THE EFFECTIVE SATO-TATE CONJECTURE AND DENSITIES PERTAINING TO LEHMER-TYPE QUESTIONS

BY

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Abstract

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Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\mathrm{new}}(\Gamma_0(N))$ be a normalized Hecke eigenform with squarefree level N. For a prime p, define $\theta_p \in [0, \pi]$ to be the angle for which $a(p) = 2p^{(k-1)/2}\cos(\theta_p)$. Let $I \subset [0, \pi]$ be a closed subinterval, and let $\mu_{ST}(I) = \int_I \frac{2}{\pi}\sin^2(\theta)\,d\theta$ be the Sato-Tate measure of I. Assuming that the symmetric power L-functions of f are automorphic and satisfy the Generalized Riemann Hypothesis, we prove that

$$|\#\{p \in [x, 2x] : \theta_p \in I\} - (\pi(2x) - \pi(x))\mu_{ST}(I)| = O\left(\frac{x^{3/4}\log(Nkx)}{\log(x)}\right),$$

where the implied constant is $\frac{5}{2}$. This bound decreases by a factor of $\sqrt{\log(x)}$ if we let $I = [\frac{\pi}{2} - \frac{1}{2}\delta, \frac{\pi}{2} + \frac{1}{2}\delta]$, where δ is small. We can use this to compute a lower bound for the density of $n \in \mathbb{N}$ for which $a(n) \neq 0$. In particular, we prove that if τ is the Ramanujan tau function, then

$$\lim_{x\to\infty} \frac{\#\{n \in [1, x] : \tau(n) \neq 0\}}{x} > 0.99999.$$

Chapter 1: Introduction

Let $\pi(x) = \#\{p \in [1,x] : p \text{ is prime}\}$. By the modified proof of the Prime Number Theorem due to de la Vallée-Poussin in 1899, there exists a constant $c > 0$ such that

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log(x)}}\right), \qquad \mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}.$$

This result follows from detailed analysis of the nontrivial zeroes of the Riemann zeta function $\zeta(s)$. In particular, Hadamard and de la Vallée-Poussin proved that if ρ is a nontrivial zero of $\zeta(s)$ in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$, then ρ is not on the boundary of the critical strip. Riemann conjectured that even more is true; he predicted that $\mathrm{Re}(\rho) = \frac{1}{2}$ for all nontrivial zeroes ρ. Much evidence favors this hypothesis. For example, Platt's computation (see [20]) shows this to be true for the first 103,800,788,359 nontrivial zeros of $\zeta(s)$, and Bui, Conrey, and Young prove in [3] that over 41% of all nontrivial zeros in the critical strip lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. Assumption of the Riemann Hypothesis improves de la Vallée-Poussin's estimate to

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log(x)).$$

Though a proof of the Riemann Hypothesis has eluded mathematicians for the last 150 years, the supposition of its truth has far-reaching consequences; see [5] for further discussion.

We will use an approach similar to that of de la Vallée Poussin to count primes in a different setting. Consider the following concrete example, which follows the presentation in [15]. Let

$$Q(x_1, \ldots, x_{24}) = \sum_{k=1}^{24} x_k^2,$$

and let $r_Q(n)$ be the number of ways one can write a whole number n as a sum of 24 squares. The function Q is an example of a *quadratic form*, a homogeneous polynomial of degree 2. By definition, $Q(x_1, \ldots, x_{24})$ is a function of 24 variables. If we let 23 of these variables equal 0 and the 1 of these variables equal 1, we can obtain the number 1 as a sum of 24 squares in 24 ways. If we let 23 of these variables to equal 0 and the 1 of these variables equal -1, we can obtain the number 1 as a sum of 24 squares in 24 more ways. Thus $r_Q(1) = 48$. We also have $r_Q(2) = 1104$, $r_Q(3) = 16192$, $r_Q(5) = 1362336$, etc. If we look at the data for primes p out to $p = 97$, a reasonable approximation for $r_Q(p)$ is $\frac{16}{691}(p^{11} + 1)$. It turns out that for large primes p, if $\mathcal{E}(p)$ is the error in this approximation, we have $|\mathcal{E}(p)| < \frac{66304}{691}\sqrt{p^{11}}$. This error is square-root accurate and looks like the error term in the Prime Number Theorem (assuming the Riemann Hypothesis). We may now ask other questions about both $r_Q(p)$ and $\mathcal{E}(p)$: Is our function of p that approximates $r_Q(p)$ optimal in some sense? When does our approximation over-count? When does our approximation under-count?

To answer questions of this sort, we scale the error by looking at the quantity $\mathcal{E}_{scaled}(p) = \frac{691\mathcal{E}(p)}{66304\sqrt{p^{11}}}$. This quantity is between -1 and 1, and we can study the manner in which these values are distributed between -1 and 1. These values of $\mathcal{E}_{scaled}(p)$ are related to certain functions called *modular forms*. Modular forms are series of the form $\sum_{n=0}^{\infty} a(n)q^n$ that come in two chief varieties: *Eisenstein series* and *cusp forms*. Eisenstein series typically yield good approximations of arithmetic data. Cusp forms yield square-root accurate approximations of the error given by the Eisenstein series approximation, much like the error term in the prime number theorem.

It turns out that in our particular case, $r_Q(n)$ is the n-th coefficient of a modular form which can be approximated by an Eisenstein series with the accuracy described above. The error in the Eisenstein series approximation is related to the cusp form given by

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau_{12}(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

We now see that the error in our approximation depends on the behavior of the numbers $\tau(n)$, where $\tau(n)$ is the Ramanujan tau function. The function $\Delta(q)$ is a special type of cusp form called a *newform*. The properties of newforms that we care about the most in this scenario is that $\tau(n)$ is a multiplicative function, and if p is prime, there exists an angle $\theta_p \in [0,\pi]$ such that $2\sqrt{p^{11}}\cos(\theta_p) = \tau(p)$. The values of $\cos(\theta_p)$ are between -1 and 1, and equal to $\mathcal{E}_{scaled}(p)$. Now, if x is a large number, and we look at the primes p between x and $2x$, what is the probability that $\theta_p \in I = [\alpha,\beta] \subset [0,\pi]$?

The work of Sato and Tate in the 1960's led to the conjecture that this probability equals $\int_0^\pi d\mu_{ST}$, where $\mu_{ST}([a,b]) = \frac{2}{\pi}\int_a^b \sin^2(\theta)\ d\theta$ is the Sato-Tate measure. This conjecture was proven by Barnet-Lamb, Geraghty, Harris, and Taylor in [1]:

Theorem 1.1 (The Sato-Tate Conjecture)**.** *Let* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\mathrm{new}}(\Gamma_0(N),\chi)$ *be a non-CM newform. If* $F : [0,\pi] \to \mathbb{C}$ *is a continuous function, then*

$$\sum_{\substack{p\in[1,x]\\ p\nmid N}} F(\theta_p) \sim \pi(x)\int_0^\pi F(\theta)\ d\mu_{ST},$$

where $\mu_{ST}(I) = \int_I \frac{2}{\pi}\sin^2(\theta)\ d\theta$ *is the Sato-Tate measure. In other words, the sequence* $\{\theta_p\}$ *is equidistributed in* $[0,\pi]$ *with respect to the measure* μ_{ST}*.*

One might ask for the shape of the error term in the Sato-Tate Conjecture. In [16], V. K. Murty proved the following version of the Sato-Tate Conjecture for non-CM elliptic curves.

Theorem 1.2 (Murty)**.** *Let* $f(z) \in S_2^{\mathrm{new}}(\Gamma_0(N))$ *be the newform associated to a non-CM elliptic curve* $E/\mathbb{Q}$*, and* $I \subset [0,\pi]$ *be a closed subinterval. Assume that* *all symmetric power L-functions of f are automorphic and satisfy the Generalized Riemann Hypothesis. Then*

$$\sum_{\substack{p\in[1,x]\\ p\nmid N}} \chi_I(\theta_p) = \mu_{ST}(I)\pi(x) + O(x^{3/4}\sqrt{\log(Nx)}).$$

In [2], Bucur and Kedlaya extend Murty's results to arbitrary motives with applications toward elliptic curves. In particular, let $E_1/\mathbb{Q}$ and $E_2/\mathbb{Q}$ be two nonisogenous non-CM elliptic curves with respective conductors $N_1 \neq N_2$. Bucur and Kedlaya apply their results to finding an upper bound for the smallest prime $p \nmid N_1N_2$ at which $a_p(E_1)$ and $a_p(E_2)$ are nonzero and of opposite sign.

In this thesis, we prove a completely explicit version of the Sato-Tate Conjecture that applies to every newform on $\Gamma_0(N)$ with N squarefree; all such newforms are non-CM. Furthermore, we improve the error term in Murty's result by a factor of $\sqrt{\log(x)}$. Define

$$\pi_{f,I}(x) = \#\{p \in [x, 2x] : \theta_p \in I\} = \sum_{p\in[x,2x]} \chi_I(\theta_p). \tag{1.1}$$

Our main result is as follows.

Theorem 1.3. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\mathrm{new}}(\Gamma_0(N))$ be a newform with squarefree level N, and let $I = [\alpha, \beta] \subset [0, \pi]$. Suppose that the symmetric power L-functions of f are automorphic and satisfy the Generalized Riemann Hypothesis. If $x \geq 10^{17}$ is an integer larger than any prime dividing N and x is not a prime power, then*

$$\begin{aligned} |\pi_{f,I}(x) - \mu_{ST}(I)(\pi(2x) - \pi(x))| \quad \leq \quad & \frac{5x^{3/4}}{2} - \frac{x^{3/4}\log\log(x)}{2\log(x)} + \frac{\log(N(k-1))x^{3/4}}{\log(x)} \\ & - \frac{2(7 + 10(\beta-\alpha))\log(\beta-\alpha)}{25(\beta-\alpha)}\sqrt{x}. \end{aligned}$$

Remark 1.4. *For very large values of x, the error arising from the values of N and k cancel out with higher order terms. For example, this occurs for $\Delta(z)$ when*

$x \geq 3.6 \times 10^{52}$. *Furthermore, the error depends on the length of I but not on where I lies within the interval* $[0, \pi]$.

The case when I is a short interval centered at a fixed angle $\varphi \in [0, \pi]$ is of special interest. If φ equals 0 or π, the estimate of $\pi_{f,I}(x)$ in Theorem 1.3 yields the distribution of primes p for which $|a(p)|$ is as large as possible. If $\varphi = \frac{\pi}{2}$, we can study the distribution of primes p for which $a(p) = 0$. This is particularly interesting when $f(z) \in S_{12}^{\mathrm{new}}(\Gamma_0(1))$, in which case $f(z)$ equals $\Delta(z) = \sum_{n=1}^{\infty} \tau_{12}(n) q^n$, where $\tau_{12}(n)$ is the Ramanujan tau function. In [14], D. H. Lehmer pondered whether $\tau(n) \neq 0$ for all $n \geq 1$; this question remains open.

Conjecture 1.5. *Let* $f(z) = \sum_{n=1}^{\infty} \tau_k(n) q^n \in S_k^{\mathrm{new}}(\Gamma_0(1))$ *be a newform, where* $k = 12, 16, 18, 20, 22,$ *or* 26. *For all such k and all $n \geq 1$,* $\tau_k(n) \neq 0$.

The case where f is the newform associated to a semistable elliptic curve $E/\mathbb{Q}$ with conductor N is well-explored. By the modularity theorem, the primes $p \geq 5$ such that $p \nmid N$ and $a(p) = 0$ are the primes for which E has supersingular reduction. In [7], Noam Elkies proves that infinitely many such primes exist (in contrast to what is conjectured for $\Delta(z)$), and in [8], Elkies unconditionally proves that the number of such primes less than or equal to x is $O(x^{3/4})$. In [13], Lang and Trotter conjectured that

$$\{p \in [1, x] : a(p) = c\} = (K_c + o(1)) \frac{\sqrt{x}}{\log(x)},$$

where K_c is a specific nonnegative constant.

In [26], Serre proves an unconditional result that addresses both newforms on $\Gamma_0(1)$ and newforms associated to semistable elliptic curves. It follows from this result that we can relate the density of integers n for which $a(n) \neq 0$ to the primes p for which $a(p) = 0$.

Theorem 1.6 (Serre). *If $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\text{new}}(\Gamma_0(N))$ is a newform with N squarefree, then*

$$\#\{p \in [1,x] : a(p) = 0\} = O\left(\frac{x \log\log(x)\sqrt{\log\log\log(x)}}{\log(x)^{3/2}}\right).$$

Furthermore,

$$\lim_{x\to\infty} \frac{\#\{n \leq x : a(n) \neq 0\}}{x} = \alpha_f \prod_{a(p)=0}\left(1 - \frac{1}{p+1}\right),$$

where $\alpha_f \in (0,1]$ is a constant which is given in the proof of Theorem 16 of [26].

When $f \in S_k^{\text{new}}(\Gamma_0(1))$, Serre proved that $\alpha_f = 1$. When $f \in S_2^{\text{new}}(\Gamma_0(11))$, Serre proved that $\alpha_f = \frac{14}{15}$, and he computed the upper bound

$$\lim_{x\to\infty} \frac{\#\{n \leq x : a(n) \neq 0\}}{x} < 0.847.$$

Additionally, he conjectured a lower bound of 0.845.

Our second result is a completely explicit upper bound on the function $\pi_f(x) = \#\{p \in [x, 2x] : a(p) = 0\}$ for any newform satisfying the hypotheses of Theorem 1.3. In the case of a newform associated to a non-CM elliptic curve, our result conditionally improves Elkies' estimate by a factor of $\sqrt{\log(x)}$.

Theorem 1.7. *Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform with squarefree level N, and assume that the symmetric power L-functions of f are automorphic and satisfy the Generalized Riemann Hypothesis. If $x \geq 10^{11}$ is an integer larger than any prime dividing N and x is not a prime power, then*

$$\pi_f(x) \leq 7.392\frac{x^{3/4}}{\sqrt{\log(x)}} - 7.391\frac{x^{3/4}\log\log(x)}{\log(x)^{3/2}} + (15.296 + 14.784\log(N(k-1)))\frac{x^{3/4}}{\log(x)^{3/2}}.$$

Under the same conditions as Theorem 1.7, Rouse proved in [23] that if $k \geq 4$ and $0 \leq \alpha \leq \frac{1}{8}$, then

$$\#\{p \in [1,x] : |a(p)| \leq p^{\frac{k-1}{2}-\alpha}\} \asymp \frac{x^{1-\alpha}}{\log(x)}.$$

The proof of Theorem 1.7 makes the work in [23] completely explicit.

Using Theorems 1.6 and 1.7, we establish a method of computing an explicit lower bound for the density of positive integers n for which $a(n)$ is nonzero. In the cases where $f \in S_k^{\text{new}}(\Gamma_0(1))$ and $f \in S_2^{\text{new}}(\Gamma_0(11))$, we use this method to prove the following result.

Theorem 1.8. *Let $f_k(z) = \sum_{n=1}^{\infty} \tau_k(n) q^n \in S_k^{\text{new}}(\Gamma_0(1))$ be a newform, where $k =$ 12, 16, 18, 20, 22, or 26. Suppose that the symmetric power L-functions of f are automorphic and satisfy the Generalized Riemann Hypothesis. If*

$$D_k = \lim_{x \to \infty} \frac{\#\{n \in [1,x] : \tau_k(n) \neq 0\}}{x},$$

then D_k is strictly bounded below as described in the following table.

k	Strict Lower Bound for D_k
12	0.9999912
16	0.9999911
18	0.9999951
20	0.9999973
22	0.9999985
26	0.9999909

Additionally, if $g(z) = \sum_{n=1}^{\infty} a(n) q^n \in S_2^{\text{new}}(\Gamma_0(11))$ is the newform associated to the isogeny class of elliptic curves of conductor 11, then

$$0.8353846 < \lim_{x \to \infty} \frac{\#\{n \in [1,x] : a(n) \neq 0\}}{x} < 0.8465248.$$

The proof of Theorem 1.8 relies on the fact if $\tau_k(n) = 0$, then $\tau_k(n)$ must satisfy specific congruences modulo powers of small primes (see [22]). The congruences arise from the Galois representations of f_k modulo these primes. The congruences help us raise the lower bounds on the densities in Theorem 1.8.

In Chapter 2, we provide the requisite background on modular forms and their applications, analytic number theory, and symmetric power L-functions. In Chapter 3, we prove a series of lemmas needed to prove Theorem 1.3; a proof of Theorem 1.3 follows. A sketch of the proofs of Theorems 1.7 and 1.8 are in Chapter 4.

Chapter 2: Background

2.1 Modular Forms on $\Gamma_0(N)$

Throughout this thesis, $\mathbb{N}$ will represent the set of positive integers.

In this section, we will give a brief overview of several topics from the theory of modular forms necessary to understand the Sato-Tate Conjecture. For more on modular forms, see [10], [11], [12], [18], [25], and the references therein.

2.1.1 Newforms

The theory of modular forms deals exclusively with meromorphic functions on $\mathbb{H}$, the upper half of the complex plane.

Definition 2.1. *Let $N \in \mathbb{N}$. We define $SL_2(\mathbb{Z}) = \{\gamma \in GL_2(\mathbb{Z}) : \det\gamma = 1\}$. We also define*

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

We note that $\Gamma_0(1) = SL_2(\mathbb{Z})$, and $\Gamma_0(N)$ is a proper subgroup of $SL_2(\mathbb{Z})$ for $N > 1$.

Definition 2.2. *Let $N \in \mathbb{N}$, and let $f : \mathbb{H} \to \mathbb{C}$. The weight k operator $|[\gamma]_k$ is defined by*

$$f|[\gamma]_k(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. We say that f is weakly modular on $\Gamma_0(N)$ with weight k if $f|[\gamma]_k(z) = f(z)$ for all $\gamma \in \Gamma_0(N)$.

Definition 2.3. *Let $k, N \in \mathbb{Z}$, where $k \geq 0$ and $N > 0$. We say that $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k and level N if:*

1. *f is holomorphic on $\mathbb{H}$.*
2. *f is weakly modular for $\Gamma_0(N)$.*
3. *$f|[\alpha]_k$ vanishes at ∞ for all $\alpha \in \Gamma_0(1)$.*

Let $f(z)$ be a modular form of weight k and level N. Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \Gamma_0(N)$ for all $N \in \mathbb{N}$, we have that $f(z+1) = f(z)$. Therefore, $f(z)$ has the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n, \quad q = e^{2\pi i z}.$$

We call $f(z)$ a *cusp form* on $\Gamma_0(N)$ if $f(z)$ vanishes at each cusp of $\Gamma_0(N)$. We denote the space of modular forms with weight k by $M_k(\Gamma_0(N))$ and the cuspidal subspace by $S_k(\Gamma_0(N))$.

The next several definitions are necessary for defining what it means for a modular form $f(z)$ to be a newform.

Definition 2.4. *Let $N \in \mathbb{N}$, and let $d \mid N$. We define the inclusion i_d in the following way:*

$$i_d : S_k(\Gamma_0(N/d)) \times S_k(\Gamma_0(N/d)) \to S_k(\Gamma_0(N))$$
$$(f(z), g(z)) \longmapsto f(z) + g(dz).$$

Definition 2.5. *We define the old subspace $S_k(\Gamma_0(N))$ to be*

$$S_k^{old}(\Gamma_0(N)) = \sum_{p|N} i_p(S_k(\Gamma_0(N/p)) \times S_k(\Gamma_0(N/p)).$$

Definition 2.6. *Let I_2 be the identity matrix on $GL_2(\mathbb{Z})$, and let $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I_2\}$. Let $k, N \in \mathbb{Z}$ with $k \geq 0$, $N > 0$. Let $f, g \in M_k(\Gamma_0(N))$, and let $\overline{g}$ be the complex conjugate of g. We define the Petersson inner product to be*

$$\langle f, g \rangle = \frac{1}{[PSL_2(\mathbb{Z}) : PSL_2(\mathbb{Z}) \cap \Gamma_0(N)]} \int_{\mathbb{H}/\Gamma_0(N)} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

Definition 2.7. *We define the new subspace of* $S_k(\Gamma_0(N))$ *to be*

$$S_k^{new}(\Gamma_0(N)) = (S_k^{\text{old}}(\Gamma_0(N)))^{\perp},$$

which is the orthogonal complement of $S_k^{\text{old}}(\Gamma_0(N))$ *with respect to the Petersson inner product on* $S_k(\Gamma_0(N))$.

Definition 2.8. *Let* $n, N \in \mathbb{N}$ *with* $\gcd(n, N) = 1$. *Then*

$$M_{n,N} = \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z}) : \det \gamma = n, c \equiv 0 \pmod{N} \right\} / \{\pm I_2\}.$$

The group $\Gamma_0(N)$ *acts on* $M_{n,N}$ *by left multiplication. This action gives rise to an equivalence relation on* $M_{n,N}$. *If* $x, y \in M_{n,N}$, *we say that* $x \sim y$ *if there exists* $g \in \Gamma_0(N)$ *such that* $gx = y$. *We define the Hecke operator* T_n *to be*

$$f(z)|T_n = n^{k-1} \sum_{\gamma \in M_{n,N}/\sim} f|[\gamma]_k.$$

Definition 2.9. *Let* f *be a modular form. If for every* $n \in \mathbb{N}$ *that is relatively prime to* N *there exists* $\lambda_n \in \mathbb{C}$ *such that* $T_n f = \lambda_n f$, *then we say that* f *is a Hecke eigenform. Moreover, if* $f(z) \in S_k^{new}(\Gamma_0(N))$ *is a Hecke eigenform that is normalized so that* $a(1) = 1$, *then we call* $f(z)$ *a newform.*

The following theorem follows from the properties of Hecke operators and Deligne's proof of the Weil conjectures.

Theorem 2.10. *Let* $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in S_k^{\text{new}}(\Gamma_0(N))$ *be a newform. The following are true:*

1. *If* $\gcd(m, n) = 1$, *then* $a(mn) = a(m)a(n)$.

2. *If* p *is prime and* $n \in \mathbb{Z}$ *with* $n > 1$, *then* $a(p^n) = a(p)a(p^{n-1}) - p^{k-1}a(p^{n-2})$.

3. *If p is prime, then $|a(p)| \leq 2p^{(k-1)/2}$. We can then define $\theta_p \in [0, \pi]$ to be the angle for which $a(p) = 2p^{(k-1)/2}\cos(\theta_p)$.*

We now define one last subspace of $M_k(\Gamma_0(N))$ that will be of interest in our discussion of quadratic forms.

Definition 2.11. *We define the Eisenstein subspace $E_k(\Gamma_0(N))$ of $M_k(\Gamma_0(N))$ to be the space of modular forms which are orthogonal to every cusp form in $S_k(\Gamma_0(N))$ with respect to the Petersson inner product. The elements of the Eisenstein subspace are called Eisenstein series.*

2.1.2 Elliptic Curves

Elliptic curves and newforms on $\Gamma_0(N)$ are intimately related. Their relationship was key to proving Fermat's Last Theorem and for an almost complete solution of the congruent number problem.

Definition 2.12. *Let $a, b \in \mathbb{Z}$, and let $\Delta(E) = 4a^3 + 27b^2$. Suppose that $\Delta(E) \neq 0$. An elliptic curve over $\mathbb{Q}$ (denoted $E/\mathbb{Q}$) is the projective closure of the equation*

$$E : y^2 = x^3 + ax + b.$$

While elliptic curves over $\mathbb{Q}$ are intrinsically interesting, we will only present their relationship with newforms. If we reduce the equation E modulo p, there are, on average, $p + 1$ solutions $(x, y) \in \mathbb{F}_p^2$.

Definition 2.13. *Let $E/\mathbb{Q}$ be an elliptic curve, and let $p \nmid \Delta(E)$ be prime. Let $E(\mathbb{F}_p)$ be the points on E over the field $\mathbb{F}_p$. Define $a_p(E) := p + 1 - |E(\mathbb{F}_p)|$. We say that $E/\mathbb{Q}$ is modular if $a_p(E)$ is the p-th Fourier coefficient of a nonzero newform of weight 2.*

Theorem 2.14 (The Modularity Theorem)**.** *If $E/\mathbb{Q}$ is an elliptic curve, then E is modular.*

We now define some technical terms that arise later in our discussion of the Sato-Tate Conjecture and elliptic curves.

Definition 2.15. *Let $f \in S_2^{\text{new}}(\Gamma_0(N))$ be the newform associated to an elliptic curve $E/\mathbb{Q}$ via the modularity theorem. The level N is called the conductor of $E/\mathbb{Q}$. If the conductor of $E/\mathbb{Q}$ is squarefree, then we call $E/\mathbb{Q}$ a semistable elliptic curve.*

Definition 2.16. *Let $E/\mathbb{Q}$ be an elliptic curve. If $p \neq 2, 3$ and $a_p(E) = 0$, then E has supersingular reduction modulo p. In other words, p is a supersingular prime for E.*

Theorem 2.10 tells us that for an elliptic curve $E/\mathbb{Q}$, $|a_p(E)| \leq 2\sqrt{p}$. Therefore, Theorem 1.1 tells us that if $E/\mathbb{Q}$ is a semistable elliptic curve, then the sequence $\{\theta_p\}$, where $\theta_p \in [0, \pi]$ is defined by $2\sqrt{p}\cos(\theta_p) = a_p(E)$, is equidistributed in $[0, \pi]$ with respect to μ_{ST}. Theorem 1.3 gives us an upper bound on the function that counts primes $p \in [x, 2x]$ such that for a fixed interval $[a, b] \subset [0, \pi]$, $\theta_p \in [a, b]$. Furthermore, Theorem 1.7 provides us with an explicit upper bound for the supersingular prime counting function of a semistable elliptic curve $E/\mathbb{Q}$.

2.1.3 Quadratic Forms

Modular forms also arise in finding the number of representations of a nonnegative integer given by certain types of quadratic forms. In 1834, Jacobi proved that if $r_m(n)$ is the number of representation of n as a sum of m squares and $\sigma_k(n) = \sum_{d|n} d^k$, then $r_4(n) = 8\sigma_1(n)$ if n is odd. If $n = 2^\alpha\beta$, where $\alpha > 0$ and $\beta \geq 1$, then $r_4(n) = 24\sigma_1(\beta)$. This can be proven using the relationships given in this section.

Definition 2.17. *A quadratic form is a function* $Q : \mathbb{R}^r \to \mathbb{R}$ *given by*

$$Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x},$$

where A *is an* $r \times r$ *symmetric matrix.*

Definition 2.18. *A quadratic form* Q *is positive-definite if* $Q(\vec{x}) \geq 0$ *for all* $\vec{x} \in \mathbb{R}^r$ *and* $Q(\vec{x}) = 0$ *if and only if* $\vec{x} = \vec{0}$.

Definition 2.19. *If* $Q = \frac{1}{2}\vec{x}^T A\vec{x}$, *where* $A = [a_{ij}]$, *is a positive-definite quadratic form, we say that* Q *is integer-valued if* $a_{ij} \in \mathbb{Z}$ *and* a_{ii} *is even. We say that* Q *represents* $n \in \mathbb{N}$ *if there exists* $\vec{x} \in \mathbb{Z}^r$ *such that* $Q(\vec{x}) = n$. *We let* $r_Q(n) = \#\{\vec{x} \in \mathbb{Z}^r : Q(\vec{x}) = n\}$.

The following theorem relating quadratic forms and modular forms is due to Siegel.

Theorem 2.20 (Siegel)**.** *Suppose that* $Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$ *is a positive-definite, integer-valued quadratic form. Let* N *be the smallest positive integer such that* NA^{-1} *has integer entries and even diagonal entries. Assume that* 4 *divides* r *(the number of variables in* Q*) and that* $\det(A)$ *is a square. Then*

$$\Theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n)q^n \in M_{r/2}(\Gamma_0(N)).$$

In the case of Jacobi's work, we have that

$$Q(\vec{x}) = \frac{1}{2}\vec{x}^T \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \vec{x},$$

and so

$$\Theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n)q^n \in M_2(\Gamma_0(4)).$$

Furthermore, using the orthogonality of Eisenstein series and cusp forms as presented in Definition 2.11, $\Theta_Q(z)$ can be expressed as a linear combination of an Eisenstein series $E_Q(z) = \sum_{n=0}^{\infty} a_Q(n)q^n$ and a cusp form $S_Q(z) = \sum_{n=1}^{\infty} b_Q(n)q^n$, where $a_Q(n)$ is a square-root accurate approximation of $r_Q(n)$ with error term $b_Q(n)$. Therefore, the study of newforms can lead to insight into the accuracy of $a_Q(n)$ as an approximation of $r_Q(n)$.

2.1.4 Two Motivational Examples

We now discuss an example that relates the above work on modular forms, elliptic curves, and quadratic forms in a meaningful way. Let $Q : \mathbb{R}^4 \to \mathbb{R}$ be defined by the equation $Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$, where

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & 6 \end{bmatrix}.$$

If we expand $Q(\vec{x})$, we obtain the equation $Q(x, y, z, w) = x^2+y^2+3z^2+3w^2+xz+yw$. It is easy to check that $Q(\vec{x})$ is a positive-definite, integer-valued quadratic form. The number of variables in Q is divisible by 4, and we have that $\det(A) = 121 = 11^2$. Furthermore,

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 6 & 0 & -1 & 0 \\ 0 & 6 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix},$$

and the smallest positive integer N such that NA^{-1} has integer entries and even diagonal entries is 11. By Siegel's theorem, we have

$$\Theta_Q(z) = 1 + 4q + 4q^2 + 8q^3 + 20q^4 + 16q^5 + \cdots = \sum_{n=0}^{\infty} r_Q(n)q^n \in M_2(\Gamma_0(11)).$$

Using the orthogonality of Eisenstein series and cusp forms as presented in Definition 2.11, we can represent $\Theta_Q(z)$ as the sum of the Eisenstein series

$$E(z) = 1 + \frac{12}{5}\sum_{n=1}^{\infty}\left(\sigma_1(n) - 11\sigma_1\left(\frac{n}{11}\right)\right)q^n$$

and the cusp form $\frac{8}{5}C(z)$, where

$$C_1(z) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 = \sum_{n=1}^{\infty}a(n)q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + \dots$$

However, $S_2(\Gamma_0(11))$ is a one-dimensional complex vector space, which means that $C(z)$ is a newform. Furthermore, $C(z)$ is the newform associated to the elliptic curve

$$E : y^2 - y = x^3 - x^2.$$

Theorem 1.7 gives us an upper bound on the number of primes $p \in [x, 2x]$ for which $a(p) = 0$, which corresponds to counting the number of primes $p \in [x, 2x]$ such that $r_Q(p) = \frac{12}{5}\sigma_1(p)$. We can interpret the density of positive integers n for which $a(n) \neq 0$, which is bounded in Theorem 1.8, as 1 minus the percentage of positive integers n for which $r_Q(n) = \frac{12}{5}\sigma_1(n)$. By the calculation in Theorem 1.8, this percentage is between 15.3% and 16.5%.

In the previous chapter, we introduced the quadratic form

$$Q(x_1, \dots, x_{24}) = \sum_{k=1}^{24} x_k^2.$$

We are now in a position to discuss how to express $r_Q(n)$ exactly. Let

$$\Delta(z) = \sum_{n=1}^{\infty}\tau_{12}(n)q^n,\ E_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty}\sigma_{11}(n)q^n,\ \text{and}\ \Theta(z) = \sum_{n=0}^{\infty}q^{n^2}.$$

We have the equality

$$\Theta_Q(z) = \Theta(z)^{24},$$

which can be checked by expanding the right hand side. It turns out that $\Theta(z)^{24}$ is a weight 12, level 4 modular form, and we can represent $\Theta_Q(z)$ as a sum of the Eisenstein series

$$E(z) = \frac{16}{691}E_{12}(z) - \frac{32}{691}E_{12}(2z) + \frac{65536}{691}E_{12}(4z)$$

and the cusp form

$$C_2(z) = \frac{33152}{691}\Delta(z) + \frac{1525760}{691}\Delta(2z) + \frac{135790592}{691}\Delta(4z).$$

This justifies the approximation given in Chapter 1, where for a prime p, $r_Q(p) \approx \frac{16}{691}(p^{11}+1)$ with error bounded by $\frac{66304}{691}\sqrt{p^{11}}$. One can show that the n-th Fourier coefficient of $C_2(z)$ is zero if and only if $\tau(n) = 0$. (A proof of this requires Galois theory and will not be discussed here.) Equivalently, if Conjecture 1.5 is true, then $r_Q(n)$ never equals its Eisenstein approximation. Theorem 1.8 tells us that under certain assumptions, this inequality occurs more than 99.99912% of the time.

2.2 The Prime Number Theorem

In this section we will provide a basic overview of the proof of the Prime Number Theorem with the error term. Complete details, including proofs of all statements, are available in [5]. We will use Landau's "Big O" notation: Let $f, g : U \subset \mathbb{R} \to \mathbb{R}$ be functions such that there exists $M > 0$ and $x_0 > 0$ such that if $x \geq x_0$, then $|f(x)| \leq M|g(x)|$. We write this as $f(x) = O(g(x))$.

Our goal is to briefly outline the steps in the proof of the following theorem.

Theorem 2.21 (The Prime Number Theorem with Error Term)**.** *If* $\pi(x)$ *is the prime counting function and* $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}$*, then there exists* $c > 0$ *such that*

$$\pi(x) = \mathrm{Li}(x) + O(xe^{-c\sqrt{\log(x)}}).$$

Definition 2.22. *Let* $j \in \mathbb{Z}$*. Define*

$$\Lambda(j) = \begin{cases} \log(p), & j = p^m \text{ for some prime } p \text{ and some } m \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

We can now write $\pi(x) = \sum_{p \in [1,x]} \frac{\Lambda(p)}{\log(p)}$. We now measure the error in switching from summing over primes to summing over integers, which is equivalent to summing over prime powers.

Lemma 2.23. *We have* $\left| \sum_{p \in [1,x]} \frac{\Lambda(p)}{\log(p)} - \sum_{j \in [1,x]} \frac{\Lambda(j)}{\log(j)} \right| = O(\sqrt{x})$.

We now relate $\sum_{j \in [1,x]} \frac{\Lambda(j)}{\log(j)}$ to $\sum_{j \in [1,x]} \Lambda(j)$.

Lemma 2.24. *Let* $\psi(x) = \sum_{j \in [1,x]} \Lambda(j)$*. We have*

$$\sum_{j \in [1,x]} \frac{\Lambda(j)}{\log(j)} = \frac{\psi(x)}{\log(x)} - \frac{\psi(2)}{\log(2)} + \int_2^x \frac{\psi(t)}{t \log^2(t)}\, dt.$$

At each prime power, $\psi(x)$ has a discontinuity, so look at the modified function

$$\psi_0(x) = \begin{cases} \psi(x), & x \text{ is not a prime power,} \\ \psi(x) - \frac{1}{2}\Lambda(x), & \text{otherwise.} \end{cases}$$

We now wish to relate $\psi_0(x)$ to the Riemann zeta function.

Definition 2.25. *If* $s \in \mathbb{C}$ *and* $\mathrm{Re}(s) > 1$, *define*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The function $\zeta(s)$ is intimately related with the distribution of prime numbers, but before we can utilize this relationship, we must do some analysis with $\zeta(s)$.

Theorem 2.26. *The function* $\zeta(s)$ *has an analytic continuation to* $\mathbb{C}\backslash\{1\}$. *(At* $s = 1$, $\zeta(s)$ *is the harmonic series, which diverges.)*

When we refer to $\zeta(s)$ henceforth, we refer to its analytic continuation.

If we define the function

$$\xi(s) = \frac{s(1-s)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

then $\xi(s) = \xi(1-s)$. The functional equation and normalization of $\zeta(s)$ tells us that since $\zeta(s)$ converges absolutely to a nonzero number when $\mathrm{Re}(s) > 1$, the same must be true when $\mathrm{Re}(s) < 0$. Furthermore, there are two types of zeros of $\zeta(s)$: the zeros that correspond with the poles of $\Gamma(\frac{s}{2})$ (the trivial zeros) and the zeros in the region $S = \{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) \leq 1\}$ (the nontrivial zeros); this region is called the critical strip.

The following deep theorem is due independently to Hadamard and de la Vallée Poussin, which is the crux of the proof of the Prime Number Theorem.

Theorem 2.27. *If* $s \in S$ *and* $\zeta(s) = 0$, *then* $\mathrm{Re}(s) < 1$.

We now state an estimate of the distribution of nontrivial zeros of $\zeta(s)$ in S.

Lemma 2.28. *Let* $N(T)$ *be the number of zeros of* $\zeta(s)$ *in* S *with imaginary part between 0 and* $T > 0$. *Then* $N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log(T))$.

We are now in a position to relate $\psi_0(x)$ to the zeros of $\zeta(s)$.

Lemma 2.29 (The Explicit Formula). *If $x \geq 1$ is not a prime power, then*

$$\psi_0(x) = x - \sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}.$$

Proof. We provide a sketch of the proof. If we look at the series representation of $\zeta(s)$ and take the logarithmic derivative of $\zeta(s)$, we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Using the discontinuous integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s}\, ds = \begin{cases} 0, & 0 < y < 1 \\ \frac{1}{2}, & y = 1 \\ 1, & 1 < y, \end{cases}$$

where $c > 0$, we obtain the equality

$$\psi_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}\, ds.$$

The right-hand side of the Explicit Formula now follows from an application of the argument principle from complex analysis.. $\square$

Since $\mathrm{Re}(\rho) < 1$, we have that $\psi_0(x) \sim x$, which is equivalent to the statement $\pi(x) \sim \mathrm{Li}(x)$. While this statement alone suffices for many applications, we want to estimate the error in this approximation. The sum over nontrivial zeros converges conditionally, so in order to make a useful estimate, we must truncate the sum to include zeros with imaginary part less than T and estimate the resulting error. Using the results on the number of zeros in S up to a heigh T given earlier and setting $T = e^{\sqrt{\log(x)}}$ (this minimizes the error term), we obtain the following estimate.

Lemma 2.30. *If $x > 1$ is not a prime power, there exists $c > 0$ such that*

$$\psi_0(x) = x + O\left(xe^{-c\sqrt{\log(x)}}\right).$$

The Prime Number Theorem now follows from a series of substitutions relating $\psi_0(x)$ to $\sum_{p\in[1,x]} \frac{\Lambda(p)}{\log(p)}$.

It was crucial that $\mathrm{Re}(\rho) < 1$ for all nontrivial zeros ρ, but more is thought to be true. When Riemann first investigated the zeros of $\zeta(s)$, he believed that all nontrivial zeros of $\zeta(s)$ were such that $\mathrm{Re}(\rho) = \frac{1}{2}$. This is the Riemann Hypothesis. If we repeat the above estimates assuming the Riemann Hypothesis, then the accuracy of $\mathrm{Li}(x)$ as an approximation of $\pi(x)$ increases dramatically.

Theorem 2.31 (Prime Number Theorem with the Riemann Hypothesis)**.** *If the Riemann Hypothesis is true, then*

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log(x)).$$

2.3 Symmetric Power L-functions

Let N be squarefree, and let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\mathrm{new}}(\Gamma_0(N))$ be a newform. As a consequence of Deligne's proof of the Weil conjectures, if p is prime, then $|a(p)| \leq 2p^{(k-1)/2}$. We define θ_p to be the angle in $[0, \pi]$ such that $a(p) = 2p^{(k-1)/2}\cos(\theta_p)$. We will study the distribution of the θ_p within $I = [\alpha, \beta] \subset [0, \pi]$. We accomplish this by studying the zeroes of L-functions constructed from these θ_p.

The normalized L-function associated to f is given by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{\frac{k-1}{2}+s}} = \prod_{p \text{ prime}} \prod_{j=0}^{1} (1 - \alpha_p^j \beta_p^{1-j} p^{-s})^{-1}.$$

It is known that $L(f, s)$ converges absolutely for $\mathrm{Re}(s) > 1$. By Theorem 2.10, we have $\alpha_p = e^{i\theta_p}$ and $\beta_p = e^{-i\theta_p}$ when $p \nmid N$.

Langlands functoriality predicts that the L-functions of degree $n+1$ given by

$$L(\mathrm{Sym}^n f, s) = \prod_{p \text{ prime}} \prod_{j=0}^{n} (1 - \alpha_p^j \beta_p^{n-j} p^{-s})^{-1}$$

are automorphic, i.e. have an analytic continuation on $\mathbb{C}$ and a functional equation of the usual type, for all positive integers n. We know this to be the case when $n = 1, 2, 3, 4$. In particular, if we let $q_{\mathrm{Sym}^n f}$, $\gamma(\mathrm{Sym}^n f, s)$, and $\epsilon_{\mathrm{Sym}^n f}$ be the prescribed conductor, gamma factor, and root number of $L(\mathrm{Sym}^n f, s)$, respectively, then we define

$$\Lambda(\mathrm{Sym}^n f, s) = q_{\mathrm{Sym}^n f}^{s/2} \gamma(\mathrm{Sym}^n f, s) L(\mathrm{Sym}^n f, s) \tag{2.1}$$

to be the completed symmetric power L-function of f. The functional equation

$$\Lambda(\mathrm{Sym}^n f, s) = \epsilon_{\mathrm{Sym}^n f} \Lambda(\mathrm{Sym}^n f, 1-s).$$

now holds; it follows that the distribution of the zeroes of $L(\mathrm{Sym}^n f, s)$ is determined by $\gamma(\mathrm{Sym}^n f, s)$ and $q_{\mathrm{Sym}^n f}$.

Assuming a global lifting map on automorphic representations that is compatible with the local Langlands correspondence, Cogdell and Michel compute the predicted equations for the conductor, gamma factor, and root number of $L(\mathrm{Sym}^n f, s)$ in [4]. The equation for the conductor is

$$q_{\mathrm{Sym}^n f} = N^n, \tag{2.2}$$

and the equation for the gamma factor is

$$\gamma(\mathrm{Sym}^n f, s) = \begin{cases} \displaystyle\prod_{j=0}^{(n-1)/2} \Gamma_{\mathbb{C}}\left(s + \frac{(2j+1)(k-1)}{2}\right) & \text{if } n \text{ is odd,} \\ \Gamma_{\mathbb{R}}(s+r) \displaystyle\prod_{j=1}^{n/2} \Gamma_{\mathbb{C}}(s + j(k-1)) & \text{if } n \text{ is even,} \end{cases} \tag{2.3}$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, and r equals the parity of $n/2$. (The function $\Gamma(s)$ denotes the usual Γ-function). We now summarize the conjectures in our hypothesis:

Conjecture 2.32. *Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform with squarefree level N. For each nonnegative integer n, the following are true.*

(a) The conductor and gamma factor of $L(\text{Sym}^n f, s)$ are as stated in Equations 2.2 and 2.3, respectively.

(b) If $n \geq 1$, the resulting completed L-function $\Lambda(\text{Sym}^n f, s)$ is an order 1 entire function. If $n = 0$, then $L(\text{Sym}^n f, s)$ is the Riemann zeta function $\zeta(s)$, so the resulting completed L-function times $\frac{1}{2}s(1-s)$ is an order 1 entire function (see [5]).

(c) The zeroes of $\Lambda(\text{Sym}^n f, s)$ have real part equal to $\frac{1}{2}$ (the Generalized Riemann Hypothesis).

Define the numbers $\Lambda_{\text{Sym}^n f}(j)$ by

$$-\frac{L'}{L}(\text{Sym}^n f, s) = \sum_{j=1}^{\infty} \frac{\Lambda_{\text{Sym}^n f}(j)}{j^s}.$$

A straightforward computation shows that if $U_n(x)$ is the n-th Chebyshev polynomial of the second type, then

$$-\frac{L'}{L}(\text{Sym}^n f, s) = \sum_{p \nmid N} \sum_{m=1}^{\infty} U_n(\cos(m\theta_p)) \log(p) p^{-ms} + \sum_{p|N} \frac{L_p'}{L_p}(\text{Sym}^n f, s).$$

Therefore, for any integer j greater than the largest prime dividing N, we have

$$\Lambda_{\text{Sym}^n f}(j) = \begin{cases} U_n(\cos(m\theta_p)) \log(p) & \text{if } j = p^m \text{ for some } p, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Furthermore, it follows from basic properties of Chebyshev polynomials of the second type that if p is a prime greater than any prime dividing N, then

$$\Lambda_{\mathrm{Sym}^n f}(p) - \Lambda_{\mathrm{Sym}^{n-2} f}(p) = 2\cos(n\theta_p)\log(p), \quad n \geq 2. \tag{2.5}$$

The following definitions will be used frequently henceforth.

Definition 2.33. *We have*

$$\Theta_{\mathrm{Sym}^n f}(x) = \sum_{p \in [x,2x]} \frac{\Lambda_{\mathrm{Sym}^n f}(p)}{\log(p)}, \tag{2.6}$$

$$\Psi_{\mathrm{Sym}^n f}(x) = \sum_{j \in [x,2x]} \frac{\Lambda_{\mathrm{Sym}^n f}(j)}{\log(j)}, \tag{2.7}$$

$$\psi_{\mathrm{Sym}^n f}(x) = \sum_{j \in [1,x]} \Lambda_{\mathrm{Sym}^n f}(j). \tag{2.8}$$

Chapter 3: The Error Term in the Sato-Tate Conjecture

3.1 Preliminary Lemmas

In this section, we will prove a series of lemmas that will enable us to prove Theorem 1.3. Let $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$ be a newform with squarefree level N satisfying Conjecture 2.32, and let $I = [\alpha, \beta] \subset [0, \pi]$ be fixed. We approximate the characteristic function $\chi_I(\theta_p)$ with a C^1 function via Lemma 12 of [28].

Lemma 3.1 (Vinogradov)**.** *Let R be a positive integer, and let $a, b, \delta \in \mathbb{R}$ satisfy*

$$0 < \delta < \frac{1}{2}, \quad \delta \leq b - a \leq 1 - \delta.$$

Then there exists a periodic function $g(y)$ with period 1 satisfying

(a) $g(y) = 1$ *when* $y \in [a + \frac{1}{2}\delta, b - \frac{1}{2}\delta]$,

(b) $g(y) = 0$ *when* $y \in [b + \frac{1}{2}\delta, 1 + a - \frac{1}{2}\delta]$,

(c) $0 \leq g(y) \leq 1$ *when* y *is in the remainder of the interval* $[a - \frac{1}{2}\delta, 1 + a - \frac{1}{2}\delta]$, *and*

(d) $g(y)$ *has the Fourier expansion*

$$g(y) = b - a + \sum_{n=1}^{\infty} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x)),$$

where for all $n \geq 1$,

$$|a_n|, |b_n| \leq \min \left\{ 2(b - a), \frac{2}{n\pi}, \frac{2}{n\pi} \left(\frac{R}{\pi n \delta} \right)^R \right\}.$$

In particular, for $n \geq 1$,

$$a_n = \frac{1}{n\pi} (\sin(2\pi n b) - \sin(2\pi n a)) \left(\frac{\sin(\pi n \delta / R)}{\pi n \delta / R} \right)^R,$$

$$b_n = \frac{1}{n\pi}(\cos(2\pi na) - \cos(2\pi nb))\left(\frac{\sin(\pi n\delta/R)}{\pi n\delta/R}\right)^R.$$

Let $g(\theta)$ be defined as in Lemma 3.1, where $R = 2$, $a = \frac{\alpha}{2\pi} - \frac{1}{2}\delta$, and $b = \frac{\beta}{2\pi} + \frac{1}{2}\delta$. Then $g_{I,\delta}(\theta) = g(\frac{\theta}{2\pi}) + g(-\frac{\theta}{2\pi})$ equals 1 when $\theta \in I$, equals 0 when $\theta \in [0, \alpha - \pi\delta] \cup [\beta + \pi\delta, \pi]$, and is between 0 and 1 elsewhere in the interval $[0, \pi]$, making $g_{I,\delta}(\theta)$ a pointwise upper bound for $\chi_I(\theta)$. Similarly, we can produce a lower bound $g_{I,-\delta}(\theta)$ for $\chi_I(\theta)$ by letting $a = \frac{\alpha}{2\pi} + \frac{1}{2}\delta$ and $b = \frac{\beta}{2\pi} - \frac{1}{2}\delta$.

The Fourier expansion of $g_{I,\pm\delta}(\theta)$ is

$$g_{I,\pm\delta}(\theta) = a_0(I, \pm\delta) + 2\sum_{n=1}^{\infty} a_n(I, \pm\delta)\cos(n\theta), \tag{3.1}$$

where

$$a_n(I, \pm\delta) = \begin{cases} \dfrac{\beta - \alpha}{\pi} \pm 2\delta & \text{if } n = 0, \\ \dfrac{\sin(n(\beta \pm \pi\delta)) - \sin(n(\alpha \mp \pi\delta))}{n\pi}\left(\dfrac{\sin(\pi n\delta/2)}{\pi n\delta/2}\right)^2 & \text{if } n > 0. \end{cases} \tag{3.2}$$

We now present some necessary bounds on the Fourier coefficients of $g_{I,\pm\delta}(\theta)$.

Lemma 3.2. *Assume the above notation. When $0 < \delta < \frac{1}{1000}$, we have the following bounds:*

(a) $\displaystyle\sum_{n=0}^{\infty} |a_n(I, \pm\delta) - a_{n+2}(I, \pm\delta)| \leq 2\log\left(\frac{1}{\delta}\right)$,

(b) $\displaystyle\sum_{n=0}^{\infty} |a_n(I, \pm\delta) - a_{n+2}(I, \pm\delta)|\log(n + 1) \leq 2\log^2\left(\frac{1}{\delta}\right) - 2\log(\beta - \alpha)$,

(c) $\displaystyle\sum_{n=0}^{\infty} |a_n(I, \pm\delta) - a_{n+2}(I, \pm\delta)|n \leq \frac{2}{\delta}$, *and*

(d) $\sum_{n=0}^{\infty} |a_n(I, \pm\delta) - a_{n+2}(I, \pm\delta)| n \log(n+1) \leq \frac{2}{\delta} \log\left(\frac{1}{\delta}\right) - \frac{2\log(\beta - \alpha)}{\beta - \alpha}.$

Proof. These bounds follow from the bounds on the Fourier coefficients given in Part (d) of Lemma 3.1. □

We will eventually need an upper bound on $|a_0(I, \pm\delta) - a_2(I, \pm\delta) - \mu_{ST}(I)|$, so we provide it here.

Lemma 3.3. *Assume the above notation. If* $0 < \delta < \frac{1}{1000}$, *then*

$$|a_0(I, \pm\delta) - a_2(I, \pm\delta) - \mu_{ST}(I)| \leq 4\delta.$$

Proof. This is a straightforward two-variable maximization problem, requiring us to take the partial derivatives of $(a_0(I, \pm\delta) - a_2(I, \pm\delta) - \mu_{ST}(I))^2$ with respect to α and β using Equation 3.2. □

Our goal is to bound $\pi_{f,I}(x)$ by analyzing $\Theta_{\mathrm{Sym}^n f}(x)$. The next lemma relates these two quantities in a useful way.

Lemma 3.4. *Assume the above notation. We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}(a_n(I, -\delta) - a_{n+2}(I, -\delta))\Theta_{\mathrm{Sym}^n f}(x) \\
= &\sum_{p \in [x, 2x]} \left(a_0(I, -\delta) + 2\sum_{n=1}^{\infty} a_n(I, -\delta)\cos(n\theta_p)\right) \\
\leq \; &\pi_{f,I}(x) \\
\leq &\sum_{p \in [x, 2x]} \left(a_0(I, \delta) + 2\sum_{n=1}^{\infty} a_n(I, \delta)\cos(n\theta_p)\right) \\
= &\sum_{n=0}^{\infty}(a_n(I, \delta) - a_{n+2}(\delta))\Theta_{\mathrm{Sym}^n f}(x).
\end{aligned}$$

Proof. The two inequalities follow from the construction in Lemma 3.1 and Equation 1.1. For the two equalities, notice that $g_{I,\pm\delta}(\theta)$ is C^1. Thus the Fourier expansions in Equation 3.1 converge absolutely. Therefore, when we relate $\cos(n\theta_p)$ to $\Lambda_{\mathrm{Sym}^n f}(p)$ using Equation 2.5, we can switch the order of summation and use Definition 2.33 to obtain the two equalities. $\square$

Now, we bound the error obtained by switching from summing over primes to summing over prime powers.

Lemma 3.5. *Assume the above notation. For $x \geq 10^{17}$, we have*

$$|\Theta_{\mathrm{Sym}^n f}(x) - \Psi_{\mathrm{Sym}^n f}(x)| < \frac{8}{5}(n+1)\frac{\sqrt{x}}{\log(x)}.$$

Proof. For any positive integer j, $\frac{\Lambda_{\mathrm{Sym}^n f}(j)}{\log(j)}$ is a sum of $n+1$ roots of unity if j is a prime power and 0 otherwise. Thus for any positive integer j,

$$\left|\frac{\Lambda_{\mathrm{Sym}^n f}(j)}{\log(j)}\right| \leq (n+1).$$

Therefore, by Definition 2.33,

$$\left|\Theta_{\mathrm{Sym}^n f}(x) - \Psi_{\mathrm{Sym}^n f}(x)\right| \leq (n+1)\sum_{j=2}^{\lfloor \log_2(2x) \rfloor}\sum_{p^j \in [x,2x]} 1 \leq (n+1)\sum_{j=2}^{\lfloor \log_2(2x) \rfloor}\sum_{p^j \leq 2x}\frac{\log(p)}{\log(x)}.$$

Let $\theta(x) = \sum_{p \leq x}\log(p)$. By [21], $\theta(x) < 1.001102x$ for all $x > 1$. Thus

$$\sum_{j=2}^{\lfloor \log_2(2x) \rfloor}\sum_{p^j \leq 2x}\log(p) \leq \theta(\sqrt{2x}) + \log_2(2x)\theta(\sqrt[3]{2x}) \leq 1.001102(\sqrt{2x} + \log_2(2x)\sqrt[3]{2x}),$$

which is bounded above by $\frac{8}{5}\sqrt{x}$ for all $x \geq 10^{17}$. $\square$

If we calculate the sum $\Psi_{\mathrm{Sym}^n f}(x)$ using Abel summation, we obtain the equality

$$\Psi_{\mathrm{Sym}^n f}(x) = \frac{\psi_{\mathrm{Sym}^n f}(2x)}{\log(2x)} - \frac{\psi_{\mathrm{Sym}^n f}(x)}{\log(x)} + \int_x^{2x}\frac{\psi_{\mathrm{Sym}^n f}(t)}{t\log^2(t)}\,dt. \tag{3.3}$$

Therefore, an estimate of $\psi_{\mathrm{Sym}^n f}(x)$ yields an estimate of $\Theta_{\mathrm{Sym}^n f}(x)$. We estimate $\psi_{\mathrm{Sym}^n f}(x)$ by relating $\psi_{\mathrm{Sym}^n f}(x)$ to the zeroes and poles of $L(\mathrm{Sym}^n f, s)$. Before we do so, we need to know something about the trivial zeroes of $L(\mathrm{Sym}^n f, s)$. By the formula for the gamma factors of $L(\mathrm{Sym}^n f, s)$, if $n > 0$ and $4|n$, then $L(\mathrm{Sym}^n f, s)$ has a simple zero at $s = 0$. Let $B_\delta(0)$ be a ball of radius δ centered at $s = 0$. There exists a function $w(\mathrm{Sym}^n f, s)$ such that $w(\mathrm{Sym}^n f, 0) \neq 0$ and for all $s \in B_\delta(0)$, $w(\mathrm{Sym}^n f, s)$ is holomorphic and $L(\mathrm{Sym}^n f, s) = s \cdot w(\mathrm{Sym}^n f, s)$. In $B_\delta(0)$,

$$\frac{L'}{L}(\mathrm{Sym}^n f, s)\frac{x^s}{s} = \frac{x^s}{s^2} + \frac{w'}{w}(\mathrm{Sym}^n f, s) \cdot \frac{x^s}{s}. \tag{3.4}$$

We now present the explicit formula for $L(\mathrm{Sym}^n f, s)$.

Lemma 3.6. *Assume the above notation. Let* $\mathrm{Res}(f, s_0)$ *equal the residue of a function* f *at* $s = s_0$. *Let* $w(\mathrm{Sym}^n f, s)$ *be a function such that* $w(\mathrm{Sym}^n f, 0) \neq 0$, $w(\mathrm{Sym}^n f, s)$ *is holomorphic in a neighborhood of* $s = 0$, *and* $w(\mathrm{Sym}^n f, s)$ *satisfies Equation 3.4. If* $T \geq 10^{12}$, *and if* x *is not a prime power, then*

$$|\psi_{\mathrm{Sym}^n f}(x) - J_{\mathrm{Sym}^n f}(T, x)| \leq (n+1)R(T, x),$$

where $R(T, x) = \frac{e\log(x)}{2T}(8T + 9x + 4x\log(x))$,

$$J_{\mathrm{Sym}^n f}(T, x) = \delta_{0,n}x - \sum_{|\Im(\rho)|<T} \frac{x^\rho}{\rho} - \mathrm{Res}\left(\frac{L'}{L}(\mathrm{Sym}^n f, s) \cdot \frac{x^s}{s}, 0\right),$$

and

$$\mathrm{Res}\left(\frac{L'}{L}(\mathrm{Sym}^n f, s) \cdot \frac{x^s}{s}, 0\right) = \begin{cases} \log(x) + \dfrac{w'}{w}(\mathrm{Sym}^n f, s) & \textit{if } n > 0 \textit{ and } 4 \mid n, \\ \dfrac{L'}{L}(\mathrm{Sym}^n f, 0) & \textit{otherwise.} \end{cases}$$

Proof. The proof of this is very similar to the proof given in Chapter 17 of [5] with two exceptions. First, to bound the difference in the lemma statement, we make use of the fact that for integers $j > 1$, $|\Lambda_{\mathrm{Sym}^n f}(j)| \leq (n+1)\Lambda(j)$, where $\Lambda(j)$ is the classical von

Mangoldt function. This follows from Equation 2.4 and basic properties of Chebyshev polynomials. Second, when $4|n$ and $n > 0$, the double pole of $L(\mathrm{Sym}^n f, s)\frac{x^s}{s}$ at $s = 0$ contributes the $\log(x) + \frac{w'}{w}(\mathrm{Sym}^n f, s)$ term. This follows from taking the Laurent expansion centered at $s = 0$ of the right hand side of Equation 3.4. When $4 \nmid n$ or $n = 0$, the pole at $s = 0$ is simple and contributes the $\frac{L'}{L}(\mathrm{Sym}^n f, 0)$ term. □

To estimate the sum over zeroes, we look at the sum over trivial zeroes and nontrivial zeroes separately.

Lemma 3.7. *Assume the above notation. Let ρ be a zero of $L(\mathrm{Sym}^n f, s)$. If $x \geq 10^{17}$, then*

$$\sum_{\substack{\rho \text{ trivial} \\ \rho \neq 0}} \left| \frac{x^\rho}{\rho} \right| \leq \frac{n+3}{\sqrt{x}}.$$

Proof. The trivial zeroes of $L(\mathrm{Sym}^n f, s)$ occur at the poles of the gamma factors. When n is even, these poles are at $s = -m - \frac{(2j+1)(k-1)}{2}$, where $0 \leq j \leq \frac{n-1}{2}$ and m is a nonnegative integer. When n is even, these poles are at $s = -2m - \epsilon$ and $s = -m - j(k-1)$, where $1 \leq j \leq \frac{n}{2}$, ϵ is the parity of $\frac{n}{2}$, and m is a nonnegative integer except in the case where $m = \epsilon = 0$ (the contribution from this case is already given in the statement of the explicit formula). The multiplicity of any of these zeroes is bounded above by $\frac{n+2}{2}$, so we have

$$\sum_{\substack{\rho \text{ trivial} \\ \rho \neq 0}} \left| \frac{x^\rho}{\rho} \right| \leq \frac{n+2}{2} \sum_{m=1}^{\infty} \frac{x^{-m/2}}{m/2} \leq \frac{n+3}{\sqrt{x}}.$$

□

Before we can estimate the sum over nontrivial zeroes, we need to understand their distribution in the critical strip.

Lemma 3.8. *Assume the above notation. Suppose that $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$ with square-free level N satisfies Conjecture 2.32, and let $N_n(j)$ be the number of nontrivial zeroes of $L(\mathrm{Sym}^n f, s)$ with height less than j. Then*

$$N_n(j+1) - N_n(j) \leq \frac{5n}{6} \log\left(\frac{N(k-1)}{2}\right) + 2\log(4j+k+7) + \frac{5(n+5)}{6} \log\left(\frac{n}{2} + \frac{7}{2} + j\right).$$

Proof. We prove an upper bound in the case where the symmetric power is even; the proof is similar when the symmetric power is odd. By Equations 2.1, 2.2, and 2.3, the completed L-function of $L(\mathrm{Sym}^n f, s)$ is

$$\Lambda(\mathrm{Sym}^n f, s) = \left(N^{ns/2}\Gamma_{\mathbb{R}}(s+r)\prod_{j=1}^{n/2}\Gamma_{\mathbb{C}}(s+j(k-1))\right) L(\mathrm{Sym}^n f, s), \tag{3.5}$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, and r equals the parity of $n/2$. By Theorem 5.6 of [9], there exist constants A_n and B_n such that $\Lambda(\mathrm{Sym}^n f, s)$ has the Hadamard product representation

$$\Lambda(\mathrm{Sym}^n f, s) = e^{A_n + B_n s} \prod_{\rho \neq 0,1} (1 - s/\rho)e^{s/\rho}, \tag{3.6}$$

where the product is over all zeros of $\Lambda(\mathrm{Sym}^n f, s)$ not equal to 0 or 1. These zeros are the nontrivial zeros of $L(\mathrm{Sym}^n f, s)$. Taking the real part of the logarithmic derivative of $\Lambda(\mathrm{Sym}^n f, s)$ in Equations 3.5 and 3.6 and setting them equal to each other gives us

$$\begin{aligned}\mathrm{Re}(B_n) + \sum_{\rho} \mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) &= \frac{n}{2}\log\left(\frac{N}{2\pi}\right) - \frac{1}{2}\log(\pi) + \frac{1}{2}\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+r}{2}\right)\right) \\ &\quad + \mathrm{Re}\left(\frac{L'}{L}(\mathrm{Sym}^n f, s)\right) + \sum_{j=1}^{n/2} \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}(s+j(k-1))\right).\end{aligned}$$

By Proposition 5.7 on page 103 of [9], the sum on the left hand side converges absolutely, and

$$\mathrm{Re}(B_n) = -\sum_{\rho} \mathrm{Re}(\rho^{-1}).$$

Thus

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{n}{2}\log\left(\frac{N}{2\pi}\right) - \frac{1}{2}\log(\pi) + \frac{1}{2}\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\epsilon}{2}\right)\right)$$
$$+\mathrm{Re}\left(\frac{L'}{L}(\mathrm{Sym}^n f, s)\right) + \sum_{j=1}^{n/2} \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}(s+j(k-1))\right).$$

Using Equation 5.31 on page 103 of [9], we have

$$\mathrm{Re}\left(\frac{L'}{L}(\mathrm{Sym}^n f, s)\right) \leq \left|\frac{L'}{L}(\mathrm{Sym}^n f, s)\right| \leq (n+1)\frac{|\zeta'(2)|}{\zeta(2)},$$

where $\zeta(s)$ is the Riemann zeta function. We now estimate the digamma terms. Let $z = a + bi$, where $a, b \in \mathbb{R}$ and $a > 0$. It follows from Lemma 4 of [19] and the bound $\frac{\Gamma'}{\Gamma}(a) \leq \log(a)$ that

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}(z)\right) \leq \frac{b^2}{a|z|^2} + \log|z|. \tag{3.7}$$

Now, let $s = 2 + iT$, where $T > 0$. Using Equation 3.7, we bound the digamma terms above by

$$\log(T+1) + 1 + \frac{n}{2}\log(k-1) + \log\left|\frac{\Gamma\left(\frac{n}{2} + 1 + \frac{2}{k-1} + i\frac{T}{k-1}\right)}{\Gamma\left(1 + \frac{2}{k-1} + i\frac{T}{k-1}\right)}\right|.$$

We now use the inequalities

$$|\Gamma(a+bi)| \geq \Gamma(a)\sqrt{\mathrm{sech}(\pi b)}, \quad a \geq \frac{1}{2}$$

and

$$|\Gamma(a+bi)| \leq \sqrt{2\pi}|a+bi|^{a-1/2}\exp\left(\frac{1}{6|a+bi|} - \frac{\pi|b|}{2}\right), \qquad a \geq 0$$

given in [17] to bound the digamma terms by

$$\log(T+1) + \frac{n}{2}\log(k-1) + \frac{n+5}{2}\log\left(\frac{n}{2} + T + 3\right) + 2.216.$$

We finally have

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{2+iT-\rho}\right) \leq \frac{n}{2}\log\left(\frac{N(k-1)}{2}\right) + \log(T+1) + \frac{n+5}{2}\log\left(\frac{n}{2}+T+3\right) + 3.$$

If we assume the Generalized Riemann Hypothesis, then each nontrivial zero $\rho = \beta + i\gamma$ of $L(\mathrm{Sym}^n f, s)$ has $\beta = \frac{1}{2}$. If we consider the zeros for which $T - \frac{1}{2} < \gamma < T + \frac{1}{2}$, then

$$\mathrm{Re}\left(\frac{1}{2+iT-\rho}\right) = \frac{6}{9+4(T-\gamma)^2} \geq \frac{3}{5}.$$

Now,

$$N_n(j+1) - N_n(j) \leq \frac{5}{3} \sum_{\substack{\rho \\ j \leq \gamma \leq j+1}} \mathrm{Re}\left(\frac{1}{2+iT-\rho}\right) \leq \frac{5}{3}\sum_{\rho} \mathrm{Re}\left(\frac{1}{2+iT-\rho}\right).$$

Therefore, when n is even, setting $T = j + \frac{1}{2}$ and substituting our upper bound for $\sum_\rho \mathrm{Re}(\frac{1}{2+iT-\rho})$ produces the bound

$$N_n(j+1) - N_n(j) \leq \frac{5n}{6}\log\left(\frac{N(k-1)}{2}\right) + \frac{5(n+5)}{6}\log\left(\frac{n+7+2j}{2}\right) + \frac{5}{3}\log\left(j+\frac{3}{2}\right) + 3.$$

When n is odd, a similar computation yields the bound

$$\begin{aligned} N_n(j+1) - N_n(j) \quad &\leq \quad \frac{5n}{6}\log\left(\frac{N(k-1)}{2}\right) + \frac{5}{3}\log\left(\left(j+\frac{3}{2}\right)(2j+k+4)\right) \\ &\quad + \frac{5(n+5)}{6}\log\left(\frac{n+7+2j}{2}\right) - \frac{1}{3}. \end{aligned}$$

The bound given in the statement of the lemma is a uniform bound for the even and odd cases. □

We can now estimate the sum over the nontrivial zeroes.

Lemma 3.9. *Assume the above notation. Assume that $L(\mathrm{Sym}^n f, s)$ satisfies the Generalized Riemann Hypothesis. If $T \geq 10^{12}$, then*

$$\sum_{\substack{\rho \text{ nontrivial} \\ |\Im(\rho) < T|}} \left| \frac{x^\rho}{\rho} \right| \leq \sqrt{x}\log(T) n \log(n+1) + \frac{1}{2}\sqrt{x}\log(T)\log(TN(k-1))$$

$$+\sqrt{x}\log(T)\log(n) + \frac{9}{2}\sqrt{x}\log(T)\log(T(k+7)).$$

Proof. Assuming the Generalized Riemann Hypothesis, if $\rho = \beta + i\gamma$ is a nontrivial zero of $L(\mathrm{Sym}^n f, s)$, then $\beta = \frac{1}{2}$. Thus

$$\left| \frac{x^\rho}{\rho} \right| \leq \begin{cases} 2\sqrt{x} & \text{if } 0 < \gamma < 1, \\ \frac{1}{\gamma}\sqrt{x} & \text{if } 1 \leq \gamma. \end{cases}$$

Using Lemma 3.8, if $T \geq 10^{12}$, then

$$\sum_{\substack{\rho \text{ nontrivial} \\ |\Im(\rho) < T|}} \left| \frac{x^\rho}{\rho} \right| \leq 2\sqrt{x}(N_n(1) - N_n(0)) + \sqrt{x}\sum_{j=1}^{T} \frac{N_n(j+1) - N_n(j)}{j}$$

$$\leq \sqrt{x}\log(T) n \log(n+1) + \frac{1}{2}\sqrt{x}\log(T)\log(TN(k-1))$$

$$+\sqrt{x}\log(T)\log(n) + \frac{9}{2}\sqrt{x}\log(T)\log(T(k+7)).$$

□

By Equation 3.3 and Lemma 3.6, we have

$$\left| \Psi_{\mathrm{Sym}^n f}(x) - \left(\frac{J_{\mathrm{Sym}^n f}(T, 2x)}{\log(2x)} - \frac{J_{\mathrm{Sym}^n f}(T, x)}{\log(x)} + \int_x^{2x} \frac{J_{\mathrm{Sym}^n f}(T, t)}{t \log^2(t)}\, dt \right) \right|$$

$$\leq (n+1) \left(\frac{R(T, 2x)}{\log(2x)} + \frac{R(T, x)}{\log(x)} + \int_x^{2x} \frac{R(T, t)}{t \log^2(t)}\, dt \right).$$

The estimates from Lemmas 3.6, 3.7, 3.8, and 3.9 produce an upper bound for $\Psi_{\mathrm{Sym}^n f}(x)$. Substituting $20\sqrt{x}\log(x)$ for T and incorporating the error from Lemma 3.5 proves the following result.

Lemma 3.10 (The Explicit Formula for $L(\mathrm{Sym}^n f, s)$)**.** *Assume the above notation. Assume that $L(\mathrm{Sym}^n f, s)$ satisfies Conjecture 2.32. Define*

$$\Theta^*_{\mathrm{Sym}^n f}(x) = \Theta_{\mathrm{Sym}^n f}(x) - \delta_{0,n}(\mathrm{Li}(2x) - \mathrm{Li}(x)).$$

If $x \geq 10^{17}$ is an integer larger than any prime dividing N and x is not a prime power, then

$$\begin{aligned} |\Theta^*_{\mathrm{Sym}^n f}(x)| &\leq \frac{7}{25} n \log(n+1)\sqrt{x} + \left(\frac{1}{8}\log(x) + \frac{1}{7}\log(N(k-1))\right) n\sqrt{x} \\ &\quad + \frac{2}{5}\log(n+1)\sqrt{x} + \left(\log(x) + \frac{7}{5}\log(k+7)\right)\sqrt{x}. \end{aligned}$$

3.2 Proof of Theorem 1.3

We now present a proof of Theorem 1.3.

Proof. By [24], if $x \geq 2657$, then $|\pi(x) - \mathrm{Li}(x)| \leq \frac{1}{8\pi}\sqrt{x}\log(x)$. Thus if $x \geq 10^{17}$, then $|\pi(2x) - \pi(x) - \mathrm{Li}(2x) + \mathrm{Li}(x)| \leq \frac{1}{10}\sqrt{x}\log(x)$. Now, using the estimate from Lemma 3.3,

$$\begin{aligned} & \mu_{ST}(I)\left(\pi(2x) - \pi(x) - \frac{1}{10}\sqrt{x}\log(x)\right) - 4\delta\left(\mathrm{Li}(2x) - \mathrm{Li}(x)\right) \\ & - \sum_{n=0}^{\infty} |a_n(I,-\delta) - a_{n+2}(I,-\delta)||\Theta^*_{\mathrm{Sym}^n f}(x)| \\ \leq\ & (\mathrm{Li}(2x) - \mathrm{Li}(x))(\mu_{ST}(I) - 4\delta) - \sum_{n=0}^{\infty} |a_n(I,-\delta) - a_{n+2}(I,-\delta)||\Theta^*_{\mathrm{Sym}^n f}(x)| \\ \leq\ & (\mathrm{Li}(2x) - \mathrm{Li}(x))(a_0(I,-\delta) - a_2(I,-\delta)) + \sum_{n=0}^{\infty}(a_n(I,-\delta) - a_{n+2}(I,-\delta))\Theta^*_{\mathrm{Sym}^n f}(x) \\ =\ & \sum_{n=0}^{\infty}(a_n(I,-\delta) - a_{n+2}(I,-\delta))\Theta_{\mathrm{Sym}^n f}(x). \end{aligned}$$

By Lemma 3.4,

$$\sum_{n=0}^{\infty}(a_n(I,-\delta)-a_{n+2}(I,-\delta))\Theta_{\mathrm{Sym}^n f}(x) \leq \pi_{f,I}(x) \leq \sum_{n=0}^{\infty}(a_n(I,\delta)-a_{n+2}(I,\delta))\Theta_{\mathrm{Sym}^n f}(x).$$

Using Lemma 3.3 again,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(a_n(I,\delta)-a_{n+2}(I,\delta))\Theta_{\mathrm{Sym}^n f}(x) \\
= \;& (\mathrm{Li}(2x)-\mathrm{Li}(x))(a_0(I,\delta)-a_2(I,\delta)) + \sum_{n=0}^{\infty}(a_n(I,\delta)-a_{n+2}(I,\delta))\Theta^*_{\mathrm{Sym}^n f}(x) \\
\leq \;& (\mathrm{Li}(2x)-\mathrm{Li}(x))(\mu_{ST}(I)+4\delta) + \sum_{n=0}^{\infty}|a_n(I,\delta)-a_{n+2}(I,\delta)||\Theta^*_{\mathrm{Sym}^n f}(x)| \\
\leq \;& \mu_{ST}(I)\left(\pi(2x)-\pi(x)+\frac{1}{10}\sqrt{x}\log(x)\right) + 4\delta\left(\mathrm{Li}(2x)-\mathrm{Li}(x)\right) \\
& + \sum_{n=0}^{\infty}|a_n(I,\delta)-a_{n+2}(I,\delta)||\Theta^*_{\mathrm{Sym}^n f}(x)|.
\end{aligned}$$

We now subtract $(\pi(2x)-\pi(x))\mu_{ST}(I)$ throughout all of the above inequalities to obtain

$$\begin{aligned}
|\pi_{f,I}(x)-\mu_{ST}(I)(\pi(2x)-\pi(x))| \;\leq\;& 4\delta(\mathrm{Li}(2x)-\mathrm{Li}(x)) + \frac{\mu_{ST}(I)}{10}\sqrt{x}\log(x) \\
& + \sum_{n=0}^{\infty}|a_n(I,\delta)-a_{n+2}(I,\delta)||\Theta^*_{\mathrm{Sym}^n f}(x)|.
\end{aligned}$$

Using the estimates in Lemmas 3.2 and 3.10, we obtain the desired result by letting $\delta = \frac{\sqrt{39}}{20}x^{-1/4}\log(x)$. □

Chapter 4: Applications to Lehmer-Type Questions

4.1 Proof of Theorem 1.7

Because the proof of Theorem 1.7 parallels the proof of Theorem 1.3 with slight deviations, we will only sketch the proof of Theorem 1.7.

Proof. If δ is defined as in the proof of Theorem 1.3 and we let $I = [\frac{\pi}{2} - \frac{1}{2}\delta, \frac{\pi}{2} + \frac{1}{2}\delta]$, then

$$\pi_f(x) \leq \pi_{f,I}(x) = O(\mathrm{Li}(x)\delta) = O(x^{3/4}).$$

However, this bound can be improved. The aforementioned choice of I makes the main term the size of the error term, so we can afford to be less efficient with how we approximate $\Theta_{\mathrm{Sym}^n f}(x)$. Instead of taking the Mellin transform of $\chi_{[1,x]}(y)$, truncating the sum over zeroes at a height T, and performing Abel summation, we can take the Mellin transform of a compactly supported C^∞ function that overestimates $\chi_{[x,2x]}(y)$, say

$$h_x(y) = \exp\left(\frac{4}{3} - \frac{1}{1-(y/x-3/2)^2}\right)\chi_{(\frac{1}{2}x,\frac{5}{2}x)}(y).$$

Because $h_x(y)$ is C^∞, as y tends to infinity, the Mellin transform of $h_x(y)$ decays at a rate such that the sum over zeroes in the explicit formula converges. We can then use the explicit formula given in Theorem 5.11 of [9] to produce a bound on $\pi_f(x)$ analogous to the Chebyshev bound

$$0.92\frac{x}{\log(x)} < \pi(x) < 1.106\frac{x}{\log(x)}$$

for sufficiently large x (see Chapter 7 of [5] for more details). By analysis similar to that in the proof of Theorem 1.3, we have

$$\pi_f(x) = O\left(\frac{\delta x}{\log(x)} + \frac{\sqrt{x}\log(\delta)}{\log(x)\delta}\right),$$

where the implied constant is absolute and effectively computable. If we redefine δ to be $O(x^{-1/4}\sqrt{\log(x)})$, then

$$\pi_f(x) = O\left(\frac{x^{3/4}}{\sqrt{\log(x)}}\right).$$

$\square$

4.2 Densities and Lehmer-Type Questions

We now want to use Theorem 1.7 to give a lower bound for the density of positive integers n for which $a(n) \neq 0$. Let $\pi_f^*(x) = \#\{p \in [1,x] : a(p) = 0\}$. By applying Abel summation to the log of the product in Theorem 1.6, we have the following lemma.

Lemma 4.1. *Assume the above notation. We have*

$$\prod_{a(p)=0}\left(1 - \frac{1}{p+1}\right) = \exp\left(-\int_2^\infty \frac{\pi_f^*(x)}{x^2+x}\,dx\right).$$

Using the bound for $\pi_f(x)$ in Theorem 1.7, we now can produce an explicit lower bound for the density of positive integers n such that $a(n) \neq 0$. This allows us to address analogues of Lehmer's question (cf. [14]) for any newform f with squarefree level and trivial character, assuming Conjecture 2.32.

Theorem 4.2. *Assume the above notation. Let x_0 be greater than the largest prime dividing N. Define*

$$\omega_f(x, x_0) = \sum_{j=1}^{1+\lfloor \log_2(x/x_0)\rfloor} \pi_f(x/2^j).$$

Then

$$\prod_{a(p)=0}\left(1 - \frac{1}{p+1}\right) > \exp\left(-\int_{x_0}^\infty \frac{\omega_f(x,x_0)}{x^2+x}\,dx\right)\prod_{\substack{p\le x_0\\ a(p)=0}}\left(1-\frac{1}{p+1}\right).$$

Proof. This follows directly from the definition of $\pi_{f,I}(x)$, Lemma 4.1, and the straightforward computation

$$\exp\left(-\int_2^{x_0} \frac{\pi_f^*(x)}{x^2+x}\,dx\right) = \left(1+\frac{1}{x_0}\right)^m \prod_{j=1}^{m}\left(1-\frac{1}{p_j+1}\right),$$

where $\pi_f^*(x_0) = m$ and $p_1, \ldots, p_m$ are the counted primes. □

We now sketch a proof of Theorem 1.8.

Proof. To apply the preceding theorem, it is necessary to bound the number of small primes p for which $a(p) = 0$. On page 168 of [6], Bosman repeats Serre's observation that if $\tau_{12}(p) = 0$, then $p = hM - 1$ for some $h \geq 1$ and $M = 3094972416000$. Moreover, $h \equiv 0, 30,$ or $48 \pmod{49}$ and $h+1$ is a quadratic residue modulo 23. These facts will allow us to bound the density above.

In order to obtain results of a similar quality, we need analogues of the congruences for $\tau_{12}(n)$ given in [27] for the higher weight level 1 newforms. The congruences and their proofs for $\tau_k(n)$, where $k = 16, 18, 20, 22,$ or 26, are in Theorem 5.3 of [22].

We will discuss the case of $\tau_{12}(n)$; the other cases are proven similarly. Using the congruences for Δ and the computation of the mod 11, mod 13, mod 17 and mod 19 Galois representations by Bosman, we compute using PARI/GP that there are precisely 1810 primes $p < 10^{23}$ that satisfy the conditions given by Serre, and for which $\tau_{12}(p) \equiv 0 \pmod{11 \cdot 13 \cdot 17 \cdot 19}$. Let P be the set of these primes; we then calculate $\prod_{p \in P}(1 - \frac{1}{p+1})$. The desired result now follows from using Theorem 1.7 to estimate $\omega(x, 10^{23})$ in Theorem 4.2. □

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