave is u_t, which we will again be assuming is strictly transversal. To find the total kinetic energy of our system, we will take the integral of the kinetic energy over the segment of string. Thus, the kinetic energy of a length of string from x_1 to x_2 is $E_k = \int_{x_1}^{x_2} \frac{1}{2}\rho u_t^2 \, dx$.

Now that we have a formula for kinetic energy, we shall find an equation for potential energy. Potential energy is known as 'stored energy' and depends on the relative position of various parts of the system. When a system is forced out of its rest position, potential energy increases because there is a force that is trying to restore the system to an arrangement of lower energy. To find the potential energy of our wave, we need to observe the stretched section of the wave (see Figure 1.2).

The curved section is approximately the hypotenuse of the infinitesimal triangle formed by the change in x and u. This tells us that the amount of stretching Δs of the length Δx is how much longer the hypotenuse Δl is than the base of the triangle Δx.

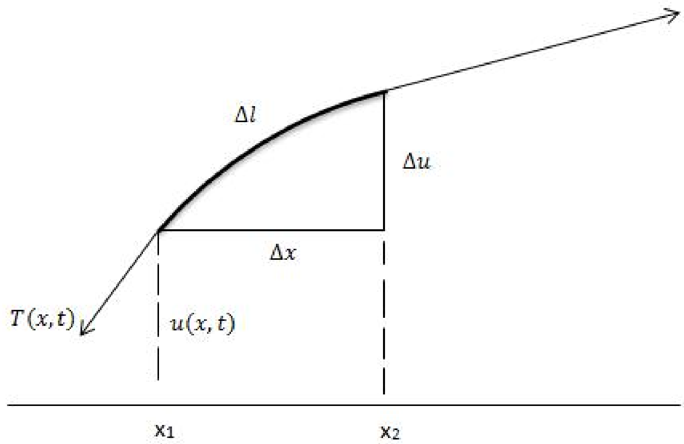


Figure 1.2: Stretched section of the wave

Thus the change of s may be approximated by the sum

$$\Delta s \approx \lim_{dx\to 0} \sum_{x=x_1}^{x=x_2} \left(\sqrt{(\Delta x)^2 + (\Delta u)^2} - \Delta x\right) dx \qquad \text{dividing by } \Delta x$$

$$= \int_{x_1}^{x_2} \left(\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} - 1\right) dx$$

$$= \int_{x_1}^{x_2} \left(\left(1 + \frac{u_x^2}{2} - \frac{u_x^4}{8} + \frac{u_x^6}{16} - ...\right) - 1\right) dx \qquad \text{by the Taylor series for } \sqrt{1+u_x^2}$$

$$= \frac{1}{2}\int_{x_1}^{x_2} \left(u_x^2 - \left(\frac{1}{2}\right)^2 u_x^4 + \left(\frac{1}{2}\right)^3 u_x^6 - ...\right) dx$$

The value of the sum of $(u_x^2 - \left(\frac{1}{2}\right)^2 u_x^4 + \left(\frac{1}{2}\right)^3 u_x^6 - ...)$ may be approximated by u_x^2 for $|u_x^2| << 1$. Thus,

$$\Delta s \approx \int \frac{1}{2} u_x^2 dx \qquad \text{for } |u_x^2| << 1.$$

Multiplying this equation by the magnitude of the tension of our small section, our potential energy has an approximate form of $E_p = \int_{x_1}^{x_2} \frac{1}{2} T u_x^2 dx$.

Now, the total energy of a line can be written as

$$
\begin{aligned}
E_W =& \int_0^l \left(\frac{1}{2}\rho u_t^2 + \frac{1}{2}T u_x^2\right) dx \\
=& \frac{1}{2}\int_0^l \left(\rho u_t^2 + T u_x^2\right) dx.
\end{aligned} \tag{1.7}
$$

To show that the total energy is conserved, we shall take the derivative of (1.7) with respect to time and show (1.6), which means that there is no change of energy:

$$
\begin{aligned}
\frac{dE_W}{dt} &= \int_0^l \frac{\partial}{\partial t}\left(\rho u_t^2 + T u_x^2\right) dx \\
&= \int_0^l \left(\rho u_t u_{tt} + T u_x u_{xt}\right) dx \\
&= \int_0^l \rho u_t u_{tt} dx + T\left[u_x u_t \mid_0^l - \int_0^l u_t u_{xx} dx\right] && \text{by integration by parts} \\
&= \int_0^l \left(\rho u_t u_{tt} - T u_t u_{xx}\right) dx + T u_x u_t \mid_0^l \\
&= \int_0^l \left(\rho c^2 u_t u_{xx} - T u_t u_{xx}\right) dx + T u_x u_t \mid_0^l && \text{since } u_{tt} = c^2 u_{xx} \\
&= \int_0^l \left(T u_t u_{xx} - T u_t u_{xx}\right) dx + T\left[u_x(l,t)u_t(l,t) - u_x(0,t)u_t(0,t)\right] && \text{by } c^2 = T/\rho.
\end{aligned}
$$

Remembering our boundary conditions, we have $u(0,t) = u(l,t)$ and $u_x(0,t) = u_x(l,t)$. From our first condition, we can see that $u_t(0,t) = u_t(l,t)$. Thus

$$
\begin{aligned}
\frac{dE_W}{dt} &= \int_0^l 0\ dx + T u_x(0,t)\left[u_t(l,t) - u_t(0,t)\right] \\
&= \int_0^l 0\ dx && \text{by } u_t(0,t) = u_t(l,t) \text{ and } u_x(0,t) = u_x(l,t) \\
&= 0.
\end{aligned}
$$

Since $\frac{dE_W}{dt} = 0$, there is no change of energy from the starting point to the ending point (i.e. no energy created nor destroyed). This shows that the law of conservation of energy holds for the wave equation with periodic boundary conditions. [1, 11]

1.4 Volleyball Background

The inspiration for this paper came from the idea that the wave propagation on the volleyball's surface, caused by the force of the server's hand, affects its path through the air when hitting a float serve. In the game of volleyball, the serve is an essential tool. The float serve is especially favoured due to its irregular and unpredictable trajectory. A float serve has no spin while it is in the air and it has a shifting path similar to the knuckleball in baseball.

An Olympic regulation volleyball has an outer layer made from a flexible leather or synthetic leather composite and must have a circumference between 25.6 and 26.4 inches. This leather exterior consists of 18 rectangular panels, which are arranged in six identical sections, and are either glued or sewn together. Underneath the surface of a volleyball is a layer of cloth. The core of a volleyball is an air-filled bladder made of rubber, the same type of material inside basketballs or soccer balls, and is covered by leather or cloth.

Bladders can be defined as either attached or floating. Attached bladders are glued to either all or some parts of the cloth layer. A floating bladder is not attached to any part of the cloth layer, except the air valve. Like other athletic balls, the core of a ball helps determine the trajectory it will travel during play. The bladder of a volleyball is filled with air and helps maintain the shape of a sphere [9]. In this paper, the type of bladder will be neglected and the volleyball will a uniform sphere with a circumference of 26 inches.

Dr. Thomas Cairns of the University of Tulsa has been conducting research of trajectories for the float and non-float serve. He fits the sixth-order nonlinear differential equation for objects moving through the air to served volleyballs videotaped at his university. This equation has two parameters that are specified to the ball: the lift coefficient and the drag coefficient. For the float serve, Dr. Cairns focused his research on the drag of the volleyball, ignoring lift because lift is produced from spin of a spherical object. The formula for drag is

$$F_{drag} = \frac{1}{2}\rho A C_D v^2, \tag{1.8}$$

where ρ is the air density, A is the ball cross sectional area, v is the speed of the ball and C_D is the drag coefficient. The drag coefficient is different for every object. A wind tunnel study by Han and McCulloch, two aeronautical engineering students at the University of Michigan, determined the drag coefficient for an 18 panel volleyball (shown in Figure 1.3).

He noticed that there was a dramatic drop in C_D at certain Reynolds Numbers, which is a quantification of speed used in fluid mechanics. These ball speeds are called critical. Highs speeds are post-critical and lower speeds are pre-critical. Observing volleyball games, Dr. Cairns saw that most of the plays in volleyball are in the critical region, which is a range of 9 m/s to 17 m/s. Also the volleyball itself is highly susceptible to aerodynamic forces. We can find this by dividing the ball's mass by the cross-sectional area. The smaller the number, the more the ball is affected by aerodynamic forces. From other sports, a table tennis ball is the the only ball that has a lower number, and a soccer ball is the next on the list after a volleyball.

Air, the medium that the volleyball is moving through, applies forces such as drag to the surface of the ball within the boundary layer, which is about 1 mm thick. The boundary layer is said to be turbulent at post-critical speeds. At pre-critical speed, the drag coefficient is nearly four times as large post-critical speeds, and is said to be laminar. As the volleyball changes speeds from the post-critical to pre-critical speeds, the boundary layer switches from turbulent to laminar. This leads into a factor that Dr. Cairns says has a major impact on determining the path of a float serve: surface roughness.

The surface of a volleyball seems to be pretty smooth, but at the boundary layer level, there are seams and indentations. The rougher the surface, the more turbulent the boundary layer will be. This means that one hemisphere of the volleyball can be experiencing a laminar boundary layer, while the other hemisphere has a turbulent boundary layer. The drag force is smaller on the turbulent hemisphere and this results in a force that causes the ball to move sideways more or less randomly. This is more commonly known as the knuckle ball effect [in baseball]. Thus, Dr. Cairns states that the irregular path of a float serve in volleyball is the result of the ball slowing into a region of speed where one hemisphere has a turbulent boundary layer while the other one is experiencing a laminar boundary layer. [2, 3].

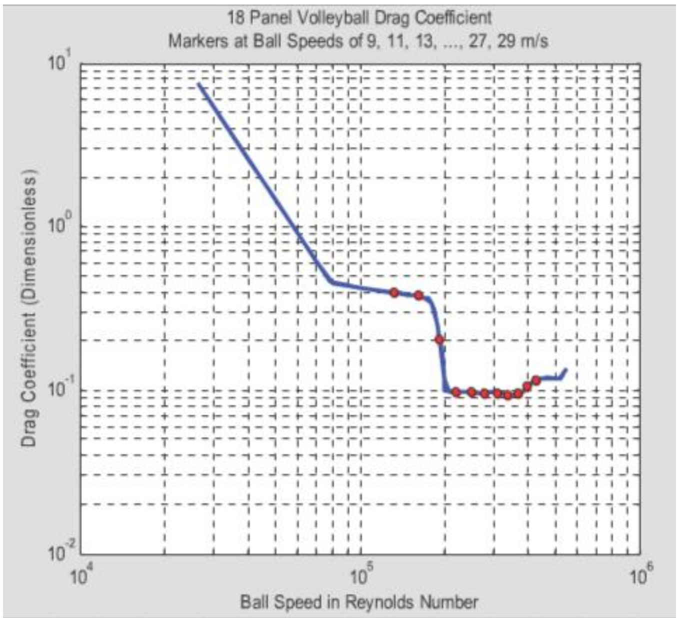


Figure 1.3: Drag Coefficient for an 18 Panel Volleyball

1.5 Organization of Thesis

This paper starts with solving the one-dimensional wave equations with periodic boundary conditions (Chapter 2). We will be using a method called separation of variables to solve the wave equation. This method splits the PDE into a product of ordinary differential equations (ODEs). ODEs are often simpler to solve since there are more known techniques for finding their solutions, which makes our PDE problem easier. In Chapter 3, we convert the three-dimensional wave equation from Cartesian to spherical coordinates. Then, we solve the three-dimensional wave equation analytically in Chapter 4. By solving for the coefficients and using spherical harmonics, we develop the solution to the wave equation on the surface of the sphere with an established radius (Chapter 5). The explanation of the *Matlab* code is shown in Chapter 6 to observe the wave propagation on the volleyball.

Chapter 2: The One-Dimensional Wave Equation with Periodic Boundary Conditions

We will use the separation of variables technique to study the wave equation on a finite interval. This method is used to transform linear homogeneous partial differential equations with certain types of boundary conditions into a finite number of associated linear ordinary differential equations with corresponding boundary conditions. We will be using periodic boundary conditions for the one-dimensional case, which can be thought of as mapping the interval to a circle. Our one-dimensional wave with periodic boundary conditions where $t \geq 0$ and l is the wavelength is

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0 \qquad 0 < x < l \quad \text{where } c \text{ is the speed of the wave} \\
u(0,t) &= u(l,t) \qquad 0 < t < \infty \\
u_x(0,t) &= u_x(l,t) \qquad 0 < t < \infty \\
u(x,0) &= \phi(x) \qquad \text{where } \phi \text{ is an arbitrary function} \\
u_t(x,0) &= \psi(x) \qquad \text{where } \psi \text{ is an arbitrary function.}
\end{aligned}$$

We are looking for a solution of the form $u(x,t) = X(x)T(t)$ for the above wave equation and its boundary conditions. This form of the solution is suggested by separation of variables. By substituting u into the one-dimensional wave equation, we see that X and T must satisfy

$$X(x)T''(t) = c^2 X''(x)T(t). \tag{2.1}$$

Dividing (2.1) by c^2TX we get

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

for some constant λ. Here λ is a constant because each expression is a function of only one distinct independent variable. This implies that

$$X''(x) + \lambda X(x) = 0$$
$$T''(t) + c^2\lambda T(t) = 0$$

which transforms our PDE into two linear ODE's. Applying our boundary conditions to the the first ordinary differential equation, we see that our function X must satisfy

$$\begin{aligned} X''(x) &= -\lambda X(x) \\ X(0) &= X(l) \\ X'(0) &= X'(l). \end{aligned} \tag{2.2}$$

If there exists some λ satisfying (2.2) for the nonzero function X, λ is called an *eigenvalue* of $\frac{d^2}{dx^2}$ on $[0, l]$ subject to periodic boundary conditions. The corresponding function $X(x)$ is called an *eigenfunction* of $\frac{d^2}{dx^2}$ on $[0, l]$, subject to periodic boundary conditions. To find the solution of this ODE we will look at the three possible types of eigenvalues.

Case 1: Suppose λ is a positive eigenvalue. We will let $\lambda = \delta^2 > 0$. Thus, we will find solutions for the equation

$$X''(x) + \delta^2 X(x) = 0 \tag{2.3}$$

with periodic boundary conditions. The solutions of (2.3) and their derivative are

$$X(x) = A\cos(\delta x) + B\sin(\delta x)$$
$$X'(x) = -A\delta\sin(\delta x) + B\delta\cos(\delta x)$$

for any constants A, B. The boundary condition $X(0) = X(l)$ gives us

$$A = A\cos(\delta l) + B\sin(\delta l).$$

The boundary condition $X'(0) = X'(l)$ gives us

$$B\delta = B\delta \cos(\delta l) - A\delta \sin(\delta l).$$

By multiplying our first boundary condition function by $B\delta$ and our second by A, we get

$$B\delta(A\cos(\delta l) + B\sin(\delta l)) = A(B\delta \cos(\delta l) - A\delta \sin(\delta l)).$$

Hence

$$\delta(B^2 + A^2)\sin(\delta l) = 0.$$

We can see that there are three possible solutions: $\delta = 0, B^2 + A^2 = 0$, or $\sin(\delta l) = 0$. $B^2 + A^2 \neq 0$ because it leads to a trivial solution, and δ must be positive, so we are left with the solution $\delta^2 = \left(\frac{\pi n}{l}\right)^2$ for some integer n. Thus we have a sequence of eigenvalues which we shall denote $\lambda_n = \left(\frac{\pi n}{l}\right)^2$. With our corresponding eigenfunction from our solution of (2.3) we have

$$X_n(x) = A_n \cos\left(\frac{\pi n x}{l}\right) + B_n \sin\left(\frac{\pi n x}{l}\right) \tag{2.4}$$

where A_n and B_n are arbitrary constants. To verify that (2.4) is a solution to our ODE (2.3) with the condition $\delta^2 > 0$, we shall take the second derivative of our solution:

$$X_n''(x) = -A_n \cos\left(\frac{\pi n x}{l}\right)\left(\frac{\pi n}{l}\right)^2 - B_n \sin\left(\frac{\pi n x}{l}\right)\left(\frac{\pi n}{l}\right)^2$$

Now we substitute it and $X_n(x)$ back into (2.3), where we get

$$\begin{aligned} &- A_n \cos\left(\frac{\pi n x}{l}\right)\left(\frac{\pi n}{l}\right)^2 - B_n \sin\left(\frac{\pi n x}{l}\right)\left(\frac{\pi n}{l}\right)^2 \\ &\qquad + \left(\frac{\pi n}{l}\right)^2\left(A_n \cos\left(\frac{\pi n x}{l}\right) + B_n \sin\left(\frac{\pi n x}{l}\right)\right) = 0. \end{aligned}$$

Everything on the left-hand side cancels out, which shows that (2.4) is a correct solution for (2.3).

We must also find the solutions of the ODE

$$T''(t) + \delta^2 c^2 T(t) = 0.$$

The solutions with our found eigenvalues are

$$T_n(t) = C_n \cos\left(\frac{\pi nct}{l}\right) + D_n \sin\left(\frac{\pi nct}{l}\right)$$

where C_n and D_n are currently arbitrary constants.

We have found that positive $\lambda_n = \left(\frac{\pi n}{l}\right)^2$ are eigenvalues for the problem (2.2). We now have to determine if there are any more eigenvalues for our one-dimensional wave equation.

Case 2: Suppose $\lambda = 0$. If our eigenvalue is equal to zero, our equations with this condition become

$$X''(x) = 0$$
$$X(0) = X(l)$$
$$X'(0) = X'(l).$$

The solutions of this eigenvalue problem and their derivatives are

$$X(x) = Ax + B$$
$$X'(x) = A.$$

The boundary conditions give us

$$X(0) = X(l) \Longrightarrow B = Al + B$$
$$X'(0) = X'(l) \Longrightarrow A = A.$$

Since $l > 0$, this leads to $A = 0$ where B is unconstrained. Thus, with $l > 0$, the case of $\lambda = 0$ has an eigenfunction of $X(x) = B$ where $B \in \mathbb{C}$. With this eigenvalue, the solution to $T''(t) = 0$ is $T(t) = Ct + D$.

Case 3: Suppose λ is a negative eigenvalue, which we denote as $\lambda = -\delta^2 < 0$. We will find solutions for the equation

$$X''(x) - \delta^2 X(x) = 0 \tag{2.5}$$

with periodic boundary conditions. The solutions of (2.5) and their derivatives are

$$\begin{aligned} X(x) &= A\cosh(\delta x) + B\sinh(\delta x) \qquad (2.6)\\ X'(x) &= A\delta\sinh(\delta x) + B\delta\cosh(\delta x) \end{aligned}$$

for any arbitrary constants A, B. Implementing our boundary conditions we get

$$\begin{aligned} X(0) = X(l) &\Longrightarrow A = A\cosh(\delta l) + B\sinh(\delta l) \\ X'(0) = X'(l) &\Longrightarrow B\delta = B\delta\cosh(\delta l) + A\delta\sinh(\delta l). \end{aligned}$$

By elimination, we develop the equation

$$0 = \delta(B^2 - A^2)\sinh(\delta l).$$

We see that $\sinh(\delta l) \neq 0$ since $\delta \neq 0$ and $l > 0$, which leads us to $B = \pm A$. By substituting these solutions back into (2.6), we get

$$\begin{aligned} A &= A\cosh(\delta l) \pm A\sinh(\delta l) \\ 1 &= \frac{e^{\delta l} + e^{-\delta l}}{2} \pm \frac{e^{\delta l} - e^{-\delta l}}{2} \\ 1 &= e^{\pm\delta l}. \end{aligned}$$

Hence $B = \pm A \neq 0$ are not solutions because $\delta \neq 0$ and $l > 0$. Therefore, there are no negative eigenvalues.

Having found the solutions to our ODE for all possible types of eigenvalues, we see that the solution for our one-dimensional wave equation in the form $u(x,t) = X(x)T(t)$ is

$$u(x,t) = B_o(C_o t + D_o) + \sum_{n=1}^{\infty} \left[\left(A_n \cos\left(\frac{\pi n x}{l}\right) + B_n \sin\left(\frac{\pi n x}{l}\right) \right) \bullet \left(C_n \cos\left(\frac{\pi n c t}{l}\right) + D_n \sin\left(\frac{\pi n c t}{l}\right) \right) \right]. \tag{2.7}$$

This is the solution for our wave equation, $u(x,t)$, on the interval $0 < x < l$. This solution satisfies the initial conditions of

$$\phi(x) = u(x,0) = B_o D_o + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{\pi n x}{l}\right) + B_n \sin\left(\frac{\pi n x}{l}\right) \right) C_n \tag{2.8}$$

and

$$\psi(x) = u_t(x,0) = B_o C_o + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{\pi n x}{l}\right) + B_n \sin\left(\frac{\pi n x}{l}\right) \right) \left(D_n \left(\frac{\pi n x}{l}\right) \right). \tag{2.9}$$

2.1 Fourier Coefficients

The equations (2.8) and (2.9) that we produced are *Fourier series* for ϕ and ψ respectively. A Fourier series decomposes periodic functions into a sum of sine and cosine functions. We will use the Fourier series to go about solving for the coefficients in the solution to our previously laid-out wave equation. We use our initial position to find our Fourier coefficients. Equation (2.8) can also be written, by combining coefficients, as

$$\phi(x) = A_o + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{\pi n x}{l}\right) + B_n \sin\left(\frac{\pi n x}{l}\right) \right) \tag{2.10}$$

$$\psi(x) = C_o + \sum_{n=1}^{\infty} \left(C_n \left(\frac{\pi n c}{l}\right) \cos\left(\frac{\pi n x}{l}\right) + D_n \left(\frac{\pi n c}{l}\right) \sin\left(\frac{\pi n x}{l}\right) \right). \tag{2.11}$$

We will first solve for the coefficients of the initial position, leading with A_n first. To do this, we multiply (2.10) by $\cos(\frac{\pi m x}{l})$ and integrate over our interval $0 < x < l$. Our equation becomes

$$\int_0^l \phi(x) \cos\left(\frac{\pi m x}{l}\right) dx = \int_0^l A_o \cos\left(\frac{\pi m x}{l}\right) dx + \int_0^l \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi m x}{l}\right) dx$$
$$+ \int_0^l \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi m x}{l}\right) dx. \tag{2.12}$$

On the right-hand side of the expression, all terms in the sum vanish unless $n = m$. This is due to the orthogonality of the trigonometric functions.

$$\begin{aligned} &\int_0^l \cos\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi m x}{l}\right) dx \\ &= \int_0^l \frac{1}{2}(\cos(\frac{\pi n x}{l} - \frac{\pi m x}{l}) + \cos(\frac{\pi n x}{l} + \frac{\pi m x}{l}))dx \\ &= \frac{1}{2(n-m)\pi} \sin\left(\frac{(m-n)\pi x}{l}\right)\bigg|_0^l + \frac{1}{2(n+m)\pi} \sin\left(\frac{(m+n)\pi x}{l}\right)\bigg|_0^l \\ &= 0. \end{aligned}$$

The different combinations of our trigonometric functions, *sin* and cos, are similarly orthogonal in this L^2 sense. These orthogonality results simplify (2.12) to

$$\int_0^l \phi(x) \cos\left(\frac{\pi n x}{l}\right) dx = A_o \int_0^l \cos\left(\frac{\pi n x}{l}\right) dx + A_n \int_0^l \cos^2\left(\frac{\pi n x}{l}\right) dx$$
$$+ A_n \int_0^l \sin\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi n x}{l}\right) dx$$

where $n = 0, 1, 2, ...$ We shall now calculate each individual integral.

$$A_o \int_0^l \cos\left(\frac{\pi n x}{l}\right) dx = A_o \left(\frac{l \sin\left(\frac{\pi n x}{l}\right)}{\pi n}\right)\Bigg|_0^l = A_o \cdot 0 = 0.$$

$$A_n \int_0^l \cos^2\left(\frac{\pi n x}{l}\right) dx = A_n \frac{1}{4}\left(\frac{l \sin\left(\frac{2\pi n x}{l}\right)}{\pi n} + 2x\right)\Bigg|_0^l = \frac{l}{2} A_n$$

$$A_n \int_0^l \sin\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi n x}{l}\right) dx = -\frac{1}{4} A_n \left(\frac{l \cos\left(\frac{2\pi n x}{l}\right)}{\pi n}\right)\Bigg|_0^l = A_n \cdot 0 = 0$$

Therefore,

$$A_n = \frac{2}{l} \int_0^l \phi(x) \cos\left(\frac{\pi n x}{l}\right) dx. \tag{2.13}$$

If we multiplied (2.10) by $\sin(\frac{\pi m x}{l})$ instead of the cosine term, we would get

$$B_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{\pi n x}{l}\right) dx. \tag{2.14}$$

To solve for the coefficient A_o we look at the case when $n = 0$:

$$\int_0^l \phi(x) dx = A_o \int_0^l 1 dx = A_o \cdot l.$$

Thus,

$$A_o = \frac{1}{l} \int_0^l \phi(x) dx. \tag{2.15}$$

Now we will solve the coefficients of our initial velocity, starting with C_n. By multiplying (2.11) by $\cos(\frac{\pi m x}{l})$ and integrating over our interval $0 < x < l$, we find that

$$C_n = \frac{2}{cn\pi} \int_0^l \psi(x) \cos\left(\frac{\pi n x}{l}\right) dx. \tag{2.16}$$

Multiplying (2.11) by $\sin(\frac{\pi m x}{l})$ instead of a cosine term, D_n is

$$D_n = \frac{2}{cn\pi} \int_0^l \psi(x) \sin\left(\frac{\pi n x}{l}\right) dx. \tag{2.17}$$

To solve for the coefficient C_o, we look at the case when $n = 0$:

$$\int_0^l \psi(x) dx = C_o \int_0^l 1 dx = C_o \cdot l.$$

This leads to

$$C_o = \frac{1}{l} \int_0^l \psi(x) dx. \tag{2.18}$$

The solved coefficients, (2.13) - (2.18) along with (2.8) and (2.9) gives us the following solution to the wave equation

$$\begin{aligned} u(x,t) = t\frac{1}{l}\int_0^l \psi(x)dx + \frac{1}{l}\int_0^l \phi(x)dx \\ + \sum_{n=1}^{\infty} \Bigg[\frac{2}{l}\cos\left(\frac{\pi n x}{l}\right)\cos\left(\frac{\pi n c t}{l}\right)\int_0^l \phi(x)\cos\left(\frac{\pi n x}{l}\right)dx \\ + \frac{2}{l}\sin\left(\frac{\pi n x}{l}\right)\cos\left(\frac{\pi n c t}{l}\right)\int_0^l \phi(x)\sin\left(\frac{\pi n x}{l}\right)dx \\ + \cos\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n c t}{l}\right)\frac{2}{cn\pi}\int_0^l \psi(x)\cos\left(\frac{\pi n x}{l}\right)dx \\ + \sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi n c t}{l}\right)\frac{2}{cn\pi}\int_0^l \psi(x)\sin\left(\frac{\pi n x}{l}\right)dx \Bigg]. \end{aligned} \tag{2.19}$$

Chapter 3: Derivation of the Laplacian in Spherical Coordinates

We have so far been working with the one-dimensional wave equation in terms of x. Now, to focus on the wave equation on a spherical membrane, we must change the Cartesian coordinates of the Laplacian to spherical coordinates.

Let $u = u(x, y, z)$ be a function where $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\phi$. By taking the partial derivatives with respect to r,ϕ, and θ, we acquire the equations

$$\frac{\partial u}{\partial r} = \sin\phi\cos\theta\frac{\partial u}{\partial x} + \sin\phi\sin\theta\frac{\partial u}{\partial y} + \cos\phi\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial \phi} = r\cos\phi\cos\theta\frac{\partial u}{\partial x} + r\cos\phi\sin\theta\frac{\partial u}{\partial y} - r\sin\phi\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial \theta} = -r\sin\phi\sin\theta\frac{\partial u}{\partial x} + r\sin\phi\cos\theta\frac{\partial u}{\partial y}.$$

Knowing that the Laplace operator in Cartesian coordinates is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, we will rewrite our newly found derivatives into functions of x, y, and z by using the process of elimination. These partial derivatives are

$$\frac{\partial u}{\partial x} = \sin\phi\cos\theta\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial \phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial u}{\partial \theta} \tag{3.1}$$

$$\frac{\partial u}{\partial y} = \sin\phi\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\phi\sin\theta}{r}\frac{\partial u}{\partial \phi} + \frac{\cos\phi}{r\sin\phi}\frac{\partial u}{\partial \theta} \tag{3.2}$$

$$\frac{\partial u}{\partial z} = \cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}. \tag{3.3}$$

We can now find the Laplacian in spherical coordinates by taking the derivative of these functions and substituting them back into the Cartesian coordinate equation.

As stated in the Introduction, the Laplacian in Cartesian coordinates is written as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Starting with the x coordinates, we shall differentiate equation (3.1).

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \left(\sin\phi\cos\theta\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial\theta}\right)\left(\sin\phi\cos\theta\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial u}{\partial\theta}\right) \\
&= \sin\phi\cos\theta\frac{\partial}{\partial r}\left(\sin\phi\cos\theta\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial u}{\partial\theta}\right) \\
&\quad + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\phi}\left(\sin\phi\cos\theta\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial u}{\partial\theta}\right) \\
&\quad - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial\theta}\left(\sin\phi\cos\theta\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial u}{\partial\theta}\right) \\
&= \sin\phi\cos\theta\left(0 + \sin\phi\cos\theta\frac{\partial^2 u}{\partial r^2}\right) + \sin\phi\cos\theta\left(\frac{-\cos\theta\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos\theta\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi}\right) \\
&\quad + \sin\phi\cos\theta\left(\frac{\sin\phi}{r^2\sin\phi}\frac{\partial^2 u}{\partial\theta} - \frac{\sin\phi}{r\sin\phi}\frac{\partial u}{\partial r\partial\theta}\right) + \frac{\cos\theta\cos\phi}{r}\left(\cos\phi\cos\theta\frac{\partial u}{\partial r} + \sin\phi\cos\theta\frac{\partial^2 u}{\partial\phi\partial r}\right) \\
&\quad + \frac{\cos\theta\cos\phi}{r}\left(\frac{-\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\phi} + \frac{\cos\phi\cos\theta}{r}\frac{\partial^2 u}{\partial\phi^2}\right) + \frac{\cos\theta\cos\phi}{r}\left(\frac{-\sin\theta}{r\cos\phi}\frac{\partial u}{\partial\theta} - \frac{\sin\theta}{r\sin\phi}\frac{\partial^2 u}{\partial\phi\partial\theta}\right) \\
&\quad - \frac{\sin\theta}{r\sin\phi}\left(-\sin\phi\sin\theta\frac{\partial u}{\partial r} + \sin\phi\cos\theta\frac{\partial^2 u}{\partial r\partial\theta}\right) - \frac{\sin\theta}{r\sin\phi}\left(\frac{-\sin\theta\cos\phi}{r}\frac{\partial u}{\partial\phi} + \frac{\cos\theta\cos\phi}{r}\frac{\partial^2 u}{\partial\phi\partial\theta}\right) \\
&\quad + \frac{\sin\theta}{r\sin\phi}\left(\frac{\cos\theta}{r\sin\phi}\frac{\partial u}{\partial\theta} + \frac{\sin\theta}{r\sin\phi}\frac{\partial^2 u}{\partial\theta^2}\right) \\
&= \sin^2\phi\cos^2\theta\frac{\partial^2 u}{\partial r^2} + \left(\frac{-2\cos^2\theta\cos\phi\sin\phi}{r^2} + \frac{\sin^2\theta\cos\phi}{r^2\sin\phi}\right)\frac{\partial u}{\partial\phi} \\
&\quad + \frac{2\cos^2\theta\cos\phi\sin\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} + \frac{\cos\theta\sin\theta}{r^2\sin^2\phi}\frac{\partial u}{\partial\theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} \\
&\quad + \frac{\sin^2\theta + \cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\phi^2} - \frac{2\cos\theta\cos\phi\sin\theta}{r^2\sin\phi}\frac{\partial^2 u}{\partial\phi\partial\theta} + \frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} \qquad (3.4)
\end{aligned}$$

We can develop the other two terms by using the same method. We obtain $\frac{\partial^2 u}{\partial y^2}$ by taking the derivative of equation (3.2).

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \sin^2\phi\sin^2\theta\frac{\partial^2 u}{\partial r^2} + \left(\frac{-2\sin^2\theta\cos\phi\sin\phi}{r^2} + \frac{\cos^2\theta\cos\phi}{r^2\sin\phi}\right)\frac{\partial u}{\partial \phi} + \frac{2\sin^2\theta\cos\phi\sin\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} \\ &\quad -\frac{\cos\theta\sin\theta}{r^2\sin^2\phi}\frac{\partial u}{\partial\theta} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta+\sin^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\phi^2} \\ &\quad +\frac{2\cos\theta\cos\phi\sin\theta}{r^2\sin\phi}\frac{\partial^2 u}{\partial\phi\partial\theta} + \frac{\cos^2\theta}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} \qquad (3.5)\end{aligned}$$

For our last term, $\frac{\partial^2 u}{\partial z^2}$, we apply the same method to equation (3.3) are obtain

$$\frac{\partial^2 u}{\partial z^2} = \cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\phi\sin\phi}{r^2}\frac{\partial u}{\partial\phi} - \frac{2\cos\phi\sin\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\phi^2} \qquad (3.6)$$

Now that we have found our partial derivatives with respect to x, y, and z, we substitute (3.4), (3.5), and (3.6) into the formula for the Laplacian in Cartesian coordinates

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{\cos\phi}{r^2\sin\phi}\frac{\partial u}{\partial\phi} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} \\ &= \frac{1}{r^2}\left(r^2\frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial u}{\partial\phi} + \sin\phi\frac{\partial^2 u}{\partial\phi^2}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}. \qquad (3.7)\end{aligned}$$

Chapter 4: Three-Dimensional Wave Equation

Now, we will examine the wave equation on a sphere. The three-dimensional wave equation is $u_{tt} = \nabla^2 u = c^2(u_{xx} + u_{yy} + u_{zz})$. The right-hand side of this equation has already been rewritten in (3.7) with spherical coordinates. So the spherical expression for the three-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial \phi} \left(\sin\phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{r^2 \sin^2\phi} \frac{\partial^2 u}{\partial \theta^2} \right)$$

where $r \in \mathbb{R},\ 0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$. Our ϕ represents the altitude, while the θ represents the azimuth of the sphere.

4.1 Solving for T(t)

As before, we will use separation of variables to find solutions to this wave equation by writing in the form $u(r, \phi, \theta, t) = R(r)\Phi(\phi)\Theta(\theta)T(t)$. By substituting this u into our three-dimensional wave equation, we see that R, Φ, Θ, and T must satisfy

$$\begin{aligned} R(r)\Phi(\phi)\Theta(\theta)T''(t) = c^2 &\left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 R'(r)\Phi(\phi)\Theta(\theta)T(t)) \right) \\ &+ c^2 \left(\frac{1}{r^2 \sin\phi} \frac{\partial}{\partial \phi} (\sin\phi R(r)\Phi'(\phi)\Theta(\theta)T(t)) \right) \\ &+ c^2 \left(\frac{1}{r^2 \sin^2\phi} (R(r)\Phi(\phi)\Theta''(\theta)T(t)) \right). \end{aligned}$$

Dividing both sides by $c^2 R(r)\Phi(\phi)\Theta(\theta)T(t)$ we get

$$\frac{1}{c^2} \frac{T''(t)}{T(t)} = \frac{1}{r^2 R(r)} \frac{\partial}{\partial r} (r^2 R'(r)) + \frac{1}{\Phi(\phi) r^2 \sin\phi} \frac{\partial}{\partial \phi} (\Phi'(\phi) \sin\phi) + \frac{1}{r^2 \sin^2\phi} \frac{\Theta''(\theta)}{\Theta(\theta)}. \quad (4.1)$$

The right-hand side of (4.1) is independent of t, so both sides of the equation are constant. Let us allow this constant to be $-\lambda$. Thus $\frac{1}{c^2} \frac{T''(t)}{T(t)} = -\lambda$, which gives us the equation $T''(t) + c^2\lambda T(t) = 0$. The solution of this differential equation depends on what value our λ is. Let us assume $\lambda = \delta^2$. As seen in Chapter 1, the different cases for the eigenvalue and their respective solutions are

$$\text{if } \delta^2 > 0 \quad T(t) = b_1 \cos(\delta ct) + b_2 \sin(\delta ct)$$

$$\text{if } \delta^2 < 0 \quad T(t) = b_1 e^{\delta ct} + b_2 e^{-\delta ct}$$

$$\text{if } \delta^2 = 0 \quad T(t) = b_1 t + b_2$$

where b_1 and b_2 are arbitrary constants. It will be shown later that $\delta^2 > 0$ is the only case that is applicable to our equation and conditions.

4.2 Solving for $\Theta(\theta)$

Now that we have found solutions for $T(t)$ depending on the value of the eigenvalue we shift our remaining eigenfunctions around and we are able to form the equation

$$-\lambda r^2 - \frac{1}{R(r)}\frac{\partial}{\partial r}(r^2 R'(r)) = \frac{1}{\Phi(\phi)\sin\phi}\frac{\partial}{\partial\phi}(\Phi'(\phi)\sin\phi) + \frac{1}{\sin^2\phi}\frac{\Theta''(\theta)}{\Theta(\theta)} = -\gamma. \quad (4.2)$$

The left-hand side of the equation depends solely on r while the right-hand side depends both on θ and ϕ. We set both of these sides to the common constant $-\gamma$. We shall focus on first finding the solution to our angle variables. Multiplying by $\sin^2\phi$ and rearranging terms, we can produce

$$\frac{\sin^2\phi}{\Phi(\phi)}\Phi''(\phi) + \frac{\sin\phi\cos\phi}{\Phi(\phi)}\Phi'(\phi) + \gamma\sin^2\phi = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \xi \quad (4.3)$$

where ξ is a constant. Let us solve for $\Theta(\theta)$ first. We have the equation

$$\Theta''(\theta) + \xi\Theta(\theta) = 0$$

where θ, being the azimuthal angle, is a value that varies from 0 to 2π. So $\Theta(\theta)$ must be a periodic function of 2π. The solutions to $\Theta(\theta)$ are similar to the function $T(t)$.

The case where $\xi = \mu^2 > 0$, produces the only nonzero periodic solutions. Below, we will show that μ must be an integer. With the periodic properties of θ conditions, we can rewrite the solution for $\Theta(\theta)$ as

$$\Theta(\theta) = e^{i\mu\theta} = e^{i\mu(\theta+2\pi)} = \Theta(\theta + 2\pi)$$
$$e^{i\mu\theta} = e^{i\mu\theta}e^{2\pi\mu i}.$$

Dividing both sides by $e^{i\mu\theta}$, we are left with

$$1 = e^{2\pi\mu i}$$
$$1 = \cos(2\pi\mu) + i\sin(2\pi\mu).$$

For this equality to be true, μ must be an integer. The relevant solution for the ODE of Θ is

$$\Theta(\theta) = b_3\cos(\mu\theta) + b_4\sin(\mu\theta), \quad \mu \in \mathbb{Z}$$

where b_3 and b_4 are arbitrary constants. For this research, we will be assuming that $b_3 = b_4$ and the term $\sin(\mu\theta)$ is complex . Therefore the solution of $\Theta(\theta)$ is

$$\begin{aligned}\Theta(\theta) &= b_3\cos(\mu\theta) + b_3 i\sin(\mu\theta), \quad \mu \in \mathbb{Z}\\ &= b_3 e^{i\mu\theta}, \quad \mu \in \mathbb{Z}.\end{aligned} \tag{4.4}$$

4.3 Solving for $\Phi(\phi)$

Let us now look at finding the solution for $\Phi(\phi)$. Using (4.3) and the solutions of our azimuthal equation, we multiply by $\Phi(\phi)$ and obtain the equation

$$\sin^2\phi\ \Phi''(\phi) + \sin\phi\cos\phi\ \Phi'(\phi) + (\gamma\sin^2\phi - \mu^2)\Phi(\phi) = 0. \tag{4.5}$$

To find the solution of the ODE (4.5), we will rewrite the equation by using a change of variables, where $s(\phi) = \cos\phi$ and $\Phi(s) = \Phi(s(\phi))$. Since ϕ is the altitude angle, it varies from 0 to π. Then, our new variable $s(\phi)$ has to lie in $[-1, 1]$. By also using the trigonometric identity of $\cos^2\phi + \sin^2\phi = 1$, we shall find the first and second derivative of $\Phi(\phi)$:

$$\frac{d\Phi}{d\phi} = \frac{d\Phi}{ds}\frac{ds}{d\phi} = \frac{d\Phi}{ds}(-\sin\phi) = \frac{d\Phi}{ds}(-\sqrt{1-s^2}) \tag{4.6}$$

$$\frac{d^2\Phi}{d\phi^2} = \frac{d}{d\phi}\left(\frac{d\Phi}{ds}\frac{ds}{d\phi}\right) = \frac{d^2\Phi}{ds^2}\frac{ds}{d\phi}\frac{ds}{d\phi} + \frac{d\Phi}{ds}\frac{d^2s}{d\phi^2} = \frac{d^2\Phi}{ds^2}(1-s^2) + \frac{d\Phi}{ds}(-s) \tag{4.7}$$

Plugging (4.6) and (4.7) into (4.5) we obtain

$$(-\sqrt{1-s^2})^2\left(\frac{d^2\Phi}{ds^2}(1-s^2) + \frac{d\Phi}{ds}(-s)\right) + \left(\frac{d\Phi}{ds}(-\sqrt{1-s^2}\right)s(-\sqrt{1-s^2})$$

$$+\left(\gamma(1-s^2)^2 - \mu^2\right) = 0$$

$$(1-s^2)^2\Phi''(s) - 2s(1-s^2)\Phi'(s) + (\gamma(1-s^2) - \mu^2)\Phi(s) = 0. \tag{4.8}$$

We want (4.2) to be a well-defined function, so we need to make sure $\Phi(s)$ is also well-defined. The endpoints $s = \pm 1$, representing the sphere's north and south poles, are potential singularities. This is due to the fact that the coefficients of $(1-s^2)$ in (4.8) vanish as we approach the endpoints. Thus, we must require $\Phi(s)$ to be bounded at the singular points. So we have the condition that $|\Phi(\pm 1)| < \infty$.

The ODE (4.8) is known as the *Associated Legendre equation of order* μ. The parameter μ controls the order of the Legendre equation, while γ has the role of an eigenvalue. We also notice that μ must be an integer to obtain a polynomial. Since this is a second order ODE, we can have two linearly independent solutions. These solutions are known as the associated Legendre functions of the first and second kind, which we will denote $P_\gamma^\mu(s)$ and $Q_\gamma^\mu(s)$ respectively. To calculate our eigenfunctions, we shall divide by $(1-s^2)$ and consider the case when $\mu = 0$ and $\gamma = \rho(\rho+1)$. This equation has the form

$$(1-s^2)\Phi''(s) - 2s\Phi'(s) + \rho(\rho+1)\Phi(s) = 0 \tag{4.9}$$

and is called the *Legendre equation*. To find solutions of our equation (4.9), we seek a series solution of differential equation and show that the series terminates after a finite number of terms. The power series expansion we will use is

$$\Phi(s) = \sum_{k=0}^{\infty} a_k s^k,$$

where k is an integer. Taking the derivatives of this power series, we will then substitute them back into (4.9).

$$\Phi(s) = \sum_{k=0}^{\infty} a_k s^k \quad \Phi'(s) = \sum_{k=0}^{\infty} k a_k s^{k-1} \quad \Phi''(s) = \sum_{k=0}^{\infty} k(k-1) a_k s^{k-2}$$

After substituting our series and its first and second derivatives into (4.9), we obtain

$$\sum_{k=0}^{\infty}(1-s^2)k(k-1)a_k s^{k-2} - \sum_{k=0}^{\infty} 2ska_k s^{k-1} + \sum_{k=0}^{\infty} \rho(\rho+1)a_k s^k = 0$$

$$\sum_{k=0}^{\infty} \left(a_k k(k-1)s^{k-2} - a_k(k(k-1))s^k - 2a_k k s^k + \rho(\rho+1)a_k s^k\right) = 0$$

$$\sum_{k=0}^{\infty}(a_k k(k-1)s^{k-2}) - \sum_{k=0}^{\infty}((k(k+1) - \rho(\rho+1))a_k s^k) = 0.$$

To combine like terms, we will replace the k in the left equation by $k+2$.

$$\sum_{k=-2}^{\infty} (a_{k+2}(k+2)(k+1)s^k) - \sum_{k=0}^{\infty}((k(k+1) - \rho(\rho+1))a_k s^k) = 0.$$

We can rewrite these series to be

$$\sum_{k=0}^{\infty}(a_{k+2}(k+2)(k+1)s^k) - \sum_{k=0}^{\infty}((k(k+1) - \rho(\rho+1))a_k s^k) = 0$$

since the terms for k values -2 and -1 are zero. Since the sum of these power series is the zero function, each of our coefficients of degree k must sum to zero. Thus solving for our coefficients we get

$$a_{k+2}(k+2)(k+1)s^k = (k(k+1) - \rho(\rho+1))a_k s^k$$

We then have a recursion relation

$$a_{k+2} = \frac{(k(k+1) - \rho(\rho+1))a_k}{(k+2)(k+1)} \quad \text{where } k = 0, 1, 2, 3.... \tag{4.10}$$

Notice that all of the coefficients a_n can be rewritten in terms of the initial conditions a_0 and a_1. Thus, the solution of (4.9) can be written as the sum of two series containing these two constants. The first few even coefficients are

$$\begin{aligned} a_2 &= \frac{-\rho(\rho+1)}{1 \cdot 2} a_0 \\ a_4 &= \frac{2 \cdot 3 - \rho(\rho+1)}{3 \cdot 4} a_2 = -\frac{\rho(\rho+1)(2 \cdot 3 - \rho(\rho+1))}{1 \cdot 2 \cdot 3 \cdot 4} a_0 \\ &\vdots \end{aligned}$$

and the first few odd coefficients are

$$\begin{aligned} a_3 &= \frac{2 - \rho(\rho+1)}{2 \cdot 3} a_1 \\ a_5 &= \frac{3 \cdot 4 - \rho(\rho+1)}{4 \cdot 5} a_3 = \frac{(2 - \rho(\rho+1))(3 \cdot 4 - \rho(\rho+1))}{2 \cdot 3 \cdot 4 \cdot 5} a_1. \\ &\vdots \end{aligned}$$

Thus

$$\begin{aligned} \Phi(s) = a_0 &\left(1 - \frac{\rho(\rho+1)}{2} s^2 + \frac{\rho(\rho+1)(\rho(\rho+1) - 2 \cdot 3)}{1 \cdot 2 \cdot 3 \cdot 4} s^4 - \dots\right) \\ + a_1 &\left(s - \frac{2 - \rho(\rho+1)}{2 \cdot 3} s^3 + \frac{(2 - \rho(\rho+1))(3 \cdot 4 - \rho(\rho+1))}{2 \cdot 3 \cdot 4 \cdot 5} s^5 \dots\right). \end{aligned} \tag{4.11}$$

Observe that for $\rho = 0$ the first term equals a_0 while the second series diverges in (4.11). When $\rho = 1$, the first series diverge at $s^2 = 1$. But the second series comes to an end at $a_1 s$. We notice that for any integer ρ, one of our series of (4.11) terminates providing a polynomial solution and the other diverges at $s^2 = 1$. Therefore, we have a collection of polynomial solutions of the Legendre equation, each containing either

a_0 or a_1. For instance, when $\rho = 0$, $\Phi(s) = a_0$; when $\rho = 1$, $\Phi(s) = a_1 s$, etc. If we select the values of a_0 and a_1 in each solution so that $\Phi(s) = 1$ when $s = 1$, the polynomials are called *Legendre polynomials*, written as $P_\rho(s)$. The first few Legendre polynomials are

$$P_0(s) = 1, \quad P_1(s) = s, \quad P_2(s) = \frac{3s^2 - 1}{2}, \quad P_3(s) = \frac{5s^3 - 3s}{2}.$$

To find Legendre polynomials in a more efficient manner, we present a general solution of the Legendre equation. An expression to represent the Legendre polynomials is

$$P_\rho(s) = \frac{1}{2^\rho \rho!} \frac{d^\rho}{ds^\rho}(s^2 - 1)^\rho, \tag{4.12}$$

which is known as *Rodrigues' formula*. The derivation of the Rodrigues' formula can be found in [6]. Knowing this formula, we will now find general solutions of our associated Legendre equation from (4.8):

$$(1 - s^2)\Phi''(s) - 2s\Phi'(s) + \left(\rho(\rho + 1) - \frac{\mu^2}{1 - s^2}\right)\Phi(s) = 0 \tag{4.13}$$

where $\mu^2 \leq \rho^2$.. We see that ρ is the maximal value for μ, which means that ρ must also be an integer.

Let $\Phi(s) = (1 - s^2)^{\mu/2} v$. Substituting our $\Phi(s)$ into (4.13) we have

$$(1 - s^2)v'' - 2(\mu + 1)sv' + (\rho(\rho + 1) - \mu(\mu + 1))v = 0. \tag{4.14}$$

Notice that if $\mu = 0$, we have the Legendre polynomials for our solutions. By differentiating (4.14), we acquire

$$(1 - s^2)(v')'' - 2((\mu + 1) + 1)s(v')' + (\rho(\rho + 1) - (\mu(\mu + 1))v' = 0. \tag{4.15}$$

We can see that (4.15) is just (4.14) with v' instead of v, and $\mu + 1$ instead of μ. So if $P_\rho(s)$ is a solution of (4.14) when $\mu = 0$, then $P'_\rho(s)$ is a solution of (4.14) when

$\mu = 1$, and $P''_\rho(s)$ is a solution of when $\mu = 2$. In general for $-\rho \leq \mu \leq \rho$, $\frac{d^\mu}{ds^\mu}(P_\rho(s))$ is a solution of (4.14). Therefore

$$P^\mu_\rho(s) = (1 - s^2)^{\mu/2} \frac{d^\mu}{ds^\mu} P_\rho(s), \tag{4.16}$$

called an *associated Legendre polynomial* and denoted $P^\mu_\rho(s)$, is a solution to (4.13). The Legendre functions of the second kind are defined by

$$Q^\mu_\rho(s) = \frac{1}{2} P^\mu_\rho(s) \ln\left(\frac{1+s}{1-s}\right) - \sum_{\mu=1}^{\rho} \left(\frac{1}{\mu} P^\mu_{\mu-1}(s) P^\mu_{\rho-\mu}(s)\right).$$

ρ	μ	$P^\mu_\rho(s)$
0	0	1
1	0	s
1	1	$-\sqrt{1-s^2}$
2	0	$\frac{1}{2}(3s^2 - 1)$
2	1	$-3s\sqrt{1-s^2}$
2	2	$3(1-s^2)$

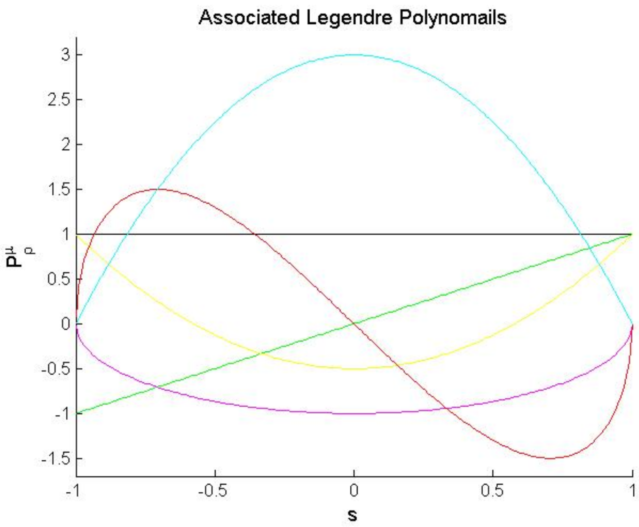


Table 4.1: A chart of the first few associated Legendre polynomials defined by the order and degree. To the right is the graph of all polynomials in the chart.

Thus, our solution for (4.3) is $\Phi(\phi) = b_5 P^\mu_\rho(\cos(\phi)) + b_6 Q^\mu_\rho(\cos(\phi))$, where b_5 and b_6 are arbitrary constants. Notice that as $|s| \to 1$, the Legendre functions of the second kind diverges. There is a pole at $s = \pm 1$ as seen by observing that as $|s| \to \pm 1$, the function approaches $+\infty$. Thus, $b_6 = 0$ and we can toss out the Legendre function of the second kind, leaving us with

$$\Phi(\phi) = b_5 P^\mu_\rho(\cos(\phi)). \tag{4.17}$$

4.4 Solving for $R(r)$

We have now discovered solutions for both of our angular variables and our time variable in our wave equation. We now only have to find the solution to the last variable of our 3-D wave equation, $R(r)$. From equation (4.2), we already have

$$-\frac{1}{R(r)}\frac{\partial}{\partial r}(r^2R'(r)) - \lambda r^2 = -\gamma = -\rho - \rho^2$$

$$r^2R''(r) + 2rR'(r) + (\lambda r^2 - (\rho + \rho^2))R(r) = 0. \tag{4.18}$$

We shall define $s = \chi r$, where $\chi = \sqrt{\lambda}$. Now the function R is dependent on s. To substitute in, we first need the derivatives of $R(r)$ written in terms of s.

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = \chi\frac{dR}{ds} \tag{4.19}$$

$$\frac{d^2R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{ds}\frac{ds}{dr}\right) = \frac{d^2R}{ds^2}\left(\frac{ds}{dr}\right)^2 + \frac{dR}{ds}\frac{d^2s}{dr^2} = \chi^2\frac{d^2R}{ds^2} + \frac{dR}{ds}\cdot 0 \tag{4.20}$$

Plugging (4.19) and (4.20) into (4.18) we get

$$\left(\frac{s}{\chi}\right)^2(\chi^2R''(s)) + 2\left(\frac{s}{\chi}\right)(\chi R'(s)) + (s^2 - (\rho+\rho^2)R(s) = 0$$

$$R''(s) + \frac{2}{s}R'(s) + \left(1 - \frac{\rho+\rho^2}{s^2}\right)R(s) = 0. \tag{4.21}$$

The solutions to (4.21) are known as the spherical Bessel functions. Similar to the associated Legendre equation, there can be two linearly independent solutions for (4.21), known as spherical Bessel functions of the first and second kind. These will be denoted as $j_\rho(s)$ and $y_\rho(s)$ respectively.

The spherical Bessel functions are written as

$$j_\rho(s) = \sqrt{\frac{\pi}{2s}}J_{\rho+1/2}(s) \quad \text{and} \quad y_\rho(s) = \sqrt{\frac{\pi}{2s}}Y_{\rho+1/2}(s)$$

where

$$J_\rho(s) = \sum_{i=0}^{\infty} (\frac{1}{2}s)^{\rho+2i} \frac{(-1)^i}{i!(\rho+i)!} \quad \text{and} \quad Y_\rho(s) = \frac{J_\rho(s)\cos(\rho\pi) - J_{-\rho}(s)}{\sin(\rho\pi)}.$$

The variables $J_\rho(s)$ and $Y_\rho(s)$ are known as the (standard) Bessel functions. The (standard) Bessel functions are the solution to the differential equation $s^2R''(s) + sR'(s) + (s^2 - \gamma^2)R(s) = 0$. Similar to the Legendre function of the second kind, the spherical Bessel function of the second kind diverges as $s \to 0$. Thus our solution for (4.16) is $R(r) = b_7 j_\rho(\chi r)$ where b_7 is an arbitrary constant.

Looking back to the first function we solved for, $T(t)$, we can determine its correct solutions from what we have learned from the other ODEs. The constant λ present in the solutions is also in the solution for $R(r)$. For the solutions of $R(r)$ to be spherical Bessel functions, λ must be a positive value. Thus the solution for our function of time is $T(t) = b_1 \cos(ct\sqrt{\lambda}) + b_2 sin(ct\sqrt{\lambda})$.

Now we combine all of the ODE solutions solved by separation of variables and let $\lambda = \rho + 1$, which was solved by Dr. Paboyle [7]. The general forms of the solution for our three-dimensional wave equation in the form $u(r, \phi, \theta, t) = R(r)\Phi(\phi)\Theta(\theta)T(t)$ and its velocity are

$$u(r, \phi, \theta, t) = \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} \Big(A_{\rho,\mu} \cos(ct\sqrt{\rho+1}) + B_{\rho,\mu} \sin(ct\sqrt{\rho+1})\Big) \bullet \Big(j_\rho(\sqrt{\rho+1}r) \Big(P_\rho^\mu(\cos(\phi))e^{i\mu\theta}\Big)\Big) \tag{4.22}$$

$$u_t(r, \phi, \theta, t) = \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} \Big(-A_{\rho,\mu} c\sqrt{\rho+1} \sin(ct\sqrt{\rho+1}) + B_{\rho,\mu} c\sqrt{\rho+1} \cos(ct\sqrt{\rho+1})\Big) \bullet \Big(j_\rho(\sqrt{\lambda}r) \Big((P_\rho^\mu(\cos(\phi))e^{i\mu\theta}\Big)\Big)$$

provided that

$$\alpha(r,\phi,\theta,0) = u(r,\phi,\theta,0) = \sum_{\rho=0}^{\infty}\sum_{\mu=-\rho}^{\rho}(A_{\rho,\mu})\left(j_\rho(r\sqrt{\rho+1})\left((P_\rho^\mu(\cos(\phi))e^{i\mu\theta}\right)\right) \tag{4.23}$$

$$\beta(r,\phi,\theta,0) = u_t(r,\phi,\theta,0) = \sum_{\rho=0}^{\infty}\sum_{\mu=-\rho}^{\rho}(B_{\rho,\mu})\left(j_\rho(r\sqrt{\rho+1})\left(P_\rho^\mu(\cos(\phi))e^{i\mu\theta}\right)\right)c\sqrt{\rho+1}. \tag{4.24}$$

Chapter 5: Fourier Coefficients

Equation (4.22) is a Fourier series and we need to solve for its coefficients. Since the main focus of this paper is the wave propagation on a volleyball, the radius r is not a function but a constant. Thus the term u measures the radial displacement of the sphere and is dependent on θ, ϕ and t. So we can leave out the spherical Bessel functions $j_\rho(\lambda r)$ and the solution becomes

$$u(\theta, \phi, t) = \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} \left(A_{\rho,\mu} \cos(ct\sqrt{\rho+1}) + B_{\rho,\mu} \sin(ct\sqrt{\rho+1}) \right) \left(P_\rho^\mu(\cos(\phi)) e^{i\mu\theta} \right)$$

with time derivative:

$$u_t(\theta, \phi, t) = \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} \left(-A_{\rho,\mu} c\sqrt{\rho+1} \sin(ct\sqrt{\rho+1}) + B_{\rho,\mu} c\sqrt{\rho+1} \cos(ct\sqrt{\rho+1}) \right)$$
$$\bullet \left(P_\rho^\mu(\cos(\phi)) e^{i\mu\theta} \right).$$

.

5.1 Coefficients on the Volleyball

We will use the initial position (4.23) and velocity (4.24) formulas to solve for the coefficients in the solutions to the three-dimensional wave equation. We shall multiply both sides of these formulas by $-\sin(\phi)$ and $P_{\rho'}^\mu(\cos(\phi))$ to get the equations

$$-\alpha(\theta, \phi) P_{\rho'}^\mu(\cos(\phi))(\sin(\phi)) = \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} A_{\rho,\mu} P_\rho^\mu(\cos(\phi)) P_{\rho'}^\mu(\cos(\phi))(-\sin(\phi)) e^{i\mu\theta} \tag{5.1}$$

$$-\beta(\theta, s) P_{\rho'}^\mu(\cos(\phi))(\sin(\phi)) = \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} B_{\rho,\mu} P_\rho^\mu(\cos(\phi)) P_{\rho'}^\mu(\cos(\phi))(-\sin(\phi))$$
$$\bullet\, e^{i\mu\theta} c\sqrt{\rho+1}. \tag{5.2}$$

Solving for the coefficient $A_{\rho,\mu}$ in (5.1), we integrate with respect to s by letting $s = \cos(\phi)$ and $ds = -\sin(\phi)d\phi$.

$$\int_{-1}^{1} \alpha(\theta, s) P_{\rho'}^{\mu}(s) ds = \int_{-1}^{1} \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} A_{\rho,\mu} e^{i\mu\theta} P_{\rho}^{\mu}(s) P_{\rho'}^{\mu}(s) ds$$

$$= \sum_{\rho=0}^{\infty} \sum_{\mu=-\rho}^{\rho} A_{\rho,\mu} e^{i\mu\theta} \int_{-1}^{1} P_{\rho}^{\mu}(s) P_{\rho'}^{\mu}(s) ds. \tag{5.3}$$

To calculate this integral, we will have to understand the orthogonality of associated Legendre polynomials.

Definition 4. The *Kronecker delta*, denoted by $\delta_{kk'}$, is defined by

$$\delta_{kk'} = \begin{cases} 1 & k = k' \\ 0 & k \neq k'. \end{cases}$$

Proposition 5.1.1. Let $P_{\rho}^{\mu}(s)$ denote the associated Legendre Polynomial of degree ρ and order μ. Then $\{P_{\rho}^{\mu}(s)\}$ is an L^2-orthogonal set over $[-1, 1]$ with respect to ρ and

$$\int_{-1}^{1} P_{\rho}^{\mu}(s) P_{\rho'}^{\mu}(s) ds = \frac{2}{2\rho + 1} \frac{(\rho + \mu)!}{(\rho - \mu)!} \delta_{\rho\rho'}.$$

Proof. To recall, the associated Legendre polynomials have a form of

$$P_{\rho}^{\mu}(s) = (1 - s^2)^{\mu/2} \frac{d^{\mu}}{ds^{\mu}} P_{\rho}(s) \tag{5.4}$$

where $P_{\rho}(s)$ is the Legendre polynomial, which is represented by the Rodrigues function as stated in (4.12). We will let $S(s) = (s^2 - 1)$ to make notation simpler and can then rewrite (5.4) as

$$P_{\rho}^{\mu}(s) = \frac{1}{2^{\rho} \rho!} (-S)^{\mu/2} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}} (S^{\rho}) \tag{5.5}$$

Our associated Legendre polynomials are defined over the range $-1 \leq s \leq 1$. Using (5.5), we have

$$\int_{-1}^{1} P_{\rho}^{\mu}(s) P_{\rho'}^{\mu}(s) ds = \frac{(-1)^{\mu}}{2^{\rho+\rho'} \rho! \rho'!} \int_{-1}^{1} S^{\mu} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \frac{d^{\rho'+\mu}}{ds^{\rho'+\mu}}(S^{\rho'})\ ds. \tag{5.6}$$

Integrating by parts we have

$$\int_{-1}^{1} P_{\rho}^{\mu}(s) P_{\rho'}^{\mu}(s) ds = \frac{(-1)^{\mu}}{2^{\rho+\rho'} \rho! \rho'!} \left[S^{\mu} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \frac{d^{\rho'+\mu-1}}{ds^{\rho'+\mu-1}}(S^{\rho'}) \Big|_{-1}^{1} - \int_{-1}^{1} \frac{d^{\rho'+\mu-1}}{ds^{\rho'+\mu-1}}(S^{\rho'}) \frac{d}{ds} \left(S^{\mu} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) ds \right].$$

Because $S = 0$ at both endpoints, the first term, the boundary term, will also be zero. If we integrate by parts again we have

$$\frac{(-1)^{\mu}}{2^{\rho+\rho'} \rho! \rho'!} \left[-\frac{d}{ds} \left(S^{\mu} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) \frac{d^{\rho'+\mu-2}}{ds^{\rho'+\mu-2}}(S^{\rho'}) \Big|_{-1}^{1} + \int_{-1}^{1} \frac{d^{\rho'+\mu-2}}{ds^{\rho'+\mu-2}}(S^{\rho'}) \frac{d^2}{ds^2} \left(S^{\mu} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) ds \right].$$

One of the derivative terms in the boundary term, $\frac{d}{ds}\left(S^{\mu} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho})\right)$, is zero because it has a factor of $S^{\mu-1}$ by the power rule, where $\mu - 1 > 0$.

$$\begin{aligned}
&\frac{d}{ds} \left(S^{\mu} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) \\
&= \frac{d}{ds}(S^{\mu}) \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) + S^{\mu} \frac{d}{ds} \left(\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) \\
&= (S^{\mu-1}) \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) + S^{\mu} \left(S^{\rho-(\rho+\mu-1)} \right) \\
&= S^{\mu-1} \left(\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) + S^{-\mu+2} \right)
\end{aligned}$$

Evaluating at the endpoints, $S^{\mu-1}$, which equals $(1 - s^2)^{\mu-1}$, is zero. Thus the integrated term is equal to zero in this integration by parts. If we continue the process of integration by parts j times, we will get

$$\frac{(-1)^\mu(-1)^j}{2^{\rho+\rho'}\rho!\rho'!}\left[\frac{d^{j-1}}{ds^{j-1}}\left(S^\mu\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)\right)\frac{d^{\rho'+\mu-j}}{ds^{\rho'+\mu-j}}(S^{\rho'})\Big|_{-1}^{1}\right.$$
$$\left.-\int_{-1}^{1}\frac{d^{\rho'+\mu-j}}{ds^{\rho'+\mu-j}}(S^{\rho'})\frac{d^j}{ds^j}\left(S^\mu\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)\right)ds\right].$$

The derivative factor in the boundary term, $\frac{d^{j-1}}{ds^{j-1}}\left(S^\mu\frac{d^{\rho+\mu}S^\rho}{ds^{\rho+\mu}}\right)$, becomes zero when evaluated at the endpoints since it has a factor of $S^{\mu-j+1}$ by the power rule, as long as $\mu+1>j$.

$$\begin{aligned}
&\frac{d^{j-1}}{ds^{j-1}}\left(S^\mu\frac{d^{\rho+\mu}S^\rho}{ds^{\rho+\mu}}\right)\\
=&S^{\mu-(j-1)}\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)+(j-1)S^{\mu-j+2}\frac{d^{\rho+\mu-1}}{ds^{\rho+\mu-1}}(S^\rho)+...\\
&+(j-1)S^{\mu-1}\frac{d^{\rho+\mu-j+2}}{ds^{\rho+\mu-j+2}}(S^\rho)+S^\mu\frac{d^{\rho+\mu-(j-1)}}{ds^{\rho+\mu-(j-1)}}(S^\rho)\\
=&S^{\mu-j+1}\left(\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)+(j-1)S\frac{d^{\rho+\mu-1}}{ds^{\rho+\mu-1}}(S^\rho)+...\right.\\
&\left.+(j-1)S^{j-2}\frac{d^{\rho+\mu-j+2}}{ds^{\rho+\mu-j+2}}(S^\rho)+S^{j-1}\frac{d^{\rho+\mu-(j-1)}}{ds^{\rho+\mu-(j-1)}}(S^\rho)\right)
\end{aligned}$$

Similar to our previous integration by parts, when we evaluate this term at the endpoints, $S^{\mu-j+1}$ will be zero. Thus, the entire boundary term will be equal to zero. Therefore, we integrate $\rho'+\mu$ times, the boundary term will be zero at every step and we have the expression

$$\int_{-1}^{1}P_\rho^\mu(s)P_{\rho'}^\mu(s)ds=\frac{(-1)^\mu(-1)^{\rho'+\mu}}{2^{\rho+\rho'}\rho!\rho'!}\int_{-1}^{1}S^{\rho'}\frac{d^{\rho'+\mu}}{ds^{\rho'+\mu}}\left(S^\mu\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)\right)ds. \tag{5.7}$$

Recall that S is a polynomial in terms of s. The degree of the polynomial S^ρ is 2ρ. Then the degree of s in $\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)$ is $s^{\rho-\mu}$. Multiplying by the highest-powered term of S^μ, we obtain a polynomial of degree $\rho + \mu$. Computing $\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)$ we get

$$\begin{aligned} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}} S^\rho &= \frac{(2\rho)!}{(2\rho - (\rho+\mu))!} s^{2\rho-(\rho+\mu)} + ... \\ &= \frac{(2\rho)!}{(\rho-\mu)!} s^{\rho-\mu} + ... \end{aligned}$$

To compute the integral in (5.7), we must examine two cases. First, when $\rho = \rho'$,

$$\begin{aligned} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}} \left(S^\mu \left(\frac{(2\rho)!}{(\rho-\mu)!} \right) s^{\rho-\mu} \right) &= \frac{(2\rho)!}{(\rho-\mu)!} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}} (s^{2\mu} s^{\rho-\mu}) \\ &= \frac{(2\rho)!}{(\rho-\mu)!} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}} (s^{\rho+\mu}) \\ &= \frac{(2\rho)!(\rho+\mu)!}{(\rho-\mu)!}. \end{aligned}$$

The second case is when $\rho \neq \rho'$. Without loss of generality (WLOG) we assume that $\rho' > \rho$ since ρ and ρ' are interchangeable in (5.6).

$$\begin{aligned} \frac{d^{\rho'+\mu}}{ds^{\rho'+\mu}} \left(S^\mu \left(\frac{(2\rho)!}{(\rho-\mu)!} \right) s^{\rho-\mu} \right) &= \frac{(2\rho)!}{(\rho-\mu)!} \frac{d^{\rho'+\mu}}{ds^{\rho'+\mu}} (s^{2\mu} s^{\rho-\mu}) \\ &= \frac{(2\rho)!}{(\rho-\mu)!} \frac{d^{\rho'+\mu}}{ds^{\rho'+\mu}} (s^{\rho+\mu}) \\ &= 0 \end{aligned}$$

Substituting the fact that $\rho = \rho'$ into (5.7) we have

$$\begin{aligned} \int_{-1}^{1} (P_\rho^\mu)^2 ds &= \frac{(-1)^{\rho+2\mu}}{2^{2\rho}(\rho!)^2} \frac{(2\rho)!(\rho+\mu)!}{(\rho-\mu)!} \int_{-1}^{1} (s^2-1)^\rho ds \\ &= \frac{(-1)^{\rho+2\mu}}{2^{2\rho}(\rho!)^2} \frac{(2\rho)!(\rho+\mu)!}{(\rho-\mu)!} (-1)^\rho \frac{(\rho!)^2 2^{1+2\rho}}{(2\rho+1)!} \\ &= \frac{2}{2\rho+1} \frac{(\rho+\mu)!}{(\rho-\mu)!}. \end{aligned}$$

Thus the normalization of the associated Legendre polynomials with respect to the L^2-inner product, where $s = \cos(\phi)$ on the interval $-1 \leq s \leq 1$, is

$$\int_{-1}^{1} P_\rho^\mu(s) P_{\rho'}^\mu(s) ds = \frac{2}{2\rho+1} \frac{(\rho+\mu)!}{(\rho-\mu)!} \delta_{\rho\rho'}. \qquad \square$$

We now have a simplified version of our equation (5.3).

$$\int_{-1}^{1} \alpha(\theta, s) P_\rho^\mu(s) ds = \sum_{\mu=-\rho}^{\rho} A_{\rho,\mu} \left(\frac{2}{2\rho+1} \right) \frac{(\rho+\mu)!}{(\rho-\mu)!} e^{i\mu\theta}.$$

To solve for our coefficient $A_{\rho,\mu}$, we shall multiply both sides by the integrating factor $e^{-i\theta\mu'}$ and integrate with respect to θ. We obtain

$$\int_0^{2\pi} \int_{-1}^{1} \alpha(\theta, s) P_\rho^\mu(s) e^{-i\theta\mu'} ds \ d\theta = \int_0^{2\pi} \sum_{\mu=-\rho}^{\rho} A_{\rho,\mu} \left(\frac{2}{2\rho+1} \right) \frac{(\rho+\mu)!}{(\rho-\mu)!} e^{i\theta(\mu-\mu')} d\theta. \quad (5.8)$$

When $\mu = \mu'$,

$$\int_0^{2\pi} e^{-i\theta(\mu-\mu')} d\theta = \int_0^{2\pi} e^{-i\theta(0)} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi.$$

If $\mu \neq \mu'$,

$$\begin{aligned} \int_0^{2\pi} e^{-i\theta(\mu-\mu')} d\theta &= \frac{-1}{i(\mu-\mu')} e^{-i\theta(\mu-\mu')} \Big|_0^{2\pi} \\ &= \frac{-1}{i(\mu-\mu')} (\cos(\theta(\mu-\mu')) + i \sin(\theta(\mu-\mu')) \Big|_0^{2\pi} \\ &= 0. \end{aligned}$$

Thus, we acquire the equation

$$\int_0^{2\pi} \int_{-1}^{1} \alpha(\theta, s) P_\rho^\mu(s) e^{-i\theta\mu'} ds \ d\theta = 2\pi \left(\frac{2}{2\rho+1} \right) \frac{(\rho+\mu)!}{(\rho-\mu)!} A_{\rho,\mu}.$$

Solving for $A_{\rho,\mu}$, we therefore have our coefficient written as

$$A_{\rho,\mu} = \frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}\int_0^{2\pi}\int_{-1}^{1}\alpha(\theta,s)P_\rho^\mu(s)e^{-i\theta\mu}ds\ d\theta. \tag{5.9}$$

Now we shall solve for our other coefficient $B_{\rho,\mu}$. We notice the difference between $\alpha(\theta,\phi)$ and $\beta(\theta,\phi)$ in equations (5.1) and (5.2) is a factor of $c\sqrt{\rho+1}$ in $\beta(\theta,\phi)$. Therefore, we can do an analogous calculation for $B_{\rho,\mu}\lambda c$, accounting for this factor, and obtain

$$B_{\rho,\mu} = \frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}\frac{1}{c\sqrt{\rho+1}}\int_0^{2\pi}\int_{-1}^{1}\beta(\theta,s)P_\rho^\mu(s)e^{-i\theta\mu}ds\ d\theta.$$

Therefore, the solution to the wave equation's initial value problem, where $s = \cos(\phi)$, $\alpha(\theta,\phi)$ is the initial position and $\beta(\theta,\phi)$ is the initial velocity, on the sphere is

$$\begin{aligned} u(\theta,\phi,t) = &\sum_{\rho=0}^{\infty}\sum_{\mu=-\rho}^{\rho}\left(\frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}\right)\left(\int_0^{2\pi}\int_{-1}^{1}\alpha(\theta,s)P_\rho^\mu(s)e^{-i\theta\mu}ds\ d\theta\right) \\ &\bullet\left(P_\rho^\mu(\cos(\phi))e^{i\mu\theta}\right)\left(\cos(ct\sqrt{\rho+1})\right) \\ &+\sum_{\rho=0}^{\infty}\sum_{\mu=-\rho}^{\rho}\left(\frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}\right)\left(\int_0^{2\pi}\int_{-1}^{1}\beta(\theta,s)P_\rho^\mu(s)e^{-i\theta\mu}ds\ d\theta\right) \\ &\bullet\left(P_\rho^\mu(\cos(\phi))e^{i\mu\theta}\right)\left(\frac{\sin(ct\sqrt{\rho+1})}{c\sqrt{\rho+1}}\right). \end{aligned} \tag{5.10}$$

5.2 Spherical Harmonics

Spherical harmonics are functions used in investigating physical problems in three dimensions that arise in mathematics and physics. They are the eigenfunctions of Laplace's equation in spherical coordinates and they form an orthogonal and complete set of functions. They are the product of our solutions (4.4) and (4.17), the trigonometric functions and the associated Legendre polynomials, which have the representation

$$\begin{aligned} Y_\rho^\mu(\theta,\phi) &= N_\rho^\mu \Theta(\theta)\Phi(\phi) \\ &= N_\rho^\mu e^{i\mu\theta} P_\rho^\mu(\cos(\phi)). \end{aligned}$$

The spherical harmonic Y_ρ^μ has an order of μ and a degree of ρ and N_ρ^μ is a normalization coefficient.

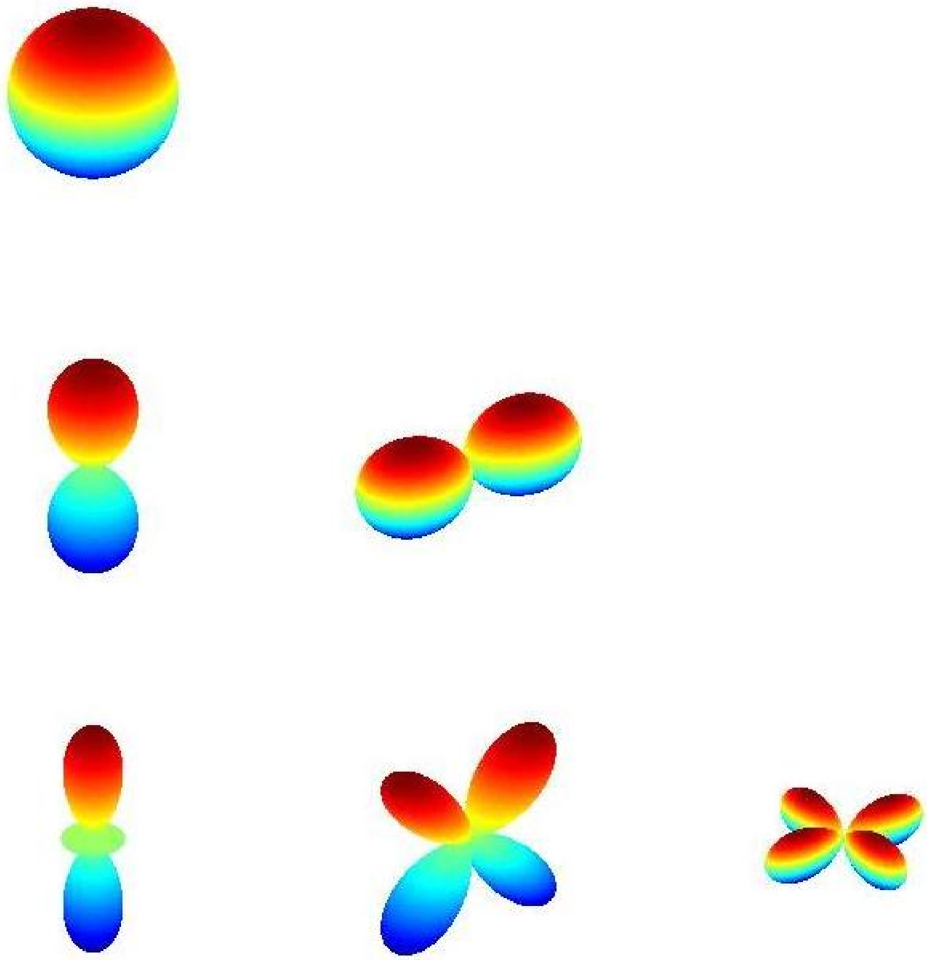


Figure 5.1: This figure illustrates the first three spherical harmonics $\rho = 0...2$ (starting at the top). The order μ is of increasing non-negative values.

Proposition 5.2.1. Spherical harmonics may be defined as

$$Y_\rho^\mu(\theta,\phi) = \sqrt{\frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}}\, e^{i\mu\theta} P_\rho^\mu(\cos(\phi)) \qquad \rho \geq 0, \quad -\mu \leq \rho \leq \mu \qquad (5.11)$$

in which the normalization is chosen such that

$$\int_0^{2\pi}\int_0^{\pi} Y_\rho^\mu(\theta,\phi)\overline{Y_{\rho'}^{\mu'}(\theta,\phi)}(-\sin(\phi))\ d\phi\ d\theta = \delta_{\rho\rho'}\delta_{\mu\mu'}$$

where the δ's are Kronecker delta functions and $\overline{Y_{\rho'}^{\mu'}}$ is the complex conjugate of our spherical harmonic.

Proof. In this case we get

$$\begin{aligned}
\delta_{\rho\rho'}\delta_{\mu\mu'} &= \int_0^{2\pi}\int_0^{\pi} Y_\rho^\mu(\theta,\phi)\overline{Y_{\rho'}^{\mu'}(\theta,\phi)}(-\sin(\phi))\ d\phi\ d\theta \\
&= \int_0^{2\pi}\int_{-1}^{1} \left(N_\rho^\mu P_\rho^\mu(s))e^{i\mu\theta}\right)\left(N_{\rho'}^{\mu'} P_{\rho'}^{\mu}(s)e^{-i\mu'\theta}\right)\ ds\ d\theta \qquad \text{where } s=\cos(\phi) \\
&= N_\rho^\mu N_{\rho'}^{\mu'} \int_{-1}^{1} P_\rho^\mu(s)P_{\rho'}^{\mu}(s)\ ds \int_0^{2\pi} e^{i\mu\theta}e^{-i\mu'\theta}\ d\theta.
\end{aligned}$$

If $\mu \neq \mu'$ or $\rho \neq \rho'$, our Kronecker delta function will be zero, and our integrals will be terms of zero. So, we let $\mu = \mu'$ and $\rho = \rho'$. From the previous chapter (6.1), we have already solved for these cases. Thus, we have

$$1 = \left(N_\rho^\mu\right)^2 \left(\frac{2}{2\rho+1}\frac{(\rho+\mu)!}{(\rho-\mu)!}\right)(2\pi).$$

Thus our normalization coefficient is

$$N_\rho^\mu = \sqrt{\frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}}.$$

□

Spherical harmonics have both real and imaginary parts, which are

$$Re(Y_\rho^\mu(\theta,\phi)) = \sqrt{\frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}}P_\rho^\mu(\cos(\phi))(\cos(\mu\theta))$$

$$Im(Y_\rho^\mu(\theta,\phi)) = \sqrt{\frac{2\rho+1}{4\pi}\frac{(\rho-\mu)!}{(\rho+\mu)!}}P_\rho^\mu(\cos(\phi))(\sin(\mu\theta)).$$

There are three classes of spherical harmonics: zonal, sectoral, and tesseral. *Zonal harmonics* are the spherical harmonics with $\mu = 0$, which take the form $Y_\rho = \sqrt{\frac{2\rho+1}{4\pi}}P_\rho(\cos(\phi))$. This means that there are no longitudinal nodes and that they are cylindrically symmetric. *Sectoral harmonics* have the form $Y_{|\mu|}^{\mu}$. When the spherical harmonic has this property, it makes the $\Phi(\phi) = \sin^2(\phi)$, which is zero at $\phi = 0$ and π. These points are the only roots of the function of θ, which are known as meridians, $\cos(\theta) + \sin(\theta)$. *Tesseral harmonics* are all the other spherical harmonics that did not fall into the previous two categories. This class of spherical harmonics have both azimuthal and co-altitude terms, which allows the harmonics to extend over the entire sphere.

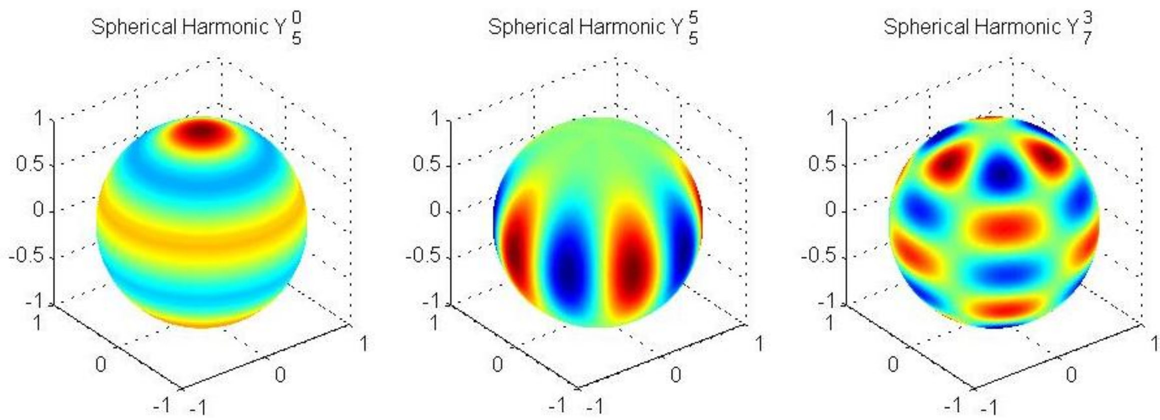


Figure 5.2: The different classes of spherical harmonics. Starting from the left, these are examples of zonal, sectoral, and tesseral spherical harmonics.

Some other properties about spherical harmonics are that we have $(\rho - \mu)$ lines of latitude and 2μ lines of longitude. If $(\rho - \mu)$ is even, then there is symmetry about the equator. , which we show in Proposition 5.3.

Proposition 5.2.2. If $\rho - \mu$ is even, then the spherical harmonics are symmetric about the equator.

Proof. From (5.11), spherical harmonics are written as

$$Y_\rho^\mu(\theta, \phi) = \sqrt{\frac{2\rho + 1}{4\pi} \frac{(\rho - \mu)!}{(\rho + \mu)!}} P_\rho^\mu(\cos(\phi))(\cos(\mu\theta)).$$

Substituting (5.5) and letting $S = s^2 - 1$ into the (5.11) we have

$$Y_\rho^\mu(\theta, \phi) = \sqrt{\frac{2\rho + 1}{4\pi} \frac{(\rho - \mu)!}{(\rho + \mu)!}} \left(\frac{1}{2^\rho \rho!} (1 - s^2)^{\mu/2} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}} (s^2 - 1)^\rho \right) \cos(\mu\theta)$$

where $s = \cos(\phi)$. Since $\rho - \mu$ is an even integer, $\rho + \mu$ is also an even integer, since $\rho - \mu = (\rho - \mu) + 2\mu$. Let $\rho + \mu = 2k$, where k is an integer. We can use the fact that cosine is an even function to show that this type of spherical harmonic reflects across the equator. Looking at the differential term $\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(s^2 - 1)^\rho$, we can see that

$$\begin{aligned}
\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(s^2 - 1)^\rho &= \frac{d^{2k}}{ds^{2k}} s^{2\rho} && \text{the degree of this polynomial is } 2\rho \\
&= c_1 s^{2k} && \text{substituting } s = \cos(\phi) \\
&= c_1 \cos^{2k}(\phi).
\end{aligned}$$

Because our cosine is a function of ϕ, our co-altitude angle, this term is shown to be symmetric over the equator. The only other term in the spherical harmonic where cosine is a function of ϕ is $(1 - s^2)^{\mu/2}$, since $s = \cos(\phi)$. The degree of this polynomial is μ. Substituting $\cos(\phi) = s$, we have this term represented as $\cos^\mu(\phi)$. Whether μ is a even or odd integer, $\cos^\mu(\phi)$ is still an even function, making the term have symmetry to the equator. Thus, if $\rho - \mu$ is even, the spherical harmonics have symmetry about the equator. $\square$

Chapter 6: Explanation of Matlab Codes

In this chapter we will be following our *Matlab* codes (listed in Appendix C) line by line to understand what is being computed and why we need to have specific commands. The initial shape α is complied in the *Matlab* code *alpha.m*. We have the examples from the first four Spherical Harmonics listed in our code (lines 4-7).

The *Matlab* code *Vballnova.m* simulates the solution of the wave equation on a volleyball. Line 3 removes all variables from the workspace and deletes them from the memory so that we do not use other codes' values in our compiling. Lines 7 through 12 provide constants that will be used throughout the code. The material, density, and type of bladder of the volleyball will be excluded from this code. We next define the sizes of the vectors that will be formulated (lines 15-20). The name *alpha* represents the initial shape and A will be the matrix holding the values of our Legendre polynomials. The coefficient of the solution of our wave equation is *integrandE* and v is the solution of our wave equation. The azimuthal angle is represented by *theta*, measured from 0 to 2π, while *phi* is the co-altitude, measured from 0 to π. Lines 21 and 22 define the mesh between our azimuth and altitude values, respectively. We need these two lines because we are using a double summation instead of integration to find our Fourier coefficients.

The loops of lines 26-30 create our initial shape that we wish to apply on the volleyball. They call the m-file '*alpha*', which houses our initial shape function [as stated earlier]. We will be using k and l as our indices for the loops that consider theta and phi, respectively, throughout the code. k is the index for theta and l is the index for phi. The maximum value of our initial shape is represented by M (line 32). We use this value to scale our initial shape on the volleyball for a more realistic visualization.

The loops started on lines 35 and 36 are values of the degrees, *rho*, and orders, *mu*, of our Spherical Harmonics. We limit the range of *rho* in the code so that

Matlab can finish compiling. From line 38, j will represent the row placement in the formation of *integrandE*. It is also the value of how many orders (mu) we are using in the loop. The if/else statement of lines 41-45 computes the Legendre polynomials for both positive (line 42) and negative orders (line 44). The *Matlab* command *legendre* command outputs a non-normalized associated Legendre functions of degree *rho* and order *mu*, evaluated for each element of cos(*phi*). Thus N from line 39 is computed and then multiplied to the Legendre functions so that they are fully normalized. *PP* is a (*rho*+1)-by-m matrix, where m is the length of *cos(phi)*. The first dimension is *rho*+1 because that is how many rhos we are considering (we start at *rho*=0). Thus, the array *PP* has one more dimension than *cos(phi)*.

From line 47 to 51, the loops form the associated Legendre nModes-by-nModes matrices. These loops first consider what degree (rho) the associated Legendre polynomial we want by looking at the absolute value of the order (mu). We defined this earlier in line 37 as p. This is because the absolute value of the order ranges from 0 to *rho*. After picking out which row we want from the first dimension (the *rho*+1), we then pick out the elements from the nModes-by-nModes matrix. Because the Legendre polynomials are evaluated with only *phi*'s, we use the l index to pull out our entries to form the nModes-by-nModes matrix A.

The next pair of loops (lines 53-58) compute the cosine and sine terms for the Fourier coefficient. Lines 60-64 compile *integrandE*, which represents the Fourier coefficient (the double integration from (5.9)) from our solution to the wave equation which is a *rho*-by-(2**rho*+1) matrix. The size of the coefficient is the amounts of *rho's* by *mu's*.

We next computing the solution to the wave equation. For this, we implement time (line 69) into our formulas. Lines 70-80 are establishing our double loop of degree and order, and also formulating the Legendre polynomials exactly in the same method used in previous lines (lines 35-45).

Line 82 represents the second part of the normalization factor, which is needed for normalizing spherical harmonics. Line 84 defines the size of *cos(mu·theta)* and line

85 redefines our Legendre matrix to erase the values from the previous loops. The formulating of the Legendre matrix (lines 56-90) and the cosine term (lines 92-96) are the same as in the previous loops (lines 47-51 and 53-58).

The double loop of lines 98-102 represents the double summation that forms the solution to the wave equation v (5.10). The values of each term [except integrandE] is formed by the values of *mu* and *rho* and is dependent on t (our time). Line 106 is the value used to resize our v to fit the scale of a volleyball. Lines 108-110 define the sizes of the Cartesian coordinates which we will be using to plot. Line 112 takes the domain specified by the vectors *theta* and *phi* and transforms them into arrays *th* and *ph*, respectively. The rows of *th* are copies of the vector *theta* and the columns of *ph* are copies of the vector *phi*. We need this because the *Matlab mesh* need its inputs to be matrices.

The loop of lines 113-115 creates the Cartesian coordinates that formulates our volleyball with v added onto the radius of the sphere. As the time interval progresses, we will see the wave move along the volleyball. The *Matlab* command *mesh* creates a wire-frame surface *x1, y1,* and *z1*. Line 120 allows the time to be shown at the top of our plot window. 2 means two digits before the left of the decimal, 3 is how many numbers there are to the right of the decimal. The command *pause* breaks the execution for a specified number of seconds before continuing. In this case, it pauses for 0.7 seconds. [5]

Chapter 7: Further Research

Chapters 4 and 5 have led us to the equations we are using to observe the wave propagation on a volleyball after the initial hit for a float serve. Using *Matlab*, we have developed a code for solving the wave equation (5.10) which utilizes spherical harmonics (5.11). The volleyball has a diameter of approximately 8 inches and we will be assuming that the width of the hand that causes the impact is roughly 4.13 inches. The code uses the initial shape from the *MATLAB* code *alpha.m*. This initial shape is modeling the shape the volleyball takes at time zero seconds, immediately after the hand strikes the ball. Also, the initial velocity is zero inches per second, which declares that each point on the surface of the volleyball is initially stationary.

The current *MATLAB* code *VBallnova.m*, as listed in Appendix C, is incomplete. For further research, the goal is to improve the *MATLAB* code so that it fully models the float serve with the mathematics that has been developed in this paper. Once that code is functioning, the cases of different initial velocities and a volleyball with a spin can be researched.

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Appendix A: Orthogonality of associated Legendre Polynomial with respect to μ

This orthogonality of the associated Legendre Polynomial with respect to its degree ρ has been shown in Chapter 5.1. The proof of its orthogonality with respect to its order μ is shown here. To see this we will start with our initial position $\alpha(\theta, \phi)$

$$\alpha(\theta,\phi) = \sum_{\lambda=0}^{\infty}\sum_{\rho=0}^{\infty}\sum_{\mu=-\rho}^{\rho} A_{\lambda,\rho,\mu}\left(P_\rho^\mu(\cos(\phi))e^{i\mu\theta}\right).$$

Multiplying both sides by $P_\rho^{\mu'}(\cos(\phi))$, we have

$$\alpha(\theta,\phi)P_\rho^{\mu'}(\cos\phi) = \sum_{\lambda=0}^{\infty}\sum_{\rho=0}^{\infty}\sum_{\mu=-\rho}^{\rho} A_{\lambda,\rho,\mu}\left(P_\rho^\mu(\cos(\phi))P_\rho^{\mu'}(\cos(\phi))e^{i\mu\theta}\right).$$

Solving for $A_{\lambda,\rho,\mu}$, we integrate with respect to s by letting $s = \cos(\phi)$ and $ds = -\sin(\phi)d\phi$.

$$\int_{-1}^{1}\frac{1}{1-s^2}\alpha(\theta,s)P_\rho^{\mu'}(s)ds = \sum_{\lambda=0}^{\infty}\sum_{\rho=0}^{\infty}\sum_{\mu=-\rho}^{\rho} A_{\lambda,\rho,\mu}e^{i\mu\theta}\int_{-1}^{1}\frac{1}{s^2-1}P_\rho^\mu(s)P_\rho^{\mu'}(s)ds$$

Focusing on the integrand of our Legendre Polynomials, we let $S = s^2 - 1$ and use (5.5) to obtain

$$\begin{aligned}\int_{-1}^{1}\frac{1}{S}P_\rho^\mu(s)P_\rho^{\mu'}(s)ds &= \frac{(-1)^{\mu+\mu'}}{2^{2\rho}(\rho!)^2}\int_{-1}^{1}\frac{1}{S}S^{\frac{\mu+\mu'}{2}}\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)\frac{d^{\rho+\mu'}}{ds^{\rho+\mu'}}(S^\rho)\ ds \\ &= \frac{(-1)^{\mu+\mu'}}{2^{2\rho}(\rho!)^2}\int_{-1}^{1}S^{\frac{\mu+\mu'-2}{2}}\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^\rho)\frac{d^{\rho+\mu'}}{ds^{\rho+\mu'}}(S^\rho)\ ds.\end{aligned}$$

The functions S^c are even functions of s, so their j^{th} derivatives are either even or odd functions depending if j is even or odd. But, no matter if μ and μ' are both even or odd, or one is even and the other is odd, the integrand will be an odd function. We can integrate by parts $\rho+\mu'$ times, where the formula is $\int_{-1}^{1} udv = uv|_{-1}^{1} - \int_{-1}^{1} vdu$. Since μ and μ' occur symmetrically, we can assume without loss of generality that $\mu' \leq \mu$. Letting

$$u = S^{\frac{\mu+\mu'-2}{2}} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho})$$

and

$$dv = \frac{d^{\rho+\mu'}}{ds^{\rho+\mu'}}(S^{\rho})\ ds$$

the uv term vanishes for the first $\mu' - 1$ integration by parts because it contains a factor of $1 - s^2$. For the μ'^{th} integration by parts, this term does not vanish if $\mu = \mu'$, and the term becomes

$$uv|_{-1}^{1} = \frac{d^{\mu-1}}{ds^{\mu-1}} \left(S^{\mu-1} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) \left(\frac{d^{\rho}}{ds^{\rho}}(S^{\rho}) \right) \delta_{\mu\mu'} \Bigg|_{-1}^{1}.$$

For the remaining ρ integrations, the uv term vanishes because there is a factor of $(1 - s^2)$ again. After $\rho + \mu$ integration by parts, we have

$$\frac{(-1)^{\mu+\mu'}}{2^{2\rho}(\rho!)^2}\, uv|_{-1}^{1} = \frac{(-1)^{\mu+\mu'}}{2^{2\rho}(\rho!)^2} \left[\frac{d^{\mu-1}}{ds^{\mu-1}} \left(S^{\mu-1} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) \left(\frac{d^{\rho}}{ds^{\rho}}(S^{\rho}) \right) \delta_{\mu\mu'} \Bigg|_{-1}^{1} \right.$$
$$\left. + \int_{-1}^{1} S^{\rho} \frac{d^{\rho+\mu'}}{ds^{\rho+\mu'}} \left(S^{\frac{\mu+\mu'-2}{2}} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho}) \right) ds \right].$$

Similar to the proof of Proposition 5.1, the highest power of s in S^{ρ} is 2ρ. After $\rho+\mu$ derivatives are taken, the highest power of becomes $\rho - \mu$. The highest power of the s in the factor $S^{\frac{\mu-\mu'-2}{2}}$ is $\mu - \mu' - 2$. Thus the highest power in

$$S^{\frac{\mu+\mu'-2}{2}} \frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho})$$

is $\rho + \mu' - 2$. This tells us that after taking $\rho + \mu' - 2$ derivatives, the highest power is 0, and this expression becomes a constant. Therefore taking 2 more derivatives of it, which gives us the $\rho+\mu$ derivatives we require, causes the term to vanish. So after $\rho+\mu$ integration by parts, we have

$$\frac{(-1)^{2\mu}}{2^{2\rho}(\rho!)^2}\,\frac{d^{\mu-1}}{ds^{\mu-1}}\left(S^{\mu-1}\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho})\right)\left(\frac{d^{\rho}}{ds^{\rho}}(S^{\rho})\right)\Bigg|_{-1}^{1}.$$

We can differentiate our $S's$ by substituting s^2-1 back into our $S's$. Using Leibniz's Rule, we let $(s^2-1)^c=(s-1)^c(s+1)^c$, where c is a positive integer. Since this function is odd, we can evaluate at the upper limit and then double the answer. We will looking at these derivative terms (starting from the right):

$$\frac{d^{\rho}}{ds^{\rho}}(s^2-1)^{\rho}=\frac{d^{\rho}}{ds^{\rho}}\left((s-1)^{\rho}(s+1)^{\rho})\right)=\rho!(s+1)^{\rho}$$

$$\begin{aligned}\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}((s^2-1)^{\rho}) &= \frac{(\rho+\mu)!}{\rho!\mu!}\left(\frac{d^{\rho}}{ds^{\rho}}(s-1)^{\rho}\right)\left(\frac{d^{\mu}}{ds^{\mu}}(s+1)^{\rho}\right)\\ &= \frac{\rho!(\rho+\mu)!}{\mu!(\rho-\mu)!}(s+1)^{\rho-\mu}\\ \frac{d^{\mu-1}}{ds^{\mu-1}}\left(S^{\mu-1}\frac{d^{\rho+\mu}}{ds^{\rho+\mu}}(S^{\rho})\right) &= \frac{\rho!(\rho+\mu)!}{\mu!(\rho-\mu)!}\frac{d^{\mu-1}}{ds^{\mu-1}}\left((s^2-1)^{\mu-1}(s+1)^{\rho-\mu}\right)\\ &= \frac{\rho!(\rho+\mu)!}{\mu!(\rho-\mu)!}(\mu-1)!(s+1)^{\rho-1}.\end{aligned}$$

When we evaluate these terms at $s=1$ and then multiply by 2 to include the lower limit, $s=-1$, we have

$$\begin{aligned}&\frac{2(-1)^{2\mu}}{2^{2\rho}(\rho!)^2}\left(2^{\rho-1}\frac{\rho!(\rho+\mu)!(\mu-1)!}{\mu!(\rho-\mu)!}\right)(2^{\rho}\rho!)\\ =&\frac{(\rho+\mu)!}{\mu(\rho-\mu)!}\end{aligned}$$

Thus the normalization of the associated Legendre polynomials with respect to the L^2-inner product and μ, where $s=\cos(\phi)$ on the interval $-1\leq s\leq 1$, is

$$\int_{-1}^{1}P_{\rho}^{\mu}(s)P_{\rho}^{\mu'}(s)ds=\frac{(\rho+\mu)!}{\mu(\rho-\mu)!}\delta_{\mu\mu'}.$$

Appendix B: Derivation of associated Legendre polynomial with negative order

The formula for the associated Legendre polynomials that we have used throughout this paper has an order μ that has a range of $-\mu \leq \rho \leq \mu$. This formula can be seen in (4.16).

$$P_\rho^\mu(s) = (1-s^2)^{\mu/2} \frac{d^\mu}{ds^\mu} P_\rho(s)$$

To show how the associated Legendre functions of positive and negative order relate, we shall combine (4.16) and (4.12).

$$P_\rho^\mu(s) = \frac{1}{2^\rho \rho!}(1-s^2)^{\mu/2} \frac{d^{\mu+\rho}}{ds^{\mu+\rho}}(s^2-1)^\rho$$

We can see that the associated Legendre formula is invariant for μ and $-\mu$. So the equations for the positive and negative order of the associated Legendre polynomials are proportional. To calculate this proportionality let us set up relationship

$$\frac{1}{2^\rho \rho!}(1-s^2)^{-\mu/2} \frac{d^{\rho-\mu}}{ds^{\rho-\mu}}(s^2-1)^\rho = c_{\rho\mu} \, \frac{1}{2^\rho \rho!}(1-s^2)^{\mu/2} \frac{d^{\mu+\rho}}{ds^{\mu+\rho}}(s^2-1)^\rho \qquad 0 \leq \mu \leq \rho$$

where $c_{\rho\mu}$ is the proportionality constant. Cancelling out like terms and moving $(1-s^2)^{\mu/2}$ to the other side we have

$$(1-s^2)^{-\mu} \frac{d^{\rho-\mu}}{ds^{\rho-\mu}}(s^2-1)^\rho = c_{\rho\mu} \, \frac{d^{\mu+\rho}}{ds^{\mu+\rho}}(s^2-1)^\rho \qquad 0 \leq \mu \leq \rho.$$

We calculate the coefficients of the highest power of s on both sides. This leads to the proportionality constant to be

$$c_{\rho\mu} = (-1)^\mu \frac{(\rho-\mu)!}{(\rho+\mu)!} \qquad 0 \leq \mu \leq \rho.$$

Therefore, the associated Legendre polynomials of positive, given by (4.16), and negative order μ are related by

$$P_{\rho}^{-|\mu|}(s) = (-1)^{\mu} \frac{(\rho - |\mu|)!}{(\rho + |\mu|)!} P_{\rho}^{|\mu|}(s).$$

Appendix C: Matlab Codes

The code below used to observe the wave propagations on a volleyball given specific initial conditions. The initial conditions are provided in the code **alpha.m** and the waves are produced in **Vballnova.m**.

```
%alpha.m  Code to input Initial Position
function [output] = alpha(th,ph)

output = 1/2*(1/sqrt(pi))+0*th+0*ph;    %Y_0^0
%output = 1/2*sqrt(3/pi)*cos(th)+0*ph;   %Y_1^0
%output = -.5*sqrt(3/(2*pi)).*sin(th).*exp(ph); %Y_1^1
%output = 1/4*sqrt(5/pi)*(3*cos(th).^2-1);   %Y_2^0
end
```

```
%Vballnova.m  Code to simulate the solution of the wave
    equation on a volleyball

clear all;
figure;

%Information
rad = 4.13;                 %The radius of our sphere in inches
time = 50;                  %How long you want the propagation to
    last
c = 1;                      %Speed of the wave
nModes = 20;                %Number of Fourier modes
nRho = 5;                   %Number of Spherical Harmonics
dt = 0.5;                   %Change in time step

%Initializations
alpha = zeros(nModes,nModes);          %Our initial position
A=zeros(nModes,nModes);                %Represents the matrix
    of the Legendre polynomials
integrandE=zeros(nRho+1,2*nRho+1);     %This will represent
    our Fourier coefficients for the solution of our wave
    equation
```

```
v=zeros(nModes,nModes);                    %The solution to our
    wave equation
theta = linspace(0,2*pi,nModes);           %Azimuth
phi = linspace(0,pi,nModes);               %Altitude
dth=2*pi/(nModes-1);                       %Distance between
    values of the azimuth
dph=pi/(nModes-1);                         %Distance between
    values of the altitude


%Calling our initial shape
for k = 1:nModes
    for l = 1:nModes
     alpha(k,l)=feval('alpha',theta(k),phi(l));
    end
end

M = max(max(abs(alpha)));

%Calculating our Fourier Coefficient
for rho=0:nRho
    for mu=-rho:rho
        p=abs(mu);
        j=mu+rho+1;
        N = (-1)^mu*sqrt((rho+1/2)*factorial(rho-mu)/
            factorial(rho+mu));

        if mu>=0
            PP=N*legendre(rho,cos(phi));
        else
            PP=N*((-1)^p)*(factorial(rho-p)/factorial(rho+p))
                *legendre(rho,cos(phi));
        end

        for k = 1:nModes
            for l = 1:nModes
                A(k,l)=PP(p+1,l);
            end
        end

        for k = 1:nModes
            for l = 1:nModes
```

```
                    cosmt(k,l)=cos(mu*theta(k));
                    sinph(k,l)=sin(phi(l));
                end
            end

            for k=1:nModes
                for l=1:nModes
                    integrandE(rho+1,j)=integrandE(rho+1,j)+dth*dph*alpha(k,l)*A(k,l)*cosmt(k,l)*sinph(k,l);
                end
            end
        end
end

%Computing the Solution to the Wave Equation
for t=0:.05:time
    for rho=0:nRho
        for mu=-rho:rho
            p=abs(mu);
            j=mu+rho+1;
            N = (-1)^mu*sqrt((rho+1/2)*factorial(rho-mu)/factorial(rho+mu));

            if mu>=0
                PP=N*legendre(rho,cos(phi));
            else
                PP=N*(-1)^p*(factorial(rho-p)/factorial(rho+p))*legendre(rho,cos(phi));
            end

            N1 = sqrt(1/(2*pi));

            cosmt=zeros(nModes,nModes);
            A=zeros(nModes,nModes);
            for k = 1:nModes
                for l = 1:nModes
                    A(k,l)=A(k,l)+PP(p+1,l);
                end
            end

            for k=1:nModes
```

```
                for l=1:nModes
                    cosmt(k,l)=cos(mu*theta(k));
                end
            end

            for k=1:nModes
                for l=1:nModes
                    v(k,l)=v(k,l)+N1*integrandE(rho+1,j).*A(k,
                       l).*cos(c*t*(sqrt(1+rho))).*(cosmt(k,l)
                       );
                end
            end
        end
    end

    abbr=0.1*M*v;

    x1=zeros(nModes,nModes);
    y1=zeros(nModes,nModes);
    z1=zeros(nModes,nModes);

    [th ph]=meshgrid(theta,phi);
    for j=1:nModes
        x1(j,:) =  (rad+abbr(j)).*sin(ph(j,:)).*cos(th(j,:));
        y1(j,:) = (rad+abbr(j)).*sin(ph(j,:)).*sin(th(j,:));
        z1(j,:) = (rad+abbr(j)).*cos(ph(j,:));
    end

   mesh(x1,y1,z1);
   title(sprintf('t = %2.3f',t));
   pause(0.7);
end
```

Vita

Katherine Novacek was born in Raleigh, North Carolina on May 9, 1987. She graduated from Lenoir-Rhyne University in May 2009 with a B.S. in Theoretical Mathematics and a B.A. in Classics. During her four years of undergraduate studies, she played on the varsity volleyball team. She will graduate from Wake Forest University in December 2012 with a M.A. in Mathematics. Her hobbies include volleyball, soccer, exercising, learning languages, and volunteering.