GENERALIZATIONS AND VARIATIONS ON GRAPH PEBBLING

BY

JOEL A. BARNETT

A Thesis Submitted to the Graduate Faculty of

WAKE FOREST UNIVERSITY GRADUATE SCHOOL OF ARTS AND SCIENCES

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF ARTS

Mathematics

May 2014

Winston-Salem, North Carolina

Approved By:

Sarah Mason, Ph.D., Advisor

Hugh Howards, Ph.D., Chair

Jason Parsley, Ph.D.

Acknowledgments

There are a number of people who deserve credit for helping me get to this point.

As my advisor, Dr. Mason, has put up with me since I started here. No matter what results didn't pan out in our research, she always stayed optimistic and found something else for us to work on. Also, I appreciate her not letting me get away with anything less than rigorous in my work.

Dr. Parsley has always been helpful and accommodating, and no matter how I may have struggled in his classes, he always displayed enthusiasm and created a classroom environment that made me always enjoyed being there.

Dr. Howards was nice enough to agree to be on and chair my committee despite me missing the first week of class and not having much of a rapport up to the point.

Of all the people from Montevallo, a few definitely need special recognition beyond the fact that they wrote recommendations for me. Dr. Sterner always challenged us at all times in class, not only making the material interesting, but connecting it to other classes and forcing us to really understand what was going on. Dr. Tyler put graph theory on my radar, but more importantly he made sure to emphasize that I should focus on the learning and the experience of a graduate program rather than going crazy trying to have perfect grades. Dr. Martin isn't a mathematician, but she's always been very supportive, looking out for us when we ought to be doing something.

If any of my classmates need calling out, it's Heather Hardeman and Jamie Eloff. Without their special brand of encouragement, I might not have moved up here at Wake at just the right time.

All my family, other friends, and former coworkers over the years. Though they may not understand much, if any, of what I'm studying, they've always been nothing but supportive and enthusiastic about me doing what I enjoy.

Table of Contents

Acknowledgments ii

Abstract iv

Chapter 1 Introduction and Background 1
- 1.1 General Graph Theory 1
- 1.2 Cartesian Products 4
- 1.3 Graph Pebbling 6

Chapter 2 Path Pebbling 9

Chapter 3 Labeled and Unlabeled Graphs 12
- 3.1 Extending Existing Results in the Labeled Pebbling View 12
- 3.2 Contrasting Labeled and Unlabeled Graphs 16

Chapter 4 Stepping into Products of Paths 25

Chapter 5 Conjectures and Future Directions 31
- 5.1 Extending the Cover Pebbling Theorem 31
- 5.2 Conjectures on Other Distributions 34
- 5.3 Conjectures on Path Products 36
- 5.4 Further Directions 38

Bibliography 40

Appendix A Labeled and Unlabeled Pebbling Numbers for Selected Graphs and Distributions 41

Curriculum Vitae 47

Abstract

Joel A. Barnett

Graph pebbling involves determining the minimum number of pebbles needed so that regardless of the initial arrangement of pebbles on a graph, a pebble can be moved to any vertex using specified "pebbling moves." This minimum number of pebbles is the pebbling number of a graph. We begin by making a brief exploration into path pebbling, which uses a sequence of pebbling moves instead of a single pebbling move. Returning to normal pebbling moves, we note that graph pebbling can be generalized by looking at a target distribution of pebbles, rather than just reaching one vertex with one pebble. We examine a contrast between pebbling on a labeled graph (where the target distribution is fixed) and an unlabeled graph (where the target distribution may be represented in multiple ways). We also seek to extend Jonas Sjöstrand's Cover Pebbling Theorem to make calculating some pebbling numbers easier.

Chapter 1: Introduction and Background

We begin with an excursion into path pebbling and prove a path pebbling number for all graphs. We follow by establishing some existing results in the view of labeled graph pebbling while recalling characteristics of Cartesian products of graphs. We then move to contrasting labeled and unlabeled pebbling, showing that labeled pebbling numbers are an upper bound for unlabeled pebbling numbers and showing when the two values are equal. We then begin constructing labeled pebbling numbers for small paths and products before making conjectures on some of the more general cases.

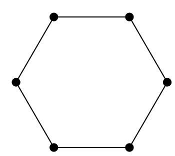
For the remainder of the first chapter, we establish some terminology, existing results, and framework for the remainder of the thesis. Chapter 2 includes the entirety of our work on path pebbling, defining the problem and establishing an encompassing result. In Chapter 3, we translate a few results into our labeled view before making the first major distinction between unlabeled pebbling and labeled pebbling. In Chapter 4, we start working in specific path cases, showing results for small cases of paths and products of paths. Chapter 5 is a list and discussion of conjectures for path graphs, separated into different types of distributions, and Chapter 6 serves as a summation of open questions from our work along with other untapped avenues of research.

1.1 General Graph Theory

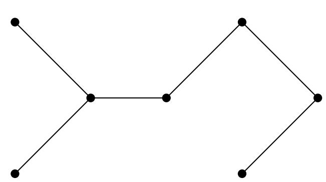
We begin by introducing basic terminology and notation from graph theory. For any graph G, we have $V(G)$ and $E(G)$, the vertex set and edge set of G, respectively. We will generally denote vertices by letters and subscripts, v, v_n, etc. For vertices x and y, we denote the edge connecting them as xy. If two vertices are connected by more than one edge, we refer to these edges as *multi-edges*. The *degree* of a vertex is the number of edges incident to it. Note that *loops*, edges with one vertex serving as both endpoints, contribute 2 to a vertex's degree.

Many simpler classes of graphs have shortened notation we will frequently use. A *path graph* on n vertices, denoted P_n, is a connected set of n vertices where the middle $n-2$ vertices have degree two while the final two end vertices have degree one.

A *cycle graph*, C_n, is a connected graph on $n \geq 3$ vertices where each vertex has degree 2.



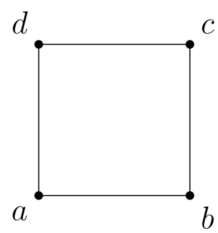
A *cycle* in any graph is a set of vertices V (along with the incident edges) such that the subgraph V is a cycle graph. The *length* of a cycle is the number of edges in the cycle. If a graph has no cycles, it is *acyclic*. A *tree*, commonly denoted T, is a connected acyclic graph.

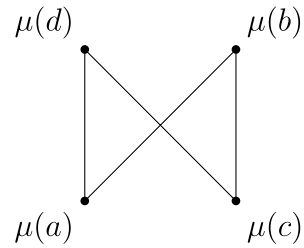


Finally, a *spanning tree* of a graph is a connected subgraph formed by deleting edges of the graph until no cycles remain.

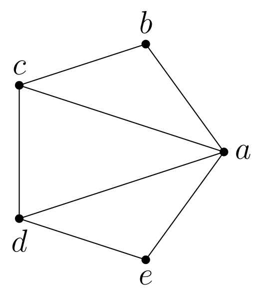
One idea in graph theory that we will make significant use of is that of graph isomorphisms. We define an *isomorphism* from a graph G to a graph H as a bijection μ from $V(G)$ to $V(H)$ and from $E(G)$ to $E(H)$ such that if $uv \in E(G)$, then

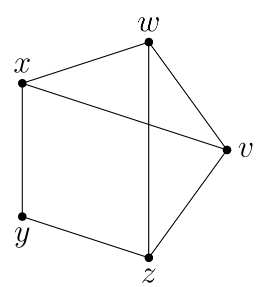
$\mu(u)\mu(v) \in E(H)$ [4]. If there is an isomorphism from G to H, we say that G and H are *isomorphic*. Consider the following two graphs:





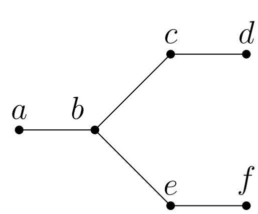
These two graphs are isomorphic, as there exists a one-to-one correspondence between the vertices and edges. Alternatively, consider the following graphs.

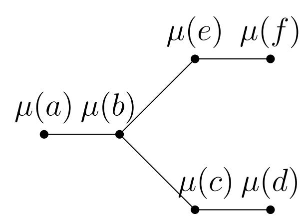




While these two graphs have the same number of vertices and edges, they are not isomorphic. Note that with any isomorphism, whichever vertex e is mapped to must also be degree four, since the same number of edges will be incident to it. However, there is no such vertex in the right graph, so these graphs cannot be isomorphic.

As another example of an isomorphism, consider the following two graphs:





These two graphs are also isomorphic, as is explicitly shown.

The two examples above of isomorphisms show that isomorphic graphs can look similar or very different. In our use of isomorphisms, we will focus on those isomorphisms which result in a graph that does look the same. This idea will factor heavily into our contrasting of labeled and unlabeled pebbling in Chapter 3.

1.2 Cartesian Products

The *Cartesian product* of a pair of graphs G and H, denoted $G\Box H$ (alternatively, $G \times H$), is defined by $V(G\Box H) = \{(g_i, h_j) \ni g_i \in G, h_j \in H\}$ and $E(G\Box H) = \{(g_i, h_j)(g_k, h_\ell)$ such that $(g_i = g_k$ and $(h_j h_\ell) \in E(H))$ or $(h_j = h_\ell$ and $(g_i g_k) \in E(G))\}$ [2].

In much of the graph pebbling literature, Cartesian products of graphs play a large role in many results and conjectures. Additionally, several significant pebbling results are dependent on some properties of graphs. In particular, we examine diameter and girth of graphs.

A *path* between u and v is a sequence of adjacent vertices between u and v with no repeated vertices. The *distance* between two vertices in a graph is the length of the shortest path between those two vertices (though the shortest path may not be unique), denoted $d(x, y)$ or $\mathrm{dist}(x, y)$ for vertices $x, y \in V(G)$. The *diameter* of a graph, $\mathrm{diam}(G)$, is the maximum of all the distances between pairs of distinct vertices in a graph. If the graph is disconnected, then the graph has infinite diameter [4]. The *girth* of a graph, $\mathrm{girth}(G)$, is the length of the shortest cycle in the graph. If the graph is acyclic, the graph is said to have infinite girth [4].

In an effort to make this paper self-contained, we include results of the effect of Cartesian products on the diameter and girth of graphs with a proof of the latter.

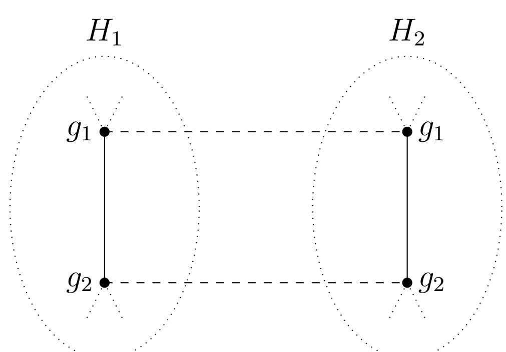
Lemma 1.1. Given two graphs G and H, each with finite diameter, $\mathrm{diam}(G\Box H) =$

$\operatorname{diam}(G) + \operatorname{diam}(H)$. [5]

Lemma 1.2. For two graphs G and H, each with nonempty edge set, we have $\operatorname{girth}(G \square H) = \min\{\operatorname{girth}(G), \operatorname{girth}(H), 4\}$.

Proof. Let G and H be graphs with nonempty edge sets. Suppose $\operatorname{girth}(G)$ and/or $\operatorname{girth}(H)$ is finite and at most 4. Note that $\operatorname{girth}(G \square H) \leq \min\{\operatorname{girth}(G), \operatorname{girth}(H)\}$, since G and H are subgraphs of $G \square H$. We first show that loops, multi-edges, and triangles in $G \square H$ imply their presence in G or H. For a loop in $G \square H$, $(g_1, h_1)(g_1, h_1) \in E(G \square H)$, and so $g_1 g_1 \in E(G)$ or $h_1 h_1 \in E(H)$. For multi-edges in $G \square H$, two edges from (g_1, h_1) to (g_1, h_2) or from (g_1, h_1) to (g_2, h_1) are required (since $(g_1, h_1)(g_2, h_2) \notin E(G \square H)$), implying $g_1 g_2$ and $g_1 g_2$ in $E(G)$ or $h_1 h_2$ and $h_1 h_2$ in $E(H)$. For triangles in $G \square H$ to not be in either G or H, we would have the vertices (g_1', h_1'), (g_1', h_2'), and (g_2', h_i'), which violates construction of $G \square H$ for $i \in \{1, 2\}$, since (g_2', h_1') is not adjacent to (g_1', h_2'), and (g_2', h_2') is not adjacent to (g_1', h_1').

Now suppose $\operatorname{girth}(G)$ and $\operatorname{girth}(H)$ are each either greater than 4 or infinite, and consider adjacent vertices $g_1, g_2 \in V(G)$ and $h_1, h_2 \in V(H)$. In $G \square H$, this yields the four-cycle with vertices (g_1, h_1), (g_2, h_1), (g_2, h_2), and (g_1, h_2). As before, no smaller cycles exist, since they would be triangles, multi-edges, or loops, and so for $\operatorname{girth}(G), \operatorname{girth}(H) > 4$, $\operatorname{girth}(G \square H) = 4$.

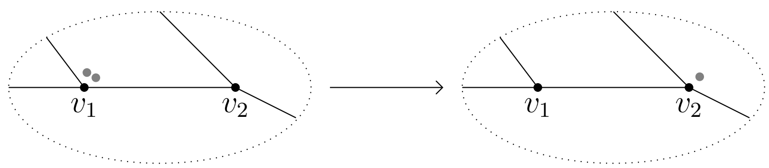


Thus, $\operatorname{girth}(G \square H) = \min\{\operatorname{girth}(G), \operatorname{girth}(H), 4\}$. □

1.3 Graph Pebbling

In graph pebbling, we assign a nonnegative integer to each vertex representing the number of pebbles placed on that vertex. We then make pebbling moves in an attempt to place a pebble onto an initially specified vertex called the *root*.

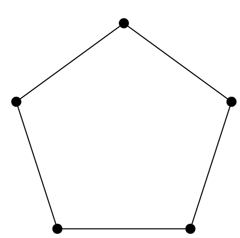
Definition 1.1. A *pebbling move* is defined as removing two pebbles from some vertex v_1 and adding one pebble on an adjacent vertex v_2, denoted $[v_1, v_2]$ [2].



Alternatively, this may be viewed as discarding one pebble as a cost or toll to move another to an adjacent vertex. One significant fact of note we will make extensive use of is that given two vertices v_i and v_j, moving one pebble from v_i to v_j requires $2^{\text{dist}(v_i,v_j)}$ pebbles.

The general problem in graph pebbling is to compute for a graph G the minimum number of pebbles p such that for any configuration C of p or more pebbles on G and any root vertex r, we may make a sequence of pebbling moves from C to reach r. This number p is called the *pebbling number* of G, generally denoted as $f(G)$ or $\pi(G)$ [2]; we adopt the latter notation.

As a simple example, consider the following graph:



The pebbling number of this graph is $\pi(G) = 5$. That is, no matter how we arrange five pebbles on the vertices of this graph, we can always move a pebble to any vertex through some sequence of moves.

Much work has been done in finding pebbling numbers of various classes of graphs. One conjecture that has received much attention was made by Ron Graham. This conjecture relates the product of the pebbling numbers of a pair of graphs with the pebbling number of their Cartesian product.

Graham's Conjecture. For all graphs G_1 and G_2, $\pi(G_1 \square G_2) \leq \pi(G_1)\pi(G_2)$ [2].

Graham's Conjecture is still an open problem, but some progress has been made. Of note, it has been proven to hold for the product of two trees and the product of two cycles, so long as for $C_m \square C_n$, m and n are not both in $\{5, 7, 9, 11, 13\}$, except for $C_5 \square C_5$, which has been proven [2].

While much of the terminology surrounding graph pebbling is similar, we will aim to make some standardization going forward. As we move into exploration of target arrangements of pebbles rather than moving a single pebble to a single vertex, there are now two arrangements which will be referenced. We will continue to use "distribution" to refer to the "target arrangement" of pebbles on a graph. That is, for a graph G, there is a function $D : \mathbb{N} \cup \{0\} \longrightarrow V(G)$, assigning a number to each vertex on G. This set of vertex/number pairs forms a distribution D. For any particular vertex v, $D(v)$ is the number of pebbles on v in D. A "configuration" C is a nearly identical idea, but we reserve it for reference to the initial arrangement of pebbles, and similarly to the notation for distributions, $C(v)$ is the initial number of pebbles on a vertex. If we begin with a configuration C and through a sequence of pebbling moves are able to reach a distribution D, then we say "we may reach D from C" or that "C is solvable" (with appropriate negations).

In [1], various generalizations of Graham's conjecture are explored. Of particular interest is the generalization of the conjecture which applies to graphs and general distributions of pebbles on those graphs, rather than just getting a single pebble to a single vertex. This forms the basis of our research as we explore relationships between the pebbling numbers of graphs when we differentiate by labeling a graph or leaving it unlabeled and how these values and relationships compare to existing conjectures. In particular, consider the following generalization of Graham's Conjecture.

Conjecture 1.1. For graphs G_1 and G_2 with distributions of pebbles D_1 and D_2, respectively, we have $\pi(G_1 \square G_2, D_1 \cdot D_2) \leq \pi(G_1, D_1)\pi(G_2, D_2)$ [1].

Our conjecture is similar, though we will use labeled graphs to make a similar claim.

Conjecture 1.2. For graphs G_1 and G_2 with distributions of pebbles D_1 and D_2, respectively, we have $\pi_L(G_1 \square G_2, D_1 \cdot D_2) \leq \pi_L(G_1, D_1)\pi_L(G_2, D_2)$.

The labeled pebbling number of a graph and distribution $\pi_L(G, D)$, which we will discuss in Chapter 3, is a variant of the original pebbling number in which we consider reaching a distribution D on a labeled graph.

Chapter 2: Path Pebbling

Before focusing on labeled and unlabeled pebbling, we will consider another pebbling variant we call path pebbling. The primary difference for this variant is how moves are defined. Contrasting to a normal pebbling move, a *path pebbling move* consists of

1. removing $|C(v_0)|$ pebbles from a starting vertex v_0
2. discarding a pebble as a cost
3. placing one pebble on each vertex of a sequence of i sequentially adjacent vertices $v_1, v_2, \ldots, v_i$ that form a path, denoted $[v_0, (v_1, v_2, \ldots, v_i)]$ where $i = |C(v_0) - 1|$.

In the case where $C(v_0) - 1 > i$ and no other adjacent vertices are available, then we simply leave the excess pebbles on v_i. Just as in normal pebbling with pebbling numbers, we define the *path pebbling number* of a graph G, $\pi_p(G)$, to be the fewest number of pebbles such that for any configuration C of size $\pi_p(G)$ or greater, we can place a pebble any root vertex r by making path pebbling moves starting from C.

We will begin with a lemma establishing a path pebbling number for trees. With that in hand, we will extend the result to all graphs. Note that a *leaf* is a vertex of degree one.

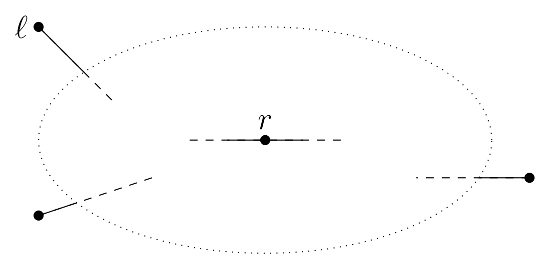
Theorem 2.1. For a tree T on n vertices, $\pi_p(T) = n$.

Proof. We will induct on the number of vertices of T. In each case, we will assume that no pebbles begin on the root vertex r, else no moves are necessary. First consider the base case $n = 2$. Note that if there is only a single pebble on T, then we may place it on the non-root vertex, and no moves are possible, so the configuration is unsolvable. Now we assume there are two pebbles on T and no pebbles on r; then there are two on the other vertex. In that case, a single path pebbling move puts a pebble on r.

Now assume that T has $n = k \geq 2$ vertices and that $\pi_p(T) = k$. Now we consider the case $n = k + 1$. As above, if there are fewer than $k + 1$ pebbles on T, then we may place one pebble on each non-root vertex. No moves are possible from this configuration, so we cannot reach r.

Now assume that there are $k + 1$ pebbles on T. Consider some leaf ℓ on T that is not the root vertex. There are two cases to consider.

- If $C(\ell) \in \{0, 1\}$, then consider the subgraph $T' = T - \{\ell\}$. Then $|C(T')| \in \{k, k+1\}$ and $|V(T')| = k$. Thus by the inductive hypothesis, we may reach r from C.
- If $C(\ell) = p > 1$, then make a path pebbling move $[\ell, (v_1, \ldots, v_{p-1})]$. Then consider the subgraph $T' = T - \{\ell\}$. After the previous move, we have either reached r, or we have $|C(T')| = k$ and $|V(T')| = k$, and so by the inductive hypothesis, we may reach r from C.



In any case, we see that we may reach r from C, and so $\pi_p(T) = n$. □

Corollary 2.1. For any graph G on n vertices, $\pi_p(G) = n$.

Proof. For any graph G, consider a spanning tree T of G. Then by Theorem 2.1, $\pi_p(T) = n$, and the moves used on T to place a pebble on the root vertex r are precisely the same as those used on G to place a pebble on r, and so $\pi_p(G) = n$. $\square$

Chapter 3: Labeled and Unlabeled Graphs

3.1 Extending Existing Results in the Labeled Pebbling View

Before beginning work contrasting labeled and unlabeled pebbling, we verify and translate some results from normal graph pebbling to labeled and unlabeled pebbling. We now introduce definitions and notations for unlabeled and labeled pebbling numbers.

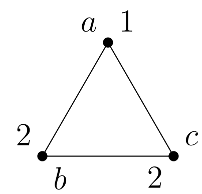
Definition 3.1. Given a graph G and a distribution D of pebbles,

- the *labeled pebbling number* $\pi_L(G, D)$ of a graph G and a distribution of pebbles D is the smallest number of pebbles such that for any configuration C with $|C| \geq \pi_L(G, D)$, we may reach D from C.
- the *unlabeled pebbling number* $\pi(G, D)$ of a graph G and a distribution of pebbles D is the smallest number of pebbles such that for any configuration C with $|C| \geq \pi(G, D)$, we may reach D or some isomorphism of D from C.

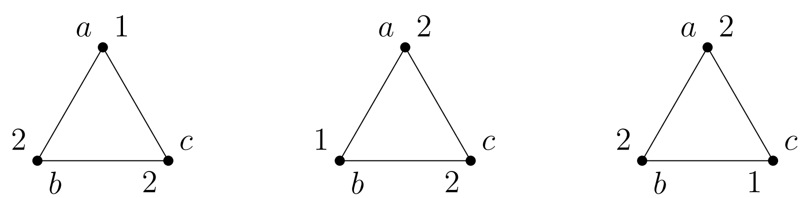
We view an isomorphism of a distribution of pebbles in a similar way to an isomorphism of a graph.

Definition 3.2. Consider a graph G and an isomorphism μ of G. An isomorphism μ_D of a distribution of pebbles D is a mapping $\mu_D : V(G) \longrightarrow V(\mu(G))$ such that for all $v \in V(G)$, $D(v) \longmapsto D(\mu(v))$.

As example of the distinction between labeled and unlabeled pebbling numbers, consider the following graph and distribution:



In calculating the labeled pebbling number of this graph and distribution, we need only examine this fixed distribution. However, when calculating the unlabeled pebbling number, it is sufficient to be able to reach any of the distributions below from configurations.



For the following propositions, we introduce the following notation and definitions. For a graph G and a set of distributions S, we define $\pi(G, S)$ as the smallest number of pebbles p such that for any configuration of p pebbles, we may reach any distribution $D \in S$. Also, note that $\pi_t(G, v)$ and $\pi_t(G)$ represent the *t-pebbling number* of a graph G and vertex v. Similar to existing definitions, $\pi_t(G, v)$ is the smallest number of pebbles p such that for any configuration C with $|C| \geq p$, t pebbles may be moved to v. $\pi_t(G)$ is defined similarly, that is, the smallest number of pebbles p such that for any configuration of p or more pebbles, we may move t pebbles to any vertex on G. Now consider Proposition 1.1 from [1].

Proposition 3.1. Let G be any graph, and let S and S' be two sets of distributions on G. Then the various pebbling numbers are related as follows.

1. We have $\pi(G, S) = \max_{D \in S} \pi(G, D)$.

2. In particular, $\pi(G) = \max_{v \in V(G)} \pi(G, v)$ and $\pi_t(G) = \max_{v \in V(G)} \pi_t(G, v)$.

3. Furthermore, if $S \subseteq S'$, then $\pi(G, S) \le \pi(G, S')$.

Now we translate and prove this proposition for labeled pebbling numbers. We establish a few definitions for use in this proposition and proof, primarily the various types of pebbling numbers.

If S is a set of distributions D on a graph G, then $\pi_L(G, S)$ is the smallest number of pebbles p such that for any configuration C of size at least p, we may reach any of the distributions in S from C. The notation $\pi_L(G, v)$ represents the smallest number of pebbles p needed so that for any configuration C with $|C| \ge p$, we may move a pebble to v; the notation $\pi(G)$ above is the original pebbling number of G, and so the notation reflects the lack of dependence on a particular vertex. We will denote $\pi_{L,t}(G, v)$ and $\pi_{L,t}(G)$ with similar labeled notation for the t-pebbling number.

Proposition 3.2. Let G be any graph, and let S and S' be two sets of distributions on G. Then the various pebbling numbers are related as follows.

1. We have $\pi_L(G, S) = \max_{D \in S} \pi_L(G, D)$.

2. In particular, $\pi_L(G) = \max_{v \in V(G)} \pi_L(G, v)$ and $\pi_{L,t}(G) = \max_{v \in V(G)} \pi_{L,t}(G, v)$.

3. Furthermore, if $S \subseteq S'$, then $\pi_L(G, S) \le \pi_L(G, S')$.

Proof. 1. Let $D \in S$. Then $\pi_L(G, S) \ge \pi_L(G, D)$, since we need a sufficient number of pebbles to solve D. Now let $M = \max_{D \in S} \pi_L(G, D)$. Let D_0 be a distribution such that $\pi_L(G, D_0) = M$. Since $\pi_L(G, S) \ge \pi_L(G, D_0)$, then $\pi_L(G, S) \ge \max_{D \in S} \pi_L(G, D)$.

Now let $M = \max\limits_{D \in S} \pi_L(G, D)$. Then $M \geq \pi_L(G, D)$ for all $D \in S$. That is, given M pebbles, we may reach D for all $D \in S$. Then by definition, $M = \max\limits_{D \in S} \pi_L(G, D) \geq \pi_L(G, S)$. So $\pi_L(G, S) = \max\limits_{D \in S} \pi_L(G, D)$.

2. Let $v \in V(G)$. Then $\pi_L(G) \geq \pi_L(G, v)$, since we must have a sufficient number of pebbles to reach v. Now let $v' \in V(G)$ be a vertex such that $\pi_L(G, v') = \max\limits_{v \in V(G)} \pi_L(G, v)$. Then $\pi_L(G) \geq \pi_L(G, v') = \max\limits_{v \in V(G)} \pi_L(G, v)$.

   Now take $M = \max\limits_{v \in V(G)} \pi_L(G, v)$. Then $M \geq \pi_L(G, v)$ for each vertex on G. That is, for a configuration of M pebbles, we may move a pebble to any vertex v on G. Thus we have $\max\limits_{v \in V(G)} \pi_L(G, v) \geq \pi_L(G)$, and so $\pi_L(G) = \max\limits_{v \in V(G)} \pi_L(G, v)$.

   Similarly, let $w \in V(G)$. Then $\pi_{L,t}(G) \geq \pi_{L,t}(G, w)$, since we must have a sufficient number of pebbles to move t pebbles to w. Now let $w' \in V(G)$ be a vertex such that $\pi_{L,t}(G, w') = \max\limits_{w \in V(G)} \pi_{L,t}(G, w)$. Then $\pi_{L,t}(G) \geq \pi_{L,t}(G, w') = \max\limits_{w \in V(G)} \pi_{L,t}(G, w)$.

   Now take $M = \max\limits_{w \in V(G)} \pi_{L,t}(G, w)$. Then $M \geq \pi_{L,t}(G, w)$ for each other vertex on G. That is, for a configuration of M pebbles, we may move t pebbles to any vertex w on G. Thus we have $\max\limits_{w \in V(G)} \pi_{L,t}(G, w) \geq \pi_{L,t}(G)$, and so $\pi_{L,t}(G) = \max\limits_{w \in V(G)} \pi_{L,t}(G, w)$.

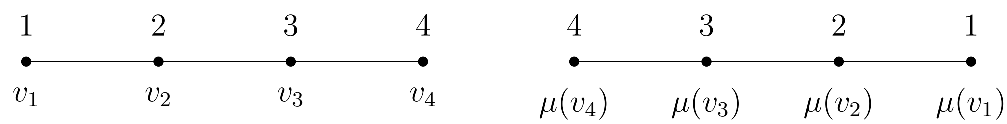
3. Assume $S \subseteq S'$. Then S' contains all distributions in S; additionally, if S is a proper subset, then S' contains other distributions. $\pi_L(G, S')$ must be at least $\pi_L(G, S)$ (since the distribution of highest pebbling number in S is also in S'), so there exists a D such that $\pi_L(G, S) = \pi_L(G, D)$. Now, since $D \in S'$, then $\pi_L(G, S') \geq \pi_L(G, D) = \pi_L(G, S)$.

$\square$

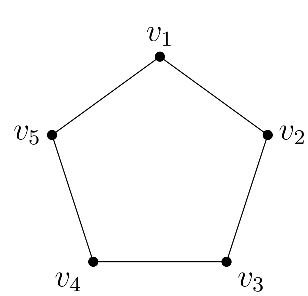
3.2 Contrasting Labeled and Unlabeled Graphs

We now return to contrasting labeled and unlabeled pebbling numbers. We begin by illustrating the difference between $\pi_L(G, D)$ and $\pi(G, D)$ (the labeled and unlabeled pebbling numbers of a graph and distribution, respectively).

In the figure below, we have a graph G and a distribution D of pebbles on the left side. The right side represents an isomorphism μ on G and D – this isomorphism is a simple reflection mapping v_i to v_{n+1-i}. When calculating the labeled pebbling number $\pi_L(G, D)$, we would simply seek to reach the left distribution from starting configurations. In contrast, when we calculate the unlabeled pebbling number $\pi(G, D)$, we would seek to reach either D or $\mu(D)$ from starting configurations.



For the above graphs, we can calculate that $\pi_L(G, D) = 49$ and $\pi(G, D) = 19$. We will explore deriving these values later in the section. An immediate takeaway from this is that calculating unlabeled pebbling numbers may be more challenging than their labeled brethren, especially for graphs with multiple isomorphisms. As the number of symmetries grows, then that many more distributions must be considered when determining whether or not a given configuration is solvable. In particular, consider the cycle graph C_5:



In this graph alone, there are five reflectional symmetries and five rotational symmetries; that is, when determining the solvability of a configuration on this graph, we could be forced to consider up to ten different distributions.

Proposition 3.3. For a graph G with a distribution of pebbles D, $\pi(G, D) \leq \pi_L(G, D)$.

Proof. Let G be a graph with a distribution of pebbles D. Consider a configuration of pebbles C with $|C| = \pi_L(G, D)$; thus we can reach D from C. Note that D is an isomorphism of itself (the identity isomorphism). Thus $\pi(G, D) \leq \pi_L(G, D)$. $\square$

In many situations, labeled pebbling numbers will be strictly greater than their unlabeled counterparts. In particular, consider the following graph and distribution of pebbles.

As we will see in the Cover Pebbling Theorem [3], we need only consider configurations with all pebbles on a single vertex. As such, we compute the minimum size of simple configurations on each vertex that will allow us to reach the given distribution. For vertex a, we see that we need $8 \cdot 3 + 4 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 31$ pebbles. By similar calculations, we see that we need 17 pebbles on b, 13 pebbles on c, and 15 pebbles on d. Thus we conclude that the pebbling number is $\pi_L(G, D) = 31$.

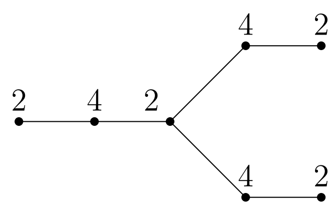
Consider vertex a. In the labeled situation, we would require 31 pebbles on a to reach the distribution. However, now consider the isomorphism μ defined by $\mu(a) = d$, $\mu(b) = c$, $\mu(c) = b$, and $\mu(d) = a$. In the labeled case, a simple configuration with 15 pebbles on a is unsolvable. However, by considering 15 pebbles on $\mu(a)$, we see that we can, in fact, reach D. Similarly, by considering a necessary configuration on

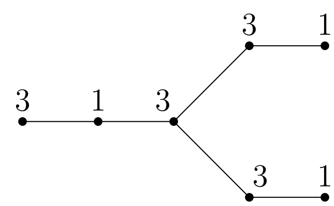
b, instead of requiring 17, we may then invoke μ to see that we actually only need 13 pebbles from $\mu(b)$. Therefore, the largest necessary simple configuration possible is 15, so the unlabeled pebbling number of this graph and distribution is $\pi(G, D) = 15$.

However, there are situations for which equality holds. In particular, *symmetric* distributions will yield equality.

Definition 3.3. A distribution D of pebbles on a graph G is *symmetric* (or invariant under isomorphism) if for any isomorphism $\mu(G)$ and all vertices v, $D(v) = D(\mu(v))$.

The use of "symmetric" is very intuitive when dealing with paths (simple reflective symmetry), cycles, and complete graphs (reflective and/or rotational symmetry), though for many other graphs, the intuitive approach may fall a bit short. Consider the tree from Chapter 1 with two different distributions shown below:





The distribution on the tree on the left is symmetric, since under isomorphism, it remains the same. However, the distribution on the tree on the right is not a symmetric distribution, as we could effectively "switch" the the pairs of vertices on the left and upper right sides to yield a different distribution.

Theorem 3.1. For a graph G and a symmetric distribution of pebbles D, $\pi(G, D) = \pi_L(G, D)$.

Proof. Let G be a graph with a symmetric distribution of pebbles D, and let C be a configuration of pebbles such that $|C| = \pi(G, D)$.

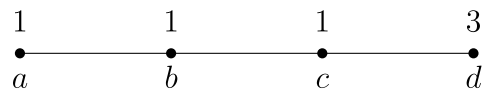
Assign labels to the vertices of G, and note the sequence of pebbling moves needed to reach D from C. Let μ be an isomorphism from G to $\mu(G)$. Consider a configuration C^* defined by $C^*(v) = C(\mu(v))$. For each pebbling move $[u, v]$ in the sequence of moves from C to D, we use the move $[\mu(u), \mu(v)]$ to reach D from C^* since $D(v) = D(\mu(v))$ for each vertex of G. Therefore, we have that $\pi_L(G, D) \leq \pi(G, D)$. From Proposition 3.3, we have $\pi(G, D) \leq \pi_L(G, D)$, so we have $\pi(G, D) = \pi_L(G, D)$. □

It is worth noting that the converse of the previous result isn't true in general; that is, given a graph G and distribution of pebbles D, if $\pi(G, D) = \pi_L(G, D)$, then the distribution of pebbles isn't necessarily symmetric.

Before constructing a counterexample for the converse of Theorem 3.1, we will reproduce Jonas Sjöstrand's Cover Pebbling Theorem and its proof [3]. This allows for significantly simpler calculation of pebbling numbers during construction. First, we begin with a few definitions local to the theorem and proof. We will modify the original statement and proof to be in line with our terminology along with minor elaboration on some points.

Definition 3.4. Definitions used in the theorem and proof:

- A *cover distribution* D on a graph G is a distribution such that $D(v) > 0$ for all $v \in V(G)$.
- Let w be a distribution of pebbles. A *w-cover* is a distribution of pebbles such that every vertex has at least as many pebbles as in w. $w(v)$ is the number of pebbles on v in w. We generally assume that w is positive, that is, for all $v \in V(G)$, we have $w(v) > 0$. For example, consider this previous graph and distribution:



In this case, a w-cover for this graph is $w(a, b, c, d) = (1, 1, 1, 3)$, adopting a compact notation similar to the normal notation for distributions.

- The *w-cover pebbling number* is the smallest n such that from any configuration of n pebbles, it is possible to obtain a w-cover via a sequence of pebbling moves.
- A vertex v is *fat*, *thin*, or *perfect* if the number of pebbles on it is greater than, less than, or equal to $w(v)$, respectively.
- A *simple configuration* is one in which all pebbles begin on a single vertex.
- The *cost* from a vertex v of a pebble on a vertex u is $2^{d(u,v)}$, and the sum of the costs from v of all pebbles in w is the cost of cover pebbling from v.
- The *value* of a pebble is the sum of the value of the pebbles removed to place it. For example, in a pebbling move, removing two pebbles of value 1 from a vertex places a pebble of value 2 on an adjacent vertex. Removing this new pebble and an existing one of value 1 on the same vertex places a pebble of value 3 on an adjacent vertex.

Cover Pebbling Theorem (Sjöstrand)**.** Let w be a positive distribution. To determine the w-cover pebbling number of a (directed or undirected) connected graph, it is sufficient to consider simple configurations. In fact, for any unsolvable configuration of size p, we may place p pebbles on one of the fat vertices *[Footnote: Of course, this is not true if there are no fat vertices, but then any vertex will do.]* with this new simple configuration still unsolvable. [3]

For the proof of this theorem, we will be using much of the original text, adding in some elaboration and details on certain points.

Proof. Start with a configuration that admits no cover pebbling. The goal will be to approach the contrapositive of the original statement, that is, if an arbitrary configuration cannot be solved, then no simple configuration of the same size can be solved. An arbitrary unsolvable configuration C will be assumed, and it will be shown that no simple configuration of the same size can be solved.

Assume there are no fat vertices in C. Then $|C| < |w|$, since each vertex v may have at most $w(v)$ pebbles on it, and at least one vertex must have fewer (else $C = w$). We can remove all $|C| = J$ pebbles from the graph and place J pebbles on any vertex. The cost of cover pebbling from this vertex is no less than the number of pebbles in w, since for any vertex v of distance d from our starting vertex, we require 2^d pebbles to move one pebble to v. As such, we lack sufficient pebbles to get back to our non-simple configuration, much less reach w.

If some vertex is fat, we will have to do some pebbling. At the beginning, all pebbles will have a value of one. During the pebbling we will always maintain the following efficiency condition: *Every pebble has a value no greater than the cost from its nearest fat vertex (the fat vertex that minimizes this cost).* The value of a pebble on a fat vertex will therefore be bounded by the nearest fat vertex that it isn't on. Note that the efficiency condition is trivially satisfied before any pebbling is done (since all the pebbles have value of one).

Now pebble like this: Among all pairs (f, t) of a fat and a thin vertex, take one that minimizes the distance $d(f, t)$. Let $fp_1p_2 \cdots p_{d-1}t$ be a minimal path from f to t. Every inner vertex p_i of this path must be perfect, since if it were thin, then (f, p_i) would be a (fat, thin)-pair with $d(f, p_i) < d(f, t)$, and if it were fat, then (p_i, t) would be a (fat, thin)-pair with $d(p_i, t) < d(f, t)$. Furthermore, f must be a nearest fat vertex to t and to every p_i. Now we use two pebbles from f to move to p_1. p_1 must have at least one other pebble on it (since w is a positive distribution), so we use the new pebble on p_1 together with any other pebble from p_1 to move a pebble to p_2; then we use the new pebble and another one on p_2 to move to p_3, repeating this process until we move a pebble onto t.

The value of the new pebble on t is 2 plus the sum of the values of the old pebbles on $p_1, \ldots, p_{d-1}$ that were consumed. By the efficiency condition this is no greater than $2 + 2^1 + 2^2 + \cdots + 2^{d-1}$ which equals 2^d. Recall that the cost to pebble from f to t is $2^{\mathrm{dist}(f,t)}$, and thus the condition is satisfied even after this operation. It is possible that f is no longer fat, but this only makes it easier to fulfill the condition, since any other fat vertices on the graph must be at least as distant as f.

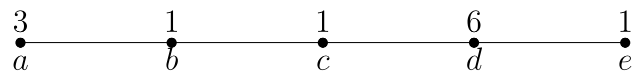
We iterate the above procedure (choosing a new pair (f, t) and so on) until no vertex is fat. During each iteration the total number of pebbles on fat vertices decreases, and no new fat vertices are created, since any thin vertices either remain thin or become perfect during the algorithm (since an iteration places a pebble on a thin vertex, the vertex will have at most as many pebbles as in w), and any perfect vertices between a fat-thin pair become thin (since each will gain one pebble and then lose two during the algorithm), so we cannot continue forever.

Let f' be the fat vertex that survived the longest, and denote the current configuration as C'. Each pebble remaining on the graph has value at most equal to its cost from f'. But there are still thin vertices (since we assumed the initial configuration was unsolvable). Then by placing v pebbles (where v is the total value of all pebbles on the graph) on f', we can reach C', but not the distribution w; thus the cost of cover pebbling from f' exceeds the total value of the pebbles. Therefore, cover pebbling is not possible with all pebbles initially on f', since an arbitrary configuration being unsolvable implies that a simple configuration of the same size is also unsolvable. $\square$

We note that by considering a graph with labeled vertices, the above proof undergoes no alteration. Translation of the result gives us the following corollary.

Corollary 3.1. If D is a cover distribution on a labeled graph G, then it is only necessary to consider simple configurations in calculating $\pi_L(G, D)$.

With this in hand, we return to the converse of Theorem 3.1. Consider the following graph and distribution:



Note that this distribution is not symmetric. We assert that for the above graph and distribution, $\pi_L(G, D) = \pi(G, D) = 73$.

First note that there exists an unsolvable distribution of 72 pebbles on the labeled graph. Consider placing 72 pebbles on a. We require 16 pebbles to move one pebble to e, $6 \cdot 8 = 48$ pebbles to move six to d, 4 to move one to c, and 2 to move one to b. However, the listed moves require 70 pebbles, leaving two on a, one short of the three we require.

With the above process in mind, we note that 73 is precisely enough to reach D from a, and from Sjöstrand's theorem, we need only consider simple configurations. Additionally, note that a simple configuration on e large enough to reach D requires 73 pebbles as well by the same process. Observe that we require $3 \cdot 16 = 48$ pebbles to move three to a, 8 to move one to b, 4 to move one to c, $6 \cdot 2 = 12$ to move six to d, and one for e. The sum of these required values is 73. Through similar calculations we can see that these values are strictly greater than the minimum sizes of required simple configurations on vertices b, c, and d.

Now consider the unlabeled pebbling number of the graph. As noted before, we aim to reach D or D_μ for some isomorphism μ. That is, we aim to reach D or the reflection of D. Again, as before, there exists an identical unsolvable distribution of 72 pebbles on a. Even by considering a configuration beginning on $\mu(a)$, we see that

72 pebbles is still insufficient to reach D. For a simple configuration on either a or e, then, we require 73 pebbles, even if we try to reach $\mu(D)$ instead of D. Thus, $\pi(G, D) = 73$ as well.

Chapter 4: Stepping into Products of Paths

Before continuing, notation and a definition must be introduced. Given two graphs G and H with respective distributions D_G and D_H, denote the product of the graphs and distributions as $(G \square H, D_G \cdot D_H)$. For arbitrary vertices $g \in V(G)$ and $h \in V(H)$, we define $D_G \cdot D_H$ by $D((g,h)) = D_G(g)D_H(h)$.

With this, we move on to pebbling numbers for particular paths. We begin with P_2 and then consider the product $P_2 \square P_2$.

Proposition 4.1. For P_2 with a distribution D of pebbles on vertices v_1 and v_2, then $\pi_L(P_2, D) = \max\{D(v_1) + 2D(v_2), D(v_2) + 2D(v_1)\}$.

Proof. Consider P_2 and a distribution D. Without loss of generality, assume $D(v_1) \leq D(v_2)$, so $D(v_1) + 2D(v_2) \geq D(v_2) + 2D(v_1)$. Then we assert that $\pi_L(P_2, D) = D(v_1) + 2D(v_2)$.

We begin by establishing an unsolvable configuration of greatest size. Consider a configuration C defined by $C(v_1) = D(v_1) + 2D(v_2) - 1, C(v_2) = 0$. In one case, if $D(v_1) = 0$, then we may make $D(v_2) - 1$ moves to put $D(v_2) - 1$ pebbles on v_2, leaving one pebble on v_1. D has not been reached, and no moves towards D are possible. In the second case, if $D(v_1) \neq 0$, then we may make $D(v_2)$ moves to put $D(v_2)$ pebbles on v_2, leaving $D(v_1) - 1$ pebbles on v_1. As before, D has not been reached, and no moves remain to do so. Thus $\pi_L(P_2, D) \geq D(v_1) + 2D(v_2)$.

Now we show that $D(v_1) + 2D(v_2)$ pebbles in any configuration is sufficient to reach D. In the first case, if $D(v_1), D(v_2) \neq 0$, then Sjöstrand's theorem [3] gives us the desired result, as a simple configuration on v_1 will require $D(v_1) + 2D(v_2)$ pebbles (greater than $D(v_2) + 2D(v_1)$ by assumption).

As a second case, assume that $D(v_1) = 0$. Then we claim $\pi_L(P_2, D) = 2D(v_2)$. Note a simple configuration C defined by $C(v_1) = 2D(v_2), C(v_2) = 0$ is solvable. Now consider a configuration C' defined by $C(v_1) = 2D(v_2) - x, C(v_2) = x$ for $x < D(v_2)$ (else the configuration is trivially solvable). If $2D(v_2) - x$ is even, then $x = 2n$ for some $n \geq 1$, and we may move $D(v_2) - n$ pebbles from v_1 to v_2. As a result, v_2 now has $D(v_2) - n + 2n \geq D(v_2)$ pebbles on it, and so C' is solvable. In the case where $2D(v_2) - x$ is odd, then $x = 2m - 1$ for some $m \geq 1$. We may then move $D(v_2) - m$ pebbles to v_2. Similarly to the previous case, we find that v_2 now has $D(v_2) - m + 2m - 1 \geq D(v_2)$ pebbles on it, and so C' is solvable. □

With this result in hand, we now move to the next result.

Theorem 4.1. Given P_2, the path on two vertices, and distributions D_1 and D_2, $\pi_L(P_2 \square P_2, D_1 \cdot D_2) = \pi_L(P_2, D_1)\pi_L(P_2, D_2)$.

Proof. Consider two P_2 graphs with v_1, v_2 and w_1, w_2 as each graph's respective vertices. Let D_1 and D_2 be distributions on each graph. Then by Proposition 4.1, we have $\pi_L(P_2, D_1) = \max\{D_1(v_1) + 2D_1(v_2), D_1(v_2) + 2D_1(v_1)\}$ and $\pi_L(P_2, D_2) = \max\{D_2(w_1) + 2D_2(w_2), D_2(w_2) + 2D_2(w_1)\}$. Without loss of generality, assume that $D_1(v_1) \leq D_1(v_2)$ and $D_2(w_1) \leq D_2(w_2)$, so $\pi_L(P_2, D_1) = D_1(v_1) + 2D_1(v_2)$ and $\pi_L(P_2, D_2) = D_2(w_1) + 2D_2(w_2)$. This gives us $\pi_L(P_2, D_1)\pi_L(P_2, D_2) = (D_1(v_1) + 2D_1(v_2))(D_2(w_1) + 2D_2(w_2))$.

Now consider the graph $P_2 \square P_2$ with distribution $D = D_1 \cdot D_2$. Then $D((v_i, w_j)) = D_1(v_i)D_2(w_j)$ for $i, j \in \{1, 2\}$. Since $D_1(v_1) \leq D_1(v_2)$ and $D_2(w_1) \leq D_2(w_2)$, then $D((v_1, w_1)) \leq D((v_1, w_2)) \leq D((v_2, w_2))$ and $D((v_1, w_1)) \leq D((v_2, w_1)) \leq D((v_2, w_2))$. There are three cases to consider: (1) D_1 and D_2 are both covers, (2)neither D_1 nor D_2 is a cover, and (3) D_1 is not a cover, but D_2 is.

Case 1: If D_1 and D_2 are covers, then so is D. Since D is a cover, then the

Cover Pebbling Theorem allows us to restrict our approach to simple configurations. We choose the initial configuration C given by $C((v_1, w_1)) = D_1(v_1)D_2(w_1) + 2(D_1(v_2)D_2(w_1) + D_1(v_1)D_2(w_2)) + 4D_1(v_2)D_2(w_2)$. We note that C' defined by $C'((v_1, w_1)) = C((v_1, w_1)) - 1$ is unsolvable. We could move $D_1(v_2)D_2(w_2)$ pebbles to (v_2, w_2), $D_1(v_1)D_2(w_2)$ pebbles to (v_1, w_2), and $D_1(v_2)D_2(w_1)$ pebbles to (v_2, w_1). However, this leaves us with $D_1(v_1)D_2(w_1) - 1$ pebbles on (v_1, w_1), and so we cannot reach D. A similar issue arises if we make moves last to some other vertex.

Observe that we may reach D from C by moving $2D((v_2, w_2))$ pebbles from (v_1, w_1) to (v_1, w_2) or (v_2, w_1), and then $D((v_2, w_2))$ pebbles from there (v_2, w_2), thus reaching D. We claim that this simple configuration is the maximum of the four possible configurations. If we assume another was larger (the one beginning on (v_1, w_2), for example), then we will show that a contradiction arises.

First note that for a simple configuration beginning on (v_1, w_1), we require (as noted above) $D((v_1, w_1)) + 2D((v_2, w_1)) + 2D((v_1, w_2)) + 4D((v_2, w_2))$ pebbles to reach D. If the configuration was on (v_1, w_2), then we require $D((v_1, w_2)) + 2D((v_1, w_1)) + 2D((v_2, w_2)) + 4D((v_2, w_1))$ pebbles to reach D. Consider

$$\begin{aligned} C((v_1, w_1)) &= D((v_1, w_1)) + 2D((v_2, w_1)) + 2D((v_1, w_2)) + 4D((v_2, w_2)) \\ &= D(v_1)D(w_1) + 2D(v_2)D(w_1) + 2D(v_1)D(w_2) + 4D(v_2)D(w_2) \\ &= (D(v_1) + 2D(v_2))(D(w_1) + 2D(w_2)), \end{aligned}$$

and similarly,

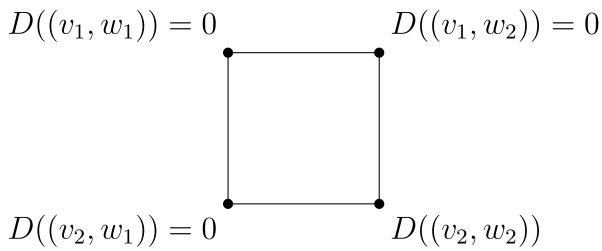
$$\begin{aligned} C((v_1, w_2)) &= D((v_1, w_2)) + 2D((v_1, w_1)) + 2D((v_2, w_2)) + 4D((v_2, w_1)) \\ &= D(v_1)D(w_2) + 2D(v_1)D(w_1) + 2D(v_2)D(w_2) + 4D(v_2)D(w_1) \\ &= (D(v_1) + 2D(v_2))(D(w_2) + 2D(w_1)). \end{aligned}$$

Now, if we assume that $C((v_1, w_1)) < C((v_1, w_2))$, we see that

$$\begin{aligned} C((v_1, w_1)) &< C((v_1, w_2)) \\ (D(v_1) + 2D(v_2))(D(w_1) + 2D(w_2)) &< (D(v_1) + 2D(v_2))(D(w_2) + 2D(w_1)) \\ D(w_1) + 2D(w_2) &< D(w_2) + 2D(w_1), \end{aligned}$$

which contradicts our assumption that $D(w_1) \leq D(w_2)$. The other cases result in similar contradictions. Thus $\pi_L(P_2 \square P_2, D_1 \cdot D_2) = \pi_L(P_2, D_1)\pi_L(P_2, D_2)$.

Case 2: In the case where neither D_1 nor D_2 is a cover distribution (assuming that $D(v_1) = D(w_1) = 0$), we end up with a situation like

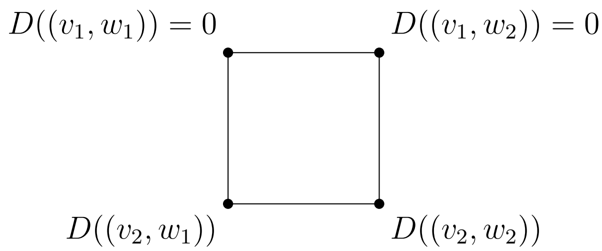


We claim that the pebbling number is $D((v_1, w_1)) + 2D((v_1, w_2)) + 2((v_2, w_1)) + 4D((v_2, w_2)) = 4D((v_2, w_2))$ (the other three terms are zero), and we will induct on the size of the distribution. Begin with the base case that $D((v_2, w_2)) = 1$. Note that an initial configuration C given by $C((v_1, w_1)) = 3$ and 0 otherwise is not solvable, as a single move puts one pebble on $((v_1, w_2))$ or $((v_2, w_1))$, leaving no possible moves with D not having been reached. Now consider four pebbles on the graph. If there are 4 on (v_1, w_1) or 2 or more on (v_1, w_2) or (v_2, w_1), then we may reach D. If these conditions are not true, then there are 2 or 3 pebbles on (v_1, w_1) and 1 pebble on (v_2, w_1) or (v_1, w_2) (or both). Make one pebbling move from (v_1, w_1) to an adjacent vertex with a pebble on it, then make a move from that vertex to (v_2, w_2). Thus we reach D.

Now assume that for some $k \geq 1$, we may begin with a configuration of $4k$ pebbles on the graph and move k pebbles to (v_2, w_2). Now consider the case for $k+1$. First note that placing $4k+3$ pebbles on (v_1, w_1) results in an unsolvable configuration, since we can use $4k$ pebbles to move k pebbles to (v_2, w_2) as per the inductive hypothesis, but the remaining three are not sufficient to move another pebble to (v_2, w_2). Now we place $4k + 4$ pebbles on the graph. The inductive hypothesis allows us to use $4k$

pebbles to move k pebbles to (v_2, w_2), and by our base case, we may use the remaining four to place one more pebble on (v_2, w_2). Thus we have reached D.

Case 3: For the final case, assume that D_1 is not a cover distribution, but D_2 is. Additionally, assume that $D(v_1) = 0$ and $D_2(w_2) \geq D_2(w_1)$. Now our arrangement looks like



We will perform two separate inductions to handle this final case. In the first, we will assume that $D((v_2, w_1)) = D((v_2, w_2)) = k$ and induct on k. For the second, we assume $D((v_2, w_1)) = m \leq D((v_2, w_2)) = k$ and induct on k.

Case 3a: First assume that $D((v_2, w_1)) = D((v_2, w_2))$. Now consider a base case for the first induction. Then $D((v_2, w_1)) = D((v_2, w_2)) = 1$. We claim that the pebbling number for this graph and distribution is 6. Placing 5 pebbles on (v_1, w_1) is an unsolvable configuration, since using 4 pebbles to move one to (v_2, w_2) or 2 pebbles to (v_2, w_1) leaves too few to move a pebble to the other vertex.

Now we look at some configuration of six pebbles on the graph. If we begin with one or two pebbles on either (v_2, w_1) or (v_2, w_2), then by the previous case, we may use four of the remaining pebbles to put one on the other vertex, reaching D. If three or more pebbles begin on either (v_2, w_1) or (v_2, w_2), then a single move to the other vertex allows us to reach D.

Now we assume that the six pebbles are arranged between (v_1, w_1) and (v_1, w_2). If each vertex has two or more pebbles on it, then the moves $[(v_1, w_1), (v_2, w_1)]$ and $[(v_1, w_2), (v_2, w_2)]$ allow us to reach D. If (v_1, w_1) (the case for (v_1, w_2) will be similar) only has one pebble, then move one pebble from (v_1, w_2) to it, and then the moves $[(v_1, w_1), (v_2, w_1)]$ and $[(v_1, w_2), (v_2, w_2)]$ will allow us to reach D. If $C(v_1, w_1) = 0$ (as before, the case for (v_1, w_2) is similar), then we may use four pebbles to move one to (v_2, w_1) and two pebbles to move one to (v_2, w_2). Thus we have reached D.

Now assume the result holds for $D((v_2, w_1)) = D((v_2, w_2)) = k$ and $6k$ pebbles. Now we look at the $k + 1$ case with $6k + 6$ pebbles. First note that a configuration on (v_1, w_1) defined by $C((v_1, w_1)) = 6k + 5$ is unsolvable. We could use $6k$ pebbles to move k pebbles to (v_2, w_1) and (v_2, w_2). The remaining 5 will not be sufficient to move a pebble to each of those vertices. Now we look at $6k+6$ pebbles on the graph. By the inductive hypothesis, we may use $6k$ pebbles to move k pebbles to (v_2, w_1) and (v_2, w_2). By the base case, the remaining 6 pebbles may be used to move one more pebble to each of (v_2, w_1) and (v_2, w_2), and so we reach D.

Case 3b: Now we induct on $D((v_2, w_2)) = k$. We begin by fixing $D((v_2, w_1)) = m$. In the base case where $k = m$, then Case 3a yields $\pi_L(P_2 \square P_2, D_1 \cdot D_2) = 2m + 4k$.

Now consider the case $D((v_2, w_2)) = k + 1$. Note that a simple configuration on (v_1, w_1) of $2m+4k+3$ pebbles is unsolvable, as we may use $2m+4k$ pebbles to move m pebbles to (v_2, w_1) and k pebbles to (v_2, w_2). However, the remaining three pebbles are insufficient to move another pebble to (v_2, w_2). Now consider some configuration of $2m + 4(k + 1) = 2m + 4k + 4$ pebbles on the graph. By the inductive hypothesis, we may move m pebbles to (v_2, w_1) and k pebbles to (v_2, w_2). Furthermore, by the base case for Case 1, we may use the remaining four pebbles to move one additional pebble to (v_2, w_2).

In any situation, then, we see that the result holds as desired. $\square$

Chapter 5: Conjectures and Future Directions

Our goal is to establish some conjectures on the various types of pebble distributions on path graphs. Utilizing the Cover Pebbling Theorem, we begin with an initial conjecture on paths with cover distributions.

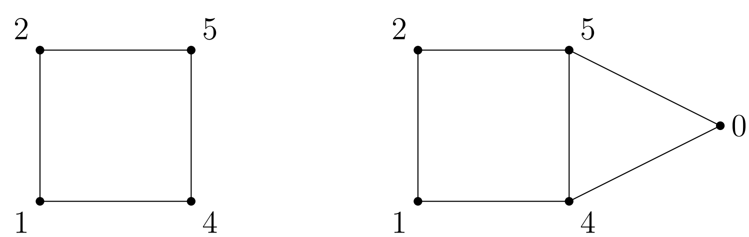
Conjecture 5.1. For P_n and a distribution D of pebbles, if D is a cover, then

$$\pi_L(P_n) = \max \left\{ \begin{array}{l} D_1(v_1) + 2D_1(v_2) + \cdots + 2^{n-1}D_1(v_n), \\ D_1(v_n) + 2D_1(v_{n-1}) + \cdots + 2^{n-1}D_1(v_1) \end{array} \right\}.$$

Our motivation for this is to extend Sjöstrand's result to some non-cover distributions with the next conjecture (an extension to Conjecture 5.1), so we remain in the simple case of a path, and then we will examine other distributions to which his result will not extend.

5.1 Extending the Cover Pebbling Theorem

First consider a graph G and a cover distribution D. By adding a set of vertices $\{v_i\}$ with $D(v_i) = 0$ and edges to the graph, we can create a graph with a largest unsolvable configuration that is not a simple configuration. Consider the following example of adding a vertex to C_4 (where the numbers represent the distribution on each vertex):



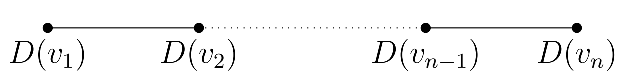
Note that on the original graph, our greatest unsolvable configuration is obtained by placing 32 pebbles on the bottom-left vertex. Since the distribution is a cover, we need only consider simple configurations. Then for the bottom-left vertex, we require $1 \cdot 1 + 2 \cdot (4 + 2) + 4 \cdot 5 = 33$. In clockwise order, simple configurations on the other vertices require 30, 21, and 24 pebbles, respectively. However, by adding the fifth vertex, we can place on pebble on it, creating a non-simple configuration with 33 pebbles that is unsolvable. Note that this gives us a greatest unsolvable configuration that is not simple, as a simple configuration on the new vertex only needs to have 30 pebbles on it to be solvable.

So we see that it does not generalize immediately, but we seek to place conditions on distributions so that the result does generalize to some degree.

Now consider some distribution D on a path of length n. We will separate the various distributions into three general types – a distribution will have zero pebbles on neither, one, or both end vertices. We make conjectures about the labeled pebbling numbers on paths with specific types of distributions. The accompanying discussions aren't intended as proofs, but rather as intuitive motivations for the reasoning behind the conjectures. During the discussions, we will take a constructive approach, building the largest unsolvable configuration possible on the given graphs and distributions. From there, we will argue that placing one more pebble anywhere on the graph will render the configuration solvable.

Conjecture 5.2. For a path P_n with $V(P_n) = \{v_1, \ldots, v_n\}$ and a distribution D of pebbles such that $D(v_1), D(v_n) \neq 0$, then

$$\pi_L(P_n, D) = \max \left\{ \begin{array}{l} D(v_1) + 2D(v_2) + \cdots + 2^{n-1}D(v_n), \\ D(v_n) + 2D(v_{n-1}) + \cdots + 2^{n-1}D(v_1) \end{array} \right\}.$$



Our conjecture is that this type of distribution will behave like a cover distribution, even if it isn't one, at least for the purposes of calculating pebbling numbers. We will begin construction by considering simple configurations on the end vertices v_1 and v_n (taking a cue from the original pebbling number calculations for paths), as they have been the vertices for simple configurations requiring the most pebbles to reach D. For such a configuration on v_1, we would require $D(v_1)+2D(v_2)+\cdots+2^{n-1}D(v_n)$ pebbles to reach D, and one fewer pebble renders the configuration unsolvable.

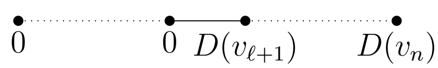
Changing the configuration by moving a pebble to another vertex v_i from the initial simple configuration will reduce the number of pebbles necessary to reach D. Since we must move at least one pebble to v_n, then we have effectively moved a pebble closer "for free," retaining the $2^{\text{dist}(v_1,v_i)}-1$ pebbles that would have been discarded moving it there. Additionally, as the the configuration is weighted towards v_n (the more pebbles we move towards v_n in the initial configuration), then we should require fewer pebbles, since, assuming that a solvable simple configuration on v_1 is of greater size than one on v_n, $D(v_1) + 2D(v_2) + \cdots + 2^{n-1}D(v_n) > D(v_n) + 2D(v_{n-1}) + \cdots + 2^{n-1}D(v_1)$ (symmetric distributions notwithstanding).

We view this particular conjecture as a partial extension of the Cover Pebbling Theorem. However, while the above discussion should provide some level of intuition for the result, there are subtleties that arise that cause this extension to be nontrivial. The idea for a how a proof might proceed is to structure an argument very similar to that in the original theorem, making some allowances for fat vertices with one pebble and perfect vertices with none. The crux of the conclusion of this proof relies on resolving the effect of fat vertices with only one pebble.

5.2 Conjectures on Other Distributions

Conjecture 5.3. For a path P_n with $V(P_n) = \{v_1, \ldots, v_n\}$ and a distribution D of pebbles such that $D(v_i) = 0$ for $i \in \{1, \ldots, \ell\}$ and $D(v_{\ell+1}), D(v_n) \neq 0$, then

$$\pi_L(P_n, D) = \max \left\{ \begin{array}{l} 2^\ell D(v_{\ell+1}) + \cdots + 2^{n-1} D(v_n), \\ D(v_n) + \cdots + 2^{n-(\ell+1)} D(v_{\ell+1}) + 2^\ell - 1 \end{array} \right\}.$$



With this type of distribution, our process begins similarly to the previous type. We begin by examining the required number of pebbles to reach D via simple configurations on v_1 or v_n (assuming for simplicity that v_n is the end vertex with a nonzero distribution). For a simple configuration on v_1, we would require $2^\ell D(v_{\ell+1}) + \cdots + 2^{n-1} D(v_n)$ pebbles to reach D. As before, one fewer is unsolvable, and moving a pebble in the initial configuration to any other vertex yields unnecessary pebbles and a smaller necessary starting configuration as per the previous type of distribution.

Alternatively, we consider a simple configuration on v_n. For such a configuration, we require $D(v_n) + \cdots + 2^{n-(\ell+1)} D(v_{\ell+1})$ pebbles to reach D. By removing a pebble, we create an unsolvable configuration. At this point our construction changes; this configuration is unsolvable, but it is not maximal. As such, we now we seek to add more pebbles to the configuration to make it maximal while leaving it unsolvable. Consider vertex v_1. To get a pebble from v_1 to $v_{\ell+1}$, we would require 2^ℓ pebbles. Then if we put $2^\ell - 1$ pebbles on v_1 in our previous unsolvable configuration, it remains unsolvable, and adding a pebble to any vertex in the initial configuration renders it solvable.

We will consider a simple specific example for illustration.

Lemma 5.1. For a path P_{2n} with $V(P_{2n}) = \{v_1, \ldots, v_{2n}\}$ and

$$D(v_i) = \begin{cases} 0 & \text{if } i \in \{1, \ldots, n\} \\ 1 & \text{if } i \in \{n+1, \ldots, 2n\} \end{cases}$$

then $\pi_L(P_{2n}, D) = 2^n + 2^{n+1} + \cdots + 2^{2n-1}$.

Proof. Assume the given hypotheses. Note that by placing $2^n + 2^{n+1} + \cdots + 2^{2n-1} - 1$ pebbles on v_1, then for any n, we cannot reach D. Now assume that there are $2^n + 2^{n+1} + \cdots + 2^{2n-1}$ pebbles on P_{2n}. We will first assume that no pebbles begin on v_i for $i \in \{n+1, \ldots, 2n\}$.

We begin by examining v_1. If there are zero pebbles or one pebble on v_1, then delete it. If there are more pebbles, then perform $[v_1, v_2]$ for each two pebbles on v_1. Afterward, delete v_1. Note that the number of pebbles remaining on $P_{2n} - \{v_1\}$ is at least $2^{n-1} + 2^n + \cdots + 2^{2n-2}$ (since we "spend" at most half of the original configuration during these moves). Repeat the process for v_2, performing the move $[v_2, v_3]$ for each two pebbles on v_2, and then delete v_2. Similarly, we will be left with at least $2^{n-2} + 2^{n-1} + \cdots + 2^{2n-3}$ pebbles on $P_{2n} - \{v_1, v_2\}$.

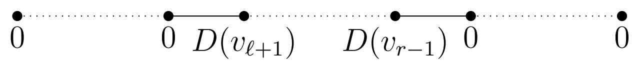
Repeat this process until we have deleted v_n. Now there must be at least $2^0 + 2^1 + \cdots + 2^{n-1}$ pebbles on the graph $P_{2n} - \{v_1, v_2, \ldots, v_n\}$. Recall that to move a pebble from u to v, we require $2^{\text{dist}(u,v)}$ pebbles. That is, the minimum number of remaining pebbles on the graph is precisely enough to move one pebble to each v_i for $i \in \{n+1, \ldots, 2n\}$.

Now we consider the case if some pebbles to begin on v_i for $i \in \{n+1, \ldots, 2n\}$. We repeat the above process until we are only left with the vertices $v_{n+1}, \ldots, v_{2n}$. The remaining distribution is a cover distribution, so we may invoke the Cover Pebbling Theorem. By comparing the simple configurations, we can see that $1 + 2 + \cdots + 2^{n-1}$ is the largest number of pebbles necessary to reach the distribution D, corresponding to a simple configuration on v_{n+1} or v_{2n}.

Thus, in either case, we have $\pi_L(P_{2n}, D) = 2^n + 2^{n+1} + \cdots + 2^{2n-1}$. $\square$

Conjecture 5.4. Given a path P_n with $V(P_n) = \{v_1, \ldots, v_n\}$ and a distribution of pebbles D such that $D(v_i) = 0$ for $i \in \{1, \ldots, \ell, r, \ldots, n\}$, then

$$\pi_L(P_n, D) = \max \left\{ \begin{array}{l} 2^{\ell} D(v_{\ell+1}) + \cdots + 2^{r-2} D(v_{r-1}) + 2^{n-(r-1)} - 1, \\ 2^{n-(r-1)} D(v_{r-1}) + \cdots + 2^{n-(\ell+1)} D(v_{\ell+1}) + 2^{\ell} - 1 \end{array} \right\}.$$



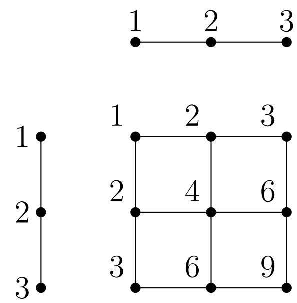
With this type of distribution, our process will be similar to the previous type. To maximize the size of an unsolvable configuration, most of the pebbles will be originally placed on either v_1 or v_n. For notational simplicity, we'll take $2^{\ell} D(v_{\ell+1}) + \cdots + 2^{r-2} D(v_{r-1}) + 2^{n-r} - 1$ to be our maximum, so for a solvable simple configuration on v_1, we would require $2^{\ell} D(v_{\ell+1}) + \cdots + 2^{r-2} D(v_{r-1})$ pebbles. Removing one from this configuration renders it unsolvable, and moving a pebble to a different starting position – any vertex v_i for $i \in \{2, \ldots, r-1\}$ – will leave us with extra pebbles leftover after reaching D.

Starting with the unsolvable configuration, then, we seek to add more pebbles to the starting configuration and it still be unsolvable. If we have $C(v_n) = 2^r - 1$, then the configuration remains unsolvable. As per the previous case, adding a pebble to any vertex will cause this to become a solvable configuration.

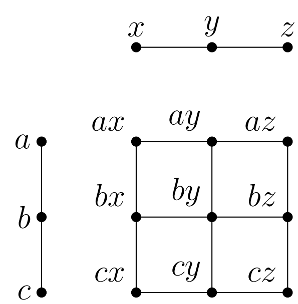
5.3 Conjectures on Path Products

Conjecture 5.5. For paths P_n and P_m with cover distributions of pebbles D_1 and D_2, $\pi_L(P_n \square P_m, D_1 \cdot D_2) = \pi_L(P_n, D_1)\pi_L(P_m, D_2)$.

Consider the following example for motivation:



A quick calculation with aid from the Cover Pebbling Theorem shows that the pebbling number of each path graph is 17. Our conjecture, then, is that the pebbling number of their Cartesian product is 289. With the Cover Pebbling Theorem, we need only consider simple configurations, and by calculating the required number of pebbles from each vertex in the product, we can verify this result. Now we look at a more general case of $P_3 \Box P_3$.



First note that the largest necessary configuration of pebbles for a cover distribution on P_3 occurs on an end vertex – we will assume for simplicity that this configuration begins on a and that $a + 2b + 4c \geq c + 2b + 4a \implies 3c \geq 3a$. If the largest necessary configuration occurred on the middle vertex, then $a + 2b + 4c < 2a + b + 2c \implies b + 2c < a \implies 3b + 6c < 3a$, contradicting our initial assumption. Similarly, we will also consider the maximum necessary configuration on the other path as the one beginning on x.

Consider some initial simple configuration on $P_3 \Box P_3$ above. We claim that the largest necessary configuration begins on ax. We first note that $ax + 2(ay + bx) +$

$4(az+by+cx)+8(bz+cy)+16cz-1$ pebbles on ax is an unsolvable configuration. Now consider a simple configuration of $ax+2(ay+bx)+4(az+by+cx)+8(bz+cy)+16cz$ pebbles on ax, noting that this is precisely enough to reach the target distribution.

Consider some other initial configuration, say one on by, and assume the required number of pebbles is greater than the previous one on ax. Then $|C(by)| = by+2(ay+bz+cy+bx)+4(ax+az+cz+cx) = 2a(2x+y+2z)+b(2x+y+2z)+2c(2x+y+2z) = (2a+b+2c)(2x+y+2z) > (a+2b+4c)(x+2y+4z)$, contradicting our initial assumption. Choosing starting configurations on other vertices results in similar contradictions.

Our more general conjecture is that this pattern holds on paths and path products in general. That is, for two paths with cover distributions, the largest necessary configuration to reach $D_1 \cdot D_2$ on $P_n \square P_m$ begins on the vertex corresponding to those on each individual path where the largest necessary configuration begins under a similar line of reasoning as the $P_3 \square P_3$ case above (though clearly displaying the contradictions with the inequalities is nontrivial in the general case). Additionally, following from the conjectured extension of the Cover Pebbling Theorem on paths with certain distributions, we further conjecture that products of such paths will follow a similar pattern as well.

5.4 Further Directions

In addition to existing graph pebbling problems and conjectures, this new approach in contrasting unlabeled and labeled graph pebbling opens up even more opportunities for work.

Obviously, the earlier portion of this chapter contains a number of conjectures dealing with paths and various distributions to be further explored and proved. The first approach would likely be to consider the conjecture on P_n with a cover distribution. Of particular interest beyond that is the extension of the Cover Pebbling

Theorem to particular distributions on paths. Beyond that, questions arise about extending it farther, exploring what restrictions must be made on graphs and distributions so that it may be applied. It also may be worth contrasting the $P_2 \square P_2$ case with the full C_4 case, as there are distributions that need to be considered for C_4 that cannot occur on $P_2 \square P_2$.

Most of the work in this paper has been on paths and products thereof, but similar results could be explored for other "nice" classes of graphs; in particular, cycle graphs and complete graphs seem the next logical step up in complexity. There is a pebbling number for trees in general, but working with a target distribution could prove exponentially difficult with the compounded variability of tree-distribution combinations.

While we have established a few foundational results in contrasting labeled pebbling and unlabeled pebbling (bounds, effects of symmetry), most of our work has focused on the labeled approach. As of yet, the effects of allowing for isomorphisms of distributions been only minimally explored, and we have few specific results. This is an obvious open area with as many questions, and as mentioned before, is definitely a more challenging question. Even returning to moving a single pebble to a single vertex could make for some interesting questions (at least for non-cycle and non-complete graphs). It is worth noting that although we've placed bounds on the unlabeled pebbling number, finding a formula for it is another issue; furthermore, the process for finding it may require a different approach than that of the labeled pebbling number (consider an unlabeled P_n and a cover distribution, for example).

As it was, graph pebbling was rife with interesting conjectures and difficult open questions. Extending from a root vertex to a distribution only increased the volume of material to explore. Now, by establishing a distinction between unlabeled and labeled pebbling, we have further bifurcated and vastly increased the possible avenues of results.

Bibliography

[1] David S. Herscovici, Benjamin D. Hester, and Glenn H. Hurlburt. Generalizations of Graham's pebbling conjecture. *Discrete Mathematics*, 2009.

[2] Glenn H. Hurlburt. A survey of graph pebbling. *Congressus Numerantium 139*, 1999.

[3] Jonas Sjöstrand. The cover pebbling theorem. *The Electronic Journal of Combinatorics*, 2005.

[4] Douglas B. West. *Introduction to Graph Theory.* Prentice Hall, Upper Saddle River, 2001.

[5] Min Xu, Jun-Ming Xu, and Xin-Min Hou. Fault diameter of Cartesian product graphs. *Information Processing Letters*, 2005.

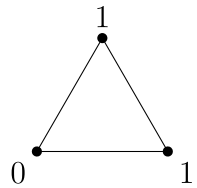
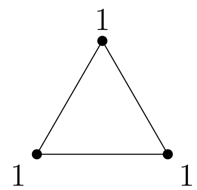
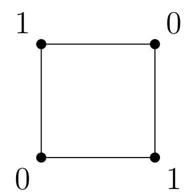
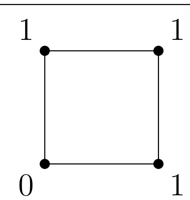
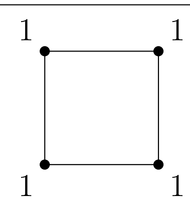
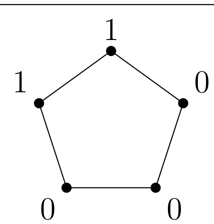
Appendix A: Labeled and Unlabeled Pebbling Numbers for Selected Graphs and Distributions

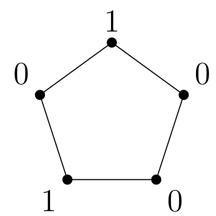
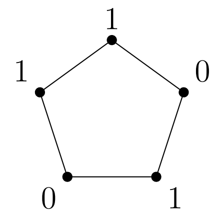
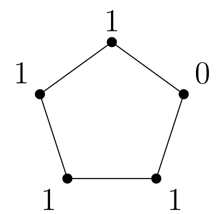
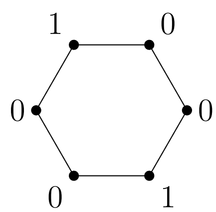
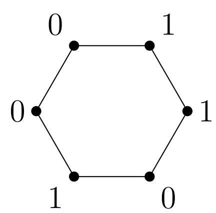
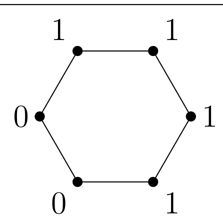
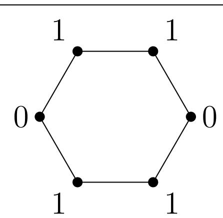
Path Graphs

Graph and distribution	Unlabeled pebbling number	Labeled pebbling number
0 — 1 — 1	$\pi(G, D) = 3$	$\pi_L(G, D) = 6$
1 — 0 — 1	$\pi(G, D) = 5$	$\pi_L(G, D) = 5$
1 — 1 — 1	$\pi(G, D) = 7$	$\pi_L(G, D) = 7$
1 — 1 — 0 — 0	$\pi(G, D) = 5$	$\pi_L(G, D) = 12$
1 — 0 — 1 — 0	$\pi(G, D) = 5$	$\pi_L(G, D) = 10$
1 — 0 — 0 — 1	$\pi(G, D) = 9$	$\pi_L(G, D) = 9$
0 — 1 — 1 — 0	$\pi(G, D) = 7$	$\pi_L(G, D) = 7$
1 — 1 — 1 — 0	$\pi(G, D) = 5$	$\pi_L(G, D) = 14$
1 — 1 — 0 — 1	$\pi(G, D) = 11$	$\pi_L(G, D) = 13$
1 — 1 — 1 — 1	$\pi(G, D) = 15$	$\pi_L(G, D) = 15$
2 — 1 — 0	$\pi(G, D) = 5$	$\pi_L(G, D) = 10$
1 — 2 — 0	$\pi(G, D) = 5$	$\pi_L(G, D) = 8$
2 — 2 — 0	$\pi(G, D) = 7$	$\pi_L(G, D) = 12$
2 — 0 — 1	$\pi(G, D) = 6$	$\pi_L(G, D) = 9$
2 — 0 — 2	$\pi(G, D) = 10$	$\pi_L(G, D) = 10$
2 — 1 — 1	$\pi(G, D) = 8$	$\pi_L(G, D) = 11$

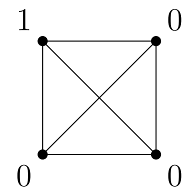
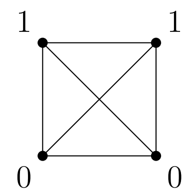
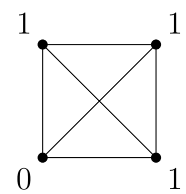
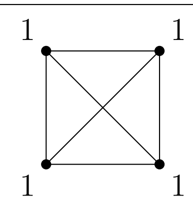
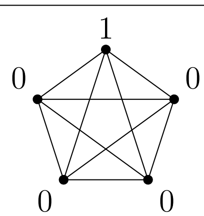
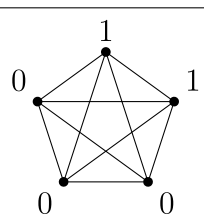
$1 - 2 - 1$	$\pi(G,D)=9$	$\pi_L(G,D)=9$
$2 - 1 - 2$	$\pi(G,D)=12$	$\pi_L(G,D)=12$
$2 - 2 - 1$	$\pi(G,D)=10$	$\pi_L(G,D)=13$
$2 - 2 - 2$	$\pi(G,D)=14$	$\pi_L(G,D)=14$
$2 - 0 - 1 - 0$	$\pi(G,D)=7$	$\pi_L(G,D)=18$
$1 - 0 - 2 - 0$	$\pi(G,D)=9$	$\pi_L(G,D)=12$
$2 - 0 - 2 - 0$	$\pi(G,D)=11$	$\pi_L(G,D)=20$
$2 - 0 - 0 - 1$	$\pi(G,D)=10$	$\pi_L(G,D)=17$
$2 - 0 - 0 - 2$	$\pi(G,D)=18$	$\pi_L(G,D)=18$
$0 - 2 - 1 - 0$	$\pi(G,D)=9$	$\pi_L(G,D)=11$
$2 - 1 - 1 - 0$	$\pi(G,D)=9$	$\pi_L(G,D)=22$
$1 - 2 - 1 - 0$	$\pi(G,D)=10$	$\pi_L(G,D)=18$
$1 - 1 - 2 - 0$	$\pi(G,D)=12$	$\pi_L(G,D)=16$
$2 - 2 - 1 - 0$	$\pi(G,D)=11$	$\pi_L(G,D)=26$
$2 - 1 - 2 - 0$	$\pi(G,D)=13$	$\pi_L(G,D)=24$
$1 - 2 - 2 - 0$	$\pi(G,D)=14$	$\pi_L(G,D)=20$
$2 - 2 - 2 - 0$	$\pi(G,D)=15$	$\pi_L(G,D)=28$
$2 - 1 - 1 - 1$	$\pi(G,D)=16$	$\pi_L(G,D)=23$
$1 - 2 - 1 - 1$	$\pi(G,D)=17$	$\pi_L(G,D)=19$
$2 - 2 - 1 - 1$	$\pi(G,D)=18$	$\pi_L(G,D)=27$
$2 - 1 - 2 - 1$	$\pi(G,D)=20$	$\pi_L(G,D)=25$
$2 - 2 - 2 - 1$	$\pi(G,D)=22$	$\pi_L(G,D)=29$
$2 - 2 - 1 - 2$	$\pi(G,D)=26$	$\pi_L(G,D)=28$

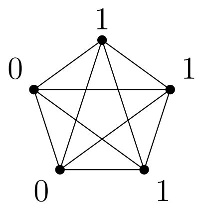
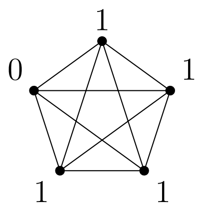
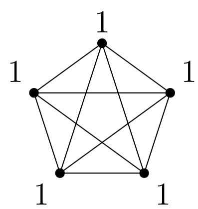
Cycle Graphs

Graph and distribution	Unlabeled pebbling number	Labeled pebbling number
	$\pi(G, D) = 3$	$\pi_L(G, D) = 4$
	$\pi(G, D) = 5$	$\pi_L(G, D) = 5$
	$\pi(G, D) = 4$	$\pi_L(G, D) = 5$
	$\pi(G, D) = 5$	$\pi_L(G, D) = 8$
	$\pi(G, D) = 9$	$\pi_L(G, D) = 9$
	$\pi(G, D) = 3$	$\pi_L(G, D) = 8$

	$\pi(G, D) = 4$	$\pi_L(G, D) = 7$
	$\pi(G, D) = 7$	$\pi_L(G, D) = 9$
	$\pi(G, D) = 9$	$\pi_L(G, D) = 12$
	$\pi(G, D) = 6$	$\pi_L(G, D) = 9$
	$\pi(G, D) = 7$	$\pi_L(G, D) = 14$
	$\pi(G, D) = 8$	$\pi_L(G, D) = 18$
	$\pi(G, D) = 12$	$\pi_L(G, D) = 15$

Complete Graphs

Graph and distribution	Unlabeled pebbling number	Labeled pebbling number
	$\pi(G, D) = 1$	$\pi_L(G, D) = 4$
	$\pi(G, D) = 3$	$\pi_L(G, D) = 5$
	$\pi(G, D) = 5$	$\pi_L(G, D) = 6$
	$\pi(G, D) = 7$	$\pi_L(G, D) = 7$
	$\pi(G, D) = 1$	$\pi_L(G, D) = 5$
	$\pi(G, D) = 3$	$\pi_L(G, D) = 6$

	$\pi(G, D) = 5$	$\pi_L(G, D) = 7$
	$\pi(G, D) = 7$	$\pi_L(G, D) = 8$
	$\pi(G, D) = 9$	$\pi_L(G, D) = 9$

Curriculum Vitae

Education

- M.A., Mathematics, Wake Forest University, May 2014
- B.S., Mathematics (major), Summa Cum Laude, University of Montevallo, December 2011

Academic Awards and Organizations

- Membership in the math honors societies Kappa Mu Epsilon (UM) and Pi Mu Epsilon (WFU)
- AMS member since Fall 2013
- Montevallo Organization of Gaming – President (Fall 2010 – Spring 2011), Vice President (Fall 2011)

Research History

- Master's thesis at Wake Forest University, March 2013 to May 2014
  - Topic: Graph theory, graph pebbling
  - Advisor: Dr. Sarah Mason
  - Explored new variations of graph pebbling problems.
  - Presented at University of Montevallo, Math Club Colloquium (March 2014)
  - Thesis defended April 2014
- Senior seminar project at University of Montevallo, September 2010 – April 2011

* Topic: Complex analysis, Möbius transformations
* Advisor: Dr. Michael Sterner
* Investigated properties and applications of the group of Möbius functions.
* Presented at University of Montevallo to faculty and classmates as major requirement.

Work History

- Teaching Assistant – Wake Forest University, Department of Mathematics (January 2013 – May 2014)
  * Led study sessions for Calculus I and Discrete Mathematics
  * Tutored students individually in Precalculus and Trigonometry, Calculus I, and Discrete Mathematics
  * Graded homework and tests in Calculus I and Discrete Mathematics
- Tutor – University of Montevallo, Learning Enrichment Center (September 2011 – December 2011)
  * Tutored students individually in College Algebra, Precalculus and Trigonometry, Finite Mathematics, and Calculus I

Computer Skills

- LaTeX
- MAPLE (basic)
- Mathematica (basic)