CURVE SKETCHING

BY

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Abstract

This paper serves as an extension of Eigencurves for Two-Parameter Sturm-Liouville Equations by Binding and Volkmer [1]. Facts about specific sequences of eigenvalues of linear equations developed in Bases and Comparison Results for Linear Elliptic Eigenproblems by Auchmuty [2] are used in order to characterize (λ, μ) eigencurves of the Sturm-Liouville ODE. A detailed discussion of those tools and their application is given.

Chapter 1: Introduction

A classical Sturm-Liouville Equation, named after Jacques Charles Francois Sturm (1803-1855) and Joseph Liouville (1809-1882), is a real second-order linear differential equation of the form

$$-\left(p(x)y'\right)' + q(x)y = \lambda r(x)y \tag{1.1}$$

where y is a function of x and the functions $p(x), q(x)$, and $r(x)$ as well as boundary conditions are specified.

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \alpha_1^2 + \alpha_2^2 > 0$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1^2 + \beta_2^2 > 0$$

The value of λ is not specified in the equation. Finding the values of λ for which there exists a non-trivial solution y satisfying the given boundary conditions is part of the problem called the Sturm-Liouville problem. Values of λ, when they exist, are called the eigenvalues of the boundary value problem defined by (1.1). The corresponding solutions, y, for a given λ, are called the eigenfunctions of this problem.

We study the two-parameter Sturm-Liouville problem

$$-(p(x)y')' + q(x)y = (\lambda r(x) + \mu)y, \qquad 0 \leq x \leq 1$$

$$\cos(\alpha)y(0) - \sin(\alpha)p(0)y'(0) = 0,$$

$$\cos(\beta)y(1) - \sin(\beta)p(1)y'(1) = 0.$$

In [1], the authors show that the set of $(\lambda, \mu) \in \mathbb{R}^2$ for which there exists a nontrivial y satisfying the differential equation and boundary conditions turns out to be a countable union of graphs of analytic functions. Our focus is on these graphs, which are commonly called eigencurves. Shown in [1] is that the eigencurves are smooth and in particular derive a formula for their slopes which provides a method of finding regions of increase and decrease. This leads to finding a second derivative and classifying the concavity of eigencurves at local extrema. Also shown in [1] is that the intersection of any line with the m^{th} eigencurve can have at most $2m$ intersections. The culmination of the facts allow for plotting of eigencurves for specific examples.

In this paper all of the above properties of eigencurves are proved using methods developed in [2]. The method of proving existence of solutions involves fixing a parameter λ, reducing the problem to the well known single parameter Sturm-Liouville problem. We offer alternative proofs of the theory developed in [2], and finally consider a classic example developed in [1] and explicitly plot eigencurves.

Chapter 2: Bases and Comparisons (Alternative proofs to [2])

This section describes alternative proofs of the construction, and some properties, of specific orthogonal sequences of eigenvectors of the pair (a, m) of continuous symmetric bilinear forms on a real Hilbert space V. This theory is developed in [2]. We present a subset of that material without reference to the sub-proximal gradient. These facts will be used to prove the existence of solutions to the Sturm-Liouville 2 parameter ODE. The inner product and norm on V are denoted $\langle ., . \rangle_V$ and $\| \ . \ \|_V$ respectively. Our interest is in finding eigenpairs $(\mu, e) \in \mathbb{R} \times V$ such that

$$a(e, v) = \mu m(e, v) \text{ for all } v \in V$$

has a non trivial solution $e \in V$. This will be called the (a, m) *eigenproblem* (EV) where $a : V \times V \longrightarrow \mathbb{R}$ is a continuous symmetric bilinear form which is an inner product on V and $m : V \times V \to \mathbb{R}$ is a weakly continuous symmetric bilinear form on V.

Define

$$A(u) := a(u, u) \text{ and } M(u) := m(u, u).$$

The operator $m(., .)$ is weakly continuous if $\langle u_n, v \rangle \to \langle u, v \rangle \ \ \forall v \in V$ implies $M(u_n) \to M(u)$. The results about the eigenproblem for (a, m) hold subject to the following conditions.

(A1): $a(., .)$ is a continuous, symmetric bilinear form that is V-*coercive*. That is there are $0 < k_0 \leq k_1 < \infty$ such that

$$k_0 \parallel u \parallel_V^2 \leq A(u) \leq k_1 \parallel u \parallel_V^2 \ \ \textit{for all } u \in V$$

(A2): $m(.,.)$ is a weakly continuous symmetric bilinear form on V.

(A3): $M(u) > 0$ for some $u \in V$

$(A1)$ implies $a(.,.)$ is an equivalent inner product on V. For our purposes we will think of $a(.,.)$ as the inner product on V. We construct the eigenpairs (μ, e) by maximizing $m(.,.)$ over a closed unit ball with respect to the $a(.,.)$ inner product. Let $C_1 = \{u \in V : a(u,u) \leq 1\}$. The next few lemmas establish this set is convex and some properties of convex sets in a Hilbert space.

Lemma 2.0.1. *C_1 is convex.*

Proof. Suppose $u, v \in C_1$. Will show $\alpha u + (1-\alpha)v \in C_1$ where $0 \leq \alpha \leq 1$. Let $(1-\alpha) = \beta$. Note $\alpha + \beta = 1$. Then,

$$\begin{aligned} a(\alpha u + \beta v, \alpha u + \beta v) &= \alpha^2 a(u,u) + 2\alpha\beta a(u,v) + \beta^2 a(v,v) \\ &\leq \alpha^2 + 2\alpha\beta a(u,v) + \beta^2 \text{ since } u, v \in C_1 \\ &\leq \alpha^2 + 2\alpha\beta + \beta^2 \text{ by Cauchy Schwartz } u, v \in C_1 \\ &= (\alpha+\beta)^2 = 1 \end{aligned}$$

□

We find the first eigenpair by maximizing $m(.,.)$ over C_1. We find the subsequent eigenpairs by maximizing over unit balls in linear subspaces of V.

Lemma 2.0.2. *The intersection of convex sets is convex and any linear subspace of V is convex.*

Proof. To see the first fact, let U, C be convex. Let $x, y \in U \cap C$. Then since $U \cap C \subset U$, $x, y \in U$. Then the line connecting x, y lies entirely in $U \cap C$. For the second fact, let W be a closed linear subspace of V. Let x, y be two points in W. Since W a subspace $ax + by \in W$ for all $a, b \in \mathbb{R}$. Then $\alpha x + (1 - \alpha)y \in W$ for all $\alpha \in [0, 1]$, and W is convex. □

Lemma 2.0.3. *C_1 is weakly compact.*

Proof. By lemma 3.0.21 we may assume $V = \ell^2$. let $(u_n) \subset C_1$. Then (u_n) is bounded since $a(u_n, u_n) \leq 1$ for all n. Without loss of generality assume (u_n) is component-wise convergent to $u \in \ell^2$, i.e $u(i) = \lim_{n\to\infty} u_n(i)$ for every i. We must show $u \in \ell^2$.

$$\sum_{i=1}^{N} u(i)^2 = \lim_{n\to\infty} \sum_{i=1}^{N} u_n(i)^2 \text{ but } \sum_{i=1}^{N} u_n(i)^2 \leq \sum_{i=1}^{\infty} u_n(i)^2 \leq M$$

where M is the bound on u_n. Then , $\sum_{i=1}^{N} u_n(i)^2 \leq M$ for all N. Therefore,

$$\sum_{i=1}^{\infty} u^2(i) \leq M \text{ and } u \in \ell^2$$

Since $(u_n) \subset C_1$, it follows that $u \in C_1$.

Let $v \in \ell^2$. We must show $< u_n, v > \longrightarrow < u, v >$. Consider,

$$\begin{aligned} | < u_n - u, v > | &= |\sum_{i=1}^{\infty}(u_n(i) - u(i))v(i)| \\ &= |\sum_{i=1}^{N}(u_n(i) - u(i))v(i) + \sum_{i=N+1}^{\infty}(u_n(i) - u(i))v(i)| \\ &\leq \sum_{i=1}^{N}|(u_n(i) - u(i))v(i)| + \sum_{i=N+1}^{\infty}|(u_n(i) - u(i))v(i)| \end{aligned}$$

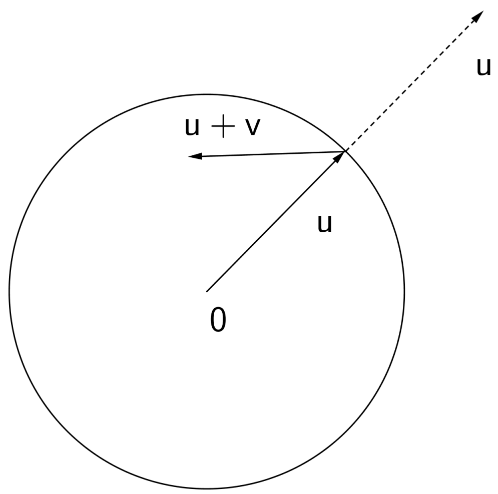
Let $\epsilon > 0$ be given. Choose N such that $\left(\sum_{i=N+1}^{\infty} v^2(i)\right)^{\frac{1}{2}} < \frac{\epsilon}{2M}$. Then,

$$\sum_{i=N+1}^{\infty} |u_n(i) - u(i)||v(i)| \leq \left(\sum_{i=N+1}^{\infty} |u_n(i) - u(i)|^2\right)^{\frac{1}{2}} \left(\sum_{i=N+1}^{\infty} |v(k)|^2\right)^{\frac{1}{2}} \text{ by Hölder}$$
$$\leq 2M\frac{\epsilon}{2M}$$
$$= \epsilon$$

Therefore, $| < u_n - u, v > | \leq |\sum_{i=1}^{N}(u_n(i) - u(i))v(i)| + \epsilon$. Letting $n \to \infty$. We get $\limsup | < u_n - u, v > | \leq \epsilon$ for all N. Letting $\epsilon \to 0$ implies $\limsup | < u_n - u, v > | = 0$. Hence, $< u_n, v > \to < u, v >$ for all $v \in V$. □

Since $a(.,.)$ is an inner product we may measure angles. It turns out the maximum of M over C_1 occurs on the boundary of C_1. We get the following useful properties of unit balls in a Hilbert space.

Lemma 2.0.4. *If* $a(u, u) = 1$ *then* $a(u, v) \leq 0$ *for all* $v \in V$ *such that* $u + v \in C_1$



The angle between u and v is greater than $\frac{\pi}{2}$ and hence $a(u, v) \leq 0$.

Proof. Since $u + v \in C_1$ it follows that $0 \leq a(u + v, u + v) \leq 1$.
But $a(u + v, u + v) = a(u, u) + 2a(u, v) + a(v, v)$. Thus

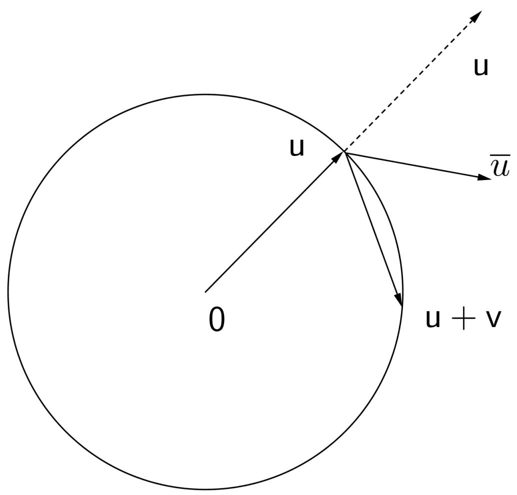
$$2a(u, v) + a(v, v) \leq 0 \text{ and hence } a(u, v) \leq \frac{-1}{2} a(v, v) \leq 0$$

for all $v \in V$ such that $u + v \in C_1$. Since $a(.,.)$ an inner product, $a(v, v) > 0$ for all $v \in V$ such that $u + v \in C_1$ and the claim follows.

□

Note, This proof holds for all nonnegative scalar multiples of u i.e if $a(u, v) \leq 0$ for all $v \in V$ such that $u + v \in C_1$, and if $\gamma \geq 0$, then $a(\gamma u, v) \leq 0$ for all $v \in V$ such that $u + v \in C_1$.

Lemma 2.0.5. *If $\overline{u} \in V$ such that $a(\overline{u}, v) \leq 0$ for all v such that $u + v \in C_1$ for some $u \in \partial C_1$, then $\overline{u} = \gamma u$ for some $\gamma \geq 0$.*



Idea: If $\overline{u}$ is not a multiple of u, then we can find $u + v \in \partial C_1$ such that the angle between $\overline{u}$ and v is less than $\frac{\pi}{2}$.

Proof. (Contradiction) Suppose that $a(u,u) = 1$ and $\overline{u}$ is not a multiple of u. Then it is possible to write $\overline{u} = \alpha u + \beta u^{\perp}$, where $u^{\perp}$ is a unit vector a-orthogonal to u and $\beta > 0$. Let $v = -\epsilon u + \delta u^{\perp}$, where $\epsilon, \delta > 0$ such that $(1-\epsilon)^2 + \delta^2 = 1$. Then $u + v = (1-\epsilon)u + \delta u^{\perp}$ satisfies

$$\begin{aligned} a(u+v, u+v) &= a((1-\epsilon)u + \delta u^{\perp}, (1-\epsilon)u + \delta u^{\perp}) \\ &= (1-\epsilon)^2 a(u,u) + 2(1-\epsilon)\delta a(u, u^{\perp}) + \delta^2 a(u^{\perp}, u^{\perp}) \\ &= (1-\epsilon)^2 + \delta^2 \\ &= 1 \end{aligned}$$

Therefore $u + v \in C_1$. By assumption $a(\overline{u}, v) \leq 0$. However,

$$\begin{aligned} a(\overline{u}, v) &= a(\alpha u + \beta u^{\perp}, -\epsilon u + \delta u^{\perp}) \\ &= -\alpha\epsilon + \beta\delta \quad \text{ since } a(u,u) = a(u^{\perp}, u^{\perp}) = 1 \\ &= -\alpha\epsilon + \beta\sqrt{1-(1-\epsilon)^2} \\ &= \sqrt{\epsilon}(-\alpha\sqrt{\epsilon} + \beta\sqrt{2-\epsilon}) \end{aligned}$$

Note, $\lim_{\epsilon \to 0} -\alpha\sqrt{\epsilon} + \beta\sqrt{2-\epsilon} = \sqrt{2}\beta$. Therefore $(\sqrt{\epsilon}(-\alpha\sqrt{\epsilon} + \beta\sqrt{2-\epsilon}) > 0$ for small ϵ, which is a contradiction. Hence $\overline{u} = \gamma u$ for some $\gamma \geq 0$ □

The preceding lemmas lead to the following useful theorem.

Theorem 2.0.6. *Let $u \in V$ such that $a(u,u) = 1$. Then, $a(\overline{u}, v) \leq 0$ for all $v \in V$ such that $u + v \in C_1$ if and only if $\overline{u} = \gamma u$ for some $\gamma \geq 0$.*

We generalize the analysis above to the following situation. Suppose that V can be decomposed into orthogonal subspaces. Let W be a subspace of V. Let

$$W^\perp := \{v \in V : a(v,w) = 0 \quad \forall w \in W\} \text{ and } C_w = \{w \in W : a(w,w) \leq 1\}$$

As before let $u \in C_W$ with $a(u,u) = 1$ or, similarly, $u \in \partial C_W$ in W. Assume that $a(\overline{u}, v) \leq 0$ for all $v \in V$ such that $u + v \in C_W$. We may apply the previous theorem to the part of $\overline{u}$ that lies in W. Write $\overline{u} = \overline{w} + \overline{w}^\perp$ where $\overline{w} \in W$ and $\overline{w}^\perp \in W^\perp$. If $v \in W$ then,

$$a(\overline{u}, v) = a(\overline{w} + \overline{w}^\perp, v) = a(\overline{w}, v) + a(\overline{w}^\perp, v) = a(\overline{w}, v) \tag{2.1}$$

Thus, $a(\overline{w}, v) \leq 0$ for all $v \in W$ such that $u + v \in C_W$. Therefore, $\overline{w} = \gamma u$ some $\gamma \geq 0$ and therefore $\overline{u} = \gamma u + w^\perp$ for some $\gamma \geq 0$ and $\overline{w}^\perp \in W^\perp$.

Recall, $M(u) = m(u,u)$. We will maximize $M(u)$ by standard tools of calculus. The next lemmas establish that the weakly continuous $m(.,.)$ is differentiable. The Gâteaux derivative is a generalization of the concept of directional derivative in differential calculus. Let X and Y be Banach spaces and let $f : X \to Y$ be a function between them. f is said to be Gâteaux differentiable if there exists an operator $T_x : X \to Y$ such that, for all v in X,

$$\lim_{t \to 0} \frac{1}{t}(f(x + tv)) - f(x)) = T_x(v)$$

Lemma 2.0.7. *M is Gâteaux-differentiable with $Dm(u) \cdot v = 2m(u,v)$*

Proof.

$$\begin{aligned} M(u + tw) - M(u) &= m(u + tw, u + tw) - m(u,u) \\ &= m(u,u) + tm(v,w) + tm(w,v) + t^2 m(w,w) - m(u,u) \\ &= 2tm(u,w) + t^2 m(w,w) \end{aligned}$$

Therefore $\lim\limits_{t\to 0^+} \frac{1}{t}\left(M(u+tw)-m(u)\right) = 2m(u,w)$ for all $w \in V$. $\square$

Hence M is differentiable with $DM(u)\cdot v = 2m(u,v)$ for all $v \in V$. Let $Lv = 2m(u,v)$. Then $L : V \to R$ is a continuous linear functional. It follows that M is Fréchet differentaible.

Lemma 2.0.8. *M is Fŕechet Differentiable.*

Proof. From above, The Gâteaux derivative $DM(u)\cdot v = 2m(u,v)$. Then M is Fréchet differentiable if

$$m(u+v,u+v) = m(u,u) + 2m(u,v) + o(\| v \|)$$

$$\text{Where} \qquad \lim_{v\to 0} \frac{o(\| v \|)}{\| v \|} = 0$$

Note,

$$o(\| v \|) = m(u+v,u+v) - m(u,u) - 2m(u,v) = m(v,v)$$

Since M us weakly continuous it follows that M is continuous. Then $\exists C > 0$ such that $|m(v,v)| \leq C \| v \|^2$. Thus

$$\lim_{v\to 0} \frac{m(v,v)}{\| v \|} \to 0$$

$\square$

Theorem 2.0.9 (Riesz Representation Theorem)**.** *If $L : V \to \mathbb{R}$ is a bounded linear functional, then there is a unique vector $v_0 \in V$ such that $L(v) = \langle v_0, v\rangle$ for all $v \in V$.*

Proof. Let $K = \ker L = L^{-1}(\{0\})$. Since L is bounded it follows that L is continuous. If $v, w \in K$ then $L(v + w) = L(v) + L(w) = 0 + 0 = 0$. Hence K is a closed linear subspace of V. We may assume L is non trivial, i.e $K \neq V$, otherwise $v_0 = 0$. Thus $K^{\perp} \neq \{0\}$. Then there is a vector $f_0 \in K^{\perp}$ such that $L(f_0) = 1$. Now if $h \in V$ and $L(h) = a$ then $L(h - af_0) = L(h) - a = 0$. So $h - L(h)f_0 \in K$. Thus,

$$\begin{aligned} 0 &= \langle h - L(h)f_0, f_0 \rangle \\ &= \langle h, f_0 \rangle - L(h)\langle f_0, f_0 \rangle \\ &= \langle h, f_0 \rangle - L(h) \parallel f_0 \parallel_H^2 \end{aligned}$$

If $v_0 = f_0 \parallel f_0 \parallel^{-2}$, then $L(v) = \langle v_0, v \rangle$ for all $v \in V$. Thus we have shown the existence of v_0. To prove uniqueness: If $\overline{v} \in V$ such that $\langle v_0, v \rangle = \langle \overline{v}, v \rangle$ for all $v \in V$, then $(v_0 - \overline{v}) \perp v$ for all $v \in V$ and so $v_0 - \overline{v} = 0$ or $v_0 = \overline{v}$. □

Recall that $DM(u) \cdot v = 2m(u, v)$ is a bounded linear functional on V. Since $a(.,.)$ is an inner product on V, it follows from the Riesz Representation theorem that there is an element $v_0 \in V$ such that $DM(u) \cdot v = a(v_0, v)$ for all $v \in V$. We can adopt the notation $\nabla M(u)$ for the vector v_0. Hence $DM(u) \cdot v = a(\nabla M(u), v)$ for all $v \in V$

In order to maximize M we first need to know the weakly continuous M actually attains a maximum on C_1.

Lemma 2.0.10. *M achieves an absolute maximum on $C_1 = \{v \in V : a(v, v) \leq 1\}$*

Proof. Let $s = \sup\{M(v) : v \in C_1\}$. Let $(v_n) \subset C_1$ be a maximizing sequence i.e $\lim_{n\to\infty} M(v_n) = s$. Since C_1 is weakly compact (v_n) has a subsequence (v_{n_k}) such that $v_{n_k} \rightharpoonup v$ for some $v \in C_1$. Note, $\lim_{k\to\infty} M(v_{n_k}) = s$ and by weak continuity of M, $\lim_{k\to\infty} M(v_{n_k}) = M(v)$. Therefore $M(v) = s < \infty$. So M achieves its supremum and s is finite. □

Lemma 2.0.11. *M achieves its absolute maximum on the boundary of C_1, i.e at a point $u \in C_1$ such that $a(u,u) = 1$.*

Proof. (Contradiction) Suppose that M achieves a maximum at u such that $a(u,u) < 1$. By hypothesis $M(u) > 0$ and u is clearly nonzero hence $a(u,u) > 0$. Then $\tilde{u} = \frac{u}{\sqrt{a(u,u)}}$ satisfies $a(\tilde{u},\tilde{u}) = 1$ i.e $\tilde{u} \in C_1$ and $M(\tilde{u}) = \frac{M(u)}{a(u,u)} > M(u)$. Which contradicts that $M(u)$ is the maximum.

□

Lemma 2.0.12. *If M achieves its maximum on C_1 at $u \in \partial C_1$, then $\bigtriangledown M(u) = \gamma u$ for some $\gamma > 0$*

Proof. Let $v \in V$ such that $u + v \in C_1$. By convexity of C_1, $u + tv \in C_1$ for all $t \in [0,1]$. Since $M(u)$ is a max $M(u+tv) - M(u) \leq 0$ for all $t \in [0,1]$. Hence, $DM(u) \cdot v = \lim_{t \to 0^+} \frac{1}{t}\left(M(u+tv) - M(u)\right) \leq 0$. Hence $a(\nabla M(u), v) \leq 0$ for all $v \in V$ and thus for all $v \in V$ such that $u + v \in C_1$. By Theorem 2.0.6, $\nabla M(u) = \gamma u$ for some $\gamma \geq 0$. Hence, $2m(u,v) = a(\gamma u, v)$ for all $v \in V$. Therefore $2m(u,v) = \gamma a(u,v)$ for all $v \in V$ and if $v = u$ then $2M(u) = \gamma$. Thus, $\gamma > 0$. □

We have found the first or principal solution of EV. Let $e_1 = u$ and $\mu_1 = \frac{2}{\gamma}$. Hence,

$$a(e_1, v) = \mu_1 m(e_1, v) \quad \forall v \in V$$

Lemma 2.0.13.

$$\frac{1}{\mu_1} = \sup_{u \in C_1} M(u)$$

Proof. From above we have that e_1 maximizes M over C_1 and $a(e_1, e_1) = 1$. Then,

$$m(e_1, e_1) = \frac{\gamma}{2} a(e_1, e_1) = \frac{\gamma}{2} = \frac{1}{\mu_1}$$

□

In order to construct the second eigenpair (μ_2, e_2) we reduce the dimension of the problem by one and apply the same maximization argument. We reduce the dimension by maximizing over the $a(.,.)$ unit ball of the subspace orthogonal to the principal eigenvalue e_1. Let $V_1 := [e_1]$ the span of e_1, $W_1 := \{v \in V : a(e_1, v) = 0\}$, and $C_2 := \{w \in W_1 : a(w, w) \leq 1\}$. Assume that $M(w) > 0$ for some $w \in W_1$. All previous assumptions hold in that $a(.,.)$ is an inner product on W_1, $m(.,.)$ is a symmetric and weakly continuous bilinear form with $M(u) > 0$ for some $u \in W_1$. Hence there is an $e_2 \in \partial C_2$ such that $a(e_2, w) = \mu_2 m(e_2, w)$ for all $w \in W_1$ with $\mu_2 \in \mathbb{R}$ and $\mu_2 > 0$. What is left to show is this holds for all $v \in V$.

Lemma 2.0.14. *$a(e_2, v) = \mu_2 m(e_2, v)$ for all $v \in V$.*

Proof. From previous derivative computations, see lemma 2.0.7 and 2.09, we know that $a(\bigtriangledown M(e_2), v) = 2m(e_2, v)$ for all $v \in V$. Break the vector $\bigtriangledown M(e_2)$ into it's components in V_1 and W_1 by writing $\bigtriangledown M(e_2) = \overline{v} + \overline{w}$. Since V_1 is the span of $\{e_1\}$ it must be that $\overline{v} = \alpha e_1$ for some $\alpha \in \mathbb{R}$, i.e $\bigtriangledown M(e_2) = \alpha e_1 + \overline{w}$. Choose $v = e_1$ in the equation above to get $a(\alpha e_1 + \overline{w}, e_1) = 2m(e_2, e_1)$. Using orthogonality and the fact that $a(e_1, e_1) = 1$. we get that

$$
\begin{aligned}
a(\bigtriangledown M(e_2), e_1) &= a(\alpha e_1 + \overline{w}, e_1) \\
&= \alpha a(e_1, e_1) + a(\overline{w}, e_1) \\
&= \alpha \\
&= 2m(e_2, e_1)
\end{aligned}
$$

Thus, $2m(e_2, e_1) = 2m(e_1, e_2) = \frac{1}{\mu_1} a(e_1, e_2) = 0$ and $\alpha = 0$. Thus we have $a(\overline{w}, v) = 2m(e_2, v) \quad \forall v \in V$. Therefore $a(\overline{w}, w) = 2m(e_2, w) = \frac{2}{\mu_2} a(e_2, w)$ for all

$w \in W_1$ and so $a(\overline{w} - \frac{2}{\mu_2}e_2, \overline{w} - \frac{2}{\mu_2}e_2) = 0$. Hence $\overline{w} = \frac{2}{\mu_2}e_2$, and $\bigtriangledown M(e_2) = \frac{2}{\mu_2}e_2$. Recall $a(\bigtriangledown M(e_2), v) = 2m(e_2, v)$. Note, $a(\bigtriangledown M(e_2), v) = 2m(e_2, v) = a(\frac{2}{\mu_2}e_2, v) = \frac{2}{\mu_2}a(e_2, v)$. Hence, $\frac{2}{\mu_2}a(e_2, v) = 2m(e_2, v)$ or $a(e_2, v) = \mu_2 m(e_2, v)$ for all $v \in V$. □

We have found the second solution to EV. It follows that given a sequence of eigenpairs, we can always find the next eigenpair as long as M is still positive.

Lemma 2.0.15. *Suppose we have found $n-1$ eigenpairs $(\mu_1, e_1), \ldots (\mu_{n-1}, e_{n-1})$. Let $V_n = [e_1, \ldots, e_{n-1}]$, $W_n = \{w \in V : a(w, e_i) = 0$ for $i = 1, \ldots, n-1\}$, and $C_n = \{w \in W_n : a(w, w) \leq 1\}$. Then there is a $e_n \in \partial C_n$and $M(u) > 0$ for some $u \in \partial W_n$ such that*

$$a(e_n, v) = \mu_n m(e_n, v) \textit{ for all } v \in V$$

Proof. As before $(A1), (A2)$ are met. Assuming $(A3)$ is met thus we may immediately conclude there is an $e_n \in \partial C_n$ such that

$$a(e_n, w) = \mu_n m(e_n, w) \text{ for all } w \in W_n$$

What is left to show is that this holds for all $v \in V$. Notice for any v we may write $v = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_{n-1} e_{n-1} + w$. Then,

$$\begin{aligned} a(e_n, v) &= a(e_n, \alpha e_1 + \beta e_2 +, \ldots, +\gamma e_{n-1} + w) \\ &= a(e_n, \alpha_1 e_1) + a(e_n, \alpha_2 e_2) +, \ldots, + a(e_n, \alpha_{n-1} e_{n-1}) + a(e_n, w) \\ &= a(e_n, w) \end{aligned}$$

Similarly using bilinearity of $m(.,.)$.

$$\begin{aligned} m(e_n, v) &= m(e_n, \alpha_1 e_1 + \alpha_2 e_2 +, \ldots, +\alpha_1 n - 1 e_{n-1} + w) \\ &= m(e_n, \alpha_1 e_1) + m(e_n, \alpha_2 e_2) +, \ldots, + m(e_n, \alpha_{n-1} e_{n-1}) + m(e_n, w) \\ &= m(e_n, w) \end{aligned}$$

Since any v can be written as linear multiples of vectors in C_n and W_n it follows that

$$a(e_n, v) = \mu_n m(e_n, v) \text{ for all } v \in V$$

□

This process of maximizing M on C_1 leads to some useful inequalities and comparisons between eigenvalues of (a, m). We focus here on the existence of and properties associated with a sequence of eigenvalues.

Lemma 2.0.16. *Let e_1 be a maximizer of M, the associated quadratic form of $m(.,.)$. Then,*

$$A(u) \geq \mu_1 M(u) \textit{ for all } u \in V$$

Proof. Lemma 2.0.14 shows $\frac{1}{\mu_1} = M(e_1)$. If $u \in C_1$, then $\frac{1}{\mu_1} = M(e_1) \geq M(u)$. Let $u \in V$. Then for $\tilde{u} = \frac{u}{\sqrt{a(u,u)}}$, we have $a(\tilde{u}, \tilde{u}) \leq 1$. Then,

$$\frac{1}{\mu_1} \geq M(\tilde{u}) = \frac{1}{a(u,u)} M(u)$$

$$\text{and thus } A(u) \geq \mu_1 M(u)$$

□

The preceding maximization problem may be iterated to generate an a-orthonormal sequence of eigenvectors of (a, m). Suppose we have found $k-1$ a-orthonormal eigenvectors $\{e_1, e_2, \ldots, e_k\}$. Then we have the following property of the eigenvalues associated.

Lemma 2.0.17. *Assume $(A1), (A2)$ hold and that $(A3)$ holds for all nontrivial $u \in V$. Suppose we have k orthonormal eigenvectors of (a, m) corresponding to the first k eigenvalues of (a, m). Then*

$$0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k$$

Moreover

$$A(u) \geq \mu_k M(u) \text{ for all } u \in W_k$$

Proof. Clearly $0 < \mu_k \leq \mu_{k+1}$ for all k since the domain of the last maximization is a subset of the preceding domain. By homogeneity, as in lemma 2.0.14 the maximizer lies in W_k and thus the claim holds. □

It is a well known fact, for which we do not offer proof, that $\lim_{n\to\infty} \mu_n = \infty$.

Chapter 3: Weak Convergence

This section describes some interesting properties of Hilbert spaces, i.e. weak convergence of a sequence in a Hilbert space. We explore the relationship between convergence and weak convergence and establish the equivalent of Bolzano-Weierstrass Theorem for weak convergence in an infinite dimensional Hilbert space. Interestingly, in many cases, it suffices to study ℓ^2 in order to prove a general claim, and we prove this fact. The following will be useful for establishing the necessary assumptions to guarantee we can solve an eigenvalue problem derived from the Sturm-Liouville ODE.

Definition 3.0.1. *A sequence $(u_n) \subset V$ converges weakly to $u \in V$, $u_n \rightharpoonup u$ if*

$$\lim_{n\to\infty} \langle u_n, v\rangle = \langle u, v\rangle \quad \textit{for all } v \in V$$

In a finite dimensional vector space convergence and weak convergence are equivalent. The following lemmas exhibit the difference of the two in an infinite dimensional case.

Example 3.0.1. *Consider the sequence of standard basis vectors $(e_i) \subset \ell^2$. This sequence is not Cauchy and thus does not converge. To see that $e_n \rightharpoonup 0$ let $v \in \ell^2$ where $v(i)$ represents the i^{th} component of v. Then $\langle e_n, v\rangle = v(n)$. Since $\sum_{i=0}^{\infty} v^2(i) < \infty$ it follows that $\lim_{n\to\infty} v(n) = 0$. Hence $\lim_{n\to\infty} \langle u_n, v\rangle = 0$.*

Lemma 3.0.18. *Convergence in V implies weak convergence.*

Proof. suppose $(v_n) \subseteq V$ such that $v_n \to v$. Let $u \in V$. Consider,

$$|\langle v_n, u\rangle - \langle v, u\rangle| = |\langle v_n - v, u\rangle| \leq \| v_n - v \| \| u \| \text{ by Cauchy-Schwartz.}$$

For every fixed u, $\| v_n - v \| \| u \| \to 0$, thus $\langle v_n, u \rangle \to \langle v, u \rangle$.

$\square$

Lemma 3.0.19. *Let* $(u_n) \subset V$ *Then,*

$$u_n \rightharpoonup u \iff L(u_n) \to L(u) \textit{ for all } L \in V^*$$

Proof. Suppose $L(u_n) \longrightarrow L(u)$ for all $L \in V^*$. Let $L(v) = \langle v, w \rangle$ for some $w \in V$. Hence $\langle u_n, w \rangle \longrightarrow \langle v, w \rangle$. Since w arbitrary we have $\langle u_n, w \rangle \longrightarrow \langle u, w \rangle$ for all $w \in V$.

Suppose $\langle u_n, v \rangle \longrightarrow \langle u, v \rangle$ for all $v \in V$. Let $L \in V^*$. By the Riesz Representation Theorem, there is a unique $w \in V$ such that $L(v) = \langle v, w \rangle$ for all $v \in V$. Hence $L(u_n) = \langle u_n, w \rangle \longrightarrow \langle u, w \rangle = L(u)$. $\square$

Lemma 3.0.20. *Bounded sequences in* ℓ^2 *have weakly converging subsequences.*

Proof. Let $\epsilon > 0$ Without loss of generality we may take $(u_n) \subset \ell^2$. It follows that (u_n) has a subsequence, which we will call (u_n), that converges component-wise to some $u \in \ell^2$. We will show $\langle u_n, v \rangle \to \langle u, v \rangle$ for all $v \in \ell^2$. Let $v \in \ell^2$. Then,

$$|\langle u_n, v \rangle| = |\sum_{i=1}^{\infty} u_n(i)v(i)| \leq \sum_{i=1}^{N} |u_n(i)v(i)| + \sum_{i=N+1}^{\infty} |u_n(i)v(i)|$$

Let M be the bound on (u_n). Pick N large so that $(\sum_{i=N+1} v(i)^2)^{\frac{1}{2}} < \frac{\epsilon}{M}$. Thus,

$$\langle u_n, v \rangle = \sum_{i=1}^{N} |u_n(i)v(i)| + \epsilon \text{ for all N large}$$

Note, $\lim \sum_{i=1}^{N} u_n(i)v(i) = \sum_{i=1}^{N} u(i)v(i)$ for all N by component wise convergence. Let $N \to \infty$. Therefore $\limsup \langle u_n, v \rangle \leq \langle u, v \rangle + \epsilon$. Similarly, $\liminf \langle u_n, v \rangle \geq \langle u, v \rangle - \epsilon$. Letting $\epsilon \to 0$ finishes the claim. $\square$

Lemma 3.0.21. *Let (u_n) be bounded in a Hilbert space V. We can construct a closed subspace of V which is isomorphic to ℓ^2 and contains (u_n).*

Proof. Begin by producing an orthonormal set (e_n) by applying the Grahm-Schmidt process to (u_n). Then let, $H := \{\sum_{i=1}^{\infty} \alpha_i e_i \, : \, \sum_{i=1}^{\infty} |\alpha_i|^2 < \infty\}$. It follows that H is complete and that H is isomorphic to ℓ^2. To see the second fact, define $\phi : \ell^2 \longrightarrow H$ such that $\phi(x) = \sum_{i=1}^{\infty} x(i)e_i$ or in other words $x = (x_1, x_2, x_3, \ldots) \mapsto \sum_{i=1}^{\infty} x(i)e_i$. Suppose $\phi(x) = 0$. Then, $\sum_{i=1}^{\infty} x_i e_i = 0$. Which is true if and only if $x_i e_i = 0$ for all i. Thus $\ker \phi = \{0\}$ so ϕ is one to one. ϕ is clearly linear by properties of sums. Let $\sum_{i=1}^{\infty} \beta_i e_i \in H$. Then, $\sum_{i=1}^{\infty} |\beta|^2 < \infty$. Therefore, $(\beta_i) \in \ell^2$ such that $\phi(\beta_i) = \sum_{i=1}^{\infty} \beta_i e_i$. Thus ϕ is a linear isometry between H and ℓ^2 and for $x \in H$ we have that $\| \, x \, \|_V = \sum_{i=1}^{\infty} \alpha_i^2$. □

Lemma 3.0.22. *The Hilbert space V is isomorphic to V^* the set of all continuous linear functionals from V to $\mathbb{R}$.*

Proof. Let $L \in V^*$. By Riesz Representation Theorem there is a unique $h_0 \in V$ such that $L(v) = \langle v, h_0 \rangle$ for all $v \in V$. We study $L \longrightarrow h_0$ as a function from V^* to V. Let $\psi : V^* \longrightarrow V$ such that $\psi(L) = h_0$. Let $\alpha \in \mathbb{R}$ and consider

$$\langle \psi(\alpha L), v \rangle = \alpha L(v) = \alpha \langle h_0, v \rangle = \langle \alpha h_0, v \rangle$$

Therefore $\psi(\alpha L) = \alpha \psi(L)$. It follows similarly that if $L_1, L_2 \in V^*$ then,

$$\begin{aligned} \langle \psi(L_1 + L_2), v \rangle = (L_1 + L_2)(v) L_1(v) + L_2(v) &= \langle \psi(L_1), v \rangle + \langle \psi(L_2), v \rangle \\ &= \langle \psi(L_1) + \psi(L_2), v \rangle \end{aligned}$$

Thus ψ is linear. Suppose $\psi(L_1) = \psi(L_2)$. Then,

$$\langle \psi(L_1), v \rangle = \langle \psi(L_2), v \rangle \text{ and } L_1(v) = L_2(v) \text{ for all } v \in V$$

Therefore $L_1 = L_2$ and ψ is linear and one to one. Further, from the Riesz Representation Theorem we get that $L(v) = \langle h_0, v \rangle$ for every $v \in V$. Then $L(h_0) = \langle h_0, h_0 \rangle$. Then,

$$L\left(\frac{h_0}{\| h_0 \|_V}\right) = \langle \frac{h_0}{\| h_0 \|_V}, h_0 \rangle = \frac{1}{\| h_0 \|_V} \| h_0 \|_V^2 = \| h_0 \|_V$$

Note, $|Lv| = |\langle h_0, v \rangle| \leq \| h_0 \| \| v \|$ for all $v \in V$ so $\| L \| \leq \| h_0 \|$. Also, $|Lh_0| = \| h_0 \| \| h_0 \|$ so $\| L \| \geq \| h_0 \|$

$$\text{Therefore, } \| L \| = \| h_0 \|_V \text{ so } V \text{ and } V^* \text{ are isometrically identified.}$$

□

Lemma 3.0.23. *If $u_n \rightharpoonup u$ then $u_n - u \rightharpoonup 0$.*

Proof.

$$\begin{aligned} \lim_{n\to\infty} \langle u_n - u, v \rangle &= \lim_{n\to\infty} \langle u_n, v \rangle - \langle u, v \rangle \\ &= \langle u, v \rangle - \langle u, v \rangle \text{ by weak convergence} \\ &= 0 \end{aligned}$$

□

Lemma 3.0.24. *If $(u_n) \subset V$ such that $u_n \rightharpoonup 0$, then there is a subsequence u_{n_k} such that $\langle u_{n_i}, u_{n_j} \rangle \leq \frac{1}{i}$ for all $j > i$.*

Proof. Let $u_{n_0} = u_0$. Since $u_n \rightharpoonup 0$. There is a m_1 such that $\langle u_{n_0}, u_{m_1}\rangle \leq 1$. Let $u_{n_1} = u_m$. Similarly there is a $m_2 > m_1$ such that $\langle u_{n_i}, u_{m_2}\rangle \leq \frac{1}{2}$ for $i = 0, 1$. Let $u_{n_2} = u_{m_2}$. Continuing in this fashion, for u_{n_k} there is a $m_{k+1} \geq m_k \geq m_{k-1} \geq \dots$ such that $\langle u_{n_i}, u_{m_{k+1}}\rangle \leq \frac{1}{k}$ for $i = 0, \dots, k$ which finishes the claim.

□

Lemma 3.0.25. *Weakly convergent sequences are bounded.*

Proof. (Contradiction) Suppose (u_n) is a weakly convergent sequence in V and that $\| u_n \|_V \to \infty$. Without loss of generality we may assume $V = \ell^2$ and that $u_n \rightharpoonup 0$. By passing to a sub-sequence, we may assume $\| u_n \| \geq n$. for all n. Any further selection of subsequence will retain this property, since $\| u_{n_k} \| \geq n_k \geq k$. By lemma 3.0.24 we make a further selection of sub-sequence so that if $n > m$ then $\langle u_n, u_m\rangle \leq \frac{1}{n}$. We reach a contradiction by constucting a v such that $\langle u_n, v\rangle \to c > 0$. Let $v := \sum_{i=1}^{\infty} \frac{u_i}{\|u_i\|^2}$. Claim $v \in \ell^2$. The partial sums of v form a Cauchy sequence where $v_n = \sum_{i=1}^{n} \frac{u_i}{\|u_i\|^2}$. Then for $n > m$ we have,

$$
\begin{aligned}
\| v_n - v_m \|^2 &= \| \sum_{i=m+1}^{n} \frac{u_i}{\| u_i \|^2} \|^2 \\
&= \langle \sum_{i=m+1}^{n} \frac{u_i}{\| u_i \|^2}, \sum_{i=m+1}^{n} \frac{u_i}{\| u_i \|^2} \rangle \\
&= \sum_{i=m+1}^{n} \frac{1}{\| u_i \|^2} + 2 \sum_{i=m+1}^{n-1} \sum_{j=m+1}^{n} \frac{\langle u_i, u_j \rangle}{\| u_i \|^2 \| u_j \|^2} \\
&\leq \sum_{i=m+1}^{n} \frac{1}{i^2} + 2 \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \frac{1}{i^2 j^2} \\
&= \sum_{i=m+1}^{n} \frac{1}{i^2} + 2 \sum_{i=m+1}^{n} \frac{1}{i^2} \left(\sum_{j=m+1}^{n} \frac{1}{j^2} \right)
\end{aligned}
$$

Given any $\epsilon > 0$ we can choose $N > 0$ such that $\sum_{i=N}^{\infty} \frac{1}{i^2} < \epsilon$. It follows that for $n, m > N$ we have $\| v_n - v_m \|^2 \leq \epsilon + 2\epsilon^2$. Hence v is the limit of a Cauchy sequence and thus $v \in \ell^2$. Finally, we note that,

$$
\begin{aligned}
\langle u_n, v \rangle &= \sum_{i=1}^{\infty} \frac{\langle u_n, u_i \rangle}{\| u_i \|^2} \\
&= \sum_{i=1}^{n-1} \frac{\langle u_n, u_i \rangle}{\| u_i \|^2} + \frac{\langle u_n, u_n \rangle}{\| u_n \|^2} + \sum_{i=n+1}^{\infty} \frac{\langle u_n, u_i \rangle}{\| u_i \|^2} \\
&\leq \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{i^2} + 1 + \sum_{i=n+1}^{\infty} \frac{1}{i^3}
\end{aligned}
$$

As a consequence we have $\langle u_n, v \rangle \longrightarrow 1$ which contradicts $u_n \rightharpoonup 0$. □

Lemma 3.0.26. *If $u_n \rightharpoonup u$ in V, Then there is a subsequence (u_{n_k}) such that the sequence of averages, $(\overline{u}_k)$, defined by $\overline{u}_k := \frac{1}{k} \sum_{i=1}^{k} u_{n_k}$, converges to u.*

Proof. Without loss of generality we assume $u_n \rightharpoonup 0$. Let $M > 0$ such that $\| u_i \| \leq M$ for all i. By lemma 3.0.24 we may select a sub-sequence, and then rename it (u_n) again, such that if $n > m$ then $\langle u_n, u_m \rangle \leq \frac{1}{n}$. let $v_k := \frac{1}{k} \sum_{i=1}^{k} u_i$. Then

$$\begin{aligned}
\| v_k \|^2 &= \langle \frac{1}{k} \sum_{i=1}^{k} u_n, \frac{1}{k} \sum_{i=1}^{k} u_n \rangle \\
&= \frac{1}{k^2} \sum_{i=1}^{k} \langle u_i, u_i \rangle + 2 \frac{1}{k^2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \langle u_i, u_j \rangle \\
&\leq \frac{1}{k^2} \sum_{i=1}^{k} M^2 + 2 \frac{1}{k^2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \frac{1}{j} \\
&\leq \frac{1}{k} M^2 + 2 \frac{1}{k^2} \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \frac{1}{j} \\
&= \frac{1}{k} M^2 + 2 \frac{1}{k}
\end{aligned}$$

Therefore as $k \to \infty$ we have $\| v_k \|_H \to 0$. $\square$

To see the last equality of the argument above consider

$$\begin{array}{cccc}
\frac{1}{2}+ & \frac{1}{3}+ & \ldots & +\frac{1}{k} \\
 & \frac{1}{3}+ & \ldots & +\frac{1}{k} \\
 & & & \vdots \\
 & & & \frac{1}{k}
\end{array}$$

By adding the row corresponding to $i = 0$, and switching the order of the sum, adding by column and then row we get,

$$
\begin{array}{lcccc}
 & 1+ & \frac{1}{2}+ & \ldots & +\frac{1}{k} \\
 & & \frac{1}{2}+ & \ldots & +\frac{1}{k} \\
 & & & & \vdots \\
+ & & & & \frac{1}{k} \\
\hline
\end{array}
$$

$$1 + 1 + \ldots + 1 + 1 = k$$

Chapter 4: Compact Embeddings

The Sobolev space, $H^1 := W^{1,2}[0,1]$, is the completion of $(C^1[0,1], \| \cdot \|_{1,2})$. In this section we show that H^1 can be compactly embedded into $C[0,1]$. For the space of differentiable functions $C^1[0,1]$ we can equip a norm $\| \cdot \|_{1,2}$ defined by

$$\| f \|_{1,2} = \left(\int_0^1 f^2 \right)^{\frac{1}{2}} + \left(\int_0^1 (f')^2 \right)^{\frac{1}{2}}$$

Let $(X_1, \| \cdot \|_{X_1})$ and $(X_2, \| \cdot \|_{X_2})$ be normed linear spaces. If $\psi : X_2 \to X_1$ is linear, one-to-one, and continuous, then we say that ψ is an embedding of X_2 into X_1. We say that ψ is a compact embedding if any sequence $(f_n) \subset X_2$ which is bounded with respect to the $\| \cdot \|_{X_2}$ is mapped to a sequence $(\psi(f_n))$ which has a converging subsequence in X_1.

Lemma 4.0.27. *If $(u_n) \subset C^1[0,1]$ is bounded with respect to $\| \cdot \|_{1,2}$, then (u_n) is bounded in $C[0,1]$.*

Proof. If (u_n) bounded in $C^1[0,1]$ then there is an $M > 0$ such that $\| u_n \|_{1,2} \leq M$ for all n. Let $u(x) \in C[0,1]$. Note, $u(x) = u(0) + \int_0^x u'(t)dt$ for all x. So,

$$\begin{aligned} |u(0)| &\leq |u(x)| + \int_0^1 |u'(t)|dt \\ &\leq |u(x)| + \left(\int_0^1 |u'(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^1 1^2 \right)^{\frac{1}{2}} \quad \text{By Hölder's Inq.} \\ &\leq |u(x)| + \| u' \|_2 \end{aligned}$$

Thus

$$|u(0)| \leq \int_0^1 |u(x)| + \int_0^1 \parallel u' \parallel_2$$

$$\leq (\int_0^1 |u(x)|^2)^{\frac{1}{2}} (\int_0^1 1^2)^{\frac{1}{2}} + \int_0^1 \parallel u' \parallel_2 \text{ By Hölder's Inq.}$$

$$= \parallel u \parallel_2 + \parallel u' \parallel_2 = \parallel u \parallel_{1,2}$$

Therefore

$$|u(x)| \leq |u(0)| + \int_0^1 |u'(t)| dt$$

$$\leq \parallel u \parallel_{1,2} + \int_0^1 |u'(t)| dt$$

$$\leq \parallel u \parallel_{1,2} + \parallel u \parallel_{1,2} = 2 \parallel u \parallel_{1,2} \text{ for all } x \in [0,1]$$

$$\therefore \quad \parallel u \parallel_\infty \leq 2 \parallel u \parallel_{1,2} \leq 2M$$

□

Lemma 4.0.28. *If (u_n) is bounded in $C^1[0,1]$ then (u_n) equicontinuous.*

Proof. As before (u_n) bounded in $C^1[0,1]$ implies there is an $M > 0$ such that $\parallel u_n \parallel_{1,2} \leq M$ for all n. Let $(u_n(x)) \subset C^1[0,1]$. Then

$$
\begin{aligned}
|u_n(x) - u_n(y)| &= |u_n(0) + \int_0^x u_n'(t)dt - (u_n(0) + \int_0^y u_n'(t)dt)| \\
&= |\int_0^x u_n'(t)dt + \int_y^0 u_n'(t)dt| \\
&\leq \int_y^x |u_n'(t)|dt \\
&\leq \left(\int_y^x |1|^2 dt\right)^{\frac{1}{2}} \left(\int_y^x |u_n'(t)|dt\right)^{\frac{1}{2}} \\
&\leq \| u_n \|_{1,2} |x-y|^{\frac{1}{2}} \\
&\leq M|x-y|^{\frac{1}{2}} \text{ for all } n
\end{aligned}
$$

Thus if $\epsilon > 0$ then we may choose $\delta > 0$ such that $M\delta^{\frac{1}{2}} < \epsilon$ and we get equicontinuity. □

The following lemma holds in a more general setting. We present it here in terms of the spaces needed for work in later sections.

Lemma 4.0.29. *If $(u_n) \rightharpoonup u$ in $(C^1[0,1], \| \cdot \|_{1,2})$ then (u_n) has a subsequence which converges uniformly in $C[0,1]$.*

Proof. $(u_n) \rightharpoonup u$ in $C^1[0,1]$ implies (u_n) is bounded in $C^1[0,1]$. Then there is a $M > 0$ such that $\| u_n \|_{1,2} \leq M$ for all n. Since, $\| u_n \|_\infty \leq 2 \| u_n \|_{1,2}$ it follows that (u_n) is bounded in $C[0,1]$. Also, there is an $M > 0$ such that $|f_n(x) - f_m(y)| \leq M|x-y|^{\frac{1}{2}}$ for all n. Hence (u_n) is bounded and equicontinuous. Thus we may apply Arzela-Ascoli to get that (u_n) has a uniformly convergent sub-sequence in $C[0,1]$. □

Lemma 4.0.30. *If $(u_n) \subset C[0,1]$ is bounded then (u_n) has a pointwise convergent subsequence on $\mathbb{Q} \cap [0,1]$.*

Proof. Let $(r_n) = \mathbb{Q} \cap [0,1]$. Since (u_n) is bounded $|u_n(x)| \leq \| u_n \|_\infty$ for all $x \in [0,1]$. Then $(u_n(r_1)) \subset \mathbb{R}$ is bounded and by Bolzano Weirsrauss has a convergent subsequence $(u_{1,n}(r_1))$. It follows that $(u_{1,n}(r_2))$ is bounded in $\mathbb{R}$ and has a convergent subsequence $(u_{2,n}(r_2))$ such that $(u_{2,n}(r_1))$ and $(u_{2,n}(r_2))$ oth converge. Proceeding in this fashion we obtain a countable collection of nested subsequences of our original sequence

$$\begin{array}{cccc} u_{1,1} & u_{1,2} & u_{1,3} & \cdots \\ u_{2,1} & u_{1,2} & u_{2,3} & \cdots \\ u_{3,1} & u_{3,2} & u_{3,3} & \cdots \\ . & . & . & \ddots \\ . & . & . & \end{array}$$

where the sequence in the n^{th} row converges at the points $r_1, \ldots, r_n$ and each row is a subsequence of the one above it. Thus the diagonal sequence $(u_{n,n})$ is a sub-sequence of the original sequence (u_n) that converges at each point of (r_n).

□

Theorem 4.0.31. Arzela-Ascoli: *If a sequence (u_n) in $C[0,1]$ is bounded and equicontinuous then it has a uniformly convergent sub-sequence.*

Proof. Assume (u_n) is pointwise convergent on $(r_n) = \mathbb{Q} \cap [0,1]$. Let $\epsilon > 0$ be given. Choose $\delta > 0$ by equicontinuity of (u_n) so that $|x - y| < \delta$ implies $|u_n(x) - u_n(x)| < \frac{\epsilon}{3}$ for all $x, y \in [0,1]$ and for all n. Assume, $0 = r_0 < r_1 < r_2 < \ldots < r_k = 1$ such that $|r_{i+1} - r_i| < \delta$. Choose N_i such that $|u_n(r_i) - u_m(r_i)| < \frac{\epsilon}{3}$ for all $n, m \geq N_i$. Let $N = \max\{N_i\}$. Let $x \in [0,1]$. Then, there is an $r_i \in (r_n)$ such that $|x - r_i| < \delta$. Then,

$$|u_n(x) - u_m(x)| \leq |u_n(x) - u_n(r_i)| + |u_n(r_i) - u_m(r_i)| + |u_m(r_i) - u_m(x)|$$
$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{for all } m, n \geq N \text{ for all } x \in [0, 1]$$

Thus (u_n) is uniformly Cauchy and therefore uniformly convergent

□

Theorem 4.0.32. *(Relich-Kondrakov Theorem) There is a compact embedding of H^1 into $C[0, 1]$*

The next series of lemmas will establish the theorem above. If we consider an element $f \in H^1$ then $f = [(f_n)]$ is an equivalence class of sequences in $C^1[0, 1]$ that are Cauchy with respect to the $\| \cdot \|_{1,2}$ norm. Since $\| \cdot \|_\infty \leq \| \cdot \|_{1,2}$, it follows that Cauchy sequences in $C^1[0, 1]$ are Cauchy in $C[0, 1]$ and thus converge since $C[0, 1]$ is complete. Thus, it is reasonable to take a limit of a representative sequence to arrive at a function that can represent f. The next lemma shows this limit is independent of the choice of the Cauchy sequence selected from the equivalence class.

Lemma 4.0.33. *Let $(f_n), (\overline{f}_n) \subset C^1[0, 1]$ such that $(f_n) \sim (\overline{f}_n)$, i.e $\| f_n - \overline{f}_n \|_{1,2} \longrightarrow 0$. Then (f_n) and $(\overline{f}_n)$ converge uniformly to an $f \in C[0, 1]$.*

Proof. $(f_n) \equiv (\overline{f}_n)$ implies $\| f_n - \overline{f}_n \|_{1,2} \to 0$. The claim holds since $\| \cdot \|_\infty \leq 2 \| \cdot \|_{1,2}$. □

If $f \in H^1$, then $f = [(f_n)]$. We have a natural choice for a function to represent f. Let

$$\psi : H^1 \longrightarrow C[0, 1] \text{ such that } \psi(f) = \lim_{n \to \infty} f_n \text{ in } C[0, 1]$$

Lemma 4.0.34. *ψ is linear.*

Proof. This is clear from the fact that limits are linear. □

Lemma 4.0.35. *ψ is 1-1.*

Proof. Let $f \in H^1$ such that $\psi(f) = 0$, and let $f = [(f_n)]$. Then it follows that $(f_n) \subset C^1[0,1]$ is Cauchy with respect to $\| \cdot \|_{1,2}$ and converges uniformly to 0 with respect to $\| \cdot \|_\infty$. Note,

$$\| f_n \|_2 = (\int_0^1 |f_n|^2)^{\frac{1}{2}} \leq (\int_0^1 \| f \|_\infty^2)^{\frac{1}{2}} = \| f_n \|_\infty \text{ thus } \| f_n \|_2 \to 0$$

What remains to show is that $\| f_n' \|_2 \to 0$. Notice, from the Fundamental Theorem of Calculus, for any interval $[a,b] \subset [0,1]$, we get

$$\int_a^b f_n' dt = f_n(b) - f_n(a) \to 0 \text{ by uniform convergence.}$$

Let $\phi := \sum_{i=1}^k \alpha_i \chi_{[a_i,b_i]}$ be a step function. Then

$$\int_a^b \phi f_n' dt = \sum_{i=1}^k \alpha_i \int_{a_i}^{b_i} f_n' dt \longrightarrow 0$$

Let $\phi \in C[0,1]$. Let $\epsilon > 0$ be given and choose $\delta > 0$ such that $|x - y| < \delta$ implies $|\phi(x) - \phi(y)| < \epsilon$. Let $\frac{1}{n} < \delta$. Consider intervals $[\frac{k}{n}, \frac{k+1}{n}] = I_k$ for $k = 0, \ldots, n-1$. Let $\beta_k = \max\{\phi(x) : x \in I_k\}$. Then

$$\phi \leq \sum_{i=1}^k \beta_i \chi_{i_k} \leq \phi + \epsilon$$

Let $\phi_1 = \sum\limits_{i=1}^{k} \beta_i \chi_{[a_i,b_i]}$. Then it follows that

$$|\int\limits_a^b \phi f_n' dt - \int\limits_a^b \phi_1 f_n' dt| = |\int\limits_a^b (\phi - \phi_1) f_n' dt|$$

$$\leq \int\limits_a^b |\phi - \phi_1||f_n'| dt$$

$$\leq \| \phi - \phi_1 \|_2 \| f_n' \|_2 \text{ By Hölder's Inq.}$$

$$\leq \epsilon C.$$

Note, the fact that $\| f_n' \|_2 < C$ follows since (f_n) is Cauchy with respect to $\| \cdot \|_2$. Therefore $\limsup |\int\limits_0^1 \phi f_n' dt - \int\limits_0^1 \phi_1 f_n' dt| \leq eC$. But $\lim \int\limits_0^1 \phi_1 f_n' dt = 0$. Therefore, $\limsup |\int\limits_0^1 \phi f_n' dt| \leq \epsilon C$. Letting $\epsilon \to 0$ we get $\lim \int\limits_0^1 \phi f_n' dt = 0$.

Suppose, without loss of generality, that $\lim \| f_n' \|_2 = c > 0$ for all n. Choose $N > 0$ such that $\| f_n' \|_2 > \frac{c}{2}$ for all $n > N$ and $\| f_n' - f_m' \|_{1,2} < \frac{c}{2}$. Then, since $\| f_n' - f_m' \|_2^2 = \langle f_n' - f_m', f_n' - f_m' \rangle$

$$\frac{c^2}{4} > \| f_n' - f_m' \|_2^2 = \| f_n' \|_2^2 + \| f_m' \|_2^2 - 2 \int\limits_0^1 f_n' f_m' dt$$

$$> \frac{c^2}{2} - 2 \int\limits_0^1 f_n' f_m' dt.$$

Let m be fixed. Then $f_m' \in C[0,1]$. Thus, by the work above, and letting $n \to \infty$ implies the integral term above goes to zero and we get a contradiction. Therefore $\| f_n' \|_2 \to 0$ and we have $\| f_n' \|_{1,2} = 0$. So $\ker \psi = \{0\}$ and ψ is one to one. □

Lemma 4.0.36. *Let $X := \psi(H^1)$ and let $\| \psi(f) \|_X := \| f \|_{1,2}$. Then, $(X, \| \cdot \|_X)$ is a normed linear space and $\psi : H^1 \to X$ is a linear isometry.*

Proof. The lemmas above show ψ is linear and one to one and thus ψ is invertible. Then for $f \in X$ we have $\| f \|_X = \| \psi^{-1}(f) \|_{1,2}$

i) Let $f \in X$. Then $\| f \|_X = \| \psi^{-1}(f) \|_{1,2}$. It follows that $\| f \|_X = 0$ if and only if $\psi^{-1}(f) = 0$. Then, since ψ invertible, $\psi(\psi^{-1}(f)) = f = \psi(0) = 0$. Therefore, $\| f \|_X > 0$ for all nonzero $f \in X$ and $\| f \|_X = 0$ if and only if $f = 0$.

ii) Let $\alpha \in \mathbb{R}$. Then for $f \in X$ we have

$$\| \alpha f \|_X = \| \psi^{-1}(\alpha f) \|_{1,2} = \| \alpha \psi^{-1}(f) \|_{1,2} = |\alpha| \, \| \alpha \psi^{-1}(f) \|_{1,2} = |\alpha| \, \| f \|_X$$

iii) Let $f, g \in X$ then

$$\begin{aligned} \| f + g \|_X = \| a\psi^{-1}(f + g) \|_{1,2} &= \| \psi^{-1}(f) + \psi^{-1}(g) \|_{1,2} \\ &\leq \| \psi^{-1}(f) \|_{1,2} + \| \psi^{-1}(g) \|_{1,2} \\ &= \| f \|_X + \| g \|_X \end{aligned}$$

The fact that ψ is a linear isometry follows directly from previous lemmas. □

Lemma 4.0.37. *Let $\phi : (C^1[0,1], \| \cdot \|_{1,2}) \to H^1$ be the isometry associated with the completion process. Then $\psi \circ \phi(f) = f$ for all $f \in C^1[0,1]$*

Proof. ϕ is the map which take a function $f \in C^1[0,1]$ to the Cauchy sequence $(f, f, f, \ldots)$. Then clearly the limit of this sequence is f. Then by definition of ψ it follows that $\psi(\phi(f)) = f$.

□

Lemma 4.0.38. *There exists positive constants C and D such that for every $f \in X$ we have*

$$\| f \|_\infty \leq C \| f \|_X \quad and \tag{4.1}$$

$$|f(x) - f(y)| \leq D \| f \|_X \, |x - y|^{\frac{1}{2}} \text{ for all } x, y \in [0,1] \tag{4.2}$$

Proof. To see inequality 4.1 let $f \in X$. Then $\psi^{-1}(f) \in H^1$. So $\psi^{-1}(f) = (f_n)$ is a Cauchy sequence where $f_n \in C^1[0,1]$ for each n. Then $\| f_n \|_\infty \leq 2 \| f_n \|_{1,2}$. Letting $n \to \infty$ we get $\| f \|_\infty \leq 2 \| f \|_{1,2}$. Inequality 4.2 follows similarly. □

Lemma 4.0.39. *If (f_n) is a bounded sequence in $(X, \| \cdot \|_X)$, then (f_n) has a subsequence that converges in $C[0,1]$.*

Proof. Inequality 0.1 implies (f_n) is bounded in $C[0,1]$. Inequality 0.4 implies (f_n) equicontinuous. Applying Arzela-Ascoli proves the claim.

□

Lemma 4.0.40. *The identity mapping $i : (X, \| \cdot \|_X) \to C[0,1]$ such that $i(f) = f$ is a compact embedding.*

Proof. i is clearly one to one, continuous, and linear. Let $(f_n) \subset X$ be bounded. Thus applying lemma 4.0.38, we get that $i((f_n)) = (f_n)$ has a converging subsequence in $C[0,1]$. □

Lemma 4.0.41. *The mapping $\psi \circ i : H^1 \to C[0,1]$ is a compact embedding.*

Proof. The composition of linear isometries is a linear isometry. Let $(f_n^k) \subset H^1$ be bounded. Then $\psi((f_n^k)) = (f^k)$. Since (f_n^k) bounded we get that $\| f^k \|_{1,2} \leq M$ for all k. So, $\| f^k \|_\infty \leq 2M$ for all k. (f^k) is clearly equicontinuous and since $i(f^k) = (f^k)$ it follows that (f^k) has a converging sub-sequence in $C[0,1]$.

□

Corollary 4.0.42. *H^1 can be compactly embedded in $L^2[0,1]$*

Proof. L^2 is the completion of $C[0,1]$ with respect to $\| \cdot \|_2$. Let ϕ be the map associated with the completion process. Then the composition of $\phi \circ \psi \circ i$ gives a compact embedding of H^1 into L^2.

□

Lemma 4.0.43. *If $u_n \rightharpoonup u$ in H^1, then $u_n \to u$ in $C[0,1]$.*

Proof. (contradiction) Without loss of generality we assume that $u_n \rightharpoonup 0$ in H^1. Suppose (u_n) does not converge to 0 in $C[0,1]$. By passing to successive sub-sequences we may assume there is an $\epsilon > 0$ such that $\| u_n \|_\infty \geq \epsilon$ for all n. and that $u_n \to u \neq 0$ in $C[0,1]$. Thus we have $u_n \rightharpoonup 0$ in H^1 and $u_n \to u \neq 0$ in $C[0,1]$. Let v_k be a sequence of averages so that $v_k \to 0$ in H^1. Averaging preserves convergence so we also have $v_k \to u \neq 0$ in $C[0,1]$. But $\| u_n \|_\infty \leq 2 \| u_n \|_{1,2}$ implying that $0 \neq \| u \|_\infty \leq 0$ a contradiction. □

Chapter 5: Sturm-Liouville ODE

The culmination of all previous sections is to analyze the following two parameter Sturm-Liouville eigenvalue problem.

$$-(p(x)y')' + q(x)y = (\lambda r(x) + \mu)y, \quad 0 \leq x \leq 1 \tag{5.1}$$

with boundary conditions

$$\cos(\alpha)y(0) - \sin(\alpha)p(0)y'(0) = 0 \quad \cos(\beta)y(1) - \sin(\beta)p(1)y'(1) = 0 \tag{5.2}$$

This is an example of what are called separated boundary conditions. This assumption is important because the sin and cos guarantee we have non-trivial boundary conditions and that the weak formulation of (5.1), (5.2) is symmetric. We assume that p is continuously differentiable and positive on $[0, 1]$ and that q and r are piecewise continuous on $[0, 1]$. We say (λ, μ) is an eigenpair when there exists a non-trivial solution, y, satisfying (5.1), (5.2). Our goal is to characterize the graph of μ as a function of λ, which we will call eigencurves. We do so by applying the theory developed in in previous sections. A few simple calculations put equation (5.1),(5.2) into the weak formulation, a form satisfying the necessary assumptions to guarantee the existence of a sequence of eigenvalues, with corresponding eigenfunctions, which in this case are solutions to (5.1),(5.2).

Let $\psi \in C^1[0, 1]$ and consider,

$$-\int_0^1 \psi(py')' + \int_0^1 \psi qy = \lambda \int_0^1 \psi ry + \mu \int_0^1 \psi y$$

Integration by parts yields:

$$-[py'\psi]\Big|_0^1 + \int_0^1 py'\psi + \int_0^1 \psi q y = \lambda \int_0^1 \psi r y + \mu \int_0^1 \psi y$$

Since

$$\cos(\alpha)y(0) - \sin(\alpha)y'(0) = 0 \text{ and } \cos(\beta)y(1) - \sin(\beta)y'(1) = 0$$

we get that

$$y'(0) = \cot(\alpha)y(0) \text{ and } y'(1) = \cot(\beta)y(1)$$

Hence,

$$-py'\psi|_0^1 = p(0)y'(0)\psi(0) - p(1)y'(1)\psi(1) = \cot(\alpha)y(0)\psi(0) - \cot(\beta)y(1)\psi(1)$$

Let $b(y,\psi) := \cot(\alpha)y(0)\psi(0) - \cot(\beta)y(1)\psi(1)$ Then,

$$b(y,\psi) + \int_0^1 py'\psi + \int_0^1 \psi q y = \lambda \int_0^1 \psi r y + \mu \int_0^1 \psi y \text{ for all } \psi \in C^[0,1] \tag{5.3}$$

Equation (5.3) is what we call the weak formulation of (5.1) and (5.2). Unlike (5.1), equation (5.3) still makes sense if we know only that $y \in C^1[0,1]$. Recall, H^1, the completion of $C^1[0,1]$ with respect to the $\|\cdot\|_{1,2}$ norm has inner product:

$$\langle f, f\rangle_{1,2} = \int_0^1 f^2 + \int_0^1 (f')^2$$

We will show that the left hand side of (5.3) is an equivalent inner product over H^1, and that the right hand side is weakly continuous and positive for some $u \in H^1$. With these facts we will be able to guarantee a sequence of eigenfunctions, which in this case, are solutions to (5.3) with corresponding parameters. First consider the following theorem which shows that the weak solutions found later in the section are classical solutions to (5.1).

Theorem 5.0.44. *(Interior Regularity). Assume $U \subset R$ is a bounded open, set. Let L have the following form*

$$Lu = -(pu_x)_x + (q - \lambda r + \mu)u \tag{5.4}$$

Assume

$$p \in C^1, (q - \lambda r + \mu) \in L^\infty \tag{5.5}$$

Suppose furthermore that $u \in H^1(U)$ is a weak solution of

$$Lu = 0 \tag{5.6}$$

Then, $u \in H^2_{loc}(U)$.

This is only a start to guaranteeing that u is in fact a classical solution. The idea of this proof is that we can restrict u to a domain for which we can show u has a derivative. Therefore u is locally smooth. We can repeat this process again to show that u is locally as smooth as required and thus a solution to (5.1). For proof of this statement and further insight on regularity theory for weak solutions see [3]. Note, for certain examples, see Richardson's Equation in section 6, the solutions are not twice differentiable at zero

Lemma 5.0.45. *Without loss of generality q is strictly positive with $q \geq c > 0$ on $[0, 1]$.*

Proof. q is bounded since q is piecewise continuous. Let k, c be positive such that $q + k \geq c > 0$. The ODE becomes

$$-(py')' + (q+k)y = \lambda ry + (\mu + k)y$$

Thus, given any piecewise continuous q, we can shift the solution of ODE up by a factor of k to guarantee q is positive.

$\square$

Lemma 5.0.46. *Without Loss of generality* $p(x) \geq 1$.

Proof. $p(x)$ is positive and continuous and thus bounded on $[0, 1]$. Assuming $p < 1$. Let $p_0 = \inf\{p(x) : x \in [0, 1]\}$. Then

$$-(\frac{p}{p_0}y')' + (\frac{q}{p_0})y = (\frac{\lambda}{p_0}\lambda r + \frac{\mu}{p_0}u)y$$

defines an equivalent problem. $\square$

We next simplify the problem by fixing λ, and collecting the λ term on the left hand side of the equation.

$$b(y,\psi) + \int_0^1 py'\psi' + \int_0^1 \psi qy - \lambda \int_0^1 \psi ry = \mu \int_0^1 \psi y \tag{5.7}$$

Lemma 5.0.47. *Let* $\overline{a}(y, \psi) = \int_0^1 py'\psi' + \int_0^1 qy\psi$. *There exist* $c_1, c_2 > 0$ *such that* $c_1\langle y, y\rangle_{H^1} \leq \overline{a}(y, y) \leq c_2\langle y, y\rangle_{H^1}$ *for some* $c > 0$.

Proof. Without loss of generality $p, q \geq 1$. Note, since p continuous and q piecewise continuous on $[0, 1]$ p and q attain a supremum on $[0, 1]$. Let m, n denote the supremum of p, q respectively. Let $t = \max\{m, n\}$. Hence

$$\langle u, u\rangle \leq \overline{a}(u, u) \leq t\langle u, u\rangle$$

□

Lemma 5.0.48. *$b(u, u) = \cot(\alpha)u^2(0) - \cot(\beta)u^2(1)$, is weakly continuous as a function on H^1.*

Proof. Suppose $u_n \rightharpoonup u$ in H^1. Then by lemmas 3.0.26, $u_n \to u$ uniformly in $C[0, 1]$ and it follows that $u_n^2 \to u^2$ uniformly and then clearly $b(u_n, u_n) \to b(u, u)$. □

Lemma 5.0.49. *There exist a $0 \leq k < \infty$ such that for all $\lambda \in [\lambda_0, \lambda_1]$,*

$$b(y, y) + \int_0^1 p(y')^2 + \int_0^1 qy^2 - \lambda \int_0^1 ry^2 + k \int_0^1 y^2 \geq \frac{t}{2}||y||_{1,2}^2$$

Proof. (Contradiction)

Suppose $b > 0$, $(k_n) \subset \mathbb{R}$ such that $(k_n) \nearrow \infty$ and $(y_n) \subset H^1$ such that $||y||_{1,2}^2 = 1$, and

$$\overline{a}(y_n, y_n) + b(y_n, y_n) - \lambda \int_0^1 ry_n^2 + k_n \int_0^1 y_n^2 < \frac{t}{2}$$

Since $\overline{a}(y_n.y_n) \leq t\langle y_n, y_n\rangle_{H^1} = t$. It follows that $t + b(y_n, y_n) - \lambda \int_0^1 ry_n^2 + k_n \int y_n^2 \leq \frac{t}{2}$. Without loss of generality we may assume $y_n \rightharpoonup y$ in H^1. Then $y_n \to y$ uniformly. Thus, by weak continuity, $b(y_n, y_n) \to b(y, y)$, $\int_0^1 ry_n^2 \to ry^2$, and $\int_0^1 y_n^2 \to \int_0^1 y^2$. Since $(k_n) \to \infty$, If $\int_0^1 y_n^2 \to \int_0^1 y^2 > 0$ then we get a contradiction. The only possibility to avoid a contradiction is that $y = 0$. Then $u_n \to 0$ and $u_n \rightharpoonup 0$. By weak continuity

$b(y_n, y_n) \to 0$ and $\int_0^1 y_n^2 \to 0$. Also, since r is piece-wise continuous and thus bounded $\int_0^1 r y_n^2 \to 0$. But, for n large, we get that $t \leq \frac{t}{2}$ and thus a contradiction. Therefore there exist a finite k such that

$$a_{\lambda,k}(y, y) := b(y, y) + \overline{a}(y, y) - \lambda \int_0^1 ry^2 + k \int_0^1 y^2 \geq \frac{t}{2}||y||_{H^1}^2 > 0$$

To see that there is a single k for all $\lambda \in [\lambda_0, \lambda_1]$. Consider the proof above for the case $\lambda = \lambda_1$. Then there is a finite k such that $a_{\lambda,k}(u, u) > 0$ and it follows that the same k guarantees positivity for all λ to the left of λ_1 □

Note, b, p are continuous and q, r are piecewise continuous. Thus we may let b_0, p_0, q_0, r_0 be the supremum of b, p, q, r respectively.

Lemma 5.0.50. *There is a $c_0 > 0$ such that $a_{\lambda,k}(y, y) \leq c_0 \langle y, y \rangle_{H^1}$*

Proof.

$$\begin{aligned}
a_{\lambda,k}(y,y) &= b(y,y) + \int_0^1 p(y')^2 + \int_0^1 qy^2 - \lambda \int_0^1 ry^2 + k\int_0^1 y^2 \\
&\leq b_0\left(\int_0^1 (y')^2 + \int_0^1 y^2\right) + p_0\int_0^1 (y')^2 + q_0\int_0^1 y^2 - \lambda r_0\int_0^1 y^2 + k\int_0^1 y^2 \\
&= b_0\left(\int_0^1 (y')^2 + \int_0^1 y^2\right) + p_0\int_0^1 (y')^2 + (q_0 - \lambda r_0 + k)\int_0^1 y^2 \\
&\leq b_0\left(\int_0^1 (y')^2 + \int_0^1 y^2\right) + s_0\left(\int_0^1 (y')^2 + \int_0^1 y^2\right) \text{ with } s_0 = \max\{p_0, q_0 - \lambda r_0 + k\} \\
&\leq c_0\left(\int_0^1 (y')^2 + \int_0^1 y^2\right) \text{ with } c_0 = \max\{b_0, s_0\} \\
&= c_0\langle y,y\rangle_{H^1}
\end{aligned}$$

Then, for every fixed λ, we have there is a k such that $a_{\lambda,k}(y,y)$ is an equivalent inner product over H^1. □

Define, from the right hand side of (5.3), $m_2(u,v) = \int_0^1 uv$ and $m_1(u,v) = \int_0^1 ruv$. Recall $M_1(u) = m_1(.,.)$ and $M_2(u) = m_2(.,.)$. To establish the remaining assumptions to ensure a sequence of eigenvalues for this problem we need to establish that $M_1(u)$ is weakly continuous and positive for some $u \in H^1$. $M_1(u)$ is clearly positive for all non-zero $u \in H^1$.

Lemma 5.0.51. *$M_2(u) = \int_0^1 u^2$ is weakly continuous.*

Proof. Suppose $u_n \rightharpoonup u$ in H^1. Then by lemma 3.0.27, $u_n \to u$ uniformly in $C[0,1]$. It follows that $u_n^2 \to u^2$ uniformly. And since,

$$|\int_0^1 u_n^2 - \int_0^1 u^2| \le \int_0^1 |u_n^2 - u^2|$$

the claim follows from uniform convergence of u_n^2 to u^2.

□

Corollary 5.0.52. *$M_1(u) = \int_0^1 ru^2$ is weakly continuous.*

Thus, we have that, $a_{\lambda,k}(.,.)$ is an equivalent inner product over H^1, $M_2(u) = \int_0^1 u^2$ is a weakly continuous bilinear form, and $M_2(u) > 0$ for all $u \neq 0$. The culmination of these facts satisfy the necessary assumptions to attain, for every fixed λ, the existence of a sequence of eigenpairs (μ_n, e_n). such that

$$a_{\lambda,k}(e_n, v) = \mu_n m(e_n, v) \text{ for every } v \in H^1$$

Recall that in order to guarantee $a_\lambda(.,.) > 0$ we shifted the problem by a factor of k. Thus, $\tilde{\mu}_n = \mu_n + k$. We will not explicitly construct this sequence for any examples but rather we use the existence to describe properties of eigencurves, μ plotted as a function of λ. We first describe some well known results about classical Sturm-Liouville theory. If $(\lambda, \mu) \in R^2$ is an eigenpair, then there is a corresponding eigenfunction which has a finite number of zeroes in the open interval $(0, 1)$. This number of zeroes is defined to be the *oscillation count* of (λ, μ). By definition, the n^{th} eigencurve, $n = 1, 2, \ldots$ consists of all eigenpairs (λ, μ) having oscillation count $n-1$. If $\lambda \in \mathbb{R}$ is fixed, then (5.1), (5.2) poses a regular Sturm-Liouville problem with eigenvalue parameter μ. It is well known that it possesses exactly one real eigenvalue μ with a corresponding eigenfunction which has exactly $n-1$ zeroes in (a, b) for each $n = 1, 2, \ldots$. This eigenvalue is denoted $\mu = \mu_n(\lambda)$. The n^{th} eigencurve is the graph of the function $\mu = \mu_n(\lambda)$. It also follows from classical Sturm-Liouville theory that $\mu_1(\lambda) < \mu_2(\lambda) < \ldots \mu_n(\lambda)$. In this section we show the eigencurves differentiable, and that the intersection of the m^{th} eigencurve with any line can have no more than $2m$ intersections. Also, the principal eigencurve is convex.

Differentiability

This section shows the (λ, μ) eigencurves are differentiable and the derivative is explicitly computed. We first explore the Implicit Function Theorem which we apply to prove differentiability of eigencurves. To understand the Implicit Function Theorem we first compare an implicit function to an explicit function. A non-implicit function, is the kind we are accustomed to, which looks like, $y = \phi(x)$. We can also express this same relationship as a zero of the implicit function F. Consider,

$$F(x, y) = y - \phi(x) = 0$$

Thus we have a natural converse to the statement above. If we have an implicit function of the form $F(x, y) = 0$ for $x, y \in \mathbb{R}$, then we can view y as a function of x and thus $g(x, \psi(x)) = 0$. Thus we get the following theorem.

Theorem 5.0.53. *2-variable Implicit Function Theorem Let $F : D \subset \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be an interior point of D with $F(x_0, y_0) = 0$. Suppose both first order partial derivatives of F exist in D and are continuous at (x_0, y_0) with $F_y(x_0, y_0) \neq 0$. Then there is an interval $I \subset \mathbb{R}$ with x_0 an interior point of I and a function $\phi : I \to \mathbb{R}$ such that ϕ is differentiable on I, $\phi(x_0) = y_0$, and $F(x, \phi(x)) = 0$ for each $x \in I$.*

We can easily verify the derivative of ϕ. The idea is that since we can view ϕ as a function of x differentiating F with respect to x we get,

$$\frac{d}{dx}\left[F(x,\phi(x))\right] = \frac{d}{dx}\left[0\right]$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial x} = 0$$

$$\text{Therefore } \phi'(x) = -\frac{\partial F/\partial x}{\partial F/\partial y}.$$

Recall the Sturm-Liouville ODE.

$$(ODE) : -(py')' + qy = (\lambda r + \mu)y \text{ in } (0,1)$$

$$(BC1) : \cos(\alpha)y_1 - \sin(\alpha)p_0y_0' = 0$$

$$(BC2) : \cos(\beta)y_1 - \sin(\beta)p_0y_0' = 0$$

Let $y(x,\lambda,\mu)$ be the unique solution of the initial value problem (IVP),

$$(ODE) : -(py')' + qy = (\lambda r + \mu)y \text{ in } (0,1)$$

$$y(0) = y_0 = y(0,\lambda,\mu) = \sin(\alpha)$$

$$p(0)y'(0) = p_0y'(0,\lambda,\mu) = \cos(\alpha)$$

Then,

$$\cos(\alpha)y_0 - \sin(\alpha)p_0y_0' = \cos(\alpha)\sin(\alpha) - \sin(\alpha)\cos(\alpha) = 0$$

Therefore y satisfies the first boundary condition, (BC1). Also, any solution of (ODE), (BC1), (BC2) can be rescaled so that it is a solution of (IVP). Let

$$D(\lambda,\mu) = \cos(\beta)y_1(\lambda,\mu) - \sin(\beta)p_1y_1'(\lambda,\mu)$$

Then, by definition, y satisfies BC2 if and only if $D(\lambda,\mu) = 0$ and thus (λ,μ) is an eigenpair if and only if $D(\lambda,\mu) = 0$. Note, $y_1(\lambda,\mu) = y(1,\lambda,\mu)$, $y'(\lambda,\mu)_1 = y'(1,\lambda,\mu)$.

We analyze this equation with the Implicit Function Theorem. As shown in [1] $y(x,\lambda,\mu)$, the solution to (IVP), is analytic with respect to all three variables and thus is continuous and differentiable of all orders with respect to all three variables. The derivatives of D with respect to λ and μ are

$$\frac{\partial D}{\partial \lambda} = \cos(\beta)\left(\frac{\partial y}{\partial \lambda}\right)(1,\lambda,\mu) - \sin(\beta)p_1\left(\frac{\partial y}{\partial \lambda}\right)'(1,\lambda,\mu)$$

$$\frac{\partial D}{\partial \mu} = \cos(\beta)\left(\frac{\partial y}{\partial \mu}\right)(1,\lambda,\mu) - \sin(\beta)p_1\left(\frac{\partial y}{\partial \mu}\right)'(1,\lambda,\mu)$$

Note, $\frac{\partial D}{\partial \lambda}$ and $\frac{\partial D}{\partial \mu}$ are continuous. Let $\frac{\partial y}{\partial \lambda} = z$ and $\frac{\partial y}{\partial \mu} = w$. To better understand z we differentiate the ODE with respect to λ, by equality of mixed partials we get

$$(ODE_z): \quad -(pz')' + qz = ry + (\lambda r + \mu)z$$

We can also differentiate $y(0,\lambda,\mu) = \sin(\alpha)$ and $p(0)y'(0,\lambda,\mu) = \cos(\alpha)$ to get

$$z_0 = \frac{\partial}{\partial \lambda}y(0,\lambda,\mu) = \frac{\partial}{\partial \lambda}\sin(\alpha) = 0 \text{ and } z_0' = 0$$

Multiply ODE by z and ODE_z by y to yield the following

$$-(py')'z + qyz = (\lambda r + \mu)yz \qquad -(pz')'y + qyz = ryz + (\lambda r + \mu)zy$$

Integration by parts and applying the boundary conditions $z(0) = 0$ and $z'(0) = 0$ yields

$$1) - y_1 p_1 z_1' + \int_0^1 y'p'z' + \int_0^1 qyz = \lambda \int_0^1 ryz + \mu \int_0^1 yz$$

$$2) - z_1 p_1 y_1' + \int_0^1 z'p'y' + \int_0^1 qzy = \int_0^1 ry^2 + \lambda \int_0^1 ryz + \mu \int_0^1 y^2$$

Subtracting 1) from 2), by symmetry, we get

$$p_1(z_1 y_1' - y_1 z_1') = \int_0^1 ry^2$$

Similarly by differentiating ODE with respect to μ and letting $\frac{\partial y}{\partial \mu} = w$, integrating and subtracting, we get

$$p_1(w_1 y_1' - y_1 w_1') = \int_0^1 y^2$$

Lemma 5.0.54. *$\frac{\partial D}{\partial \mu} \neq 0$ when (λ, μ) is an eigenpair. i.e D and $\frac{\partial D}{\partial \mu}$ cannot both be zero.*

Proof. (contradiction) Assume $D = 0$ and $\frac{\partial D}{\partial \mu} = 0$. Then,

$$\cos(\beta) y_1 - \sin(\beta) p_1 y_1' = 0$$
$$\cos(\beta) z_1 - \sin(\beta) p_1 z_1' = 0$$

Which gives us the system

$$\begin{pmatrix} y_1 & -p_1 y_1' \\ w_1 & -p_1 w_1' \end{pmatrix} \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note, $\begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix}$ is non trivial. Hence the matrix above must be singular with determinant zero. So

$$p_1(y' w_1 - y_1 w_1') = 0 \text{ which implies } \int_0^1 y^2 = 0 \text{ a contradiction}$$

□

Therefore If $D = 0$ i.e $(BC2)$ is satisfied, then $\frac{\partial D}{\partial \mu} \neq 0$. Thus we can apply the Implicit Function Theorem to get that μ is a differentiable function of λ in that neighborhood. Note, that y satisfies (BC2), so $y_1 = c\sin(\beta)$, $p_1 y_1' = c\cos(\beta)$ for some $c \neq 0$.

$$\begin{aligned} \frac{\partial \mu}{\partial \lambda} &= -\frac{\partial D/\partial \lambda}{\partial D/\partial \mu} \\ &= -\frac{\cos(\beta) z_1 - \sin(\beta) z_1'}{\cos(\beta) w_1 - \sin(\beta) w_1'} \\ &= \frac{p_1 y_1' z_1 - p_1 y_1 z_1'}{p_1 y_1' w_1 - p_1 y_1 w_1'} \\ &= -\int_0^1 r y^2 \Big/ \int_0^1 y^2 \end{aligned}$$

An application of the quotient rule gives an expression for $\frac{\partial^2 \mu}{\partial \lambda^2}$. This expression simplifies if we assume $\frac{\partial \mu}{\partial \lambda} = 0$ to get

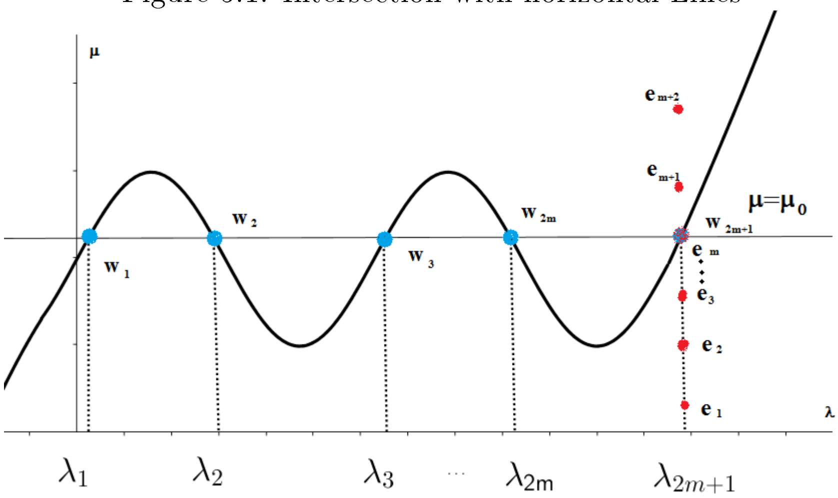
$$\frac{\partial^2\mu}{\partial\lambda^2} = -((\int_0^1 y^2)(2\int_0^1 ryzdx) - (\int_0^1 ry^2)(2\int_0^1 yz))/\int_0^1 y^2$$

$$= -2\int_0^1 ryz/\int_0^1 y^2$$

Intersection with Horizontal Lines

In this section we show that the m^{th} eigencurve can have at most $2m$ intersections with any horizontal line. We argue by contradiction. Assume there are $2m+1$ intersections of some horizontal line, $\mu = \mu_0$, with the m^{th} eigencurve.

Figure 5.1: Intersection with horizontal Lines



By assumption, the graph of $\mu = \mu_m(\lambda)$ and $\mu = \mu_0$ must appear similarly to the figure 5.1. The argument is similar for each case. As in 5.1, at each intersection point, we have an associated slope of the eigencurve. Thus, the assumption there are $2m+1$ intersections implies the sign of the slope of the eigencurve at each intersection.

Assume

$$\frac{\partial \mu}{\partial \lambda_1}, \frac{\partial \mu}{\partial \lambda_3}, \ldots, \frac{\partial \mu}{\partial \lambda_{2m+1}} \geq 0 \text{ and } \frac{\partial \mu}{\partial \lambda_2}, \frac{\partial \mu}{\partial \lambda_4}, \ldots, \frac{\partial \mu}{\partial \lambda_{2m}} \leq 0$$

Let $w_1, \ldots, w_{2m+1}$ be corresponding linearly independent eigenfunctions i.e solutions to the Sturm-Liouville ODE for corresponding (λ, μ) pairs $(\lambda_1, \mu_0), \ldots (\lambda_{2m+1}, \mu_0)$. And, let $e_1, \ldots, e_m$ be the first m linearly independent eigenfunctions corresponding to the eigenvalue problem for $a_{2m+1,k}(.,.)$. Recall,

$$M_1(u) = \int_0^1 ru^2 \text{ and } M_2(u) = \int_0^1 u^2$$

Lemma 5.0.55. *w_i, w_k are m_1 and $a - \mu m_2$ perpendicular for $i \neq k$*

Proof. Since (λ_j, μ_j) is an eigenpair we have that there is a solution to the differential equation w_j such that

$$a(w_j, v) = \lambda_j m_1(w_j, v) + \mu m_2(w_j, v) \text{ for all } v \in H^1$$

Thus, it follows that

$$a(w_i, w_k) = \lambda_i m_1(w_i, w_k) + \mu m_2(w_i, w_k)$$
$$a(w_k, w_i) = \lambda_k m_1(w_k, w_i) + \mu m_2(w_k, w_i)$$

Subtracting we get

$$0 = (\lambda_i - \lambda_k) m_1(w_i, w_k)$$

Since, $\lambda_i \neq \lambda_k$ it must be that $m_1(w_i, w_k) = 0$ which shows the first part of the claim. To see the second, Consider the first equation, noting that w_i, w_k are m_1 orthogonal.

$$a(w_i, w_k) = \mu m_2(w_i, w_k)$$
$$a(w_i, w_k) - \mu m_2(w_i, w_k) = 0$$

□

Consider $w = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_{2m+1} w_{2m+1}$. Then,

$$
\begin{aligned}
& a(w,w) - \overline{\lambda} m_1(w,w) + \mu m_2(w,w) \\
& = \alpha_1^2 a(w_1,w_1) + \ldots + \alpha_{2m+1}^2 a(w_{2m+1},w_{2m+1}) \\
& - \overline{\lambda} \left(\alpha_1^2 M_1(w_1,w_1) + \ldots + \alpha_{2m+1}^2 m_1(w_{2m+1},w_{2m+1}) \right) \\
& - \mu \left(\alpha_1^2 m_2(w_1,w_1) + \ldots + \alpha_{2m+1}^2 m_2(w_{2m+1},w_{2m+1}) \right)
\end{aligned}
$$

Substituting $a(w_i,w_i) = \lambda_i m_1(w_i,w_i) + \mu m_2(w_i,w_i)$

$$
\begin{aligned}
& = \alpha_1^2(\lambda_1 m_1(w_1,w_1) + \mu m_2(w_1,w_1)) + \ldots \\
& + \alpha_{2m+1}^2(\lambda_{2m+1} m_1(w_{2m+1},w_{2m+1}) + \mu m_2(w_{2m+1},w_{2m+1})) \\
& - \overline{\lambda} \left(\alpha_1^2 m_1(w_1,w_1) + \ldots + \alpha_{2m+1}^2 m_1(w_{2m+1},w_{2m+1}) \right) \\
& - \mu \left(\alpha_1^2 m_2(w_1,w_1) + \ldots + \alpha_{2m+1}^2 m_2(w_{2m+1},w_{2m+1}) \right) \\
& = \alpha_1^2(\lambda_1 - \overline{\lambda}) m_1(w_1,w_1) + \ldots + \alpha_{2m+1}^2(\lambda_{2m+1} - \overline{\lambda}) m_1(w_{2m+1},w_{2m+1})
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
& a(w,w) - \overline{\lambda} m_1(w,w) + \mu m_2(w,w) \\
& = \alpha_1^2(\lambda_1 - \overline{\lambda}) m_1(w_1,w_1) + \ldots + \alpha_{2m+1}^2(\lambda_{2m+1} - \overline{\lambda}) m_1(w_{2m+1},w_{2m+1})
\end{aligned}
$$

Consider $\alpha_i^2(\lambda_i - \overline{\lambda}) m_1(w_i,w_i)$. $\alpha_i^2 \geq 0$ for all $i = 1, \ldots, 2m+1$. Assume $\lambda_i \leq \overline{\lambda}$. Then we have

$$
\frac{\partial \mu}{\partial \lambda}(\lambda_{2i}) \leq 0 \quad \text{with} \quad \frac{\partial \mu}{\partial \lambda}(\lambda_{2i}) = \frac{-m_1(w_{2i},w_{2i})}{M_2(w_{2i},w_{2i})} \quad \text{hence} \quad m_1(w_{2i},w_{2i}) \leq 0
$$

Similarly,

$$\frac{\partial\mu}{\partial\lambda}(\lambda_{2i-1}) \geq 0 \quad \text{implies} \quad m_1(w_{2i+1}, w_{2i+1}) \leq 0$$

Lemma 5.0.56. *There is a* $\overline{\lambda}$ *and* $\alpha_1, \ldots, \alpha_{2m+1} \in \mathbb{R}$ *such that*

$$a(w,w) - \overline{\lambda} m_1(w,w) - \mu m_2(w,w) \leq 0$$

Proof. Consider $\overline{\lambda} = \lambda_{2m+1}$ and $\alpha_i = 0$ for $i = 2, 4, \ldots, 2m$ Then,

$$\begin{aligned}
&a(w,w) - \overline{\lambda} m_1(w,w) + \mu m_2(w,w) \\
&= \alpha_1^2(\lambda_1 - \lambda_{2m+1}) m_1(w_1, w_1) + \ldots + \alpha_{2m+1}^2(\lambda_{2m+1} - \lambda_{mn+1}) m_1(w_{2m+1}, w_{2m+1})
\end{aligned}$$

From previous derivative computations we know $m_1(w_i, w_i) \leq 0$ for all $i = 1, 3, 5, \ldots, 2m+1$. Since $\lambda_{2m+1} > \lambda_{2m} > \ldots > \lambda_1$ we have

$$a(w,w) - \lambda_{2m+1} m_1(w,w) - \mu m_2(w,w) \leq 0 \text{ for all } \alpha_1, \alpha_3, \ldots, \alpha_{2m+1} \in \mathbb{R}$$

□

By adding $\mu_0 m_2(w,w) + k m_2(w,w)$ to both sides we get,

$$a(w,w) - \lambda_{2m+1} m_1(w,w) + k m_2(w,w) \leq (\mu_0 + k) m_2(w,w)$$

Since $a_{\lambda_{2m+1},k}(.,.)$ is an equivalent inner product over H^1 we may normalize w so that the left hand side above is 1. Let $\tilde{w} = \frac{w}{\sqrt{a_{\lambda_{2m+1},k}(w,w)}}$. Then

$$1 \leq (\mu_0 + k) m_2(\tilde{w}, \tilde{w}) \quad \text{therefore} \quad \frac{1}{\mu_0 + k} \leq m_2(\tilde{w}, \tilde{w})$$

Recall, the $m+1$ eigenvalue is the maximum of m_2 over $[e_1, e_2, \ldots, e_m]^\perp$ where e_i is the i^{th} eigenvalue corresponding to the eigenvalue problem for λ_{2m+1}. Thus if we can show $\tilde{w} \in [e_1, e_2, \ldots, e_m]^\perp$, we will have the following inequality,

$$\frac{1}{\mu_0 + k} \leq \frac{1}{\tilde{\mu}_{m+1}} = \frac{1}{\mu_{m+1} + k}$$

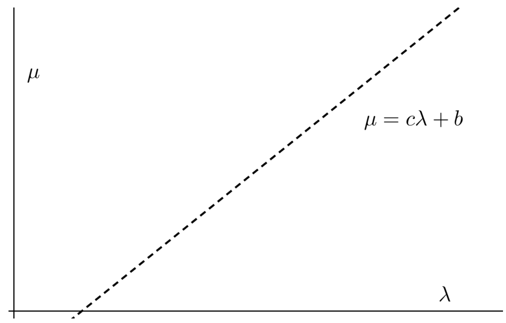
A contradiction since by lemma 2.0.17 we get $\mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \mu_m \leq \mu_{m+1}$

To finish the claim we need to know that there exists constants $\alpha_1, \alpha_3, \ldots, \alpha_{2m+1} \in \mathbb{R}$ such that $w = \alpha_1 w_1 + \alpha_3 w_3 + \ldots + \alpha_{2m+1} w_{2m+1}$ and

$$\begin{array}{c}(a - \lambda_{2m-1} m_1 + k m_2)(w, e_1) = 0 \\ (a - \lambda_{2m-1} m_1 + k m_2)(w, e_2) = 0 \\ \vdots \\ (a - \lambda_{2m-1} m_1 + k m_2)(w, e_m) = 0\end{array}$$

which is a system of m equations in $m+1$ unknowns. Thus we are guaranteed a non-trivial solution.

We can also allow for the intersection of non-horizontal lines. Consider the line $\mu = c\lambda + b$.



Suppose $a(u, v) - \lambda m_1(u, v) - \mu m_2(u, v) = 0$ for all $v \in V$. Then it follows that

$$a(u, v) - \lambda m_1(u, v) - (c\lambda + b) m_2(u, v) = 0 \text{ for all } v \in V \text{ and}$$

$$a(u,v) - \lambda(m_1(u,v) - cm_2(u,v)) - bm_2(u,v) = 0 \text{ for all } v \in V$$

Define $\tilde{m}_1(u,v) = m_1(u,v) - cm_2(u,v)$. Then $a(u,v) - \mu\tilde{m}_1(u,v) - bm_2(u,v)$ defines an equivalent problem. Thus we can apply the previous argument to get that the m^{th} eigencurve can have no more than $2m$ intersections with any line.

Chapter 6: Richardson's Equation

Consider Richardson's equation, an example of the two parameter Sturm-Liouville eigenvalue problem. Our goal is to graph what we will refer to as the eigencurves, i.e. μ plotted as a function of λ.

Richardson's Equation

$$-y'' = (\lambda\,\mathrm{sgn}(x) + \mu)y, \quad y(\pm 1) = 0 \tag{6.1}$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & : x \geq 0 \\ -1 & : x < 0 \end{cases}$$

(6.1) is example of the Sturm-liouville 2 parameter ODE. To see this, starting with (5.1), (5.2), let $p \equiv 1$, $q(x) \equiv 0$, and $r(x) = \mathrm{sgn}(x)$ defined above, with $\alpha = 0$ and $\beta = 2\pi$. Then clearly p is continuous, and q, r are piecewise continuous. Thus, (6.1) satisfies the necessary assumptions to guarantee, for every fixed λ, there is a sequence of corresponding μ's and solutions to the differential equation satisfying the given boundary conditions. We may also immediately conclude that the eigencurves are continuous and differentiable.

Rectangular Decomposition

In order to simplify the problem we separate the ODE to the positive and negative x axis and solve the two problems separately while ensuring that the two solutions meet smoothly. The separated problems then become:

i) $-y_1'' = (\mu - \lambda)y_1,$ such that $y_1(-1) = 0$ and $y_1'(0) = 0$

ii) $-y_2'' = (\mu + \lambda)y_2,$ such that $y_2(1) = 0$ and $y_2'(0) = 0$

For i) we get solutions:

$$y_1 = \cos(\sqrt{\mu - \lambda} x)$$

Applying the boundary condition $y(-1) = 0$ we get that $\cos(-\sqrt{\mu - \lambda}) = 0$ which implies that

$$-\sqrt{\mu - \lambda} = (2k+1)\frac{\pi}{2} \text{ for } k = 1, 2, \ldots$$

Hence

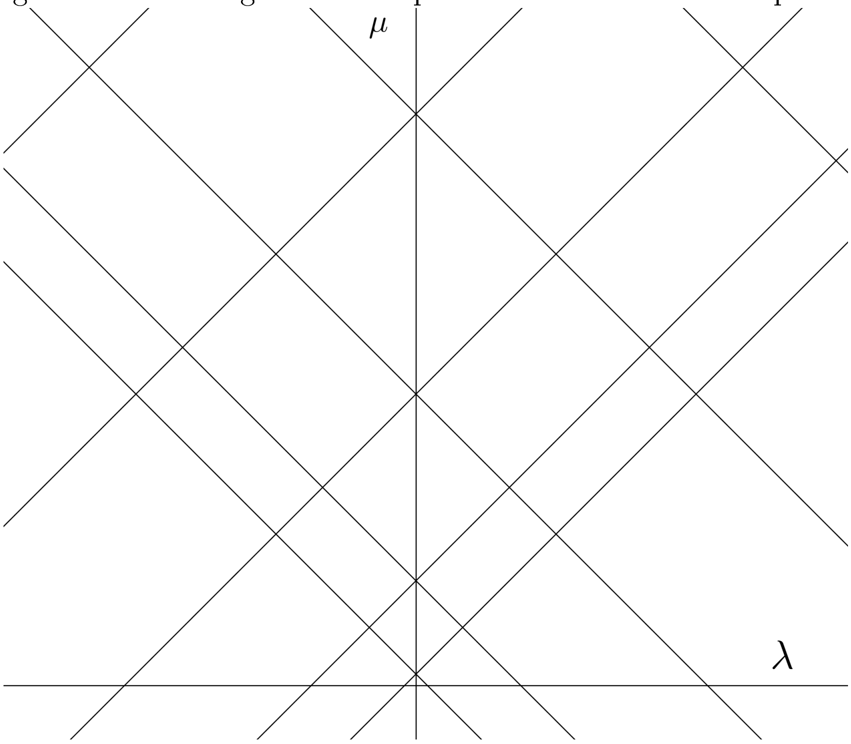
$$\begin{aligned} \mu - \lambda &= (2k+1)^2 \frac{\pi^2}{4} \text{ for } k = 1, 2, \ldots \\ &= \frac{\pi^2}{4}, 9\frac{\pi^2}{4}, \ldots \end{aligned}$$

Similarly, the solution to 2) yields, $y_2 = \cos(\sqrt{\mu + \lambda} x)$, where clearly $y_2'(0) = 0$. Applying the boundary condition $y(1) = 0$ we get that

$$\begin{aligned} \mu + \lambda &= (2k+1)^2 \frac{\pi^2}{4} \text{ for } k = 1, 2, \ldots \\ &= \frac{\pi^2}{4}, 9\frac{\pi^2}{4}, \ldots \end{aligned}$$

Every intersection gives a solution to y_1 and y_2 which meet smoothly given the boundary condition $y_i'(0) = 0$ for $i = 1, 2$. From the analysis above we have that if there exist a solution y satisfying both of the separated equations then it is a solution to (6.1). Also, if $\mu - \lambda = (2k+1)^2 \frac{\pi^2}{4}$ then we have a solution y if and only if $\mu + \lambda = (2k+1)^2 \frac{\pi^2}{4}$ and vice versa. This means that the only eigenpairs on the lines are at the crossing points. It follows that for $\mu < \lambda + \frac{\pi^2}{4}$ and $\mu < -\lambda + \frac{\pi^2}{4}$ the eigencurves are trapped in their respective corridors. We can now use the derivative to

Figure 6.1: Rectangular Decomposition of Richardson's equation



classify regions of increase and decrease and then graph the eigencurves. We can also compute a second derviative to classify concavity at local max/min. The following claims can be found in [1]. We offer details that the authors do not include.

Lemma 6.0.57. *The eigencurves of Richardson's equation are symmetric with respect to the* μ *axis.*

Proof. If $\mu = \mu_n(\lambda)$ and y is a corresponding eigen-function solving (6.1), then $y(-x)$ solves (6.1) with (λ, μ) replaced by $(-\lambda, \mu)$. Since $y(x)$ and $y(-x)$ have $n-1$ zeros, it follows that $\mu = \mu_n(-\lambda)$.

□

Regions of Increase and Decrease

Note, the following are details missing from the claims made in [1].

Lemma 6.0.58. *For $\mu < \lambda + \frac{\pi^2}{4}$ all eigen-curves are decreasing.*

Proof. We know the eigen-curves are differentiable with derivative

$$\frac{\partial \mu_n}{\partial \lambda} = -\frac{\int_{-1}^{1} \operatorname{sgn}(x) y^2 dx}{\int_{-1}^{1} y^2 dx}$$

Thus, we will prove the claim if we can show $\int_{-1}^{1} \operatorname{sgn}(x) y^2 dx > 0$ i.e

$$\int_{-1}^{0} -y(x)^2\, dx + \int_{0}^{1} y(x)^2\, dx > 0 \tag{6.2}$$

whenever y is a nontrivial solution of the original boundary value problem, and $\omega = \mu - \lambda < \frac{\pi^2}{4}$. To prove this, we break the argument into to cases, $i)$ when $\mu - \lambda \in (0, \frac{\pi^2}{4})$ and $ii)$ when $\mu - \lambda < 0$. For case $i)$. First note that any solution to (6.1) also satisfies

$$y'' + \omega y = 0, y(-1) = 0, \text{ on the interval } x \in [-1, 0], \text{ where } \omega \in (0, \frac{\pi^2}{4}) \tag{6.3}$$

and for any solution to (6.2) we claim that the following two inequalities hold

$$\int_{-1}^{0} y(x)^2\, dx < \frac{1}{2} y(0)^2 \text{ and } y(0)y'(0) > 0.$$

Our solutions are of the form

$$y = \sin(\sqrt{\omega}\,(x+1))\text{with}$$
$$y' = \sqrt{\omega}\cos(\sqrt{\omega}\,(x+1))$$

Thus

$$y(0)y'(0) = \sqrt{\omega}\sin(\sqrt{\omega})\cos(\sqrt{\omega})$$

which is strictly positive on $\omega \in (0, \frac{\pi^2}{4})$. This proves the second part of the claim. To see the first. Note,

$$\int_{-1}^{0} \sin^2(\sqrt{\omega}\,(x+1)) = \int_{-1}^{0} \frac{1}{2} - \frac{1}{2}\cos(2\sqrt{\omega}\,(x+1)) = \frac{1}{2} - \frac{1}{4\sqrt{\omega}}\sin(2\sqrt{\omega})$$

$$\text{and } \frac{1}{2}y^2(0) = \frac{1}{2}\sin^2(\sqrt{\omega}) = \frac{1}{2}\left(1 - \cos^2(\sqrt{\omega})\right)$$

Thus the claim is equivalent to showing

$$\frac{1}{2}\left(1 - \frac{1}{\sqrt{\omega}}\sin(\sqrt{\omega})\cos(\sqrt{\omega})\right) < \frac{1}{2}\left(1 - \cos^2(\sqrt{\omega})\right)$$

which, if we simplify a bit, noting that $\cos[\sqrt{\omega}] > 0$ for all $\omega \in (0, \frac{\pi^2}{4})$, reduces to

$$\sin(\sqrt{\omega}) > \sqrt{\omega}\cos(\sqrt{\omega})$$

Observe that $\sin(\sqrt{0}) = \sqrt{0}\cos(\sqrt{0}) = 0$. Then we differentiate both expressions with respect to ω to get

$$\frac{d}{d\omega}\sin(\sqrt{\omega}) = \frac{1}{2\sqrt{\omega}}\cos(\sqrt{\omega}) \text{ and } \frac{d}{d\omega}\sqrt{\omega}\cos(\sqrt{\omega}) = \frac{1}{2\sqrt{\omega}}\cos(\sqrt{\omega}) - \frac{1}{2}\sin(\sqrt{\omega})$$

Since $\sqrt{\omega}\sin(\sqrt{\omega}) > 0$ for all $\omega \in (0, \frac{\pi^2}{4})$, it follows that

$$\frac{d}{d\omega}\sin(\sqrt{\omega}) > \frac{d}{d\omega}\sqrt{\omega}\cos(\sqrt{\omega})$$

Therefore $\sin(\omega)$ and $\sqrt{\omega}\cos(\omega)$ start at the same value, but $\sin(\omega)$ always increases faster. Thus, we can conclude that the claim holds. Note, as before, any solution to (6.1) also satisfies

$$y'' + \gamma^2 y = 0 \text{ where } y(1) = 0, \text{ on the interval } [0,1] \text{ where } \gamma > 0 \tag{6.4}$$

Then for every solution to (6.4), the following inequality holds

$$\int_0^1 y(x)^2\,dx = \frac{1}{2}C^2 + \frac{1}{2\gamma^2}y(0)y'(0) > \frac{1}{2}y(0)^2$$

where C is a constant that comes from solving the differential equation guaranteeing that the left and right hand solutions meet smoothly. To prove the claim we note that, in this case, solutions take the form

$$\begin{aligned} y &= C\sin(\gamma\,(x-1)) \quad \text{with} \\ y' &= C\gamma\cos(\gamma\,(x-1)) \end{aligned}$$

Thus

$$y(0)y'(0) = -C^2\gamma\sin(\gamma)\cos(\gamma)$$

Computation of the integral yields

$$\int_0^1 y(x)^2\,dx = \int_0^1 C^2\sin^2(\gamma\,(x-1))\,dx = \frac{C^2}{2} - \frac{C^2}{2\gamma}\sin(\gamma)\cos(\gamma)$$

which demonstrates the equality on the left side of the claim. Now we compute the right side of the inequality to get

$$\frac{1}{2}y(0)^2 = {}^{C}\!/_{2}\sin(\gamma\,(-1)) = \frac{C^2}{2} - \frac{C^2}{2}\cos^2(\gamma)$$

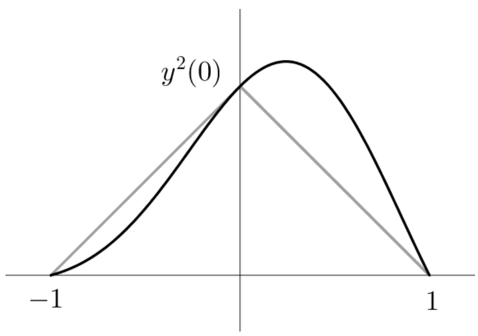
A similar argument to the one given in the proof of the first claim shows that proving the inequality above is equivalent to proving

$$\sin(\gamma) > \gamma\cos(\gamma)$$

Since this is the same inequality, but with γ's instead of $\sqrt{\omega}$'s, proven in the last section the argument given there applies. This, finishes the claim and with the previous, finishes the proof for case i).

We can take a geometric perspective of the claims proved above. We are making a comparison of the area under curve of the square of a solution to the left and right hand ODE's with the area of a triangle with height $y^2(0)$ and base 1. Thus, a solution to (5.1) must have shape similar to the following.

(6.5)



For case ii) Note that any solution to (6.1) also satisfies (6.3) where $\omega < 0$. write $\omega = -\gamma^2$. The we get the following general solution

$$y = \alpha_1 e^{\gamma(x+1)} + \alpha_2 e^{-\gamma(x+1)} \quad \text{where } y(-1) = 0$$

Applying the boundary condition $y(-1) = 0$ we get that $\alpha_1 = -\alpha_2$. Letting $\alpha_1 = 1$ yields,

$$y = e^{\gamma(x+1)} - e^{-\gamma(x+1)} \quad \text{where } y(-1) = 0$$

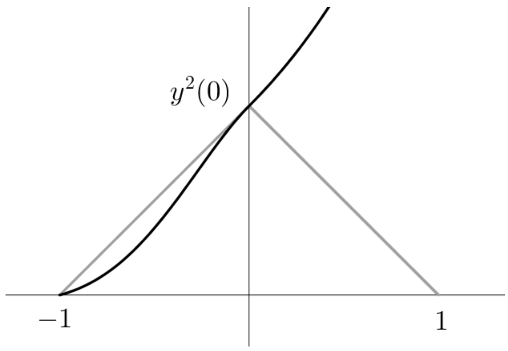
Notice

$$\begin{aligned} \frac{d}{dx} y^2 &= 2yy' \text{ and} \\ \frac{d^2}{dx^2} y^2 &= 2(y')^2 + 2yy" \\ &= 2(y')^2 - 2y^2 \text{ since } y" = -\omega y \end{aligned}$$

Notice, since $\omega < 0$, it follows that $\frac{d^2}{dx^2} y^2 > 0$. Thus y^2 is concave up with $y^2(-1) = 0$. We may then conclude that the graph of y^2 is concave up on $[-1, 0]$ must fall below the line connecting $(-1, 0)$ and $(0, y^2(0))$. We would like to claim that a solution to (6.1) also satisfies

$$\int_0^1 y^2 > \frac{1}{2} y^2(0)$$

whenever $\omega < 0$. It follows that y solves $y" + \omega y = 0$ such that $y(1) = 0$. If $\omega < 0$ then this differential equation has exponential solution and must meet smoothly with the left hand solution. Thus the graph of y^2 on $[-1, 1]$ must have the following form



Thus it is impossible, in this case, for y to satisfy the right boundary condition $y(1) = 1$. Thus we rule out the possibility for $\omega < 0$. Thus $\omega > 0$ on $[0, 1]$ and y has the form sin or cos. Then the argument to finish the claim is similar to before. This finishes the claim for both cases and we may conclude that in the region $\mu - \lambda < \frac{\pi^2}{4}$ all eigencurves are decreasing, and by symmetry, in the region $\mu + \lambda < \frac{\pi^2}{4}$ all eigencurves are increasing.

$\square$

Concavity at Critical Points

From the rectangular decomposition above we get the existence of solutions to (6.1) at every crossing where

$$\lambda + (2k-1)^2 \frac{\pi^2}{4} = -\lambda + (2l-1)^2 \frac{\pi^2}{4} \tag{6.5}$$

which implies $\lambda = ((2k-1)^2 - (2l-1)^{)} \frac{\pi^2}{8}$ and $\mu = ((2k-1)^2 + (2l-1)^2) \frac{\pi^2}{8}$. It is clear from the intersection points of these lines that the (λ, μ) pairs above are eigenpairs of (6.1), but every other point on this line is not an eigenpair. Thus an eigencurve can enter and leave a rectangle only through one of it's corners and an eigenfunction corresponding to one of these pairs is

$$y(x) = \begin{cases} \cos((2k-1)\frac{\pi}{2}x) & x \leq 0 \\ \cos((2l-1)\frac{\pi}{2}x) & x \geq 0 \end{cases} \tag{6.6}$$

Counting the number of zeroes of (5.6) we see that the eigenpair (6.5) lies on the $(k+l-1)^{th}$ eigencurve.

Lemma 6.0.59. *The n eigenpairs* (6.5) *with* $(k,l) = (1,n), (2,n-1), \ldots, (n,1)$ *are local maxima of the n^{th} eigencurve of* (6.1).

[1] offer the following proof:

Proof. Let (λ_0, μ_0) be one of the pairs (6.5). Then the corresponding eigenfunction satisfies

$$\begin{aligned} \int_{-1}^{1} \operatorname{sgn}(x) y^2(x) dx &= \int_{-1}^{0} \cos^2((2l-1)\frac{\pi}{2}x)dx - \int_{-1}^{0} \cos^2((2k-1)\frac{\pi}{2}x)dx \\ &= \frac{1}{2(2l-1)\pi} \sin((2l-1\pi x)\Big|_0^1 + \frac{1}{2(2k-1)\pi} \sin((2k-1\pi x)\Big|_{-1}^0 \\ &= 0 \end{aligned}$$

Thus $\mu_n'(\lambda_0) = 0$. To find the second derivative we calculate $z = \frac{\partial}{\partial} y$ by solving the linear homogeneous equation

$$-z" = \lambda_0 \operatorname{sgn}(x) z + \mu_n(\lambda_0) z + \operatorname{sgn}(x) y, \quad z(-1) = z'(-1) = 0 \tag{6.7}$$

Although we do not show it here, this differential equation can be solved by variation of parameters to get solution

$$z(x) = \begin{cases} \frac{1}{\pi(2k-1)}(1+x)\sin((2k-1)\frac{\pi}{2}x) + y(x) & x < 0 \\ \frac{1}{\pi(2l-1)}(1-x)\sin((2l-1)\frac{\pi}{2}x) + y(x) & x \geq 0 \end{cases} \tag{6.8}$$

From the previous section, we get the the second derivative, at a local maximum, simplifies to

$$\frac{\partial^2}{\partial \lambda^2}\mu_n(\lambda) = -2\int_{-1}^{1} ryz / \int_{-1}^{1} y^2$$

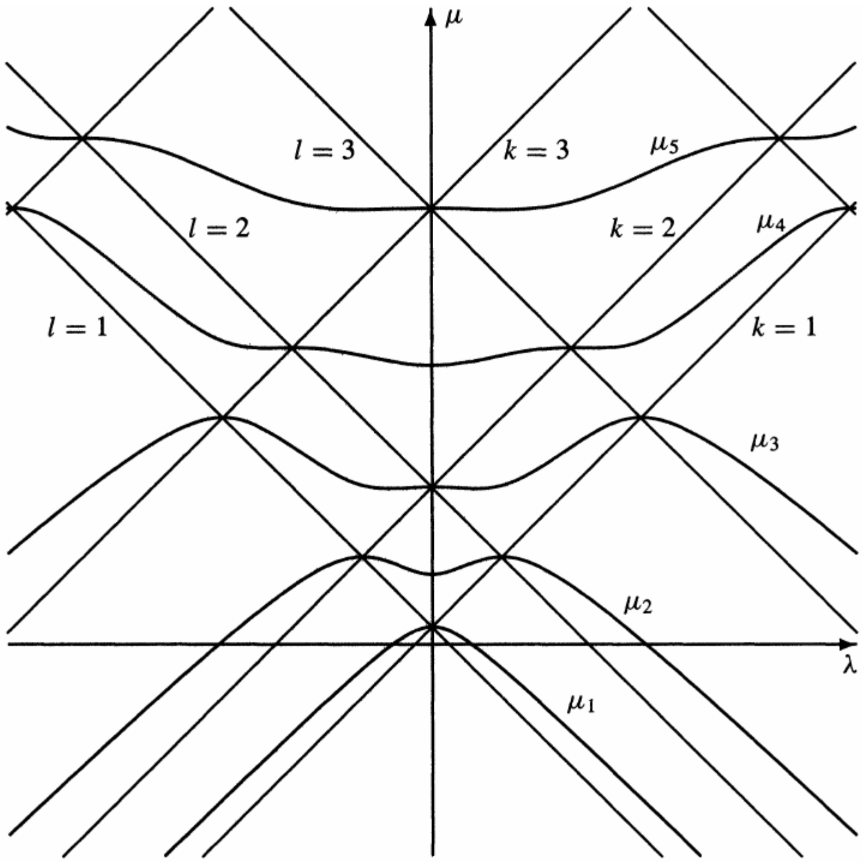
Then from the definitions of y and z above we get that

$$\frac{\partial^2}{\partial \lambda^2}\mu_n(\lambda) = -\frac{1}{\pi^2}\left(\frac{1}{(2k-1)^2} + \frac{1}{(2l-1)^2}\right) < 0$$

Thus, the eigen-curves are concave down at each intersection point. □

Thus, in conclusion, we may apply all previous arguments to get that the eigen-curves of Richardson's equation are continuous, differentiable, and the intersection of the m^{th} eigencurve can have at most, $2m$ intersections with any horizontal or diagonal line. Given, that at each intersection of the rectangular decomposition we have a local maximum and that for $\mu - \lambda < \frac{\pi^2}{4}$ the eigencurves are increasing, and for $\mu + \lambda < \frac{\pi^2}{4}$ the eigencurves are increasing, we may then accurately plot eigencurves which finishes our analysis. See figure (6.1) for a graph of the first five eigencurves of Richardson's Equation.

Figure 6.2: First 5 eigencurves of Richardson's equation [1]



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