ON THE STABILITY OF SOLUTIONS TO A PHASE TRANSITION MODEL

BY

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Abstract

Heather Hardeman

We will discuss the stability of certain solutions to a phase transition model. This model is typically expressed as a partial differential equation:

\[ u_t = \epsilon^2 u_{xx} - F'(u), \]
\[ u_x(0) = u_x(1) = 0, \]

where \( F(u) \) is a so-called “double-well” potential. We consider both classical and nonclassical examples for \( F \). Furthermore, the main method we use in this discussion of stability is that of upper and lower solutions.
Chapter 1: Introduction

In this paper, we study the functional

\[ J_\epsilon(u) := \frac{\epsilon^2}{2} \int_0^1 |u_x|^2 \, dx + \int_0^1 F(u) \, dx, \quad u \in W^{1,2}(0,1) \]

under suitable assumptions on \( F : \mathbb{R} \to \mathbb{R} \). This functional represents the total free energy of a phase transition model. In nature, energy seeks a minimum. As such, we can see that the first integral wants the slope of \( u \) to be flat whereas the second integral is pushing towards the minimum values of \( F \). With this in mind, we want to consider a function \( F \) which has the following shape:

![Graph of a function with two minima](image)

A function with this shape captures the idea of an object with two different states of being. A good example of phase transition is water phasing from a liquid to a solid.

For our purposes, two different cases of \( F \) which have this shape are the classical case \( F(u) = (1 - u^2)^2 \) and the non-classical case \( F(u) = (1 - u^2)^\alpha \) where \( 1 < \alpha < 2 \). While the classical and non-classical cases appear indistinguishable from a modeling perspective, they have very different consequences. In fact, the non-classical case has features which the classical case lacks. These features of the non-classical case may help explain observed phenomena such as “slow dynamics.” As such, our focus will be on the non-classical case of \( F \).
In [1], Drabek, Manasevich, and Takac discovered all of the critical points of the functional $J_\epsilon$. These critical points of the functional correspond to solutions of the Neumann boundary value problem

$$\begin{align*}
-\epsilon^2 u_{xx} + F'(u) &= 0, \\
u_x(0) = u_x(1) &= 0.
\end{align*}$$

(1.1)

These solutions are also all the stationary solutions of the bistable equation

$$\begin{align*}
\frac{\partial u}{\partial t} - \epsilon^2 u_{xx} + F'(u) &= 0, \\
u_x(0,t) = u_x(1,t) &= 0,
\end{align*}$$

(1.2)

with the initial condition $u(x,0) = u_0(x)$.

Note that $\pm 1$ are global minima of $J_\epsilon$. It is not hard to see that 0 is a saddle point. Since the critical point 0 is a saddle point, we also know that it is an unstable solution of (1.2). In [1], the authors also describe manifolds of solutions corresponding to single node solutions, two-node solutions, etc. The set of single node solutions for example can be visualized as a trough. In [2], Drabek and Robinson verified that the interior of each of these manifolds consists of local minima.

The image above depicts a “graph” of $J_\epsilon$ with critical points lying along the base of the trough. These critical points are the one-node solutions of (1.1). They are solutions which cross the $x$-axis only once at some point $x_0$. The solution is equal to 1 on $[1 - \delta_1, 1]$ and it is equal to $-1$ on $[0, \delta_2]$ for $\delta_1, \delta_2 > 0$. The figure below gives an example of a solution located along the bottom of the trough.
By moving the point $x_0$ where the solution crosses the $x$-axis, we obtain these other solutions which lie along the base of the trough. For each $x_0$, there is a unique solution by Theorem 7.1 in [1]. The image below depicts two different cases. One is a solution located at the left side of the trough and the other is a solution from the right side of the trough. The rest of the solutions which are in the base reside in the shaded region between the two solutions.

Given that the local geometry of $J_\epsilon(u)$ takes the shape of a trough for the one-node solutions, the question of stability arises. As such, the purpose of this thesis is to study the stability of these solutions. In particular, we want to know what set of initial conditions for (1.2) will be drawn to the bottom of the trough. Another way to say this is that we are studying the stable manifold for the trough.

Our study will be developed over several cases. We will begin with the case when the solution is symmetric via $u(1 - x) = -u(x)$. This occurs when $x_0 = \frac{1}{2}$. Our consideration of the symmetric case results in the following theorem.
Theorem 1.1 (Hardeman, Robinson (2013)). Let $u$ be the one-node solution of the time independent problem (1.1) satisfying $u(\frac{1}{2}) = 0$ and $u(0) = -1$. Assume $-1 < u_0(x) \leq \beta u(x)$ for $x \in [0,1/2]$ and some $0 < \beta < 1 - \frac{1}{n}$. Assume $u_0(x) = -u_0(1-x)$. Then, the solution, $h(x,t)$, of (1.2) satisfies $\lim_{t \to \infty} h(x,t) = u(x)$ (uniformly).

Theorem 1.1 shows that the symmetric initial condition is in the stable manifold, i.e. the solution corresponding to the symmetric initial condition converges to a point in the trough.

We will utilize the method of upper and lower solutions to prove Theorem 1.1. A common assumption for this method is that the forcing function satisfies a Lipschitz condition, but that is not the case for our problem. Given the nature of the non-classical $F$, we will need to deal with the unboundedness of the slope of the function $F'$ near $-1$ and $1$ which we will do by truncating problem (1.2).

We will also consider the case when the solution is asymmetric. The solution is asymmetric when it crosses the $x$-axis at a point $x_0 \neq \frac{1}{2}$. We can see an example of an asymmetric solution in the image below.

In both the symmetric case and the asymmetric case, we are interested in the symmetry of the initial condition. While we employ upper and lower solutions for the symmetric case, we utilize a different approach for the asymmetric case. In fact, we can prove that there are asymmetric initial conditions which are drawn to the base of the trough. The proof of this result will rely on continuous dependence. Our consideration of the asymmetric case results in the following theorem.
Theorem 1.2 (Hardeman, Robinson (2014)). There is at least one asymmetric initial condition $u_0(x)$ such that the solution $h(x,t)$ of (1.2) satisfies $\lim_{t \to \infty} h(x,t) = v(x)$, where $v(x)$ is an asymmetric one-node solution of (1.1).

This paper is organized as follows. We begin by proving a version of the Maximum Principle from [5] in chapter 2. In chapter 3, we consider preliminary materials starting with an existence theorem and the method of upper and lower solutions; then, ending chapter 3 with a discussion of existence, uniqueness, and continuous dependence. In chapter 4, we prove the main theorems of this paper starting with the symmetric case and then proving the theorem for the asymmetric case. We conclude with a brief discussion of open problems.
Chapter 2: Maximum Principle

The Maximum Principle is an important theorem for our discussion of upper and lower solutions. The following version of the Maximum Principle comes from [5].

For our purposes, we will consider only the special case $A$ as the operator $Au := cu_{xx} + du - u_t$ where $d \leq 0$ and $c > 0$. We will also consider $D = (0,1) \times (0,\infty)$.

**Theorem 2.1** (Maximum Principle). Let $u$ be a smooth function. Suppose that $Au \geq 0$ in $D$. Let $\sup_D u = M \geq 0$. If $u(x,t) = M$ for some $(x,t) \in D$, then $u(x,t) = M$ for all $(x,t) \in D$ with $t \leq \tilde{t}$.

**Theorem 2.2.** Let $u$ be a smooth function. Suppose that $Au \geq 0$ in $D$. Let $\sup_D u = M \geq 0$. Suppose that $u(0,t_0) = M$ ($u(1,t_0) = M$) and that there is a ball $B_r(\bar{x},\bar{t}) \subset D$ such that $u < M$ in $B_r(\bar{x},\bar{t})$ and $(0,t_0) \in \partial B_r(\bar{x},\bar{t})$ ($(1,t_0) \in \partial B_r(\bar{x},\bar{t})$). Then $u_x(0,t_0) < 0$ ($u_x(1,t_0) > 0$.)

Before we investigate the Maximum Principle, we need to consider the following two lemmas.

**Lemma 1.** If $v$ is a smooth function, and $Av > 0$ in the interior of $D$, then $v$ cannot have a non-negative interior maximum.

**Proof.** If $v$ has a non-negative maximum $M$ at $(x_0,t_0) \in D$, then $v_t = 0$, $v_{xx} \leq 0$, and $dv \leq 0$ since $d \leq 0$ by hypothesis. So

$$Av := v_{xx} + dv - v_t = v_{xx} + dv \leq 0$$

since $v_{xx} \leq 0$ and $dv \leq 0$, which is a contradiction. \qed

Note that if we do not assume that $d \geq 0$, then we get the following result.

**Lemma 2.** If $v$ is a smooth function and $Av > 0$ in the interior in $D$, then $v$ cannot have an interior maximum of 0.
Proof. Suppose \( v \) has an interior maximum of 0 at \( (x_0, t_0) \in D \). Then \( v_t = 0 \) and \( v_{xx} \leq 0 \). Also, \( dv = 0 \). Thus, \( Av := v_{xx} + dv - v_t = v_{xx} \leq 0 \), which is a contradiction.

The full proof of the Maximum Principle can be found in [5], but we provide several important cases of the proof below.

Proof. Assume the hypotheses hold.

Case 1: First, consider a ball \( K = B_r(\bar{x}, \bar{t}) \) with \( \overline{K} \subset D \). Assume \( u(x, t) < M \) for \( (x, t) \in K \). Let \( u(x_0, t_0) = M \) for some \( (x_0, t_0) \in \partial K \). Suppose \( (x_0, t_0) \) is the only point in \( \overline{K} \) such that \( u(x, t) = M \). Let \( \rho = \frac{1}{2}|x_0 - \bar{x}| \). Let \( K_1 = B_{\rho}(x_0, t_0) \). Let \( \partial K_1 = \Gamma_1 \cup \Gamma_2 \) where \( \Gamma_1 = \overline{K} \cap \partial K_1 \) and \( \Gamma_2 = K^C \cap \partial K_1 \).
Note that $\Gamma_1$ is closed and bounded, hence, compact. Therefore, $u$ achieves a maximum on $\Gamma_1$, but $u < M$ on $\Gamma_1$ so $u \leq M - \delta$ for some $\delta > 0$ on $\Gamma_1$. Also, observe $u \leq M$ on $\Gamma_2$ because $\sup_{\Gamma_1} u = M$.

Let

$$h(x, t) = e^{-\alpha[(x-x)^2+(t-t)^2]} - e^{-\alpha r^2}$$

where $\alpha > 0$. Notice $h(x, t) = 1 - e^{-\alpha r^2} > 0$ since $e^{-\alpha r^2} < 1$. For $(x, t) \in \partial K$, we have $h(x, t) = 0$ since $(x-x)^2 + (t-t)^2 = r^2$. Also, for $(x, t) \in K^C$, $h(x, t) < 0$.

**Claim:** For appropriate choice of $\alpha > 0$, we have $Ah > 0$ in $K_1$.

Let $\epsilon = \min_{K_1} |x-x|$ and let $T = \max |t-t|$ for all $(x, t) \in K$. Then, we have

$$h_t = -2\alpha(t-t)e^{-\alpha[(x-x)^2+(t-t)^2]},$$

$$h_x = -2\alpha(x-x)e^{-\alpha[(x-x)^2+(t-t)^2]},$$

$$h_{xx} = -2\alpha e^{-\alpha[(x-x)^2+(t-t)^2]} + 4\alpha^2(x-x)^2e^{-\alpha[(x-x)^2+(t-t)^2]}.$$ 

Therefore, we have

$$Ah = c\alpha e^{-\alpha[(x-x)^2+(t-t)^2]}[-2 + 4\alpha(x-x)^2] + d(e^{-\alpha[(x-x)^2+(t-t)^2]} - e^{-\alpha r^2})$$

$$-\alpha e^{-\alpha[(x-x)^2+(t-t)^2]}[-2(t-t)]$$

$$\geq \alpha e^{-\alpha[(x-x)^2+(t-t)^2]} \left[4\alpha \epsilon^2 - 2T - 2c + \frac{d}{\alpha}\right] - d\epsilon^{-\alpha r^2} > 0$$

when $\alpha > \frac{2T + 2c - d}{4\epsilon \epsilon^2}$. Thus, we obtain the claim.

Now, we consider the case when $v = u + \sigma h$. As we have previously shown, $Ah > 0$ in $K_1$ for large enough $\alpha$. Then, by linearity, we have

$$Av = Au + \sigma Ah > 0 \text{ in } K_1$$

since $Au \geq 0$ by hypothesis. Thus, $v$ cannot have a positive interior maximum in $K_1$.

Recall $u \leq M - \delta$ on $\Gamma_1$ and $u \leq M$ on $\Gamma_2$. Let $\sigma > 0$ such that $\sigma h \leq \frac{\delta}{2}$ on $\Gamma_1$. Then,

$$v = u + \sigma h \leq M - \delta + \frac{\delta}{2} = M - \frac{\delta}{2}.$$
Also, since $\sigma h < 0$ on $\Gamma_2$, then $v = u + \sigma h$ implies $v < u \leq M$ on $\Gamma_2$. Hence, $\sup_{\Gamma_1} v < M$. But

$$v(x_0, t_0) = u(x_0, t_0) + \epsilon h(x_0, t_0) = M + \epsilon \cdot 0 = M$$

so $v$ achieves an interior maximum greater than or equal to $M$ in $K_1$, a contradiction.

The result in Case 1 can be used to show that if $u(x, t) < M$ at an interior point then we get...

**Case 2:** Assume $u(x, t) < M$ for $x \in (0, 1)$ and $t \in (t_0, t_1)$. Let $K$ be the half circle centered at the point $(\overline{x}, t_1)$ with radius less than $t_1 - t_0$. Consider the parabola

$$(x - \overline{x})^2 + \alpha(t - T) = 0.$$

We define

$$h(x, t) = e^{-\|x-\overline{x}\|^2 + \alpha(t-T)} - 1,$$

and let $H$ be the open region determined by the parabola. Then, assume $\Gamma_1 = \partial K \cap H$ and $\Gamma_2 = \partial H \cap K$. Let $u < M$ for all $(x, t) \in H$ and suppose $u(\overline{x}, t_1) = M$. 
Notice

\[ Ah = h_{xx} + dh - h_t \]
\[ = [4(x - \bar{x})^2 - 2]e^{-[(x-\bar{x})^2+\alpha(t-T)]} - de^{-[(x-\bar{x})^2+\alpha(t-T)]} - d + \alpha e^{-[(x-\bar{x})^2+\alpha(t-T)]}[4(x-\bar{x})^2 - 2 + d + \alpha] - 2d \]

Thus, \( Ah > 0 \) for large enough \( \alpha > 0 \) since \( d \leq 0 \) by hypothesis. Then, for \( v = u + \epsilon h \), we have

\[ Av = Au + \epsilon Ah > 0 \]

since \( Au \geq 0 \) by hypothesis and \( \epsilon > 0 \) implies \( \epsilon Ah > 0 \) in \( D \). Hence, \( v \) does not have an interior non-negative maximum in \( D \) by Lemma 1. Note that \( v \equiv u \) on \( \Gamma_2 \), so \( v \leq M \) on \( \Gamma_2 \). Also \( u \leq M - \delta \) on \( \Gamma_1 \). Thus, there exists an \( \epsilon > 0 \) such that \( v \leq M \) on \( \Gamma_1 \). Therefore, \( v_t \geq 0 \) at \((\bar{x}, t_1)\). But \( u_{xx} \leq 0 \) and \( au \leq 0 \) at \((\bar{x}, t_1)\) since \( u(\bar{x}, t_1) = M \). Hence, \( Au = u_{xx} + au - u_t < 0 \), a contradiction.

The arguments above are the main parts in the proof of the Maximum Principle. For more details, see [5]. Now consider what might happen with a maximum at the boundary.

**Case 3:** Assume the hypotheses of Theorem 2.2 hold. Define \( h, v, \Gamma_1 \) similarly to Case 1. Let \( \Gamma_2 = K_1 \cap \partial K \).
Similarly to Case 1, \( v \leq M - \frac{\delta}{2} \) on \( \Gamma_1 \) and \( v \leq M \) on \( \Gamma_2 \) and \( v \) has no positive interior maximum in \( K \cap K_1 \). Then, \( v \) achieves a maximum at \((0, \bar{t})\). So,

\[
0 \geq v_x = u_x + \epsilon h_x
\]

\[
= u_x + \epsilon \left( 2\alpha_\infty e^{-a(x^2 + (\bar{t} - \bar{t})^2)} \right)
\]

since \( \epsilon h_x > 0 \), then \( u_x < 0 \).

**Case 4**: When \((x_0, t_0) = (1, \bar{t}_0)\) such that \( u(1, \bar{t}_0) = M \) we can apply a similar argument to the argument in Case 2.
3.1 Method of Upper and Lower Solutions

In the next two sections, we will utilize the Maximum Principle several times to prove results about upper and lower solutions as well as existence. These results will be helpful in proving our main results. We will consider each of these definitions and theorems in terms of the following generalized problems:

\[
\epsilon^2 u_{xx} + f(u) = 0 \quad \text{in } (a,b) \\
\alpha \frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{at } x = a,b
\]  

(3.1)

where \( f \) is Lipschitz continuous and \( \alpha, \beta \geq 0 \) such that \( \alpha + \beta > 0 \). Also, we want to consider

\[
u_t - \epsilon^2 u_{xx} - f(u) = 0 \quad \text{in } (a,b) \text{ and } t > 0 \\
u(x,0) = u_0(x) \\
\alpha \frac{\partial \nu}{\partial \nu} + \beta \nu = 0 \quad \text{at } x = a,b
\]  

(3.2)

with \( f, \alpha, \) and \( \beta \) defined as above.

Notice that solutions of (3.1) are time-independent solutions of (3.2), i.e. steady-state solutions. Let us begin by considering the following definition of an upper solution from [4].

**Definition 1.** A smooth function \( \overline{u} \) is an upper solution of the boundary value problem (3.2) if

\[
\overline{u}_t - \epsilon^2 \overline{u}_{xx} \geq f(\overline{u}) \quad \text{in } (a,b) \\
\alpha \frac{\partial \overline{u}}{\partial \nu} + \beta \overline{u} \geq 0 \quad \text{on the lateral boundaries, and} \\
\overline{u}(x,0) \geq u_0(x).
\]  

(3.3)

Note that by reversing the inequalities, we obtain the definition of a lower solution.
With these definitions available to us, we are now ready to prove several results using upper and lower solutions. We will begin by considering an existence theorem and some lemmas involving upper and lower solutions.

### 3.1.1 Existence

The idea of this section is to prove an existence theorem which will be important to proving results in Chapter 4.

**Theorem 3.1.** If there exists $u$, $\overline{u}$ as defined in the previous section such that $u \leq \overline{u}$, then the boundary value problem (3.1) has a solution $u$ such that $u \leq u \leq \overline{u}$.

Before we prove Theorem 3.1, we need to recall some standard theorems from analysis.

**Theorem 3.2** (Arzela-Ascoli Theorem). If $F \subset C[0, 1]$ is bounded and equicontinuous, then every sequence in $F$ has a convergent subsequence.

**Lemma 3.** Let $(f_n) \subset C[0, 1]$ where each $f_n$ is differentiable and there is some $M > 0$ such that $||f'_n||_\infty \leq M$ for all $n$, then $(f_n)$ is equicontinuous.

We will also need the Dominated Convergence Theorem from [7].

**Theorem 3.3** (Dominated Convergence Theorem). Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f_1, f_2, \ldots : X \rightarrow C$ be a sequence of measurable functions that converge pointwise $\mu$-almost everywhere to a measurable limit $f : X \rightarrow C$. Suppose that there is an unsigned absolutely integrable function $G : X \rightarrow [0, +\infty]$ such that $|f_n|$ are pointwise $\mu$-almost everywhere bounded by $G$ for each $n$. Then we have

$$\lim_{n \rightarrow +\infty} \int_X f_n d\mu = \int_X f d\mu.$$  

For the purposes of this paper, we will prove Theorem 3.1 using Neumann boundary conditions; however, the argument using general boundary conditions is similar.

**Proof.** Time Independent Equation:

Let $c > 0$ such that $f(u) + cu$ is strictly increasing (i.e. $f(x_2) + cx_2 - f(x_1) - cx_1 > 0$
for all $x_2 > x_1)$. Let $u_0 = u$. Note: $u_0$ should not be confused with the initial condition. Let

$$-u''_1 + cu_1 = f(u_0) + cu_0, \quad u'_1(a) = 0 = u'_1(b).$$

For each $n > 1$, let $u_n$ satisfy

$$-u''_n + cu_n = f(u_{n-1}) + cu_{n-1}, \quad u'_n(a) = 0 = u'_n(b).$$

Consider $w_1 = u_1 - u_0$. Then,

$$w''_1 = u''_1 - u''_0 = (cu_1 - f(u_0) - cu_0) - u''_0 \leq (cu_1 - f(u_0) - cu_0) + f(u_0) = cw_1.$$

Hence, $w''_1 \leq cw$. Also

$$w'_1(a) = u'_1(a) - u'_0(a) = -u'_0(a) \leq 0, \text{ and}$$

$$w'_1(b) = u'_1(b) - u'_0(b) = -u'_0(b) \geq 0.$$

By the Maximum Principle, $w_1 \geq 0$. Therefore, since $w_1 = u_1 - u_0$, then $u_1 \geq u_0$.

Consider $w_2 = u_2 - u_1$. Then

$$w''_2 = u''_2 - u''_1 = cu_2 - [(f(u_1) + cu_1) - (f(u_0) + cu_0)] - cu_1$$

$$= cw_2 - [f(u_1) + cu_1] - (f(u_0) + cu_0)]$$

$$\leq cw_2$$

since $u_1 \geq u_0$ and $f(u) + cu$ is increasing. Also,

$$w'_2(a) = u'_2(a) - u'_1(a) = 0, \text{ and}$$

$$w'_2(b) = u'_2(b) - u'_1(b) = 0.$$

It follows that $u_2 \geq u_1$. By induction, $u_n \geq u_{n-1}$.
Note that \( u_0 \leq \overline{u} \). Consider \( w_1 = \overline{u} - u_1 \). Then

\[
w_1'' = \overline{u}'' - u_1'' \leq -f(\overline{u}) - (c_1 - f(u_0) - cu_0) = -f(\overline{u}) - c\overline{u} + c\overline{u} - (c_1 - f(u_0) - cu_0) = [(f(u_0) + cu_0) - (f(\overline{u}) + c\overline{u})] + c(\overline{u} - u_1) \leq cw_1
\]
since the fact that \( f(u) + cu \) is strictly increasing implies \([(f(u_0) + cu_0) - (f(\overline{u}) + c\overline{u})] \leq 0 \) for \( \overline{u} \geq u_0 \). Therefore, \( w_1'' \leq cw_1 \). By a similar process to that above, we get \( w_1 \geq 0 \) which implies \( \overline{u} - u_1 \geq 0 \) which means \( \overline{u} \geq u_1 \). Similarly, \( \overline{u} \geq u_2 \). By induction \( \overline{u} \geq u_n \). Hence, we have a monotone and bounded sequence between \( \underline{u} \) and \( \overline{u} \). By the Monotone Convergence Theorem, \((u_n(x))\) converges in \( \mathbb{R} \) for all \( x \in [a, b] \). Hence, \((u_n(x))\) is pointwise convergent.

We want to show that \((u_n(x))\) and \((u'_n(x))\) are uniformly convergent. Let \( u(x) = \lim_{n \to \infty} u_n(x) \). Notice that \( u''_n = cu_n - f(u_{n-1}) - cu_{n-1} \) where \( u \leq u_n \leq \overline{u} \) for all \( n \). Hence, there is an \( M > 0 \) such that \( |u''_n(x)| \leq M \) for all \( x \) and for all \( n \). But

\[
\begin{align*}
u_n'(x) &= u_n'(a) + \int_a^x u_n''(t)dt \\
&= \int_a^x u_n''(t)dt \quad \text{since } u_n'(a) = 0.
\end{align*}
\]

So

\[
|u_n'(x)| \leq \int_a^x |u_n''(t)|dt \leq M \int_a^x dt \leq M|b-a| \quad \text{since } x \in [a, b].
\]

Hence, by Lemma 3, \((u_n)\) is equicontinuous and bounded and \((u'_n)\) is equicontinuous and bounded. It follows that \( u_n \longrightarrow u \) in \( C^1[a, b] \). Also, \( u''_n \) converges uniformly because \( u''_n = cu_n - f(u_{n-1}) - cu_{n-1} \) converges uniformly. So \( u''_n \longrightarrow u'' \) uniformly.
Note that we get a similar result when \( u_0 = \overline{u} \).

**Time Dependent Equation**

Let \( u(x) \leq \overline{u}(x) \) as before. Also, assume \( u_n(x,0) = u_0(x) \) with \( u \leq u_0(x) \leq \overline{u} \). Let \( c > 0 \) such that \( g(u) = f(u) - cu \) is strictly decreasing. Consider

\[
\begin{align*}
  u_t & = \epsilon^2 u_{xx} - cu - (f(u) - cu) \\
  & = \epsilon^2 u_{xx} - cu - g(u).
\end{align*}
\]

Let \( u_0 = u \). Let \( u_1 \) satisfy

\[
\begin{align*}
  (u_1)_t & = \epsilon^2 (u_1)_{xx} - cu_1 - g(u) \\
  (u_1)_x(a,t) & = 0 = (u_1)_x(b,t), \\
  u_1(x,0) & = u_0(x).
\end{align*}
\]

For each \( n > 1 \), let \( u_n \) satisfy

\[
\begin{align*}
  (u_n)_t & = \epsilon^2 (u_n)_{xx} - cu_n - g(u_{n-1}) \\
  (u_n)_x(a,t) & = 0 = (u_n)_x(b,t), \\
  u_n(x,0) & = u_0(x).
\end{align*}
\]

Consider \( w_1 = u_1 - u \). Then

\[
\begin{align*}
  (w_1)_t & = (u_1)_t \\
  & = \epsilon^2 (u_1)_{xx} - cu_1 - g(u) \\
  & = \epsilon^2 (u_1)_{xx} - cu_1 - f(u) + cu \\
  & \geq \epsilon^2 (u_1)_{xx} - c(u_1 - \overline{u}) - cu \\
  & \geq \epsilon^2 (u_1 - \overline{u})_{xx} - cu_1 - cu \\
  & = \epsilon^2 (w_1)_{xx} - cu_1 \\
  & = \epsilon^2 (w_1)_{xx} - cw_1.
\end{align*}
\]

By the Maximum Principle, \( w_1 \geq 0 \). Thus, \( w_1 = u_1 - u \) implies \( u_1 \geq u \).
Consider \( w_2 = u_2 - u_1 \). Then

\[
(w_2)_t = (u_2)_t - (u_1)_t \\
= \epsilon^2(u_2)_{xx} - c(u_2 - g(u_1)) - \epsilon^2(u_1)_{xx} + cu_1 + g(u) \\
= \epsilon^2(u_2 - u_1)_{xx} - c(u_2 - u_1) - (g(u_1) - g(u)) \\
\geq \epsilon^2(w_2)_{xx} - cw_2
\]

since \( g \) is strictly decreasing and \( u_1 \geq \bar{u} \) then \( g(u_1) \leq g(\bar{u}) \). It follows that \( u_2 \geq u_1 \).

By induction, \( u_n \geq u_{n-1} \).

We know \( \bar{u} \leq \bar{v} \). Consider \( w_1 = \bar{v} - u_1 \). Then

\[
(w_1)_t = -(u_1)_t \\
= -\epsilon^2(u_1)_{xx} + cu_1 + g(u) \\
\geq -\epsilon^2(u_1)_{xx} + cu_1 + g(\bar{u}) \quad \text{since} \quad \bar{u} \leq \bar{v} \quad \text{and} \quad g \text{ is decreasing} \\
= -\epsilon^2(u_1)_{xx} + cu_1 + (f(\bar{u}) - c\bar{u}) \\
\geq -\epsilon^2(u_1)_{xx} + cu_1 + (\epsilon^2\bar{u}_{xx} - c\bar{u}) \quad \text{since} \quad \epsilon^2\bar{u}_{xx} \leq f(\bar{u}) \\
= -\epsilon^2(u_1)_{xx} + cu_1 + \epsilon^2\bar{u}_{xx} - c\bar{u} \\
= \epsilon^2(\bar{u} - u_1)_{xx} - c(\bar{u} - u_1) \\
= \epsilon^2(w_1)_{xx} - cw_1.
\]

Therefore, \((w_1)_t \geq \epsilon^2(w_1)_{xx} - cw_1\). Hence, by a similar process, we have \( w \geq 0 \) implies \( \bar{u} - u_1 \geq 0 \). Thus, \( \bar{u} \geq u_1 \).

Similarly, \( \bar{u} \geq u_2 \). By induction, \( \bar{v} \geq u_n \). So, we have a monotone, bounded sequence between \( \bar{y} \) and \( \bar{v} \). By the Monotone Convergence Theorem, \((u_n(x,t))\) converges in \( \mathbb{R} \) for all \( x \in [a,b] \) and for all \( t > 0 \). Hence, \((u_n(x,t))\) is pointwise convergent.

Let \( G(x - y, t) \) denote the heat kernel \( e^{-(x-y)^2/4kt} \) as defined in [6]. Then, it is a standard theorem that

\[
u_n(x,t) = \int_{-\infty}^{\infty} G(x - y, t)u_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s)g(u_{n-1}(x))dy.
\]
See [5], pg. 195. Now, consider

\[
\lim_{n \to \infty} u_n(x, t) = \lim_{n \to \infty} \left[ \int_{-\infty}^{\infty} G(x - y, t)u_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s)g(u_{n-1})dy \right]
\]

\[
= \int_{-\infty}^{\infty} G(x - y, t)u_0(y)dy + \lim_{n \to \infty} \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s)g(u_{n-1})dy.
\]

Recall that \((u_n)\) is pointwise convergent. Therefore, since \(g\) is continuous, then \(g(u_{n-1})\) is pointwise convergent. Also, since \(G\) is bounded and \(g\) is bounded and pointwise convergent, then, by the Dominated Convergence Theorem, we have

\[
\int_{-\infty}^{\infty} G(x - y, t)u_0(y)dy + \lim_{n \to \infty} \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s)g(u_{n-1})dy
\]

\[
= \int_{-\infty}^{\infty} G(x - y, t)u_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s)g(u)dy.
\]

Recall \(\lim_{n \to \infty} u_n(x, t) = u(x, t)\). Hence,

\[
u(x, t) = \int_{-\infty}^{\infty} G(x - y, t)u_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s)g(u)dy.
\]

Therefore, \(u(x, t)\) is a solution of

\[
u_t = \epsilon^2u_{xx} - cu - g(u),
\]

\[
u_x(a, t) = 0 = u_x(b, t),
\]

\[
u(x, 0) = u_0(x).
\]

(3.4)

3.1.2 Positivity Lemma

With Theorem 3.1 and the definitions of upper and lower solutions in mind, we can consider some important lemmas from[4]. Note that the proofs of these lemmas can be found in [4]. We include them here for completeness. We will begin with the positivity lemma.
Lemma 4 (Positivity Lemma). Let \( w \in C(D_T) \cap C^{1,2}(D_T) \) be such that

\[
\begin{align*}
    w_t - \epsilon^2 w_{xx} + cw & \geq 0 \quad \text{in } D_T \\
    \alpha \frac{\partial w}{\partial \nu} + \beta w & \geq 0 \quad \text{at } x = 0, 1 \\
    w(x,0) & \geq 0 \quad \text{in } (0,1)
\end{align*}
\] (3.5)

where \( \alpha, \beta \geq 0 \), and \( \alpha + \beta > 0 \) at \( x = a, b \). Let \( c = c(x,t) \) be a bounded function in \( D_T \). Then, \( w(x,t) \geq 0 \) in \( D_T \). Moreover, \( w(x,t) > 0 \) in \( D_T \) unless \( w \equiv 0 \).

We will consider \( D_T = (0,1) \times (0,T) \) for \( T > 0 \).

Proof. Assume the hypotheses hold.

Case 1: Let \( c(x,t) > 0 \). Suppose to the contrary \( w(x_0,t_0) < 0 \) at some point \( (x_0,t_0) \in D_T \). Therefore, \( w \) must have a negative minimum somewhere in \( \overline{D_T} \). Notice that this minimum cannot occur at \( t = 0 \) since \( w(x,0) \geq 0 \) by hypothesis.

Suppose the minimum occurs in \( D_T \). Hence, by the Maximum Principle, \( w \) must be a negative constant function, but, then

\[
w_t - \epsilon^2 w_{xx} + cw = 0 - 0 + cw = cw < 0
\]

since \( c > 0 \) by hypothesis, a contradiction.

So, consider \( (x_0,t_0) \in S_T = (0,T] \times \partial \Omega \) where \( \Omega = (0,1) \). For \( (x_0,t_0) \in S_T \) with \( 0 < t_0 < T \), we have a contradiction of Theorem 2.2. Consider \( t_0 = T \): Let \( m = \min_{D_T} \{u\} \). Suppose \( w(0,t) = m < 0 \). Therefore, \( w_t(0,t_0) \leq 0 \). Also, we must have \( \alpha > 0 \) or else we get an immediate contradiction of Theorem 2.2. So \( \frac{\partial w}{\partial \nu} \geq \frac{\beta}{\alpha} w \geq 0 \), which means \( w_x(0,t_0) \leq 0 \). Notice \( \epsilon^2 w_{xx} \leq w_t + cw < 0 \) which implies \( w \) is concave down in the \( x \)-direction in a neighborhood \( C \) of \( (0,t_0) \). Thus, \( w(x,t_0) < w(0,t_0) = m \) for \( (x,t_0) \in C \), which is a contradiction. When \( (x_0,t_0) \) occurs on the line \( t = T \), we have a contradiction of Case 2 of the Maximum Principle.

Thus, \( w(x,t) \geq 0 \) for all \( (x,t) \in D_T \). Note that if \( w(x,t) = 0 \) at a point in \( D_T \), then \( w(x,t) \equiv 0 \) by the Maximum Principle. Otherwise, \( w(x,t) > 0 \) in \( D_T \).
Case 2: Let \( c = c(x,t) \) be a bounded function. Let \( w = e^{\gamma t}v \). Then,

\[
  w_t - \epsilon^2 w_{xx} + cw = \gamma e^{\gamma t}v + e^{\gamma t} - \epsilon^2 e^{\gamma t}v_{xx} + cw
  = e^{\gamma t}[v_t - \epsilon^2 v_{xx} + (\gamma + c)v]
\]

Since \( c \) is bounded, we can choose \( \gamma \) so that \( \gamma + c > 0 \). Hence,

\[
e^{\gamma t}[v_t - \epsilon^2 v_{xx} + (\gamma + c)v] \geq 0
\]

\[
\implies v_t - \epsilon^2 v_{xx} + (\gamma + c)v \geq 0.
\]

If \( w(x,0) \geq 0 \), then \( e^{0}v(x,0) \geq 0 \) so \( v(x,0) \geq 0 \). Also, note that \( \nu \) only depends on \( x \). So \( \frac{\partial w}{\partial \nu} = e^{\gamma t} \frac{\partial v}{\partial \nu} \). Hence, since

\[
v_t - \epsilon^2 v_{xx} + (\gamma + c)v \geq 0
\]

\[
v(x,0) \geq 0
\]

\[
\alpha \frac{\partial v}{\partial \nu} + \beta v \geq 0
\]

where \( \alpha, \beta \geq 0, \alpha + \beta > 0 \), then Case 1 holds for \( v \). Therefore, \( w(x,t) \geq 0 \) in \( D_T \).

Furthermore, \( w(x,t) > 0 \) in \( D_T \) unless it is identically zero.

\[\square\]

We now have the tools we need to prove the following two lemmas which will be very useful for our proof of Theorem 1.1.

**Lemma 5.** Let \( \underline{u}(x), \overline{u}(x) \) be upper and lower solutions of (3.1) and let \( \underline{U}(x,t), \overline{U}(x,t) \) be solutions of (3.2) corresponding to the initial conditions \( u_0 = \underline{u} \) and \( u_0 = \overline{u} \), respectively. Assume \( f \) is a \( C^1 \)-function in \( \langle \underline{u}, \overline{u} \rangle \). Let \( \underline{u} \leq \overline{u} \). Then, for \( x \in [a,b] \), \( \underline{U}(x,t) \) is non-increasing in \( t \), \( \overline{U}(x,t) \) is non-decreasing in \( t \), and

\[
\underline{u}(x) \leq \underline{U}(x,t) \leq \overline{U}(x,t) \leq \overline{u}(x) \quad \text{in} \ \mathcal{D}.
\]

If \( \underline{u} \) (resp. \( \overline{u} \)) is not a solution of (3.1), then \( \underline{U} \) is strictly increasing (resp. \( \overline{U} \) is strictly decreasing) in \( t \).

Note that \( \langle \underline{u}, \overline{u} \rangle \) is the interval of functions bounded below by \( \underline{u} \) and bounded above by \( \overline{u} \).
With the above picture in mind, Lemma 5 tells us that if we have an upper and a lower solution with a solution $U(x,t)$ between them, then depending on the initial condition of $U(x,t)$, $U(x,t)$ is non-increasing or non-decreasing in time. For two solutions $\overline{U}(x,t)$ and $\underline{U}(x,t)$ between $\overline{u}$ and $\underline{u}$ with $\overline{u}$ the initial condition of $\overline{U}(x,t)$ and $\underline{u}$ the initial condition of $\underline{U}(x,t)$, then Lemma 5 also tells us $\overline{u}(x) \geq \overline{U}(x,t) \geq \underline{U}(x,t) \geq \underline{u}(x)$. This result will be a key factor in proving Theorem 1.1. Now, let’s prove Lemma 5.

Proof. Assume $\underline{u}(x)$ is a lower solution of (3.1). Also, $\underline{u}(x) \leq \overline{u}(x)$ where $\overline{u}(x)$ is an upper solution of (3.1). Let $\underline{U}(x,t)$ and $\overline{U}(x,t)$ be solutions of (3.2) with $\underline{U}(x,0) = \underline{u}(x)$, and similarly $\overline{U}(x,0) = \overline{u}(x)$. Also, let $B$ be the operator such that $Bu = \alpha \frac{\partial u}{\partial \nu} + \beta u = 0$.

By the iteration argument from the proof of Theorem 3.1, $\underline{u}(x) \leq \underline{U}(x,t) \leq \overline{u}(x)$ for all $t > 0$. Let $\delta > 0$ and let $w(x,t) = \underline{U}(x,t + \delta) - \underline{U}(x,t)$.

First, note that

$$\frac{\partial w}{\partial \nu} = \frac{\partial}{\partial \nu} \underline{U}(x,t + \delta) - \frac{\partial}{\partial \nu} \underline{U}(x,t).$$

Therefore,

$$\alpha \frac{\partial w}{\partial \nu} + \beta w = \alpha \left( \frac{\partial}{\partial \nu} \underline{U}(x,t + \delta) - \frac{\partial}{\partial \nu} \underline{U}(x,t) \right) + \beta \left( \frac{\partial}{\partial \nu} \underline{U}(x,t + \delta) - \frac{\partial}{\partial \nu} \underline{U}(x,t) \right)$$

$$= \left[ \alpha \frac{\partial}{\partial \nu} \underline{U}(x,t + \delta) + \beta \underline{U}(x,t + \delta) \right] - \left[ \alpha \frac{\partial}{\partial \nu} \underline{U}(x,t) + \beta \underline{U}(x,t) \right]$$

$$= B \underline{U}(x,t + \delta) - B \underline{U}(x,t)$$
With this fact in mind, we want to show that $w$ satisfies the hypotheses of the positivity lemma. Therefore, we must verify the following:

(i) $Bw(a, t) = 0$.

Notice $Bw(a, t) = BU(a, t + \delta) - BU(a, t) = 0 - 0 = 0$ since $U$ is a solution of (3.2). Thus, $Bw(a, t) = 0$.

(ii) $Bw(b, t) = 0$.

Similarly, observe $Bw(b, t) = BU(b, t + \delta) - BU(b, t) = 0 - 0 = 0$ since $U$ is a solution of (3.2). Therefore, $Bw(b, t) = 0$.

(iii) $w(x, 0) \geq 0$

Note that $w(x, 0) = U(x, \delta) - U(x, 0) = U(x, \delta) - u(x) \geq 0$ because we know by the iteration method that $u(x) \leq U(x, t)$ for all $t > 0$.

Now, observe

$$w_t - \epsilon^2 w_{xx} = U_t(x, t + \delta) - U_t(x, t) - \epsilon^2 (U_{xx}(x, t + \delta) - U_{xx}(x, t))$$

$$= U_t(x, t + \delta) - \epsilon^2 U_{xx}(x, t + \delta) - (U_t(x, t) - \epsilon^2 U_{xx}(x, t))$$

$$= f(U(x, t + \delta)) - f(U(x, t))$$

$$= \frac{\partial f}{\partial u}(\xi(x, t))(U(x, t + \delta) - U(x, t))$$

$$= \frac{\partial f}{\partial u}(\xi(x, t))w(x, t)$$

by the Mean Value Theorem where $\xi(x, t)$ is bounded in the closed interval $[\inf u, \sup \bar{u}]$.

By the positivity lemma, either $w(x, t) \equiv 0$ for all $(x, t)$ or $w(x, t) \geq 0$ with $w(x, t) > 0$ in $D_T$. Hence, $w(x, t) = U(x, t + \delta) - U(x, t) \geq 0$ which implies $U(x, t + \delta) \geq U(x, t)$. Thus, $U$ is non-decreasing. The proof that $\overline{U}$ is non-increasing is similar.

Now we want to show that $\overline{U} \geq \underline{U}$. Consider $W(x, t) = \overline{U}(x, t) - \underline{U}(x, t)$. We want to show $W$ is a solution to the same problem as $\underline{U}, \overline{U}$. First, we must compute
the following in terms of $W$:

$$\frac{\partial W}{\partial \nu} = \frac{\partial}{\partial \nu} \overline{U}(x,t) - \frac{\partial}{\partial \nu} \underline{U}(x,t).$$

Therefore, we have

$$\alpha \frac{\partial W}{\partial \nu} + \beta W = \alpha \left( \frac{\partial}{\partial \nu} \overline{U}(x,t) - \frac{\partial}{\partial \nu} \underline{U}(x,t) \right) + \beta \left( \frac{\partial}{\partial \nu} \overline{U}(x,t) - \frac{\partial}{\partial \nu} \underline{U}(x,t) \right)$$

$$= \left[ \alpha \frac{\partial}{\partial \nu} \overline{U}(x,t) + \beta \overline{U}(x,t) \right] - \left[ \alpha \frac{\partial}{\partial \nu} \underline{U}(x,t) + \beta \underline{U}(x,t) \right]$$

$$= B \overline{U}(x,t) - B \underline{U}(x,t)$$

$$= BW(x,t).$$

Now, let us verify that $W$ satisfies the following properties of the positivity lemma:

(i) $BW(a,t) = 0$.

Notice $BW(a,t) = B \overline{U}(a,t) - B \underline{U}(a,t) = 0 - 0 = 0$ since $\overline{U}$ and $\underline{U}$ are solutions. Thus, $BW(a,t) = 0$.

(ii) $BW(b,t) = 0$.

Similarly, observe $BW(b,t) = B \overline{U}(b,t) - B \underline{U}(b,t) = 0 - 0 = 0$ since $\overline{U}$ and $\underline{U}$ are solutions of (3.2). Therefore, $BW(b,t) = 0$.

(iii) $W(x,0) \geq 0$

Note that

$$w(x,0) = \overline{U}(0,x) - \underline{U}(0,x)$$

$$= \overline{u}(x) - \underline{u}(x) \quad \text{by hypothesis}$$

$$\geq 0$$

since $\underline{u}(x) \leq \overline{u}(x,t)$ for all $t > 0$ by the iteration argument from the proof of Theorem 3.1.
Also, we have

\[ W_t - \epsilon^2 W_{xx} = \mathcal{U}_t(x,t) - \mathcal{U}_t(x,t) - \epsilon^2 (\mathcal{U}_{xx}(x,t) - \mathcal{U}_{xx}(x,t)) \]

\[ = \mathcal{U}_t(x,t) - \epsilon^2 \mathcal{U}_{xx}(x,t) - (\mathcal{U}_t(x,t) - \epsilon^2 \mathcal{U}_{xx}(x,t)) \]

\[ = f(\mathcal{U}(x,t)) - f(\mathcal{U}(x,t)) \]

\[ = \frac{\partial f}{\partial u}(\eta(x,t))(\mathcal{U}(x,t) - \mathcal{U}(x,t)) \]

\[ = \frac{\partial f}{\partial u}(\eta(x,t))W(x,t) \]

by the Mean Value Theorem where \( \eta(x,t) \) is bounded in the closed interval \([\inf \mathcal{U}, \sup \mathcal{U}]\).

Since \( W \) satisfies the previous properties, by the positivity lemma, we know that \( W \equiv 0 \) if \( W(x,0) \equiv 0 \) or \( W > 0 \) if \( W(x,0) \neq 0 \). Thus, \( W \geq 0 \) which implies \( \mathcal{U} - \mathcal{U} \geq 0 \) so \( \mathcal{U} \geq \mathcal{U} \). If \( \bar{u}, u \) are not steady-state solutions, then \( \bar{u} \neq u \implies W(x,0) \neq 0 \). So, \( \mathcal{U} \geq \mathcal{U} \) in \( D_T \).

Therefore, if \( u \) (resp. \( \bar{u} \)) is not a solution of

\[ \epsilon^2 u_{xx} + f(u) = 0 \]

\[ Bu = \alpha_0 \frac{\partial u}{\partial \nu} + \beta_0 u = 0 \]

then \( \mathcal{U} \) (resp. \( \mathcal{U} \)) is strictly increasing in \( t \) and \( \mathcal{U} \geq \mathcal{U} \) in \( D_T \). \( \square \)

**Lemma 6.** Let the hypotheses of the previous lemma hold, and let \( u(x,t) \) be the solution of (3.2) with \( u_0 \in (\underline{u}, \bar{u}) \). Then, \( \underline{U}(x,t) \leq u(x,t) \leq \bar{U}(x,t) \) in \( D \).
Now, Lemma 6 tells us that given solutions $\overline{U}$ and $\underline{U}$ of (3.2), then, for a solution $u(x,t)$ of (3.2) with $u_0 \in (\underline{u}, \overline{u})$, $u(x,t)$ lies between $\overline{U}$ and $\underline{U}$. We will see later how this lemma in conjunction with Lemma 5 helps us obtain the proof for Theorem 1.

Proof. Assume the hypotheses of Lemma 5. Let $u(x,t)$ be a solution with $u_0 \in (\underline{u}, \overline{u})$. Let $w(x,t) = \overline{U}(x,t) - u(x,t)$. Observe

$$\alpha \frac{\partial w}{\partial \nu} + \beta w = \alpha \frac{\partial}{\partial \nu} \overline{U}(x,t) - \alpha \frac{\partial}{\partial \nu} u(x,t) + \beta \overline{U}(x,t) - \beta u(x,t)$$

$$= \alpha \frac{\partial}{\partial \nu} \overline{U}(x,t) + \beta \overline{U}(x,t) - \alpha \frac{\partial}{\partial \nu} u(x,t) - \beta u(x,t)$$

$$= B\overline{U}(x,t) - Bu(x,t)$$

Hence, we want to verify that $w$ satisfies the following conditions:

(i) $Bw(a,t) = 0$

Notice $Bw(a,t) = B\overline{U}(a,t) - Bu(a,t) = 0 - 0 = 0$ since $\overline{U}$ and $u$ are solutions of (3.2). So, $Bw(a,t) = 0$.

(ii) $Bw(b,t) = 0$

Similarly, we have $Bw(b,t) = B\overline{U}(b,t) - Bu(b,t) = 0 - 0 = 0$ since $\overline{U}$ and $u$ are solutions of (3.2). Then, $Bw(b,t) = 0$.

(ii) $w(x,0) \geq 0$

We have

$$w(x,0) = \overline{U}(x,0) - u(x,0)$$

$$= \pi(x) - u_0(x)$$

$$\geq 0$$

since $u_0 \in (\underline{u}, \overline{u})$. 

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Now, we also have
\[ w_t - \epsilon^2 w_{xx} = \bar{U}_t(x, t) - u_t(x, t) - \epsilon^2 \bar{U}_{xx}(x, t) + \epsilon^2 u_{xx}(x, t) \]
\[ = \bar{U}_t(x, t) - \epsilon^2 \bar{U}_{xx}(x, t) - (u_t(x, t) - \epsilon^2 u_{xx}(x, t)) \]
\[ = f(\bar{U}) - f(u) \]
\[ = \frac{\partial f}{\partial u}(\xi(x, t)) (\bar{U}(x, t) - u(x, t)) \]
\[ = \frac{\partial f}{\partial u}(\xi(x, t)) w(x, t) \]
by the Mean Value Theorem, where \( \xi(x, t) \) is bounded in the closed interval \([\inf \bar{U}, \sup \bar{U}]\).

Thus, by the positivity lemma, \( w(x, t) \equiv 0 \) or \( w > 0 \). Hence, \( \bar{U} \geq u \). The case is similar for \( u \geq \bar{U} \). Hence, \( \bar{U}(x, t) \leq u(x, t) \leq \bar{U}(x, t) \) in \([a, b]\).

\[ \Box \]

### 3.2 Existence, Uniqueness, Continuous Dependence

In this section, we are going to be adapting some work found in [5]. This will be important in the following chapter especially our discussion of the asymmetric case.

Now, consider
\[ u_t = ku_{xx} \quad \text{in } [0, 1] \]
\[ u_x(0, t) = 0 = u_x(1, t) \]
\[ u(x, 0) = \phi(x) \quad \text{in } [0, 1]. \]

(3.6)

It will be helpful to transform (3.6) into a problem on the entire real line. So, given \( \phi(x) \) on \([0, 1]\), let
\[ \tilde{\phi}(x) = \begin{cases} \phi(x) & 0 \leq x \leq 1 \\ \phi(-x) & -1 \leq x < 0. \end{cases} \]

Let \( \Phi(x) = \tilde{\phi}(x - 2k) \) for \( x \in [-1 + 2k, 1 + 2k] \) where \( k \in \mathbb{Z} \). We are ready to prove some lemmas which will be helpful proving an existence result from [5].

**Lemma 7.** Let \( G \) be the heat kernel as defined in Chapter 3.1.1.

(i) \( \Phi(-x) = \Phi(x) \) and \( \Phi(1 + x) = \Phi(1 - x) \) for all \( x \in \mathbb{R} \).
(ii) If \( u(x, t) := (G(t) * \Phi)(x) \), then \( u(-x, t) = u(x, t) \) and \( u(1 - x, t) = u(1 + x, t) \) for all \( x, t \). Moreover, \( u_x(0, t) = 0 = u_x(1, t) \) for all \( t > 0 \).

Proof. (i) Note that

\[
\Phi(-x) = \tilde{\Phi}(-x - 2k) \text{ for } x \in [-1 + 2k, 1 + 2k]
\]

\[
= \begin{cases} 
\phi(-x - 2k) & 0 \leq -x - 2k \leq 1 \\
\phi(x + 2k) & -1 \leq -x - 2k < 0 
\end{cases} \quad k \in \mathbb{Z}
\]

\[
= \begin{cases} 
\phi(-x - 2k) & 2k \leq -x \leq 1 + 2k \\
\phi(x + 2k) & -1 + 2k \leq -x < 2k 
\end{cases} \quad k \in \mathbb{Z}
\]

\[
= \begin{cases} 
\phi(-x - 2k) & -2k - 1 \leq x \leq -2k \\
\phi(x + 2k) & -2k < x \leq -2k + 1 
\end{cases} \quad k \in \mathbb{Z}
\]

Let \( 2j = -2k \) with \( j \neq k \). So we have

\[
\Phi(-x) = \begin{cases} 
\phi(-(x - 2j)) & -1 + 2j \leq x \leq 2j \\
\phi(x - 2j) & 2j < x \leq 2j + 1 
\end{cases} \quad j \in \mathbb{Z}
\]

\[
= \begin{cases} 
\phi(x - 2j) & 2j < x \leq 2j + 1 \\
\phi(-(x - 2j)) & -1 + 2j \leq x \leq 2j 
\end{cases} \quad j \in \mathbb{Z}
\]

\[
= \tilde{\phi}(x - 2j) = \Phi(x).
\]

Hence, \( \Phi(-x) = \Phi(x) \).

Now, we claim that \( \Phi(1 - x) = \Phi(x + 1) \). To see this,

\[
\Phi(1 - x) = \begin{cases} 
\phi(1 - x - 2k) & 0 \leq 1 - x - 2k \leq 1 \\
\phi(-1 + x + 2k) & -1 \leq 1 - x - 2k < 0 
\end{cases} \quad k \in \mathbb{Z}
\]

\[
= \begin{cases} 
\phi(1 - 1 + 1 - x - 2k) & 2k - 1 \leq -x \leq 2k \\
\phi(1 - 1 - 1 + x + 2k) & 2k - 2 \leq -x < 2k - 1 
\end{cases} \quad k \in \mathbb{Z}
\]

\[
= \begin{cases} 
\phi(-(1 + x + 2k - 2)) & -2k \leq x \leq -2k + 1 \\
\phi(1 + x + 2k - 2) & 1 - 2k < x \leq 2 - 2k 
\end{cases} \quad k \in \mathbb{Z}
\]
Let $2j = -2k + 2$ with $j \neq k$. Then,

$$\Phi(1-x) = \begin{cases} 
\phi(-(1 + x - 2j)) & 2j - 2 \leq x \leq 2j - 1 \\
\phi(1 + x - 2j) & 2j - 1 < x \leq 2j 
\end{cases} \quad j \in \mathbb{Z}$$

$$= \begin{cases} 
\phi(1 + x - 2j) & 2j < x + 1 \leq 2j + 1 \\
\phi(-(1 + x - 2j)) & -1 + 2j \leq x + 1 \leq 2j 
\end{cases} \quad j \in \mathbb{Z}$$

$$= \tilde{\phi}(1 + x - 2j)$$

$$= \Phi(1 + x)$$

Thus, $\Phi(1-x) = \Phi(1+x)$.

(ii) Let $u(x, t)$ be as defined in the claim. First, we will show that $u(-x, t) = u(x, t)$ for all $x, t$. Note that

$$u(-x, t) = \int_{-\infty}^{\infty} G(-x - y, t)\Phi(y)dy$$

$$= \int_{-\infty}^{\infty} G(x + y, t)\Phi(y)dy \quad \text{since } G \text{ is symmetric,}$$

$$= \int_{-\infty}^{\infty} G(y, t)\Phi(x + y)dy$$

Let $v = -y$. Then, we have

$$u(-x, t) = -\int_{-\infty}^{\infty} G(-v, t)\Phi(x - v)dv$$

$$= \int_{-\infty}^{\infty} G(v, t)\Phi(x - v)dv \quad \text{since } G \text{ is symmetric,}$$

$$= \int_{-\infty}^{\infty} G(x - v, t)\Phi(v)dv$$

$$= u(x, t).$$

Thus, $u(-x, t) = u(x, t)$ for all $x, t$. 

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Now we will prove \( u(1-x,t) = u(1+x,t) \) for all \( x,t \). Notice

\[
\begin{align*}
  u(1-x,t) &= \int_{-\infty}^{\infty} G(1-x-y,t)\Phi(y)dy \\
  &= \int_{-\infty}^{\infty} G(y,t)\Phi(1-x-y)dy \\
  &= \int_{-\infty}^{\infty} G(y,t)\Phi(1-(x+y))dy \\
  &= \int_{-\infty}^{\infty} G(y,t)\Phi(1+(x+y))dy \quad \text{by previous claim,} \\
  &= \int_{-\infty}^{\infty} G(y,t)\Phi(1+x-(-y))dy
\end{align*}
\]

Let \( v = -y \). Then,

\[
\begin{align*}
  u(1-x,t) &= -\int_{-\infty}^{\infty} G(-v,t)\Phi(1+x-v)dv \\
  &= \int_{-\infty}^{\infty} G(1+x-v,t)\Phi(v)dv \\
  &= u(1+x,t).
\end{align*}
\]

Hence, \( u(1-x,t) = u(1+x,t) \) for all \( x,t \).

Finally, we will now prove \( u_x(0,t) = 0 = u_x(1,t) \) for all \( t > 0 \). Recall that \( u \) has reflection symmetry around \( x = 0 \) and \( x = 1 \) so \( u_x(0,t) = 0 \) and \( u_x(1,t) = 0 \) are immediate.

Now, consider the set \( B := \{j \in C(\mathbb{R}) : j(x) = j(-x), j(1-x) = j(1+x)\} \) with \( \|j\|_B := \|j\|_\infty \).

**Lemma 8.** Let \( G \) be the heat kernel as defined in Chapter 3.1.1. If \( j \in B \), then

(i) \( j \) is 2-periodic,

(ii) \( j \) is bounded,

(iii) \( \|j\|_B = \sup_{0\leq x\leq 1} |j| \), and

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(iv) \( G(t) \ast j \in B \) for all \( t \).

**Proof.** (i) Let \( j \in B \). We want to show \( j(x + 2) = j(x) \). Consider

\[
\begin{align*}
  j(x + 2) &= j((x + 1) + 1) \\
          &= j(1 - (x + 1)) \quad \text{since } j \in B, \\
          &= j(1 - x - 1) \\
          &= j(-x) \\
          &= j(x) \quad \text{since } j \in B.
\end{align*}
\]

Hence, \( j \) is 2-periodic.

(ii) Note that the functions in \( B \) are continuous in \( \mathbb{R} \) and 2-periodic. Therefore, the functions are determined by their values on the interval \([0, 2]\). Recall that a continuous function on a closed and bounded interval is bounded. Hence, we have if \( j \in B \), then \( j \) is bounded.

(iii) Recall, by definition, \( ||j||_B := ||j||_\infty \). Note that \( j \) is continuous on a closed interval. Therefore, \( j \) has a supremum on that interval, and \( \sup j = \max j \). Hence,

\[
||j||_B := \sup_{0 \leq x \leq 1} |j|.
\]

(iv) Since \( f \) is bounded and continuous, then \( G(t) \ast f \) is well-defined and integrable.

The fact that \( G(t) \ast j \) is continuous in \( \mathbb{R} \) is standard. So, we only need to show

(a) \( (G(t) \ast j)(x) = (G(t) \ast j)(-x) \), and (b) \( (G(t) \ast j)(1 - x) = (G(t) \ast j)(1 + x) \)

which is true by Lemma 7 (ii).

Before we move on, we should observe that it is well-known that \( u \) is a solution of (1.2) if and only if

\[
\begin{align*}
u(x, t) &= \int_{\mathbb{R}} G(x - y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}} G(x - y, t - s)f(u(y, s))dyds \\
&= G(t) \ast u_0 + \int_0^t G(t - s) \ast f(u(s))ds.
\end{align*}
\]

(3.7)
With this in mind, we can consider the following two lemmas which can be found in Chapter 14 on pg. 196-198 of [5]. We will provide a proof which modifies the approach found in [5] by selecting a different function space, $B$. Overall the arguments are almost identical. The first lemma gives us existence and uniqueness for the problem (1.2).

**Lemma 9.** Let $u_0 \in B$; then there exists $t_0 > 0$, where $t_0$ depends only on $f$ and $\|u_0\|_{\infty}$, such that (3.7) has a unique solution in $C([0,t_0]; B)$, and $\|u\| \leq 2 \|u_0\|$.

**Proof.** Let $|f(u)| > M$ for $-1 \leq u \leq 1$. Assume $t_0 \leq \frac{1}{M}$. Let $\Gamma := \{u \in C([0,t_0]; B) : \|u(t) - G(t) \ast u_0\| \leq 1, 0 \leq t \leq t_0\}$ for fixed $t$ with $\|\cdot\|$ on $\Gamma$ given by

$$\|u(x,t)\| = \sup_{0 \leq t \leq t_0} |u(x,t)|$$

$$= \sup_{0 \leq t \leq t_0} \sup_{0 \leq x \leq 2} |u(x,t)|.$$ 

Note that $\Gamma \neq \emptyset$ since $u_0 \in \Gamma$.

Now, we want to show that $\Gamma$ is closed. Suppose $(u_n) \subseteq \Gamma$ such that $u_n \to u$ in $C([0,t_0]; B)$ so

$$u_n(t) - G(t) \ast u_0 = u_n(x,t) - \int_{-\infty}^{\infty} G(x-y,t)u_0(y,t)dy$$

with

$$\left|u_n(x,t) - \int_{-\infty}^{\infty} G(x-y,t)u_0(y)dy\right| \leq 1,$$

for all $t \in [0,t_0]$ and $x \in (0,1)$. Let $n \to \infty$ to get $u \in \Gamma$. Therefore, $(\Gamma, \|\cdot\|)$ is complete since closed subsets of complete spaces are complete. Hence, we can apply the Contraction Mapping Theorem.

Now, consider $\Lambda : C([0,t_0]; B) \to C([0,t_0]; B)$ such that

$$\Lambda(u)(t) := \int_{-\infty}^{\infty} G(x-y,t)u_0(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)f(u(y,s))dyds.$$
By p. 49 of [3], this is a smooth function. Since $u$ has reflection symmetries so does $f(u)$. $G$ preserves reflection symmetries by Lemma 7, thus $\Lambda(u)$ has reflection symmetries. Therefore, $\Lambda(u) \in C([0,t_0]; B)$.

Let $u \in \Gamma$. Then,

$$
\left| \Lambda(u) - \int_{-\infty}^{\infty} G(x - y,t)u_0(y)dy \right| \leq \int_0^t \int_{-\infty}^{\infty} G(x - y,t - s)|j(u(y,s))|dyds
\leq Mt
\leq 1 \text{ for } t \leq \frac{1}{M}.
$$

Hence, $\Lambda : \Gamma \rightarrow \Gamma$. The rest of the proof follows as in the proof of Theorem 14.2 in [5].

The second lemma from [5] gives us continuous dependence. This will be very important to our proof of the asymmetric case.

**Lemma 10.** Let $u, v \in C([0,T]; B)$ be solutions of (3.2) on $0 \leq t \leq T$, where $\|u\|_{\infty}, \|v\|_{\infty} \leq 1$. Then there is a constant $k$ such that

$$
\|u(t) - v(t)\|_B \leq e^{kt}\|u(0) - v(0)\|_B, \quad 0 \leq t \leq T. \tag{3.8}
$$

**Proof.** Let $t \in [0,T]$. Then

$$
u(t) - v(t) = G(t) * (u(0) - v(0)) + \int_0^t G(t - s)[f(u(s)) - f(v(s))]ds.
$$

By the Lipschitz continuity of $f$, there exists a $k$ such that

$$
\|f(u(s)) - f(v(s))\|_B \leq k \|u(s) - v(s)\|_B.
$$

Therefore,

$$
\|u(t) - v(t)\|_B \leq \|u(0) - v(0)\|_B + k \int_0^t \|u(s) - v(s)\|_B ds.
$$

Hence, by Gronwall’s inequality, we get

$$
\|u(t) - v(t)\|_B \leq e^{kt}\|u(0) - v(0)\|_B.
$$

\[\square\]
Chapter 4: Stability

The main purpose of this paper is to prove stability properties of solutions to (1.2). In this chapter, we will make our argument through two cases: one when the initial condition is symmetric and one when the initial condition is asymmetric. With these two cases, we will have proven stability properties for all the solutions in the base of the trough.

4.1 One-Node Solution: Symmetric Case

Recall that the one-node solution is symmetric when it crosses the $x$-axis at $x_0 = \frac{1}{2}$.

Before we can continue with this case, we need to discuss our non-classical choice of $F$. Recall that the non-classical $F(u) = (1 - u^2)^{\alpha}$ for $1 < \alpha < 2$. Therefore, as $u$ approaches $\pm1$, the slope of $F'$ becomes unbounded. It follows that $F'$ is not Lipschitz. To avoid the unboundedness of $F'$ at these points, we will truncate $F'$. So we define

$$f_n(u) = \begin{cases} F'(u) & -1 + \frac{1}{n} \leq u \leq 0 \\ nF'(-1 + \frac{1}{n})(u + 1) & -1 < u < -1 + \frac{1}{n}. \end{cases}$$

Consider the following graph of $f_n(u)$ on the interval $(-1, 0)$. 

![Graph of f_n(u)]
The dotted line represents the point when $f_n(u)$ no longer equals $F'$. Also, $f'_n(u)$ is bounded as it approaches $u = -1$. However, it follows that as $n \to \infty$, $f_n(u)$ converges to $F'(u)$ uniformly. Observe that in the following proofs, we use the fact that $f_n \leq F'$ and $f_n \to F'$ uniformly as $n \to \infty$. Therefore we could replace $f_n$ by a smoother approximation.

First, we note that if the initial condition is symmetric around $x_0 = \frac{1}{2}$, then so is the solution of (1.2). This allows us to reduce our argument to the following problem.

Further, we truncate $F$ to get

$$u_t = \epsilon^2 u_{xx} - f_n(u)$$
$$u_x(0, t) = 0 = u \left( \frac{1}{2}, t \right)$$
$$u(x, 0) = u_0(x).$$

Note that when $u \in [-1 + \frac{1}{n}, 0]$, (4.1) is problem (3.2).

Now, in order to use Lemma 5 and Lemma 6 from Chapter 3, we must find an upper and a lower solution for (4.1). Since we are working with Neumann boundary conditions, we will prove a function is an upper solutions using the following inequalities:

$$\underline{u}(\frac{1}{2}) \geq 0$$
$$\underline{u}_x(0) \leq 0$$
$$\epsilon^2 \underline{u}_{xx} \leq 0.$$  

For the lower solutions, we only need to reverse the inequalities. Note that we have just chosen values for $\alpha$ and $\gamma$ from the upper solution definition in Chapter 3 in order
Lemma 11. Let \( u(x) \) be the symmetric one-node solution to (1.1) satisfying \( u(0) = -1 \) and \( u\left(\frac{1}{2}\right) = 0 \). Then, \( \bar{u} = \beta u(x) \) is an upper solution for the problem (4.1) with \( 0 < \beta < 1 - \frac{1}{n} \).

Before we prove this lemma, we want to recall that \( u(x) \) is unique by Theorem 7.1 from [1].

Proof. Let \( 0 < \beta < 1 - \frac{1}{n} \). Consider \( \beta u(x) \). We must show that \( \beta u(x) \) satisfies the following conditions.

First, notice \( \beta u\left(\frac{1}{2}\right) = 0 \). Therefore, \( u\left(\frac{1}{2}\right) \geq 0 \).

Now, observe \( (\beta u(x))_x = \beta u_x(x) \). Recall \( u_x(0) = 0 \) since \( u(x) \) is a solution of (1.1). Thus, \( u_x(0) \leq 0 \).

Finally, note that \( (\beta u(x))_{xx} = \beta u_{xx}(x) \), so we have \( \epsilon^2(\beta u_{xx}(x)) = \beta(\epsilon^2 u_{xx}(x)) = \beta F'(u(x)) \). Since \( 0 < \beta < 1 - \frac{1}{n} \), then

\[
(u(x))^2 \geq \beta^2 (u(x))^2,
\]

so \( -(u(x))^2 \leq -\beta^2 (u(x))^2 \).

Also, \( 1 - (u(x))^2 \leq 1 - \beta^2 (u(x))^2 \),

and \( (1 - (u(x))^2)^{\alpha - 1} \leq (1 - \beta^2 (u(x))^2)^{\alpha - 1} \) since \( 1 < \alpha < 2 \); thus, \( -2\alpha \beta u(x)(1 - (u(x))^2)^{\alpha - 1} \leq -2\alpha \beta u(x)(1 - \beta^2 (u(x))^2)^{\alpha - 1} \) since \( u(x) \leq 0 \) on \( [0, \frac{1}{2}] \).

Then \( \beta F'(u(x)) \leq F'(\beta u(x)) \).

Therefore, \( \epsilon^2 \beta u_{xx}(x) \leq f_n(\beta u_{xx}(x)) \) since \( f_n(u) = F'(u) \) for \( u \in (-1 + \frac{1}{n}, 0) \). Hence, \( \beta u(x) \) is an upper solution for (4.1) when \( 0 < \beta < 1 - \frac{1}{n} \).

Lemma 12. Define \( u \) as in Lemma 11. Then \( u = u(x) \) is a lower solution of the problem (4.1).

Proof. Let \( \epsilon > 0 \) be given. First, we notice \( u\left(\frac{1}{2}\right) = 0 \), so \( u\left(\frac{1}{2}\right) = 0 \).

Also, recall \( u_x(0) = 0 \) as a solution of (1.1). Therefore, \( u_x(0) = 0 \).
Finally, note that

\[ 0 = \epsilon^2 u_{xx}(x) - F'(u(x)) \]

since \( u(x) \) is a solution of (1.1). By a calculus argument, we have that \( F' \) is concave down on the interval \((-1, -1 + \frac{1}{n})\). Hence, \( F'(u(x)) \geq f_n(u(x)) \) for all \( x \). Therefore,

\[
0 = \epsilon^2 u_{xx}(x) - F'(u(x)) \\
\leq \epsilon^2 u_{xx}(x) - f_n(u(x)).
\]

Then, \( \epsilon^2 u_{xx}(x) \geq f_n(u(x)) \), so \( \epsilon^2 u_{xx} \geq f_n(u) \). Therefore, \( u(x) \) is a lower solution of (4.1).

We want to also note that the interval \( [\underline{u}, \overline{u}] = [u(x), \beta u(x)] \) contains no solutions to (1.1) other than \( 
\underline{u} = u(x) \). With this in mind, we are now prepared to prove the following theorem for the symmetric one-node case.

**Theorem 1.1** (Hardeman, Robinson (2013)). Let \( u \) be the one-node solution of the time independent problem (1.1) satisfying \( u(\frac{1}{2}) = 0 \) and \( u(0) = -1 \). Assume \(-1 < u_0(x) \leq \beta u(x) \) for \( x \in [0, 1/2] \) and some \( 0 < \beta < 1 - \frac{1}{n} \). Assume \( u_0(x) = -u_0(1 - x) \). Then, the solution, \( h(x, t) \), of (1.2) satisfies \( \lim_{t \to \infty} h(x, t) = u(x) \) (uniformly).

**Proof.** Let \( h_n(x, t) \) be the solution of (4.1). Let \( \underline{U}_n(x, t) \) and \( \overline{U}_n(x, t) \) be solutions of (4.1) with initial conditions \( u_0(x) = \overline{u}(x) \) and \( u_0(x) = \underline{u}(x) \) respectively. By Lemma 11 and Lemma 12, \( \beta u(x) \) is an upper solution of (4.1) for \( 0 < \beta < 1 - \frac{1}{n} \) and \( u(x) \) is a lower solution of (4.1), respectively. By Theorem 3.1, we know that such a solution \( h_n(x, t) \) exists and is bounded between \( \underline{u} \) and \( \overline{u} \). Also, note that \( u(x) \leq \beta u(x) \) since \( u(x) \leq 0 \) for \( x \in [0, \frac{1}{2}] \). Thus, by Lemma 6,

\[
\underline{U}_n(x, t) \leq h_n(x, t) \leq \overline{U}_n(x, t)
\]

for all \( x \) and \( t \). Notice \( \underline{U}_n(x, t) \geq u(x) \). So, we have

\[
u(x) \leq h_n(x, t) \leq \overline{U}_n(x, t).
\]

Observe that for any given \( n \) and all \((x, t)\) such that \( \overline{U}_n(x, t) \geq -\frac{1}{n} + 1 \) we have

\[
f_n(\overline{U}_n(x, t)) = F'(\overline{U}_n(x, t)).
\]

By uniqueness, we have \( \overline{U}(x, t) = \overline{U}_n(x, t) \) for such
$(x,t)$. Similarly, $h_n(x,t) = h(x,t)$ for such $(x,t)$. Therefore,

$$u(x) \leq h(x,t) \leq U(x,t).$$

Now, by Lemma 5, we know $U(x,t)$ is non-increasing in $t$. Also, $U(x,t)$ is bounded below by the solution $u(x)$. By [1], we know that $U$ converges uniformly to a solution of (1.1) as $t \to \infty$, and we know that $u(x)$ is the only solution in $\{v : u \leq v \leq \bar{u}\}$. Therefore, $\lim_{t \to \infty} U(x,t) = u(x)$ uniformly. Hence, by the squeeze theorem, $\lim_{t \to \infty} h(x,t) = u(x)$ uniformly. 

4.2 One-Node Solution: Asymmetric Case

With the symmetric case proven, we are now ready to consider the asymmetric case. Recall, the asymmetric solutions occur when the solution crosses the $x$-axis at a point $x_0 \neq \frac{1}{2}$.

As with the symmetric case, we are interested in the symmetry of the initial conditions. We will start with an asymmetric initial condition which converges to the base of the trough in the limit. While we know that it will converge to a critical point, we need to prove that it does not converge to $\pm 1$. To do this, we first need to create a curve $\gamma : [0, 1] \to W^{1,2}(0,1)$ such that

- $\gamma(0) = -1$ and $\gamma(1) = 1$,
- $\gamma(s)$ is an asymmetric function for every $s \in (0,1)$, i.e. every point on the curve is asymmetric except for 0 and 1, and
\( J(\gamma(s)) < L_k \) for \( k \geq 2 \) where \( L_k \) denotes the energy of the \( k \)-node solutions and \( J(\gamma(s)) < J(0) \).

So, let \( u(x) \) be our nice symmetric solution of (1.1) with \( u(\frac{1}{2}) = 0 \). Let \( x_1 \) be the first \( x \)-value for which \( u(x) = -1 \). Now, we want to extend \( u(x) \) to the whole real line. So, let

\[
  u^*(x) = \begin{cases} 
    u(x) & 0 \leq x \leq 1 \\
    -1 & x < 0 \\
    1 & x > 1.
  \end{cases}
\]

Then, let \( \xi : \mathbb{R} \rightarrow C[0,1] \) such that \( \xi(s) = u^*(x - s) \). When \( s \geq 1 - x \), then \( \xi(s) \equiv -1 \) on \([0, 1]\). Note that when \( s \leq x_1 - 1 \), then \( \xi(s) \equiv 1 \) on \([0, 1]\). Note we have such a curve \( \xi \), but it is defined on \([x_1 - 1, 1 - x_1]\). We can easily redefine this curve on \([0, 1]\).

For \( s \in [-x_1, x_1] \), our \( \xi(s) \) gives the continuum of one-node solutions to (1.1), i.e. the bottom of the trough. Also, \( J(\xi(s)) \equiv L_1 \) where \( L_1 \) denotes the energy at the single node solutions of (1.1). It is known that \( L_1 < L_k \) for all \( k \geq 2 \) and \( L_1 < J(0) \). For details, see [1].

For \( s \not\in [-x_1, x_1] \), \( J(\xi(s)) \leq L_1 \). We know this to be true for the energy at \( \xi(s) \) by the following argument. Let us first consider some images.

The images above denote the shifting entailed in \( \xi \). We can see that we remove an interval \([0, y]\) of the solution which has positive slope and \( \int_0^y F(u) > 0 \) and replace it with a constant function of 1 on the interval \([1 - y, 1]\). We know that the two
integrands of $J_\epsilon$ equal zero on $[1-y, 1]$. Hence, $J_\epsilon(\xi(-x_1-y)) < J_\epsilon(\xi(-x_1))$. Consider the following for a more rigorous argument of this point,

$$ J_\epsilon(\xi(-x_1-y)) = \int_0^1 |(\xi(-x_1-y))'|^2 + \int_0^1 F(\xi(-x_1-y)) $$

$$ = \int_0^{1-y} |(\xi(-x_1-y))'|^2 + \int_y^{1-y} F(\xi(-x_1-y)) $$

$$ = \int_y^1 |(\xi(-x_1))'|^2 + \int_y^{1} F(\xi(-x_1)) $$

$$ < \int_0^1 |(\xi(-x_1))'|^2 + \int_0^1 F(\xi(-x_1)) \quad (4.3) $$

Now, let $x_2 \in (x_1, \frac{1}{2})$. Choose $\delta > 0$ so that $(x_2 - \delta, x_2 + \delta) \subseteq (x_1, \frac{1}{2})$. Let $\eta(x)$ be a smooth function $\nu(x) > 0$ for $x \in (x_2 - \delta, x_2 + \delta)$ and zero everywhere else such that $J_\epsilon(u^* + \eta) < L_2$ and $J_\epsilon(u^* + \eta) < J_\epsilon(0)$ where $L_2$ is the energy at the two-node solutions of (1.1) [1]. Let $\gamma(s) = u^*(x-s) + \nu(x-s)$.

Notice that $\gamma(s)$ is asymmetric for all $s \in (x_1-1, x_1-1-x_1)$. Also, we can make a similar argument to (4.3) to show that $J_\epsilon(\gamma(s)) < L_2$ for all $s$. So, $\gamma(s)$ satisfies the properties we listed above and we are ready to prove the following theorem of the asymmetric case.

**Theorem 1.2 (Hardeman, Robinson (2014)).** There is at least one asymmetric initial condition $u_0(x)$ such that the solution $h(x,t)$ of (1.2) satisfies $\lim_{t \to \infty} h(x,t) = v(x)$, where $v(x)$ is an asymmetric one-node solution of (1.1).

The proof of Theorem 1.2 follows as a result from the next lemma.

**Lemma 13.** Let $\gamma : [0, 1] \to W^{1,2}(0, 1)$ be a continuous function such that $\gamma(0) \equiv -1$ and $\gamma(1) \equiv 1$. For all $s \in (0, 1)$, let $L_k > J_\epsilon(\gamma(s)) > L_1$ for $k \geq 2$ and $J_\epsilon(\gamma(s)) < J_\epsilon(0)$. Then there exists $s \in [0, 1]$ such that $\gamma(s) \to u(x)$ where $u(x)$ is an asymmetric one-node solution of (1.1).

**Proof.** Let $\gamma : [0, 1] \to W^{1,2}(0, 1)$ be a continuous function such that $\gamma(0) \equiv -1$ and $\gamma(1) \equiv 1$. Let $A_1 = \{s \mid u_s(x,t) \to 1\}$ and $A_{-1} = \{s \mid u_s(x,t) \to -1\}$. Assume
\([0, 1] = A_1 \cup A_{-1}\). Note that \(A_1\) and \(A_{-1}\) are nonempty. By the Rellich-Kondrakov Theorem,

\[
\{v \in W^{1,2}(0, 1) : ||v - 1||_{W^1} < \frac{1}{2}\} \subset \{v \in W^{1,2}(0, 1) : v \geq \frac{1}{2}\}.
\]

Let \(s_0 \in A_1\). Since \(u_{s_0}(x, t) \to 1\) in \(W^{1,2}(0, 1)\) (and \(C^1[0, 1]\)), there exists \(T_0 > 0\) such that \(u_{s_0}(x, T_0) \in \{v \in W^{1,2}(0, 1) : ||v - 1||_{W^1} < \frac{1}{2}\}\). By Lemma 10, there exists \(\delta_0 > 0\) such that \(u_{s}(x, T_0) \in \{v \in W^1 : ||v - 1||_{W^1} < \frac{1}{2}\}\) for all \(s \in (s_0 - \delta, s_0 + \delta)\).

It follows that \((s_0 - \delta, s_0 + \delta) \subseteq A_1\). Hence, \(A_1\) is open. By a similar argument, \(A_{-1}\) is open. Since \(A_{-1} = A_1^c\), then \(A_1\) is open and closed. Also, \([0, 1]\) is the union of two open sets. By connectedness, \(A_1\) must be the entire set. Hence, all initial conditions go to 1, which contradicts the fact that \(A_{-1}\) is nonempty. So, we know that for some \(s\), \(\gamma(s)\) is an asymmetric initial condition such that \(u_s\) does not converge to \(\pm 1\). The only remaining alternative is convergence to a one-node solution.
Chapter 5: Conclusion

The goal of this paper was to prove stability properties of solutions to problem (1.2). In order to achieve this objective, we opened with a discussion of the Maximum Principle in chapter 2. The Maximum Principle was very important to proving results of upper and lower solutions in chapter 3. We also proved results of existence, uniqueness, and continuous dependence. The theorems and lemmas in Chapter 2 and 3 provided us with the tools we needed in order to complete a discussion of the symmetric and asymmetric cases. Employing these tools, we were able to prove stability properties for the symmetric solution as well as the asymmetric solution.

5.1 Open Problems

While we have accomplished the main goal of the paper, there are several open problems which relate to our results. The first of these problems is a complete description of the stable manifold. In fact, a complete description of the stable manifold for the one-node case by itself would be a helpful result.

Another open problem related to our results is the multi-node case. From [2], we know that there is a set of multiple-node solutions.

These are solutions which cross the $x$-axis multiple times. Currently, we hypothesize that the case for the symmetric multiple-node solutions will be similar to the symmetric case. For instance, consider the case when the solution crosses at $x = \frac{1}{4}$ and $x = \frac{3}{4}$.
The solution on the interval \((0, \frac{1}{4})\) is symmetric (via \(u(1 - x) = -u(x)\)) to the interval \((\frac{1}{4}, \frac{1}{2})\). Furthermore, the solution on the intervals \((\frac{1}{2}, \frac{3}{4})\) and \((\frac{3}{4}, 1)\) are also symmetric to the solution on the interval \((0, \frac{1}{4})\). Thus, it should follow from the proof of Theorem 1.1 with the exception that we define

\[
u_0(x) = \begin{cases} 
-u_0\left(\frac{1}{2} - x\right) & \frac{1}{4} < x \leq \frac{1}{2} \\
-u_0\left(x - \frac{1}{2}\right) & \frac{1}{2} < x \leq \frac{3}{4} \\
u_0(1 - x) & \frac{3}{4} < x \leq 1.
\end{cases}
\]

Notice that by similarly redefining \(u_0(x)\) on specific intervals, we should be able to make a similar argument for other symmetric multiple node solutions as well.

It would be nice if this was true. However, one aspect of the multiple-node case which must be accounted for is the local geometry of the functional \(J_\epsilon\) for the multiple node solutions. From [1], we know that as the number of nodes increases, the dimension of the trough does as well. So, for the two-node solutions, we would be considering a 3-dimensional trough which resembles a triangular plateau. This fact would be important to any proof of the multiple-node case.
Bibliography


Curriculum Vitae

Education

• M.A., Mathematics, Wake Forest University, expected May 2014

• B.S., Mathematics (major), English (minor), (summa cum laude with Superior Academic Achievement), University of Montevallo, May 2012

Research Experience

1. Master Thesis at Wake Forest University, Department of Mathematics – March 2013 to May 2014

   • Project: Partial Differential Equations, Stability
   • Advisor: Dr. Stephen Robinson
   • Studied the stability of the solutions to a phase transition model using upper and lower solutions as well as other methods.
   • Presentations:
     – Presentation at MAA Southeastern Section Meeting - March 2014
     – Presentation at University of Montevallo, Math Club Colloquium - March 2014
     – Presentation at the 9th Annual UNCG RMSC conference - November 2013
     * Won the outstanding student presentation award in the graduate student category

2. REU at North Carolina State University, Department of Mathematics– June 2011 to August 2011

   • Project: Cluster Analytics
• Mentors: Dr. Carl Meyer of NCSU; Ralph Abbey

• Analyzed different methods for cluster analysis, such as k-means, non-negative matrix factorization, and Reverse Simon-Ando, and applied them to well-known data sets.

• Presentations:
  – Presentation at the UT Undergraduate Math Conference – April 2012
  – Poster presentation at the 2012 Joint Mathematics Meeting in Boston – January 2012
  – Poster presentation at NCSU Undergraduate Research Symposium – August 2011

3. REU at University of Alabama at Birmingham, Department of Physics May 2010 to July 2010

• Project: Computation Physics

• Advisor: Dr. Joseph Harrison of UAB

• Employed MOPAC2009 to analyze the structural effects of introducing nitrogen atoms into carbon and boron-nitride nanotubes in terms of total energy in periodic and non-periodic systems.

• Presentations:
  – Poster Presentation at Montevallo Research Day – March 2011
  – Poster Presentation at UAB Summer Research Expo – July 2010
    * Poster won Honorable Mention

**Job Experience**

1. Wake Forest University, Student Athlete Services

• Tutor - May 2014
  – Tutored Calculus I and Calculus II.
2. Wake Forest University, Department of Mathematics

- Teaching Assistant - August 2012 to present
  - Led study sessions for: Mathematical Statistics and the entire Calculus sequence (I,II,III)
  - Graded for Calculus I, II, and III
- Math Center tutor - August 2012 to present
  - Tutored: Mathematical Statistics, Calculus I, Calculus II, Calculus III, and Differential Equations

3. Learning Enrichment Center

- October 2010 to May 2012
- Tutor undergraduates in:
  - Introductory German
  - Pre-Calculus, Finite Mathematics, Calculus I, and Calculus II
  - Physics: Introductory, Algebra-based, and Calculus-based
  - Western Civilization I & II

4. Private Tutor

- high school student in Physics - September 2011
- undergraduate student in Algebra-based Physics - September 2010 to December 2010
- high school student in Pre-Calculus - August 2010 to May 2010

Honors and Awards

- Senior Elite in Mathematics
- Jacquita Knight Hauserman Scholarship
- Majorie Murdock Math Fund
• Kappa Mu Epsilon Award

• Highest Honors: Freshmen, Sophomore, Junior, Senior

• Presidents List – Fall Semester 2008 to Spring Semester 2011

• Academic Recognition Scholarship – August 2008

** Clubs and Organizations**

1. Pi Mu Epsilon Mathematics Honor’s Society - initiated April 2013

2. AMS member since Fall 2012

3. Sigma Tau Delta English Honors Society initiated April 2011

4. Kappa Mu Epsilon Mathematics Honors Society initiated April 2010

5. Math Club
   • Secretary – August 2010 to May 2011
   • President – August 2011 to May 2012

**Papers**

1. Final Reports

2. Publication
   • Hardeman, H. Robinson, S. “On Stability of Solutions to a Phase Transition Model.” (Accepted for publication in *Topics from the 9th Annual UNCG Regional Mathematics and Statistics Conference May 2014*).