EFFICIENT ITERATIVE ALGORITHM FOR COMPUTING QUASINORMAL MODES OF BLACK HOLES AND INFORMATION EXTRACTION FROM BLACK HOLE RINGDOWN SIGNAL

BY

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A Thesis Submitted to the Graduate Faculty of
WAKE FOREST UNIVERSITY GRADUATE SCHOOL OF ARTS AND SCIENCES
in Partial Fulfillment of the Requirements
for the Degree of
MASTER OF SCIENCE
Computer Science
May, 2014
Winston-Salem, North Carolina

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Acknowledgments

First and foremost, I would like to thank my advisors Dr Gregory B. Cook of the Department of Physics and Dr V. Paul Pauca of the department of Computer Science for their help and guidance. If there were people to whom this esoteric work is as important as it is to me, that would be them. Our interaction has not only been limited to academia. I also enjoyed playing many sports with them. Their academic excellence coupled with athleticism exemplifies a type of a person I want to be. I thank Dr Plemmons, who agreed to be on my committee and was willing to devote his time and effort during his retirement.

There are many people whom I have been fortunate to meet throughout my graduate studies. The Computer Science classes I took from Dr Todd Torgersen and Dr Errin Fulp were both entertaining and informative. My fellow graduate students (past and present) have provided me with a rich environment for my academic growth. David Sontheimer, Edison Munoz, Percy Campos, Freddie Awuah-Gyasi, Jose Picado, Dan Xue of Computer Science Department, Shuowen Wei, Katie Novacek, Kris Patton, Joe Paat, Katy Beeler of the Mathematics Department and, of course, the list would not be complete without the physics graduate students, with many of whom I have taken Computer Science classes, Nicholas Lepley, Jeremy Hohertz, Jason Bates, Richard Dudley, Xiao Xu, Joel Grim, Qi Li, Yuan Li and others. I wish you all success in your future careers.

I would also like to thank the Graduate School of Arts and Sciences for the continuing financial support provided through the Departments of Physics and Computer Science.

Last but not least, I thank my family. My mother Valentina Zalutskaya and father Pavel Zalutskiy, who understood the importance of higher education and supported my pursuit of knowledge with everything they had. I specifically thank my sister Lilya McDonald who encouraged me to apply to a graduate school in the USA.

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Abstract

The present work shows how black hole information, such as mass and angular momentum, can be extracted via a new theoretical and computational analysis of the hole’s so called ringdown radiation, which is a gravitational wave emitted by a freely oscillating black hole.

This analysis fits within the broader field of numerical relativity whose treatment requires fast and robust computational algorithms. A new spectral algorithm is presented in this work that solves the spin weight spheroidal equation, describing angular perturbations of a black hole. We then use it together with Leaver’s continued fraction method to calculate quasinormal modes (QNM) of a Kerr black hole for multipole number $l = 2$ through 12 and first 8 overtones with $\epsilon = 10^{-12}$ accuracy.

The computed QNM spectrum is then used to represent the ringdown signal of a black hole, which is a signature radiation that corresponds to the hole’s parameters. According to the no-hair theorem these parameters are limited to the mass, angular momentum and charge. We assume the ringing black hole is electrically neutral so only the information about its mass and angular momentum is present in the ringdown signal. The multimode nonlinear fitting formalism, that we developed, allows us to extract that information from the black hole.
Chapter 1: Introduction

When Einstein’s equations of general relativity are solved, they allow a very interesting solution for the spacetime metric, describing a trapped region from which nothing (even light) can escape. Such a solution is known as a *black hole*\(^1\), the term John Wheeler coined in 1967. Since then, black holes have attained the status of mainstream subjects of research and are no longer regarded as a theoretical curiosity. Due to their nature, black holes are really hard to detect and their physical reality is usually supported by an array of indirect evidence, mainly dealing with the behavior of the surroundings of a black hole candidate.

Nevertheless, the question remains: is there a way to see a black hole by a direct observation?

First of all, what does, in the context of black holes, an observation mean? According to the *no-hair theorem* [MTW73] the amount of information describing a given black hole is limited to three things: mass, angular momentum and charge. Therefore, measuring the hole’s mass, angular momentum (if it has it) and charge (if it has it) would constitute an observation.

Such an observation surely presents a challenge for a conventional astronomer, who is, limited by traditional techniques, unable to see black holes, but that is when another prediction of general relativity comes into play. Einstein’s equations can be linearized and the wave equation for the perturbation metric can be derived, thus predicting the existence of a certain kind of waves travelling at the speed of light, which are called *gravitational waves*. What kind of waves are those? These are tiny tidal perturbations of spacetime propagating outwards from the source. Such a source can be any asymmetric accelerating distribution of matter. In order for the signal to be detectable in a practical sense of the word, the matter should

---

\(^1\)The term no longer portrays the concept accurately, since the discovery of Hawking radiation[Haw74]
be quite massive. Perturbed black holes, due to their enormous mass, are very
effective as generators of gravitational waves which can be detected.

In our investigation we deal with a certain type of black hole gravitational
radiation known as ringdown, which is the characteristic radiation resultant from
the hole’s free oscillations after being perturbed from equilibrium. The black
hole is modeled as spinning, but electrically neutral (Kerr black hole) so the two
relevant pieces of information to be determined are mass and angular momentum.
The ringdown signal is mathematically represented by a set of quasinormal modes
(QNM) [Pre71, PT74]. We compute QNM for a wide set of parameters so that
the ringdown signal can be represented accurately. We use Leaver’s numerical
algorithm with QNM frequencies as roots of a continued fraction corresponding to
the equation for the radial perturbation of the black hole. We use a new method,
which we call the spectral method to find the separation variable from the angular
perturbation equation. We compare and contrast the spectral method with the
continued fraction method and show that the former has an array of advantages.

The present estimates for the mass and angular momentum of a black hole
rely on the analysis of the gravitational waveform with only one or two, usually
dominant, modes (see, for example, [Ech89, Fin92, BCW06]). It is our objective
to develop a formalism for a multimode analysis of the radiation. As part of
aforementioned objective, we developed a nonlinear fitting tool using Wolfram
Language™ of Mathematica®. With its help we can fit any ringdown signal
given in a basis of spin weight spherical harmonics (see Appendix B). As the fit
matches the data with a satisfactory accuracy, it supports our confidence that
the mass and spin parameter, which were used to generate the fit, correspond
to the hole’s true mass and spin. It is our hope, that the presented theoretical
and computational analysis will pave the way to the first experimental detection
of the gravitational wave signal, which will mark the beginning of a new field:
gravitational wave astronomy.

The following are the key contributions of the present work:
• We developed a new method of solving a spin weighted spheroidal equation that reduces the corresponding Sturm-Liouville problem to a smaller eigenvalue problem. This, so called, spectral method is compared against Leaver’s continued fraction method, showing high accuracy and convergence.

• We developed a multimode fitting formalism organised as a Mathematica® package that can be used for study of ringdown signals. This package will be made available to the gravitational wave researchers.
Chapter 2: Quasinormal modes of black holes

When perturbed, black holes vibrate with characteristic frequencies, similar to the way any other oscillating systems do in nature. However, the peculiar feature of black holes is that their modes are always damped due to the radiation loss at infinity and the horizon. This dissipation cannot be removed even in principle so the modes of black holes are always quasinormal. They were first numerically observed by Vishveshwara in 1970 [Vis70] in the problem of scattering of gravitational waves by a black hole. Since then the subject of QNM has been studied by many authors [Pre71, DRPP71, CD75, Det77, Lea85, FM84] with comprehensive reviews appearing in print after about every decade of progress [Cha83, KS99, BCS09]. The best numerical algorithm for computing QNM was developed by Leaver which he applied to two black hole geometries: Schwarzschild and Kerr.

2.1 QNM. Problem formulation

The Kerr black hole is an astrophysically very important solution to Einstein’s equations, which describes a rotating uncharged black hole. When the geometry of spacetime is perturbed with respect to the Kerr solution, it results in a complicated partial differential equation which can be separated into equations, with four functions describing the perturbation: temporal, two angular (azimuthal $\phi$ and polar $\theta$) and radial. The temporal and azimuthal functions are harmonic decompositions with solutions behaving like $e^{-i\omega t}$ ($\omega$ is QNM frequency) and $e^{im\phi}$ respectively. The radial $R(r)$ and angular polar $s_{lm}(x)$ functions satisfy the equations which were derived by Teukolsky (see [Teu72] or more detailed [Teu73]).
\[
\begin{align*}
\Delta^s \frac{d}{dr} \left[ \Delta^{s+1} \frac{dR(r)}{dr} \right] + \left[ \frac{K^2 - 2i s (r - M) K}{\Delta} + 4i s \omega r - \lambda \right] R(r) &= 0, \\
\frac{d}{dx} \left[ (1 - x^2) \frac{d}{dx} S_{\ell m}(x) \right] + \left[ (cz)^2 - 2csx + s + A_{\ell m} - \frac{(m + sx)^2}{1 - x^2} \right] S_{\ell m}(x) &= 0.
\end{align*}
\]

(2.1)

where \( \omega \) is the QNM frequency of the oscillation, which appears as a part of an oblateness parameter \( c = a \omega \) in the angular equation. \( A_{\ell m} \) is a separation constant. Also the angular variable is redefined as \( x \equiv \cos(\theta) \). Other parameters for both the equations are defined below

\[
\Delta \equiv r^2 + a^2 - 2Mr, \quad \Sigma \equiv r^2 + a^2 \cos^2 \theta, \quad \Lambda \equiv r^2 + a^2 \omega^2 - 2aM\omega.
\]

(2.2)-(2.4)

where \( M \) is the mass of the black hole and \( a \) is the spin parameter, which is related to the hole’s angular momentum.

Each equation in (2.1) presents a Sturm-Liouville type of problem with \( \omega \) being an eigenvalue of the Radial equation and \( A_{\ell m} \) being the eigenvalue of the angular equation. The differential operators in both the equations are not self-adjoint, which makes the eigenvalues complex. The equations are coupled since both \( \omega \) and \( A_{\ell m} \) appear in both equations, requiring simultaneous solution of the equations.

Let us look at each equation in (2.1). The solution to the radial equation can be written as

\[
R(r) = e^{i \omega r} (r - r_+)^{-s+1+2i \sigma_+}(r - r_-)^{-s-i \sigma_+} \sum_{n=0}^{\infty} a_n \left( \frac{r - r_+}{r - r_-} \right)^n,
\]

(2.7)

where \( \sigma_+ \equiv \frac{2 \omega r_+ - ma}{r_+ - r_-} \) with \( r_+ \) and \( r_- \) defined according to (2.5). It should be noted that the transition to dimensionless quantities has occurred by setting
mass \( M = 1 \) (also the gravitational constant and speed of light are both set to unity, \( G = c = 1 \)). The common factor of the series in (2.7) is a combination of asymptotic behaviors at the hole’s horizon and infinity. Inserting (2.7) into the radial equation in (2.1) yields

\[
a_0 D_3 + a_1 D_0 + \sum_{n=1}^{\infty} \left\{ a_{n+1} \left\{ (n^2 + (1 + D_0)n + D_0) \right\} \\
+ a_n \left\{ -2n^2 + (2 + D_1)n + D_3 \right\} \\
+ a_{n-1} \left\{ n^2 + (-3 + D_2)n \right\} \\
+ D_4 - D_2 + 2 \right\} \right\} \left( \frac{r - r_+}{r - r_-} \right)^n = 0,
\]

(2.8)

where

\[
D_0 \equiv 1 - s - 2i\sigma_+,
\]

(2.9)

\[
D_1 \equiv 4[i(\sigma_+ + \omega) - 1] + 2i(r_+ - r_-)\omega,
\]

(2.10)

\[
D_2 \equiv 3 + s - 2i(\sigma_+ + 2\omega),
\]

(2.11)

\[
D_3 \equiv (4r_+ - a^2)\omega^2 + 2\omega((r_+ - r_-)\sigma_+ + 4\sigma_+ + i\sigma_+ - (1 + s) - A_{lm}),
\]

(2.12)

\[
D_4 \equiv (1 - 4i\omega)(1 + s - 2i\sigma_+).
\]

(2.13)

Finally, since Eq. (2.8) must be true for all values of \( \left( \frac{r - r_+}{r - r_-} \right) \), we find

\[
a_0 \beta_0 + a_1 \alpha_0 = 0,
\]

(2.14)

\[
a_{n+1}\alpha_n + a_n\beta_n + a_{n-1}\gamma_n = 0,
\]

(2.15)

\[
\alpha_n \equiv n^2 + (D_0 + 1)n + D_0,
\]

(2.16)

\[
\beta_n \equiv -2n^2 + (D_1 + 2)n + D_3,
\]

(2.17)

\[
\gamma_n \equiv n^2 + (D_2 - 3)n + D_4 - D_2 + 2.
\]

(2.18)

The \( \alpha_n, \beta_n, \gamma_n \) are known functions of the complex frequencies \( \omega \). The series in solution (2.7) converges if \( \omega \) is a root of the following continued fraction (provided
or written in a standard notation as

\[ \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 - \frac{\alpha_1 \gamma_2}{\beta_2 - \frac{\alpha_2 \gamma_3}{\beta_3 - \ldots}}} = 0. \]  

(2.20)

We can apply the same approach of power series expansion to the second equation in (2.1) and get a continued fraction equation for finding \( A_{\ell m} \). However, there is a better method.

### 2.2 Spectral method

Solutions to the angular equation in (2.1) are known as spin-weighted spheroidal harmonics (SWSdH)\(^1\) (see Appendix A). They do not have an analytic representation and are always determined numerically. We calculate SWSdH using a new algorithm which we call the spectral method. It has much better convergence than the continued fraction method and it seems more stable, although that was not rigorously investigated. The main idea behind it is a new set of basis functions, which is selected with regard to the nature of the equation to be solved. Such a set can be build out of spin-weighted spherical harmonics (SWSH) (see Appendix B), which are a special case of SWSdH with an oblateness parameter \( c \equiv a \omega \) set to zero. For a general spin weight \( s \) the expansion is

\[ sS_{\ell m}(\theta, \phi; c) = \sum_{\ell' = \ell_0}^{\infty} C_{\ell' \ell m}(c) Y_{\ell' m}(\theta, \phi) \]

(2.21)

\[ = \sum_{\ell' = \ell_0}^{\infty} C_{\ell' \ell m}(c) S_{\ell' m}(x; 0), \]

\(^1\)These are only \( \theta \)-parts of the harmonics, so the more appropriate term would be spin-weighted spheroidal functions. However, throughout the body of the text the term harmonic would be used to mean both, except in places where the distinction is important.
where \( \ell_0 \equiv \max(|m|, |s|) \) and \( x \equiv \cos \theta \). SWSH \( Y_{\ell m}(\theta, \phi) \) form a complete set of orthonormal basis functions. They are defined as

\[
Y_{\ell m}(\theta, \phi) = (-1)^s \sqrt{\frac{2\ell + 1}{4\pi}} d^\ell_{m(s)}(\theta)e^{im\phi},
\]

(2.22)

where \( d^\ell_{m(s)}(\theta) \) is the Wigner d-function. Spin weight properties of the harmonics are important in describing the gravitational field, which is a spin -2 field. Moreover, SWSH have an analytical representation which simplifies the calculations. Thus, the problem of solving the angular equation is reduced to finding the coefficients \( C_{\ell' \ell m}(c) \), which have the meaning of SWSdH/SWSH scalar products. The importance of this result is going to be highlighted in the next chapter.

We substitute the expansion (2.21) in the angular perturbation equation in (2.1)

\[
\frac{d}{dx} \left[ (1 - x^2) \frac{d}{dx} [s_{\ell m}(x; c)] \right] = \sum_{\ell' = \ell_0}^\infty C_{\ell' \ell m}(c) \frac{d}{dx} \left[ (1 - x^2) \partial_x [s_{\ell m}(x; 0)] \right]
\]

\[
= - \sum_{\ell' = \ell_0}^\infty C_{\ell' \ell m}(c) \left[ s + \ell' (\ell' + 1) - s(s + 1) - \frac{(m + sx)^2}{1 - x^2} \right] , s_{\ell' m}(x; 0)
\]

\[
= - \sum_{\ell' = \ell_0}^\infty C_{\ell' \ell m}(c) [\ell' (\ell' + 1)] , s_{\ell m}(x; 0) + \left[ s^2 + \frac{(m + sx)^2}{1 - x^2} \right] , S_{\ell m}(x; c)
\]

\[
= - \left[ (cx)^2 - 2csx + s + A_{\ell m} - \frac{(m + sx)^2}{1 - x^2} \right] , S_{\ell m}(x; c).
\]

Cancelling the common term in the last two lines we have

\[
\left[ (cx)^2 - 2csx + s(s + 1) + A_{\ell m} \right] , S_{\ell m}(x; c) =
\]

\[
\sum_{\ell' = \ell_0}^\infty C_{\ell' \ell m}(c) [\ell' (\ell' + 1)] , S_{\ell m}(x; 0)
\]

(2.23)

which we can rewrite as

\[
\sum_{\ell' = \ell_0}^\infty C_{\ell' \ell m}(c) [\ell' (\ell' + 1) - s(s + 1) - (cx)^2 + 2csx - A_{\ell m}] , S_{\ell m}(x; 0) = 0.
\]

(2.24)
We then eliminate $x$ and $x^2$ dependence using recurrence relations (B.5) and (B.9), and shift the $\ell'$ indices so that all of the spin weight spherical functions $S_{\ell' m}(x; 0)$ are of the same order in $\ell'$, we get

$$
- \sum_{\ell'=\ell_0+2}^{\infty} c^2 C_{(\ell'-2)\ell m}(c) A_{s(\ell'-2)m} S_{\ell' m}(x; 0)
$$

$$
- \sum_{\ell'=\ell_0+1}^{\infty} C_{(\ell'-1)\ell m}(c) \left[ c^2 D_{s(\ell'-1)m} - 2 cs F_{s(\ell'-1)m} \right] S_{\ell' m}(x; 0)
$$

$$
+ \sum_{\ell'=\ell_0}^{\infty} C_{(\ell')\ell m}(c) \left[ [\ell' (\ell' + 1) - s(s + 1) - c^2 B_{s\ell' m} + 2 cs H_{s\ell' m}] \right] S_{\ell' m}(x; 0)
$$

$$
- \sum_{\ell'=\ell_0-1}^{\infty} C_{(\ell'+1)\ell m}(c) \left[ c^2 E_{s(\ell'+1)m} - 2 cs G_{s(\ell'+1)m} \right] S_{\ell' m}(x; 0)
$$

$$
- \sum_{\ell'=\ell_0-2}^{\infty} c^2 C_{(\ell'+2)\ell m}(c) C_{s(\ell'+2)m} S_{\ell' m}(x; 0)
$$

$$
= \sum_{\ell'=\ell_0}^{\infty} A_{\ell m} C_{\ell' \ell m}(c) S_{\ell' m}(x; 0). \quad (2.25)
$$

Since the index $\ell'$ cannot take on the value lower than $\ell_0$ we must have $F_{s(\ell_0-1)m} = 0$, which leads to $A_{s(\ell_0-1)m} = A_{s(\ell_0-2)m} = D_{s(\ell_0-1)m} = 0$. Also, for the same reason $S_{(\ell_0-1)m}(x; 0) = S_{(\ell_0-2)m}(x; 0) = 0$, which allows us to put all the terms under one sum

$$
\sum_{\ell'=\ell_0}^{\infty} \left\{ - c^2 A_{s(\ell'-2)m} C_{(\ell'-2)\ell m}(c) - \left[ c^2 D_{s(\ell'-1)m} - 2 cs F_{s(\ell'-1)m} \right] C_{(\ell'-1)\ell m}(c) \right. 
$$

$$
+ [\ell' (\ell' + 1) - s(s + 1) - c^2 B_{s\ell' m} + 2 cs H_{s\ell' m}] C_{\ell' \ell m}(c)
$$

$$
- \left[ c^2 E_{s(\ell'+1)m} - 2 cs G_{s(\ell'+1)m} \right] C_{(\ell'+1)\ell m}(c)
$$

$$
- c^2 C_{s(\ell'+2)m} C_{(\ell'+2)\ell m}(c) \right\} S_{\ell' m}(x; 0)
$$

$$
= \sum_{\ell'=\ell_0}^{\infty} A_{\ell m}(c) C_{\ell' \ell m}(c) S_{\ell' m}(x; 0). \quad (2.26)
$$
We find, then, the five-term recurrence relation for the expansion coefficients
\[ -c^2 A_{s(\ell'-2)m} C_{(\ell'-2)\ell m}(c) - [c^2 D_{s(\ell'-1)m} - 2csF_{s(\ell'-1)m}] C_{(\ell'-1)\ell m}(c) \]
\[ + \left[ \ell'(\ell' + 1) - s(s + 1) - c^2B_{s\ell' m} + 2csH_{s\ell' m} \right] C_{\ell'\ell m}(c) \]
\[ - \left[ c^2 E_{s(\ell'+1)m} - 2csG_{s(\ell'+1)m} \right] C_{(\ell'+1)\ell m}(c) - c^2C_{s(\ell'+2)m} C_{(\ell'+2)\ell m}(c) \]
\[ = A_{\ell m}(c) C_{\ell'\ell m}(c). \] (2.27)

We need to truncate the series at \( \ell_N \) so we have a finite-dimensional spectral approximation of the angular equation in (2.1). With \( N = \ell_N - \ell_0 + 1 \), then for given values of \( s \) and \( m \), we have an \( N \times N \) pente-diagonal matrix
\[
M = \begin{pmatrix}
M_{\ell_0,\ell_0} & M_{\ell_0,\ell_0+1} & M_{\ell_0,\ell_0+2} & 0 & 0 & \cdots & 0 \\
M_{\ell_0+1,\ell_0} & M_{\ell_0+1,\ell_0+1} & M_{\ell_0+1,\ell_0+2} & M_{\ell_0+1,\ell_0+3} & 0 & \cdots & 0 \\
M_{\ell_0+2,\ell_0} & M_{\ell_0+2,\ell_0+1} & M_{\ell_0+2,\ell_0+2} & M_{\ell_0+2,\ell_0+3} & M_{\ell_0+2,\ell_0+4} & \cdots & 0 \\
0 & M_{\ell_0+3,\ell_0+1} & M_{\ell_0+3,\ell_0+2} & M_{\ell_0+3,\ell_0+3} & M_{\ell_0+3,\ell_0+4} & \cdots & 0 \\
0 & 0 & M_{\ell_0+4,\ell_0+2} & M_{\ell_0+4,\ell_0+3} & M_{\ell_0+4,\ell_0+4} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & M_{\ell_N,\ell_N}
\end{pmatrix}
\] (2.28)
whose elements are given by
\[
M_{\ell\ell'} = \begin{cases}
\text{highest diagonal (} \ell' = \ell + 2 \text{)} : & -c^2 C_{s\ell' m}, \\
\text{higher diagonal (} \ell' = \ell + 1 \text{)} : & -c^2 E_{s\ell' m} + 2csG_{s\ell' m}, \\
\text{main diagonal (} \ell' = \ell \text{)} : & \ell'(\ell' + 1) - s(s + 1) - c^2B_{s\ell' m} + 2csH_{s\ell' m}, \\
\text{lower diagonal (} \ell' = \ell - 1 \text{)} : & -c^2 D_{s\ell' m} + 2csF_{s\ell' m}, \\
\text{lowest diagonal (} \ell' = \ell - 2 \text{)} : & -c^2 A_{s\ell' m}, \\
\text{otherwise} : & 0.
\end{cases}
\] (2.29)

Thus the Sturm-Liouville eigenvalue problem represented by the angular equation in (2.1) is reduced to the simpler eigenvalue equation
\[
M \cdot \vec{C}_{\ell m}(c) = A_{\ell m} \vec{C}_{\ell m}(c). \] (2.30)
Figure 2.1: The convergence of the spectral method (green triangles) and continued fraction method (blue dots) shown as a function of the natural logarithm of the accuracy vs the number of terms in the expansion.

The matrix $M$ is a complex valued symmetric matrix that is constructed out of the known functions according to (2.29), provided $s$, $m$ and $c$ are given. Its $N$ eigenvalues are the $A_{\ell m}$, where $\ell \in \{\ell_0, ..., \ell_N\}$, and the elements of the corresponding eigenvector $\vec{C}_{\ell m}(c)$ are the expansion coefficients $C_{\ell \ell m}(c)$. We are only interested in one eigenvalue at a time corresponding to a particular $\ell$ (for given $s$, $m$, and $c$), which we will select out of $\{\ell_0, ..., \ell_N\}$ set.

The accuracy with which we calculate the eigenvalue is related to the number of terms (the size of matrix $N$) that we keep in the expansion (2.21) and accuracy of the value of the QNM frequency being supplied to the Eigensystem solver [Mat14a]. Due to the exponential convergence of spectral expansions on the one hand, and the appropriate choice of the function basis set on the other hand, the matrix $M$ is not very big, yet it allows us to have $A_{\ell m}$ and $\vec{C}_{\ell m}(c)$ with an extremely high accuracy.
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2.3 Spectral method vs continued fraction method

Both Leaver’s continued fraction and our spectral methods converge exponentially, however the spectral method has better convergence as it is evident from Fig. 2.1, where we plot the natural logarithm of a difference between the true value\(^2\) of the separation constant \(A_{22}\) as a function of the number of terms in the expansion for the mode corresponding to \(l = 2, m = 2\), which is the dominant mode in problems where the ringdown is the result of a merger of a black hole binary.

In the continued fraction method we find the separation variable \(A_{lm}\) as a root of the continued fraction using a two dimensional Newton’s method, which is known to have quadratic convergence. However, the number of function evaluations needs to be taken into account in order to estimate the total time cost. With three function evaluations per iteration (two additional function evaluations are needed to calculate the derivatives of the real and imaginary parts) the total time cost of the method becomes super-linear. Also, Newton’s method is known to be sensitive to the choice of the initial guess for a root, so it might not converge at all if the function behaves badly. Under-relaxation might help the matter, but that will lead to a slow down adding to the total time cost.

In the spectral method we solve an eigenvalue problem of matrix \(M\), which happens to be i) symmetric, ii) sparse and iii) not very big. The latter is significant as it allows us to compute the eigenvalues by a direct method exactly. Hence the error in estimating the eigenvalue will only come from the truncation of the

\(^2\)True value is calculated by either method with a large number of terms \(N\)
expansion. Of course, one must take into account the error with which $\omega_{lm}$ was calculated, which is supplied to the method from the previous iteration of the radial continued fraction.

The features of both methods are summarized in Table 2.1

### 2.4 QNM. Computation

In solving the system (2.1) our primary goal is to find its eigenvalues: $\omega$ and $A_{lm}$. The former corresponds to the QNM frequency of a Kerr (rotating) black hole and the latter does not have a physical interpretation and is going to be treated as a necessary computational step. The frequency $\omega$ is the root of continued fraction (2.20) and the separation constant $A_{lm}$ can be found solving eigenvalue problem (2.30) so the system (2.1) is reduced to

\[
\begin{align*}
\text{CF}(\omega; A_{lm}) &= 0, \\
\textbf{M} \cdot \mathbf{C}_{lm}(a\omega) &= A_{lm} \mathbf{C}_{lm}(a\omega),
\end{align*}
\]

where the first equation represents a continued fraction equation on $\omega$ with $A_{lm}$ as a parameter. Each QNM frequency is labeled by three indices: multipole index $l$, azimuthal index $m$ and the overtone number $n$. The dependence on $m$ is explicit as it appears in the equations. The allowed values for $m$ are integers ranging from $-l$ to $l$. For given $l$ and $m$ the radial equation (and the corresponding continued fraction equation) allows multiple solutions for $\omega$, which are often called overtones. We are going to label those solutions with a separate index $n$, so the notation for the frequency will be $\omega_{lmn}$.

Before we proceed to solve the above system, it is useful to solve a simpler problem corresponding to a Schwarzschild (non-rotating) black hole, where the angular momentum is set to zero ($a = 0$) in which case the angular equation can be solved analytically. The radial equation gets simplified and the previously described analysis of its solution can be repeated.
Once again the radial equation has a power series solution, whose coefficients satisfy a three-term recurrence relationship. To make the power expansion converging the coefficients should obey a set of rules leading to a continued fraction equation similar to (2.20). For practical purposes the continued fraction can be inverted \( n \) times:

\[
\beta_n - \frac{\alpha_{n-1} \gamma_n}{\beta_{n-1}} - \frac{\alpha_{n-2} \gamma_{n-1}}{\beta_{n-2}} - \ldots - \frac{\alpha_1 \gamma_1}{\beta_1} = \frac{\alpha_{n+1} \gamma_{n+1}}{\beta_{n+1}} - \frac{\alpha_{n+2} \gamma_{n+2}}{\beta_{n+2}} - \ldots
\]

(2.32)

Both forms are equivalent, yet the latter is preferable due to the fact that the \( n \)-th overtone is usually the most stable root of the \( n \)-th inversion. We search for the root numerically, using two-dimensional Newton’s method with the initial guess \( \omega_0 = (l + \frac{1}{2} - (n + \frac{1}{2})i) / (27)^{1/4} \), which is the asymptotic behavior of the modes for large \( l \) [FM84]. It works as an initial guess for small \( l \) as well.

Calculating Schwarzschild modes does not present a computational challenge. We calculated the first eight overtones \((n=0\) through \(n=7)\) for \(l=2\) through \(l=12\) with the accuracy \( \epsilon = 10^{-14} \) which are plotted in Figure 2.2.
**input**: \( a_j, b_j \) of the Continued Fraction, accuracy \( \epsilon \)

**output**: the value of Continued Fraction

\[
f_1 = b_0; \\
C_1 = f_1; \\
D1 = 0; \\
\Delta = 10; \\
\text{while } \text{Abs}(\Delta - 1) > \epsilon \text{ do} \\
\quad D_2 = b_j + a_jD_1; \\
\quad C_2 = b_j + a_j/C_1; \\
\quad D_2 = 1/D_2; \\
\quad \Delta = C_2D_2; \\
\quad f_2 = f_1\Delta; \\
\quad \text{swap } D1 \text{ with } D2, C_1 \text{ with } C_2, f_1 \text{ with } f_2; \\
\text{end}
\]

Algorithm 2.1. Lenz’s method of the Continued Fractions evaluation

The accuracy of the root is related to the accuracy with which we evaluate the continued fraction (which in turn, is related to the accuracy of evaluation of the power series expansion). We need to know how far down the right hand side of (2.32) needs to go before we can truncate it and not loose accuracy. The way a continued fraction is written as either (2.20) or (2.32) assumes a bottom-up evaluation. Let us first redefine the coefficients: \( a_j \equiv -\alpha_{n+j-1}\gamma_{n+j} \) and \( b_j \equiv \beta_{j+n} \).

Then the right hand side of (2.32) becomes

\[
f_j = \frac{a_1}{b_1} \frac{a_2}{b_2} \cdots \frac{a_j}{b_j} \quad (2.33)
\]

Next, we rewrite it as a rational number \( f_j = \frac{A_j}{B_j} \), with \( A_j \) and \( B_j \) defined recurrently

\[
\begin{align*}
A_1 & \equiv 1, & B_1 & \equiv 0, \\
A_0 & \equiv b_0, & B_0 & \equiv 1, \\
A_j & = b_jA_{j-1} + a_jA_{j-2}, & B_j & = b_jB_{j-1} + a_jB_{j-2}.
\end{align*}
\]  

(2.34)

Now it becomes possible to evaluate the continued fraction in a top-down manner with a predetermined accuracy. The modified Lentz’s method of a continued fraction evaluation [Len76, PTVF07] takes advantage of this approach. Instead of
input: Schwarzschild modes for given $l$ and the overtone number $n$.
output: Kerr modes for $l$, $m$ and the overtone number $n$

$$\omega_0 = \text{SchwarzschildMode}(l, n);$$
$$c = a\omega_0;$$
$$A_0 = l(l+1) - s(s+1) + O(c);$$
$$\epsilon = 10^{-12};$$
$$\Delta A = A_0; \Delta \omega = \omega_0;$$
for $a = 0$ to $a_{max}$ do
  while $\Delta A \geq \epsilon$ or $\Delta \omega \geq \epsilon$ do
    $$\omega = \text{Newton2}(\omega_0, A_0);$$
    $$c = a\omega;$$
    $$A = \text{GetASpectral}(c);$$
    swap $A$ and $\omega$ with $A_0$ and $\omega_0$;
    update $\Delta A$ and $\Delta \omega$;
  end
  save the mode in the KerrQNMTable;
  increment $a$;
end

Algorithm 2.2. Kerr QNM algorithm

working with numerator $A_j$ and denominator $B_j$ directly, the method introduces ratios

$$C_j \equiv \frac{A_j}{A_{j-1}}, \quad D_j \equiv \frac{B_{j-1}}{B_j}$$

and the continued fraction (2.33) can be calculated using

$$f_j = f_{j-1}C_jD_j.$$  

(2.36)

The pseudo code of the method is given in Algorithm 2.1. Notice that the loop termination variable $\Delta$ is set to a high enough value so that we are guaranteed to enter into the loop. The top-down evaluation for the continued fraction via Lenz’s method makes it really easy to have a good control over the accuracy. It must be noted that it is also possible to evaluate the continued fraction from bottom-up effectively controlling accuracy via power series expansion of the residual [CZ14].

Let us now return to the system (2.31). The equations are coupled via their eigenvalues, which means they are to be solved simultaneously. To find a self-consistent solution, we iterate between the two equations until the desired accuracy
The first eight overtones of Kerr quasinormal modes for $l = 2$ through 12. Each color (shade of gray for black & white printing) corresponds to one overtone. Schwarzschild modes shown as a black dashed line. The curves are parametrized by the value of the dimensionless rotation parameter $a$.

of the solution is attained. The essence of the method is shown in Algorithm (2.2). Running the code produces set of Kerr modes for given $l$, $m$, overtone $n$ and spin parameter $a$ ranging from 0 to $a_{\text{max}}$. The value for $a_{\text{max}}$ can get very close to 1, but never is equal to 1. We start with Schwarzschild modes with fixed $l$ and the overtone $n$ slowly increasing the spin parameter $a$. We use the previous iteration as a guess for the root in Newton’s method. As $a$ gets closer to one, it becomes more difficult to find the root of the continued fraction with a high accuracy. Moreover, the Newton’s method might converge to a root corresponding to a different overtone. To counteract these challenges we found it useful to use under-relaxation in the following sense. The Newton correction is written in the form $\omega_{\text{next}} = \omega_{\text{prev}} + u\Delta\omega$, where $0 < u \leq 1$. That leads to a greater control over to which continued fraction root the method converges. Unfortunately, it also leads to a slow down in the overall convergence of the method, but that is a price one should be willing to pay. There is a way, however, to avoid under-relaxation. In calculating the derivative for the Newton’s method, the value for the separation
constant is fixed throughout. This is not necessary and if the separation variable is allowed to be recalculated at an increased value of $\omega$, that will lead to better convergence of the method [CZ14].

It must also be noted, that due to the nature of our iterative algorithm, we need to increase the precision of the computation far enough, so that the round off error would not affect the accuracy of the results. A precision of 1.5 times machine precision (about 24 correct digits) is sufficient in most cases.

The result of calculation is plotted in Figure 2.3. The modes split into $2l+1$ curves ($m$ goes from $-l$ to $l$) with the spin parameter $a$ increasing along the branches away from the Schwarzschild limit. The ”zero-damped” regions where the modes coalesce to purely real frequencies for large value of $a$ are visible around $Re(M\omega) = m/2$.

It must be noted, that the eigenvalue problem allows another solution with a negative real part of the frequency, which means that the full picture of the modes is that of Figure 2.3 reflected with respect to $Im(\omega)$ axis and the relationship between the modes is

$$Re(\omega_{lmn}) = -Re(\omega_{l(-m)n}), \quad Im(\omega_{lmn}) = Im(\omega_{l(-m)n}).$$

(2.37)
Chapter 3: Quasinormal Mode Decomposition of the ringdown signal

Having solved the previously defined perturbation problem (2.1) we can now describe the ringdown signal, that a Kerr black hole is emitting. In a most general case, the field can be written as [Teu72]

$$\psi(t, r, \theta, \phi) = \frac{1}{2\pi} \int e^{-i\omega t} \sum_{\ell,m} S_{\ell m}(\theta, \phi; a\omega) R_{\ell m}(r) d\omega,$$  

(3.1)

where the summation over $\ell$ and $m$ is always confined to $|s| < \ell < \infty$ and $|m| \leq \ell$. The angular perturbation is described by SWSdH $S_{\ell m}$ and radial perturbation is described by $R_{\ell m}$. To represent the ringdown (or any kind of a outgoing gravitational wave), we use $s = -2$ and $\psi = \rho^{-4}\Psi_4$, with $\Psi_4$ known as a Weyl scalar. It behaves like an outgoing gravitational wave at large distances and $\rho \equiv -1/(r - ia \cos \theta)$. Asymptotically, $R_{\ell m}(r) = r^3 e^{i\omega r^*}$ ($r^*$ is the “tortoise” radial coordinate), $\rho = 1/r$, and we find

$$\Psi_4 \sim \frac{1}{r^2} \frac{1}{2\pi} \int e^{-i\omega(t-r^*)} \sum_{\ell,m} S_{\ell m}(\theta, \phi; a\omega) d\omega.$$  

(3.2)

Making both sides dimensionless, we take as our general asymptotic expansion for an outgoing gravitational wave:

$$r M \Psi_4 = \frac{M}{2\pi} \int e^{-i\omega(t-r^*)} \sum_{\ell,m} S_{\ell m}(\theta, \phi; a\omega) d\omega.$$  

(3.3)

In order to represent the ringdown signal of a Kerr black hole, we carefully replace the Fourier integral with a QNM expansion making an assumption that QNM is a complete set (it isn’t). For each mode $(\ell, m)$, there are two infinite sets of QNM labeled by $n$: $\omega_{\ell mn}$ and $\omega'_{\ell mn}$ (see Eqns (3.5) and (3.6) ). When we replace
the integral with a sum over all QNM, we must include both sets of modes giving,

\[
    rM\Psi_4 = \sum_{\ell m n} \left\{ C_{\ell mn} e^{-i\omega_{\ell mn}(t-r)} S_{\ell m}(\theta, \phi; a\omega_{\ell mn}) \\
    + C'_{\ell mn} e^{-i\omega'_{\ell mn}(t-r)} S_{\ell m}(\theta, \phi; a\omega'_{\ell mn}) \right\},
\]

where \( C_{\ell mn} \) and \( C'_{\ell mn} \) are the arbitrary dimensionless complex coefficients of the QNM expansion.

The two sets of QNMs for Kerr are closely related to each other. The \( \omega \) correspond to “positive frequency” modes and the \( \omega' \) to “negative frequency” modes in the following sense. Let us redefine:

\[
    \omega_{\ell mn} \equiv \omega_{\ell mn} - \frac{i}{\tau_{\ell mn}}; \quad a\omega_{\ell mn} \equiv c_{\ell mn} : \quad \text{and} \quad \omega_{\ell mn} \geq 0, \quad (3.5)
\]

\[
    \omega'_{\ell mn} \equiv \omega'_{\ell mn} - \frac{i}{\tau'_{\ell mn}}; \quad a\omega'_{\ell mn} \equiv c'_{\ell mn} : \quad \text{and} \quad \omega'_{\ell mn} \leq 0. \quad (3.6)
\]

They are further related by the fact that

\[
    \omega'_{\ell mn} = -\omega_{(-m) n}, \quad (3.7)
\]

\[
    \tau'_{\ell mn} = \tau_{(-m)n}. \quad (3.8)
\]

In terms of these definitions, and assuming we evaluate the expansion at fixed \( r \) (and related fixed \( r^* \)), we find

\[
    rM\Psi_4 = \sum_{\ell m n} \left\{ C_{\ell mn} e^{-i\omega_{\ell mn}t} e^{-t/\tau_{\ell mn}} S_{\ell m}(\theta, \phi; c_{\ell mn}) \\
    + C'_{\ell mn} e^{i\omega_{(-m)n}t} e^{-t/\tau_{(-m)n}} S_{\ell m}(\theta, \phi; c'_{(-m)n}) \right\},
\]

where we have absorbed constant factors involving \( e^{r^*} \) into the expansion coefficients.

Using the fact that the sum over \( m \) extends from \(-\ell \cdots \ell\), we can trivially rewrite the expansion as

\[
    rM\Psi_4 = \sum_{\ell m n} \left\{ C_{\ell mn} e^{-i\omega_{\ell mn}t} e^{-t/\tau_{\ell mn}} S_{\ell m}(\theta, \phi; c_{\ell mn}) \\
    + C'_{\ell(-m)n} e^{i\omega_{(-m)n}t} e^{-t/\tau_{(-m)n}} S_{\ell(-m)}(\theta, \phi; c'_{(-m)n}) \right\},
\]

(3.10)
Furthermore, because
\[ c'_{\ell(-m)n} = a \left( \omega'_{\ell(-m)} - \frac{i}{\tau_{\ell(-m)n}} \right) = -a \left( \omega_{\ell mn} - \frac{i}{\tau_{\ell mn}} \right)^* = -c^*_{\ell mn}, \] (3.11)
and by the symmetries of the spin-weighted spheroidal harmonics given in Eqns (A.21) and (A.22), we find
\[ -2 S_{\ell(-m)}^{(-)}(\theta, \phi; c'_{\ell(-m)n}) = (-1)^l S_{\ell m}^{*}(\pi - \theta, \phi; c_{\ell mn}). \] (3.12)

Plugging those in, we get
\[ rM\Psi_4 = \sum_{\ell mn} \left\{ C_{\ell mn} e^{-i\omega_{\ell mn} t} e^{-t/\tau_{\ell mn}} S_{\ell m}^{(-)}(\theta, \phi; c_{\ell mn}) \right. \] \[ + (-1)^l C'_{\ell(-m)n} e^{i\omega_{\ell mn} t} e^{-t/\tau_{\ell mn}} S_{\ell m}^{*}(\pi - \theta, \phi; c_{\ell mn}) \} \] (3.13)
Finally, we can rewrite the two complex expansion coefficients as
\[ C_{\ell mn} \equiv A_{\ell mn} e^{i\phi_{\ell mn}}, \] (3.14)
\[ C'_{\ell mn} \equiv A'_{\ell mn} e^{-i\phi'_{\ell mn}}, \] (3.15)
where \( A_{\ell mn}, A'_{\ell mn}, \phi_{\ell mn} \) and \( \phi'_{\ell mn} \) are real parameters, and note the sign choice for \( \phi'_{\ell mn} \). The resulting equation is
\[ rM\Psi_4 = \sum_{\ell mn} \left\{ A_{\ell mn} e^{-i\omega_{\ell mn} t + i\phi_{\ell mn}} e^{-t/\tau_{\ell mn}} S_{\ell m}(\theta, \phi; c_{\ell mn}) \right. \] \[ + (-1)^l A'_{\ell(-m)n} e^{i\omega_{\ell mn} t - i\phi'_{\ell(-m)n}} e^{-t/\tau_{\ell mn}} S_{\ell m}^{*}(\pi - \theta, \phi; c_{\ell mn}) \} \] (3.16)
On the other hand, the Weyl scalar can also be decomposed using spin-weight -2 spherical harmonics. Again, assuming fixed \( r \),
\[ rM\Psi_4 = \sum_{\ell m} C_{\ell m}(t) Y_{\ell m}(\theta, \phi). \] (3.17)
The next step is to relate complex amplitudes \( C_{\ell m} \) to the parameters of the ringdown signal (3.16). To accomplish this, we first expand the spheroidal harmonics.
in terms of spherical harmonics:

\[ s S_{\ell m}^{*}(\theta, \phi; c) = \sum_{\ell'} C_{\ell' \ell m}^{*}(c) Y_{\ell' m}(\theta, \phi), \]  
(2.21)

where \( s \) is a spin weight. Using Eqns (A.16), (A.17), (A.21), and (A.22), we also have

\[ -S_{\ell m}^{*}(\pi - \theta, \phi; c) = \sum_{\ell'} (-1)^{\ell'} C_{\ell' \ell m}^{*}(c) Y_{\ell'(-m)}(\theta, \phi). \]  
(3.18)

[Note: \( C_{\ell' \ell m}^{*}(c) = (-1)^{\ell' + \ell} C_{\ell' \ell (-m)}(c^{*}). \)]

Equating Eqns. (3.16) and (3.17) results in

\[
\sum_{\ell m} C_{\ell m}(t) Y_{\ell m}(\theta, \phi) = \sum_{\ell m \pi} \left\{ A_{\ell m n} e^{-i\omega_{\ell m n} t + i\phi_{\ell m n}} e^{-i \gamma_{\ell m n}} S_{\ell m}(\theta, \phi; c_{\ell m n}) ight. \\
+ (-1)^{\ell} A'_{\ell(-m) n} e^{i\omega_{\ell(-m) n} t - i\phi_{\ell(-m) n}} e^{-i \gamma_{\ell(-m) n}} S_{\ell m}^{*}(\pi - \theta, \phi; c_{\ell m n}) \right\},
\]  
(3.19)

and using the usual definition for the inner product

\[ \langle f(\theta, \phi)|g(\theta, \phi)\rangle \equiv \int f^{*}(\theta, \phi) g(\theta, \phi) d\Omega \] along with the orthonormality of the spin-weighted spherical harmonics, Eq. (3.19) becomes

\[
C_{\ell' m'}(t) = \sum_{\ell m n} \left\{ A_{\ell m n} e^{-i\omega_{\ell m n} t + i\phi_{\ell m n}} e^{-i \gamma_{\ell m n}} \langle Y_{\ell' m'}(\theta, \phi) | S_{\ell m}(\theta, \phi; c_{\ell m n}) \rangle \\
+ (-1)^{\ell} A'_{\ell(-m) n} e^{i\omega_{\ell(-m) n} t - i\phi_{\ell(-m) n}} e^{-i \gamma_{\ell(-m) n}} \langle Y_{\ell' m'}(\theta, \phi) | S_{\ell m}^{*}(\pi - \theta, \phi; c_{\ell m n}) \rangle \right\},
\]  
(3.20)

Finally, using the expansion in Eqns. (2.21) and (3.18), and the orthonormality of the spin-weighted spherical harmonics we get

\[
C_{\ell' m'}(t) = \sum_{\ell m n} \left\{ A_{\ell m n} e^{-i\omega_{\ell m n} t + i\phi_{\ell m n}} e^{-i \gamma_{\ell m n}} \sum_{\ell''} C_{\ell'' \ell' m}^{*}(c_{\ell m n}) \langle Y_{\ell' m'}(\theta, \phi) | Y_{\ell'' m}(\theta, \phi) \rangle \\
+ (-1)^{\ell} A'_{\ell(-m) n} e^{i\omega_{\ell(-m) n} t - i\phi_{\ell(-m) n}} e^{-i \gamma_{\ell(-m) n}} \sum_{\ell''} (-1)^{\ell''} C_{\ell'' \ell' m}^{*}(c_{\ell m n}) \langle Y_{\ell'' m'}(\theta, \phi) | Y_{\ell''(-m)}(\theta, \phi) \rangle \right\}
\]

\[
= \sum_{\ell n} \left\{ A_{\ell m' n} e^{-i\omega_{\ell m' n} t + i\phi_{\ell m' n}} e^{-i \gamma_{\ell m' n}} \sum_{\ell''} C_{\ell'' \ell' m'}^{*}(c_{\ell m' n}) \\
+ (-1)^{\ell + \ell'} A'_{\ell m' n} e^{i\omega_{\ell(-m') n} t - i\phi_{\ell(-m') n}} e^{-i \gamma_{\ell(-m') n}} C_{\ell'' \ell' m'}^{*}(c_{\ell(-m') n}) \right\},
\]  
(3.21)
After relabelling indices we have

\[ C_{\ell m}(t) = \sum_{\ell' n} \left\{ A_{\ell' mn} C_{\ell' m}(c_{\ell' mn}) e^{-i\omega_{\ell' mn} t + i\phi_{\ell' mn}} e^{-t/\tau_{\ell' mn}} \right\}, \quad (3.22) \]

\[ + (-1)^{\ell' + \ell} A'_{\ell' mn} C^*_{\ell' m}(-m) e^{i\omega_{\ell' (-m)n} t - i\phi_{\ell' mn}} e^{-t/\tau_{\ell' (-m)n}} \left\}. \right. \]

Let us define the function

\[ \mathcal{R}_{\ell' mn}(t, \phi_{\ell' mn}) \equiv C_{\ell' m}(c_{\ell' mn}) e^{-i\omega_{\ell' mn} t + i\phi} e^{-t/\tau_{\ell' mn}} = C_{\ell' m}(c_{\ell' mn}) e^{-i\omega_{\ell' mn} t + i\phi}, \quad (3.23) \]

so we can write (3.22) more compactly

\[ C_{\ell m}(t) = \sum_{\ell' n} \left\{ A_{\ell' mn} \mathcal{R}_{\ell' mn}(t, \phi_{\ell' mn}) + (-1)^{\ell' + \ell} A'_{\ell' mn} \mathcal{R}^*_{\ell' (-m)n}(t, \phi_{\ell' mn}) \right\}. \quad (3.24) \]

Thus, the contribution to the wave from a given \( \ell \) and \( m \) is described by four sets of numbers \( \{A_{\ell' mn}\}, \{A'_{\ell' mn}\}, \{\phi_{\ell' mn}\}, \) and \( \{\phi'_{\ell' mn}\} \) and the final ringdown signal is constructed using Eq. (3.17). It should be noted that the scalar products \( C_{\ell' m}(c) \), which are part of \( \mathcal{R}_{\ell' mn}(t, \phi) \) definition, are the solutions of the eigenvalue problem in the spectral method Eq. (2.30). They were “naturally” computed as part of the QNM algorithm.

### 3.1 The orientation of the angular momentum

The ringdown model (3.24) or more detailed (3.22), that was derived in the previous section depends, among other parameters, on the mass and value of angular momentum. What about the direction of the angular momentum\(^1\)? It was assumed to always point in the \( z \)-direction. In this section we will determine how the ringdown model should be altered to incorporate an arbitrary direction of the angular momentum.

We must consider two separate coordinate systems because the SWSdH, needed to describe the ringdown of a Kerr black hole, are defined with respect to the

---

\(^1\)Angular momentum is a vector quantity
direction of the angular momentum of the black hole which may not align with the z-axis of the original coordinates. In order to account for that change we introduce two coordinate systems where \{x, y, z\} is fixed, \{\hat{x}, \hat{y}, \hat{z}\} is rotated. The two frames are related through Euler angles. We adopt a so called z-y-z convention (a rotation around z-axis by angle \(\alpha\) followed by a rotation around a newly formed y-axis by angle \(\beta\), followed by a rotation around a newly formed z-axis by angle \(\gamma\)). It is sometimes useful to use an alternative equivalent definition of rotations as related to the fixed axes of \{x, y, z\} frame. Accordingly, we first rotate by \(\gamma\) around the z-axis, then by \(\beta\) around the original y axis, and finally by \(\alpha\) around the original z-axis (see, for example, [MP87]). This newly adopted convention allows us to relate Euler angles to standard spherical coordinates (see Fig. (3.1) where \(\gamma = 0\)).

Rotations are a partial case of the Lorentz transformation and in its most general form it results in a complicated expression for the transformation rule between two frames. The rule is simpler if we limit ourselves to simple rotations. Before we proceed, we must note that spin-weighted quantities transform differently from spin-zero quantities. It can be shown that for the spin weight \(s = -2\) Weyl scalar, containing the gravitational wave signal, transforms as

\[
\Psi_4(\hat{\theta}, \hat{\phi}) = e^{is(\Lambda - \phi + \hat{\phi})} \Psi_4(\theta, \phi) \\
e^{i(-2)(\Lambda - \phi + \hat{\phi})} \Psi_4(\theta, \phi),
\]

(3.25)

where \(\Lambda\) is a real valued spin-rotation angle. Similarly SWSHs transform as

\[
y_{\ell m}(\hat{\theta}, \hat{\phi}) = \\
e^{is(\Lambda - \phi + \hat{\phi})} \sum_{\hat{m}} y_{\ell \hat{m}}(\theta, \phi) D_{\hat{m}m}^\ell(\alpha, \beta),
\]

(3.26)
where $D_{\ell m}^\dagger$ is a Wigner D-matrix. The above result reduces to the usual transformation rule for spherical harmonics when $s = 0$.

The Weyl scalar in each frame can be expanded in terms of its own set of spin-weighted spherical harmonics:

$$
\Psi_4(\theta, \phi) = \sum_{\ell m} C_{\ell m} Y_{\ell m}(\theta, \phi) \quad \text{and} \quad \Psi_4(\dot{\theta}, \dot{\phi}) = \sum_{\ell m} \dot{C}_{\ell m} Y_{\ell m}(\theta, \phi). \quad (3.27)
$$

This gives us

$$
\Psi_4(\theta, \phi) = \sum_{\ell m} C_{\ell m} Y_{\ell m}(\theta, \phi) = e^{-i(-2)(\Lambda - \phi + \dot{\phi})} \sum_{\ell m} \dot{C}_{\ell m} Y_{\ell m}(\theta, \phi). \quad (3.28)
$$

Using the orthonormality of the spin-weighted spherical harmonics, we find

$$
C_{\ell\dot{m}} = \oint \cdot_2 Y_{\ell m}^\ast (\theta, \phi) \left( e^{-i(-2)(\Lambda - \phi + \dot{\phi})} \sum_{\ell m} \dot{C}_{\ell m} Y_{\ell m}(\theta, \phi) \right) d\Omega. \quad (3.29)
$$

We can use the inverse of Eq. (3.26)

$$
\cdot_2 Y_{\ell m}(\theta, \phi) = e^{-i(-2)(\Lambda - \phi + \dot{\phi})} \sum_{\ell \dot{m}} D_{\ell \dot{m}}^{1\ell} Y_{\ell \dot{m}}(\theta, \phi) \quad (3.30)
$$

and trivially transforming the area element to get

$$
C_{\ell\dot{m}} = \oint \left( e^{-i(-2)(\Lambda - \phi + \dot{\phi})} \sum_{\ell \dot{m}} D_{\ell \dot{m}}^{1\ell} Y_{\ell \dot{m}}(\theta, \phi) \right)^* \times \left( e^{-i(-2)(\Lambda - \phi + \dot{\phi})} \sum_{\ell m} \dot{C}_{\ell m} Y_{\ell m}(\theta, \phi) \right) d\dot{\Omega}

= \sum_{\ell \dot{m}} \sum_{\ell m} \left( D_{\ell \dot{m}}^{1\ell} \right)^* C_{\ell m} \oint \cdot_2 Y_{\ell m}^\ast (\theta, \phi) \cdot_2 Y_{\ell m}(\theta, \phi) d\Omega. \quad (3.31)
$$

Using $D_{\ell \dot{m}}^{1\ell} = D_{\ell m}^{\ell\dagger}$ property for Wigner matrix allows us to simplify Eq. (3.31), after relabeling indices, to

$$
C_{\ell m} = \sum_{\ell \dot{m}} D_{\ell \dot{m}}^{\ell\dagger} (\alpha, \beta) \dot{C}_{\ell\dot{m}}. \quad (3.32)
$$
In particular, if our $\hat{C}_{\ell m}$ are from the mode decomposition of a perturbed Kerr black hole given in Eq. (3.22), we find that

$$C_{\ell m}(t) = \sum_{\ell' m'} D_{\ell m \ell' m'}(\alpha, \beta) \left\{ A_{\ell' m'} \mathcal{C}_{\ell' m'}(c_{\ell' m'}) e^{-i\omega_{\ell' m'}t + i\phi_{\ell' m'} e^{-t/\tau_{\ell' m'}}} + (-1)^{\ell + \ell'} A_{\ell' m'}^\ast \mathcal{C}_{\ell' m'}^\ast(c_{\ell' m'}) e^{i\omega_{\ell' m'}t - i\phi_{\ell' m'} e^{-t/\tau_{\ell' m'}}} \right\}, \quad (3.33)$$

With the Wigner rotation matrices $D_{\ell m}(\alpha, \beta)$ defined as (see also Appendix B)

$$D_{\ell m}(\alpha, \beta) \equiv e^{-ima} d_{\ell m}(\beta), \quad (3.34)$$

it is clear that the $\phi$ coordinate rotation by angle $\alpha$ gets absorbed into the definitions of the $\phi_{\ell m}$ and $\phi'_{\ell m}$ phase shifts so, in the context of trying to find the orientation of the black hole spin axis, it is not possible to determine the $\phi$ rotation.
Chapter 4:  Ringdown fitting

4.1 Wolfram language

In building our fitting tool we heavily rely on Wolfram language, which Stephen Wolfram described as a “new kind of a thing... a knowledge-based language” [Ste14]. It is a multi-paradigm programming language, which we have used mainly in three capacities: as a procedural language, functional programming language and, to a small extent, symbolic calculator. The latter is significant as it is tied to the very philosophy of the language: symbolic mathematics.

The language is also a database of a vast number of algorithms. In particular, a set of optimization and data analysis tools is of interest to us. With its help we use the best optimization algorithm available for fitting our highly nonlinear ringdown model.

Wolfram language has a great degree of automation, allowing a programmer to focus only on the task at hand. The interpretive nature of the Mathematica® environment provides a convenient way for testing and debugging one’s code: every portion of the code, however small, can be run by itself and tested. There is also a way to effectively control the precision of the computation, ranging from machine to any higher arbitrary precision or even infinite precision.

With a great many tools already available as part of Wolfram language, there is still a need for a solution to a new problem from a specialized area of study. Procedural programming, which is integrated into the language, allows a programmer to solve a vast array of new problems. There are various constructs to organize the code, e.g. blocks and modules. Finally, the large amount of functions and data can be organized as a package, that can be exported to the users.
Table 4.1: Some of the functions/data of QNMfit package

<table>
<thead>
<tr>
<th>method</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Export:</strong></td>
<td></td>
</tr>
<tr>
<td>ReadKerrQNM</td>
<td>Loads quasi-normal modes for given $l$, $m$ and $n$. The available within the package array KerrQNMTable[l, m, n, a] is created.</td>
</tr>
<tr>
<td>Clm</td>
<td>Generates $C_{\ell m}$ (3.22) of the ringdown as a symbolic expression, if the values for the fit parameters are not provided.</td>
</tr>
<tr>
<td>GenData</td>
<td>Generates the data for the ringdown based on (3.17) by calling Clm for sets of ${\ell}$ and ${m}$. Returns data in a special format, i.e. ringdown type.</td>
</tr>
<tr>
<td>MyFit</td>
<td>Fits the ringdown data with model (2.37), wrapping around Mathematica®’s NonlinearModelFit.</td>
</tr>
<tr>
<td>DataPlot</td>
<td>Plots the ringdown data with each $C_{\ell m}$ piece (real or imaginary) on an individual frame.</td>
</tr>
<tr>
<td>DataPlot2</td>
<td>Plots all the real/imaginary $C_{\ell m}$ pieces together (e.g, Figure 4.2).</td>
</tr>
<tr>
<td>DataFitPlot</td>
<td>Plots the fit with data points for given $l$ and $m$.</td>
</tr>
<tr>
<td>ModePlot</td>
<td>Plots individual modes of the fit for given $l$ and $m$.</td>
</tr>
<tr>
<td><strong>Private:</strong></td>
<td></td>
</tr>
<tr>
<td>DefineInterpolations</td>
<td>Interpolates QNM and SWSH/SWSdH scalar products as functions of $a$</td>
</tr>
<tr>
<td>ClmSplit</td>
<td>Splits the $C_{\ell m}$ piece into real and imaginary part, introducing a new auxiliary variable</td>
</tr>
<tr>
<td>RD</td>
<td>Presents the ringdown coefficients $C_{\ell m}$ as one multivariable model function appropriate for the input to NonlinearModelFit</td>
</tr>
<tr>
<td>FitParam</td>
<td>Presents the fit parameters in the form appropriate for the input to NonlinearModelFit.</td>
</tr>
<tr>
<td>Mass</td>
<td>A protected variable representing the hole’s mass in the options of MyFit.</td>
</tr>
<tr>
<td>Spin</td>
<td>A protected variable representing the hole’s spin in the options of MyFit.</td>
</tr>
<tr>
<td>Theta</td>
<td>A protected variable representing the orientation of the hole’s spin in the options of MyFit.</td>
</tr>
</tbody>
</table>
4.2 QNMfit package

We extend the existing capabilities of Wolfram language by organizing the fitting code as a Mathematica® package, following the object-oriented style of programming, especially the idea of encapsulation, where certain parts of the code are hidden from the package user.

The content of the QNMfit package is shown in Table 4.1 where there is a group of procedures which appear in the export part of the package definition. These are the procedures that provide the interface and are accessible to the user. Among them there are tools designed for the visualization of the results, mostly in the form of plots, e.g. DataPlot, DataFitPlot, ModePlot, etc. The export section of the package is followed by the private section, whose content is only available within the package.

The work of the package depends on the QNM data, which should be made accessible via QNMfit'path variable prior to the use of the package. Notice how the variable is prefixed with the name of the package. It is a convention, that all variables which appear in the package definition are put into a separate context, which is usually the name of the package. Sometimes it is necessary to protect certain names declared within the package from being used outside in the global context, e.g. variables Mass, Spin, Theta (see Table 4.1).

We keep the QNM related information in binary format, which gets loaded into memory on an as-needed basis. We represent QNM as functions of the spin parameter $a$, interpolating between the computed values. The harmonic scalar products are similarly represented as functions of the spin parameter $a$. The package can be used in two different ways. It can generate the data for an arbitrary ringdown signal. That is useful mainly for the debugging and testing purposes. After the expected behavior of the algorithm is established, we can fit our ringdown model to different sets of data points coming from a different simulation (data sets should be properly converted to the package format).
The single most important procedure of QNMfit is NonlinearModelFit procedure of Mathematica® [Mat14b], which uses a nonlinear regression analysis to find the optimizer. In that regard, the package’s function MyFit is a very sophisticated form a wrapper for NonlinearModelFit, providing the input for it. The depen-
This graph shows the intricate relationship between MyFit and other functions and procedures of the package.

### 4.3 Fitting example

In this section we show how the QNMfit functionality can be applied to the ringdown fitting. After we established the model (3.17) with coefficients defined according to (3.33) we can fit it to an arbitrary ringdown signal. Our goal is to extract the information about the hole’s parameters, which uniquely correspond to its QNM spectrum. The better the fit, the greater our confidence that the extracted values are correct.

Consider a six-term ringdown signal given by

$$rM\Psi_4 = \sum_{\ell m} C_{\ell m}(t) Y_{\ell m}(\theta, \phi) = C_{32,32} Y_{32} + C_{22,22} Y_{22} + C_{21,21} Y_{21}$$

$$+ C_{20,20} Y_{20} + C_{2(-1),2(-1)} Y_{2(-1)} + C_{2(-2),2(-2)} Y_{3(-2)}. \quad (4.1)$$

Within the package the data is presented in a way that can be used by the fitting algorithm. Different parts of the ringdown represented by coefficients $C_{\ell m}(t)$ are numbered and split into real and imaginary parts with the purpose of fitting them simultaneously as it is shown in Fig. 4.2. As you can see, the $C_{\ell m}$ number plays a role of an extra dimension (real and imaginary separation does too) so our optimization domain becomes

$$t \times (\text{number of } C_{\ell m} \text{ terms in the expansion}) \times 2. \quad (4.2)$$

Inside MyFit we supply `NonlinearModelFit` with fit parameters: amplitudes $\{A_{\ell m}\}, \{A'_{\ell m}\}$, phases $\{\phi_{\ell m}\}, \{\phi'_{\ell m}\}$, mass ratio $\delta$, spin $a$ and spin tilt $\beta$.

The routine returns a symbolic fitted model that can be compared against the original data points.

To illustrate the use of the package, let us generate an arbitrary ringdown signal data set and then fit it to get the parameters back. We will use `GenData`
Figure 4.2: The ringdown signal presented to the fitting routine as a multivariable function.
routine with two terms in the ringdown: \( l = 2, m = 2 \) and \( l = 3, m = 2 \). Each will consist of two fundamental modes: \( l = 2, m = 2, n = 0 \) and \( l = 3, m = 2, n = 0 \). The mass, spin and \( \beta \) angle are set to 0.9, 0.73 and zero respectively (see the listing below).

Listing 4.1: Testing of the fitting algorithm

```plaintext
Needs["QNMfit"];
t1 = 0; tN = 80; dt = 0.4;
M = 0.9; a = 0.73;
gRD = GenData[{{2, 2}, {3, 2}}, {{2, 2, 0, -0.5, \pi}, {3, 2, 0, -0.03, \pi/4}}, {}, M, a, 0, t1, tN, dt];
fit = MyFit[gRD, {{2, 2}, {3, 2}}, {{2, 2, 0}, {3, 2, 0}}, {}, OptRange -> {40, 80}, Theta -> {0, True}];
fit[[1]]["ParameterConfidenceIntervalTable"]
```

The result of fitting is shown in Table 4.2. We fix \( \beta \) for now (we call it Theta in the routine options). Notice how the amplitudes came out positive. The minus got absorbed into the phases (\(-0.5e^{i\pi} = 0.5e^{i\pi} \) and \(-0.03e^{i\pi/4} = 0.03e^{i5\pi/4}\)). We were able to obtain the values as fit parameters using only half of the points (40 < \( t < 80 \)). The mass and spin are denoted by \( \delta \) and \( \alpha \) respectively.

Table 4.2: The output of Listing 4.1

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>0.9</td>
<td>( 1.8557406857344626 \times 10^{-15} )</td>
<td>{0.9, 0.9}</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.73</td>
<td>( 2.3072221201412073 \times 10^{-15} )</td>
<td>{0.73, 0.73}</td>
</tr>
<tr>
<td>A220</td>
<td>0.5</td>
<td>( 6.797498768857668 \times 10^{-15} )</td>
<td>{0.5, 0.5}</td>
</tr>
<tr>
<td>A320</td>
<td>0.03</td>
<td>( 1.1405480430071444 \times 10^{-15} )</td>
<td>{0.03, 0.03}</td>
</tr>
<tr>
<td>( \phi_{220} )</td>
<td>12.5664</td>
<td>( 1.3978053761211032 \times 10^{-15} )</td>
<td>{12.5664, 12.5664}</td>
</tr>
<tr>
<td>( \phi_{320} )</td>
<td>3.92699</td>
<td>( 4.754384860099999 \times 10^{-14} )</td>
<td>{3.92699, 3.92699}</td>
</tr>
</tbody>
</table>

Now let us fit our model to the ringdown signal coming from a black hole formed as the result of the merger of a pair black holes orbiting a common center of mass. We defined this ringdown earlier (4.1) with the full set of data organized as shown in Figure 4.2. To convert the data into appropriate ringdown format we need to load another small package \texttt{RDdataManip}.
First, we read the data with `GetData` function, using every third point of the “tail” of the signal then we fit our ringdown model to these data points as shown in Listing 4.2). We only used two fundamental modes in the expansion for each $C_{lm}$ in (4.1) and for time optimization range corresponding to about two thirds of data points (see `OptRange->\{40,150\}` in the Listing 4.2), we were able to get the smallest standard error for the mass and spin of the black hole producing the given ringdown signal. Including additional modes along with extending the optimization range back in time may produce a more accurate fit. However, one must be careful increasing the number of fit parameters, as it leads to a greater statistical error.

<table>
<thead>
<tr>
<th>Table 4.3: The output of Listing 4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\delta}$</td>
</tr>
<tr>
<td>0.966282</td>
</tr>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$A_{220}$</td>
</tr>
<tr>
<td>$A_{320}$</td>
</tr>
<tr>
<td>$\phi_{220}$</td>
</tr>
<tr>
<td>$\phi_{320}$</td>
</tr>
</tbody>
</table>

The interface of `MyFit` routine allows the investigator to fix some parameters in the ringdown waveform, while letting others vary. Finding the balance between the number of fit parameters and accuracy of the waveform fitting is the subject of the present research. Four waveforms are plotted in Fig 4.3. Note how the fits have a satisfactory accuracy even for the points beyond the fitting range ($40 < t < 150$).
Figure 4.3: The fit for $(l,m) = (2,2)$ and $(l,m) = (3,2)$. The data points are shown as blue dots.
Chapter 5: Future work

The future work is two-pronged. One direction would be improving our numerical algorithm of calculating Kerr QNM, especially for higher overtones and multipole index $l$. The other direction would be the ringdown fitting to other ringdown signal data sets.

We calculated a sufficient number of Kerr QNM to accurately simulate the ringdown signal. However, QNM research subject is very broad and can be explored further. There is a potentially infinite number of QNM corresponding to higher overtones $n$ and multipole indices $l$. It becomes harder to find the solutions for higher values of those indices. One area of improvement might be the method of solving the radial equation of the perturbation problem (2.1). We use Newton’s method to find the roots of the corresponding continued fraction. The method depends on the derivative of the function, which we approximate numerically. That leads to additional function evaluations. There is a way to avoid costly derivative-related function evaluations with quasi-Newton method, which does not require calculations of derivatives.

There are also so called algebraically “special” modes, e.g. one for a Schwarzschild black hole with $l = 2$ and $n = 8$, which is purely imaginary. That corresponds to a mode without any oscillation and only damping present. What physical meaning does it have? Similarly, the Kerr modes for a certain value of the spin parameter, $l$ and overtone $n$ end up near the imaginary axis. Are they all purely imaginary or just have an extremely small real part? Although, the latter is more satisfying from the physical stand point of view, our recent investigation indicates that they are indeed purely imaginary, corresponding to the so called polynomial solution of the eigenvalue problem (2.1). This analytical solution is in good agreement with our numerical results.

More work should be done in ringdown fitting. Our multimode fitting algo-
rithm allows us to extract the information about the black hole from its ringdown radiation. It can also be used as an instrument in our understanding of the ringdown itself. Certain modes might be more important than others so the nature of the ringdown should be taken into consideration, whether it is a black hole binary merger ringdown, gravitational collapse ringdown, etc.
Appendix A: Spin-Weighted Spheroidal Harmonics

The spin-weighted spheroidal harmonics, \( s_{\ell m}(\theta, \phi; c) \), are generalizations of the spin-weighted spherical harmonics, \( Y_{\ell m}(\theta, \phi) = s_{\ell m}(\theta, \phi; 0) \), where \( s \) is the spin weight of the harmonic, and \( c \) is the oblateness parameter. The angular dependence separates as

\[
s_{\ell m}(\theta, \phi; c) \equiv \frac{1}{2\pi} s_{\ell m}(\cos \theta; c) e^{im\phi}. \tag{A.1}
\]

With \( x \equiv \cos \theta \), the spin-weighted spheroidal function, \( s_{\ell m}(x; c) \), satisfies

\[
\partial_x \left[(1-x^2)\partial_x s_{\ell m}(x; c)\right] + \left[(cx)^2 - 2csx + s + A_{\ell m}(c) - \frac{(m + sx)^2}{1-x^2}\right] s_{\ell m}(x; c) = 0, \tag{A.2}
\]

where \( A_{\ell m}(c) \) is the separation constant.

The basic symmetries inherent in the spin-weighted spheroidal functions follow from (A.2):

\[
 s \rightarrow -s; \quad x \rightarrow -x \Rightarrow \begin{cases} s_{\ell m}(x; c) \propto s_{\ell m}(-x; c) \\ A_{\ell m}(c) = A_{\ell m}(c) + 2s \end{cases} \tag{A.3}
\]

\[
 m \rightarrow -m; \quad x \rightarrow -x; \quad c \rightarrow -c \Rightarrow \begin{cases} s_{\ell(-m)}(x; c) \propto s_{\ell(-m)}(-x; -c) \\ A_{\ell(-m)}(c) = A_{\ell m}(-c) \end{cases} \tag{A.4}
\]

complex conjugation \Rightarrow \begin{cases} s_{\ell m}(x; c) \propto s_{\ell m}(x; c^*) \\ A_{\ell m}(c) = A_{\ell m}(c) \end{cases} \tag{A.5}

Additional sign and normalization conventions are chosen for consistency with common sign conventions for the angular-spheroidal functions, \( S_{\ell m}(x; c) = S_{\ell m}(x; c) \), and the spin-weighted spheroidal harmonics, \( Y_{\ell m}(\theta, \phi) \).

Angular spheroidal functions

\[
\partial_x \left[(1-x^2)\partial_x S_{\ell m}(x; c)\right] + \left[(cx)^2 + A_{\ell m}(c) - \frac{m^2}{1-x^2}\right] S_{\ell m}(x; c) = 0 \tag{A.6}
\]
\[ S_{\ell m}(x; c) = (-1)^{\ell + m} S_{\ell m}(x; c) \]  
(A.7)

\[ S_{\ell m}(x; c) = S_{\ell m}(x; c) \]  
: \[ A_{\ell m}(c) = A_{\ell m}(c) \]  
(A.8)

\[ S_{\ell(-m)}(x; c) = (-1)^m S_{\ell m}(x; c) \]  
: \[ A_{\ell(-m)}(c) = A_{\ell m}(c) \]  
(A.9)

\[ S^*_m(x) = S_{\ell m}(x; c^*) \]  
: \[ A^*_m(c) = A_{\ell m}(c^*) \]  
(A.10)

Spin-weighted spherical functions

\[ \partial_x \left[ (1 - x^2) \partial_x \left[ s_{\ell m}(x; 0) \right] \right] + \left[ s + A_{\ell m}(0) - \frac{(m + sx)^2}{1 - x^2} \right] s_{\ell m}(x; 0) = 0 \]  
(A.11)

\[ A_{\ell m}(0) = l(l + 1) - s(s + 1) \]  
(A.12)

\[ s_{\ell m}(x; 0) = (-1)^{\ell + m} s_{\ell m}(x; 0) \]  
: \[ A_{\ell m}(0) = A_{\ell m}(0) + 2s \]  
(A.13)

\[ s_{\ell(-m)}(x; 0) = (-1)^{\ell + s} s_{\ell m}(x; 0) \]  
: \[ A_{\ell(-m)}(0) = A_{\ell m}(0) \]  
(A.14)

\[ s^*_m(x) = s_{\ell m}(x; 0) \]  
: \[ A^*_m(0) = A_{\ell m}(0) \]  
(A.15)

Spin-weighted spherical harmonics

\[ s Y_{\ell m}(\theta, \phi) = (-1)^{\ell + m} s Y_{\ell m}(\pi - \theta, \phi) \]  
(A.16)

\[ Y_{\ell(-m)}(\theta, \phi) = (-1)^{s + m} Y^*_{\ell m}(\theta, \phi) \]  
(A.17)

These leave us with the following conventions for the spin-weighted spheroidal harmonics and spheroidal functions:

\[ s_{\ell m}(x; c) = (-1)^{\ell + m} s_{\ell m}(x; c) \]  
: \[ A_{\ell m}(c) = A_{\ell m}(c) + 2s \]  
(A.18)

\[ s_{\ell(-m)}(x; c) = (-1)^{\ell + s} s_{\ell m}(x; -c) \]  
: \[ A_{\ell(-m)}(c) = A_{\ell m}(-c) \]  
(A.19)

\[ s^*_m(x)c = s_{\ell m}(x; c^*) \]  
: \[ A^*_m(c) = A_{\ell m}(c^*) \]  
(A.20)

\[ s_{\ell m}(\theta, \phi; c) = (-1)^{\ell + m} s_{\ell m}(\pi - \theta, \phi; c) \]  
(A.21)

\[ s_{\ell(-m)}(\theta, \phi; c) = (-1)^{s + m} s^*_m(\theta, \phi; c^*) \]  
(A.22)
Appendix B: Spin-Weighted Spherical Harmonics

Spin-weighted spherical harmonics are the general case of scalar (spin weight zero) spherical harmonics \( Y_{\ell m}(\theta, \phi) \).

\[
Y_{\ell m}(\theta, \phi) \equiv \frac{1}{2\pi} s_s Y_{\ell m}(\cos \theta; 0)e^{im\phi} = (-1)^s \sqrt{\frac{2\ell + 1}{4\pi}} d_m(\theta)e^{im\phi}, \tag{B.1}
\]

where \( d_m(\theta) \) is the Wigner d-function defined by

\[
d_m(\theta) \equiv (-1)^\lambda \left( \frac{2\ell - k}{k + a} \right)^{\frac{1}{2}} \left( \frac{k + b}{b} \right)^{\frac{1}{2}} \left( \sin \frac{\theta}{2} \right)^a \left( \cos \frac{\theta}{2} \right)^b P_{k}^{(a,b)}(\cos \theta), \tag{B.2}
\]

where \( k \equiv \min(\ell + n, \ell - n, \ell + m, \ell - m) \) and

\[
if \ k = \begin{cases} 
\ell + n : & a = m - n, \quad \lambda = m - n \\
\ell - n : & a = n - m, \quad \lambda = 0 \\
\ell + m : & a = n - m, \quad \lambda = 0 \\
\ell - m : & a = m - n, \quad \lambda = m - n 
\end{cases} \tag{B.3}
\]

\[
b = 2\ell - 2k - a. \tag{B.4}
\]

More generally, when a non-vanishing spin-gauge \( \gamma \) is allowed,

\[
Y_{\ell m}^\gamma(\theta, \phi) = (-1)^s \sqrt{\frac{2\ell + 1}{4\pi}} (D_m^{\ell}(\phi, \theta, -\gamma))^*, \]

where the Wigner rotation matrices \( D_m^{\ell}(\phi, \theta, \gamma) \) is defined by

\[
D_m^{\ell}(\phi, \theta, \gamma) \equiv e^{-im\phi} d_m(\theta)e^{-in\gamma}. \]

The spin-weighted spherical functions, \( s_s Y_{\ell m}(x; 0) \), satisfy the recurrence relations

\[
x_s S_{\ell m}(x; 0) = \mathcal{F}_{s\ell m} S_{(\ell+1)m}(x; 0) + \mathcal{G}_{s\ell m} S_{(\ell-1)m}(x; 0) + \mathcal{H}_{s\ell m} S_{\ell m}(x; 0), \tag{B.5}
\]
where

\[ F_{\ell m} = \begin{cases} \sqrt{(\ell + 1)^2 - m^2} \left( \frac{(\ell + 1)^2 - s^2}{(2\ell + 3)(2\ell + 1)} \right), & \text{if } s \neq 0, \\ \sqrt{(\ell + 1)^2 - m^2} \left( \frac{(\ell + 1)^2 - s^2}{(2\ell + 3)(2\ell + 1)} \right), & \text{if } s = 0. \end{cases} \]  

(B.6)

\[ G_{\ell m} = \begin{cases} \sqrt{\ell^2 - m^2} \left( \frac{\ell^2 - s^2}{4\ell^2 - 1} \right), & \text{if } \ell \neq 0, \\ 0, & \text{if } \ell = 0. \end{cases} \]  

(B.7)

\[ H_{\ell m} = \begin{cases} -\frac{ms}{\ell(\ell + 1)}, & \text{if } \ell \neq 0 \text{ and } s \neq 0, \\ 0, & \text{if } \ell = 0 \text{ or } s = 0. \end{cases} \]  

(B.8)

\[ x^2 S_{\ell m}(x; 0) = A_{\ell m} S_{(\ell+2)m}(x; 0) + B_{\ell m} S_{\ell m}(x; 0) + C_{\ell m} S_{(\ell-2)m}(x; 0) + D_{\ell m} S_{(\ell+1)m}(x; 0) + E_{\ell m} S_{(\ell-1)m}(x; 0), \]  

(B.9)

where

\[ A_{\ell m} = F_{\ell m} F_{s(\ell+1)m}, \]  

(B.10)

\[ D_{\ell m} = F_{\ell m} (H_{s(\ell+1)m} + H_{s\ell m}), \]  

(B.11)

\[ B_{\ell m} = F_{\ell m} G_{s(\ell+1)m} + G_{\ell m} F_{s(\ell-1)m} + H_{s\ell m}^2, \]  

(B.12)

\[ E_{\ell m} = G_{\ell m} (H_{s(\ell-1)m} + H_{s\ell m}), \]  

(B.13)

\[ C_{\ell m} = G_{\ell m} G_{s(\ell-1)m}. \]  

(B.14)
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Specialist ¹ Physics  Volgograd State University, Russia  June 2005

Experience

Wake Forest University
Teaching Assistant  2008–Present

CJSC Columbia Telecom, Volgograd, Russia
Programmer  2006-2008
Programming the company’s help desk, arp table tool, photo web site, etc.

¹A 5-year Russian degree, B.S./M.A. equivalent
CJSC Volgograd GSM, Volgograd, Russia

Intern  

December 2005

Cellular network maintenance, monitoring, troubleshooting etc.

Honors

Sigma Pi Sigma

Languages

Russian (native), English (fluent).