THE COMPLETE AND ELEMENTARY SYMMETRIC FUNCTIONS AS QUOTIENTS OF DETERMINANTS

BY

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The standard definition of a Schur symmetric function, $s_\mu$, is as a quotient of a determinant $A_\mu$ with the Vandermonde determinant $a_\delta$. Other collections of symmetric functions include the elementary symmetric functions and the complete symmetric functions, both of which are defined in a multiplicative fashion. In this thesis we show how to construct the elementary symmetric functions, $e_\mu$, and the complete symmetric functions, $h_\mu$, as a quotient of determinants. Specifically, we create analogues of the determinant $A_\mu$ to do this. In each case, we give two different matrices that yield the required quotients. These matrices are natural analogues of the matrix $A_\mu$ that use the partially symmetric functions $e_a^{(i)}$ and $h_h^{(i)}$ in place of monomials $x_i^n$, respectively. Note that a determinant is a sum of positively and negatively signed terms. The elementary symmetric functions, $e_\mu$, and the complete symmetric functions, $h_\mu$, are non-negative sums. Thus terms in the determinants must cancel out. Our proof is combinatorial in that we show exactly how these terms cancel out, leaving the corresponding symmetric functions.

Kelly Lynn Kuykendall
Chapter 1: Introduction to Symmetric Functions

We generally follow Macdonald [2] in our development of symmetric functions. Let \( n \) be a positive integer. Let \( \mathbb{Z}[x_1, x_2, \cdots, x_n] \) be the ring of polynomials in the variables \( X = \{x_1, x_2, \cdots, x_n\} \) with coefficients from the integers \( \mathbb{Z} \). A partition \( \mu \) of \( k \), denoted by \( \mu \vdash k \), is a sequence \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \) where \( \mu_i \geq \mu_{i+1} \geq 0 \) for \( 1 \leq i \leq n-1 \) and \( \mu_1 + \mu_2 + \cdots + \mu_n = k \). In this thesis, the length of partitions will always be \( n \).

\( S_n \) denotes the symmetric group on \( n \) letters \( \{1, 2, \cdots, n\} \). The permutation

\[
\sigma = \begin{pmatrix}
1 & 2 & \cdots & n \\
\sigma_1 & \sigma_2 & \cdots & \sigma_n
\end{pmatrix} \in S_n
\]

is often represented by \( \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \). The action of \( \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in S_n \) on a polynomial

\[
P(X) = P(x_1, x_2, \cdots, x_n) \in \mathbb{Z}[x_1, x_2, \cdots, x_n]
\]

is given by

\[
\sigma P(x_1, x_2, \cdots, x_n) = P(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}).
\]

Recall that

\[
\Lambda_n = \mathbb{Z}[x_1, x_2, \cdots, x_n]^{S_n} = \{ P(X) \in \mathbb{Z}[x_1, x_2, \cdots, x_n] : \sigma P(X) = P(X) \ \forall \ \sigma \in S_n \}
\]

denotes the ring of symmetric functions; set \( \Lambda_n^k \) to be the subset of \( \Lambda_n \) of polynomial degree \( k \).

There are several well-known bases for \( \Lambda_n^k \) including the monomial symmetric functions, the elementary symmetric functions, the complete symmetric functions, and the Schur symmetric functions (see [2]). It is important to note that the Schur
functions, the elementary symmetric functions and the complete symmetric functions are defined using different techniques. The Schur functions are defined as quotients of determinants, while the elementary symmetric functions and the complete symmetric functions are defined in multiplicative fashions. The goal of this thesis is to show how the elementary symmetric functions and the complete symmetric functions can be computed in an analogous manner as the Schur functions.

In Chapter 1 Ferrers diagrams, tableaux, monomial symmetric functions, elementary symmetric functions, complete symmetric functions and Schur functions are defined; important properties are proven concerning these structures. In Chapter 2, matrices $B_\mu$ and $D_\mu$ are constructed whose quotients with the Vandermonde determinant $a_\delta$ yield the elementary symmetric function $e_\mu$ and the complete symmetric function $h_\mu$, respectively.

1.1 Ferrers Diagram and Tableaux

A Ferrers diagram is a visual representation of a given partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ composed of rows of cells, which visually resemble boxes. The first row of a partition in a Ferrers diagram has $\lambda_1$ cells in it, the second row has $\lambda_2$ cells in it and so on.

Example 1.1.1. With $n = 5$ and $\lambda = (4, 2, 2, 1)$, the Ferrers diagram of shape $\lambda = (4, 2, 2, 1, 0)$ is given by

\[ \begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & \\
\cdot & & & & \\
\cdot & & & & \\
\end{array} \]

$\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_n)$ is the conjugate of a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ if $\lambda'_i$ is the number of cells of the $i^{th}$ column from left to right of the Ferrers diagram of shape $\lambda$.

Example 1.1.2. With $n = 5$ and $\lambda = (2, 1, 1, 0)$, then $\lambda' = (3, 1, 0, 0, 0)$ since the first column of $\lambda$ has three cells, followed by the second column which has two cells.
This can be seen visually by the following Ferrers diagrams, which have the shape $\lambda$ and $\lambda'$ respectively:

Ferrers diagram of shape $\lambda = \begin{array}{c}
\end{array}$

Ferrers diagram of shape $\lambda' = \begin{array}{c}
\end{array}$

Let $c_{\lambda} = \{c_{i,j}\}$ be the collection of cells in a Ferrers diagram of shape $\lambda$ where $c_{i,j}$ is the cell in row $i$ and column $j$. A tableaux $T$ is a function $f_T : c_{\lambda} \to \{1, \ldots, n\}$. We often represent a tableaux by placing the image of the cell into the cell. Let $\mu_k$ be the number of occurances of $k$ in a tableaux $T$. The weight $\mu_T$ is defined by $\mu_T = (\mu_1, \ldots, \mu_n)$.

**Example 1.1.3.** The following is an example of a tableaux:

$$T = \begin{array}{c}
5 & 3 & 2 & 1 \\
3 & 8 \\
7
\end{array}$$

If $c_{i,j}$ is the cell in row $i$ and column $j$ then we can see $f(c_{1,1}) = 5$, $f(c_{1,2}) = 3$ and $f(c_{3,1}) = 7$. The weight of $T$ is $(1,1,2,0,1,0,1,1)$.

A tableaux $T$ is said to be semi-standard if $f_T(c_{i,j}) \leq f_T(c_{i,j+1})$ and $f_T(c_{i,j}) < f_T(c_{i+1,j})$ for $1 \leq i \leq \lambda_i$ and $1 \leq j \leq \lambda'_j$.

Define $K_{\lambda\mu}$ to be the number of semi-standard tableaux, $T$, of shape $\lambda$ and weight $\mu$. The Kostka matrix is given by

$$K = (K_{\lambda\mu})_{\lambda\mu}$$

where $\lambda$ and $\mu$ are partitions of $K$ listed in reverse lexicographic order [2].
Example 1.1.4. With \( n = 6, k = 5, \lambda = (4, 1, 0, 0, 0, 0) \) and \( \mu = (3, 1, 1, 0, 0, 0) \), then the only semi-standard fillings of the Ferrers diagram of shape \( \lambda \) and weight \( \mu \) are:

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This implies that

\[ K_{\lambda \mu} = 2. \]

1.2 Monomial Symmetric Functions

The monomial \( x^\mu \), with \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \) and \( \mu_i \geq 0 \), is defined as

\[ x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}. \]

If \( \nu = (\nu_1, \nu_2, \cdots, \nu_n) = \sigma(\mu) = (\mu_{\sigma_1}, \mu_{\sigma_2}, \cdots, \mu_{\sigma_n}) \) where \( \sigma \in S_n \) then \( \nu \) is a permutation of \( \mu \). In such circumstances \( \sigma(x^\mu) = x^\nu \).

Example 1.2.1. Let \( n = 4 \) and \( \mu = (2, 1, 0, 0) \) then

\[ x^\mu = x_1^2 x_2. \]

Note that the permutation \( \nu = (0, 2, 0, 1) \) of \( \mu \) yields

\[ x^\nu = x_2^2 x_4. \]

Note that \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \in S_n \) and \( \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \in S_n \) are elements of \( S_n \) such that \( x^\nu = \sigma(x^\mu) = \gamma(x^\mu) \).

The monomial symmetric function \( m_\mu \) is defined as

\[ m_\mu = \sum_{\nu} x^\nu \]

where the summation is over all distinct permutations \( \nu \) of the partition \( \mu \vdash k \).
Example 1.2.2. Let $n=4$ and $\mu = (2, 1, 0, 0)$. The monomial symmetric function $m_\mu$ is

$$m_\mu = x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_1x_2^2 + x_2x_3 + x_2x_4 + x_1x_3^2 + x_2x_3 + x_1x_4^2 + x_2x_4 + x_3x_4.$$  

Let $P(x_1, x_2, \cdots, x_n) = \sum_{\alpha} u_\alpha x^\alpha \in \Lambda^k_n$. If $\beta$ is a permutation of a partition $\gamma$ then since $\sigma(P(X)) = P(X)$ for all $\sigma \in S_n$, then $u_\beta = u_\gamma$. Thus, we can write

$$P(x_1, x_2, \cdots, x_n) = \sum_\psi u_\psi m_\psi.$$  

So, the monomial symmetric functions

$$\mathfrak{m}_n^k = \{m_\mu : \mu \triangleright k, \text{length}(\mu) = n\}$$

span $\Lambda^k_n$.

Suppose $\sum_\alpha u_\alpha m_\alpha = 0$ where the summation is over all partitions $\alpha$ of $k$. If $\beta$ is a distinct partition from $\alpha$ then $m_\alpha$ and $m_\beta$ are sums of distinct monomials. Thus $u_\alpha = 0$ for all partitions $\alpha$ and the monomial symmetric functions are also linearly independent. Therefore the monomial symmetric functions both span $\Lambda^k_n$ and are linearly independent in $\Lambda^k_n$; we have the following theorem.

**Theorem 1.2.3.** The collection $\mathfrak{m}_n^k$ is a basis for $\Lambda^k_n$. Thus the dimension of $\Lambda^k_n$ as a vector space equals the number of partitions of $k$ of length $n$.

1.3 Elementary Symmetric Functions

We define the elementary symmetric function $e_\mu$, $\mu = (\mu_1, \cdots, \mu_n) \triangleright k$ and $k \leq n - 1$, to be

$$e_\mu = e_{\mu_1}e_{\mu_2} \cdots e_{\mu_n}$$

with

$$e_\alpha = \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq n} x_{j_1}x_{j_2} \cdots x_{j_n}.$$
where $e_0 = 1$. Note that $e_n = m_{(1^n)}$.

**Example 1.3.1.** With $n=3$, the elementary symmetric function $e_2$ is

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3$$

and

$$e_{2,1} = e_2e_1$$

$$= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3)$$

$$= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 3x_1x_2x_3.$$

Macdonald [2] proves the following theorem that yields a more important relationship between the elementary symmetric functions and the monomial symmetric functions.

**Theorem 1.3.2.** Let $\lambda$ be a partition and $\lambda'$ its conjugate. Then

$$e_{\lambda'} = m_{\lambda} + \sum_{\mu} w_{\lambda \mu} m_{\mu}$$

where $w_{\lambda \mu}$ are non-negative integers and the sum over partitions $\mu$ are later than $\lambda$ in reverse lexicographic ordering.

Set $E^k_n = \{ e_\mu : \mu \vdash k, \text{length}(\mu) = n \}$. Theorem 1.3.2 establishes a triangular relationship between the elementary symmetric functions and the monomial symmetric functions. Note that the collection of elementary symmetric functions and the monomial symmetric functions are both indexed by partitions. Therefore we know

$$|E^k_n| = |M^k_n|.$$

Since the monomial symmetric functions form a basis for $\Lambda^k_n$, the elementary symmetric functions must also form a basis. Thus we have

**Corollary 1.3.3.** The collection $E^k_n$ is a basis for $\Lambda^k_n$. 6
1.4 Schur Functions

Set $V_n$ to be the $n \times n$ matrix $V_n = (V_{i,j})_{1 \leq i,j \leq n}$ where

$$V_{i,j} = x_i^{n-j};$$

$a_δ = \det(V_n)$ is the Vandermonde determinant. Now, let $A_μ$ be the $n \times n$ matrix $A_μ = (A_{i,j})_{1 \leq i,j \leq n}$ where

$$A_{i,j} = x_i^{n-j+\mu_j} = x_i^{-j} x_i^{\mu_j}. \quad (1.1)$$

The Schur symmetric function $s_μ$ is defined by

$$s_μ = \frac{\det(A_μ)}{a_δ}.$$

Example 1.4.1. With $n = 5$ and $μ = (3,1,1,0,0)$ then

$$A_{(3,1,1,0,0)} = \begin{bmatrix} x_1^3 & x_1^4 & x_1^3 & x_1 & 1 \\ x_2^3 & x_2^4 & x_2^3 & x_2 & 1 \\ x_3^3 & x_3^4 & x_3^3 & x_3 & 1 \\ x_4^3 & x_4^4 & x_4^3 & x_4 & 1 \\ x_5^3 & x_5^4 & x_5^3 & x_5 & 1 \end{bmatrix}.$$

The Schur function is

$$s_{3,1,1,0,0} = \frac{\det(A_{3,1,1,0,0})}{a_δ}$$

$$= x_1^3 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_2 x_5 + x_1^3 x_3 x_4 + x_1^3 x_3 x_5 + x_1^4 x_4 x_5 +$$

$$x_1^2 x_2^2 x_3 + x_1 x_2^2 x_4 + x_1^2 x_2 x_5 + x_1^3 x_3 x_5 + 3x_1^2 x_2 x_3 x_4 +$$

$$3x_1^2 x_2 x_3 x_5 + x_1^2 x_2 x_4 + 3x_1^2 x_2 x_4 x_5 + x_1^2 x_2 x_5 + x_1^3 x_3 x_4 +$$

$$x_1^3 x_3 x_5 + x_1^2 x_3 x_4 + x_1^2 x_3 x_4 x_5 + x_1^2 x_3 x_5 + x_1^2 x_4 x_5 +$$

$$x_1^2 x_4 x_5 + x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + x_1^2 x_2 x_5 + x_1^2 x_2 x_5 +$$

$$3x_1^2 x_3 x_4 + 3x_1^2 x_3 x_5 + x_1 x_2^2 x_4 + x_1 x_2^2 x_5 + x_1 x_2^2 x_5 +$$

$$3x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + x_1^2 x_2 x_5 + x_1^2 x_2 x_5 +$$

$$3x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + x_1 x_2^2 x_5 + x_1 x_2^2 x_5 +$$
\[x_1 x_2 x_3 + 3 x_1 x_2 x_3^2 x_4 + 3 x_1 x_2 x_3 x_5 + 3 x_1 x_2 x_4 x_5^2 + 6 x_1 x_2 x_3 x_4 x_5 + \]
\[3 x_1 x_2 x_3 x_5^2 + x_1 x_2 x_4^3 + 3 x_1 x_2 x_4 x_5 + 3 x_1 x_2 x_4 x_5^2 + x_1 x_2 x_5^3 + \]
\[x_1 x_3 x_4 + x_1 x_3^2 x_5 + 3 x_1 x_3 x_4 x_5 + x_1 x_3 x_4 x_5^2 + x_1 x_3 x_5^3 + x_1 x_4 x_5^3 + \]
\[x_1 x_3 x_4 + 3 x_1 x_3 x_4 x_5 + 3 x_1 x_3 x_4 x_5^2 + x_1 x_3 x_5^3 + x_1 x_4 x_5^3 + \]
\[x_1 x_4 x_5^3 + x_1 x_4 x_5^3 + x_2 x_3 x_4 + x_2 x_3 x_5 + x_2 x_4 x_5 + \]
\[x_2 x_3 x_4^2 + x_2 x_3 x_5^2 + x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_5^2 + x_2 x_3 x_5^2 + \]
\[x_2 x_4 x_5 + x_2 x_4 x_5 + x_2 x_3 x_4 + x_2 x_3 x_5 + x_2 x_4 x_5 + \]
\[3 x_2 x_3 x_4 x_5 + x_2 x_3 x_5^2 + x_2 x_3 x_4 + 3 x_2 x_3 x_4 x_5 + 3 x_2 x_3 x_4 x_5 + \]
\[x_2 x_3 x_5^3 + x_2 x_4 x_5 + x_2 x_4 x_5^2 + x_2 x_4 x_5^2 + x_2 x_4 x_5 + \]
\[x_2 x_4 x_5 + x_2 x_4 x_5^2 + x_3 x_4 x_5 + x_3 x_4 x_5^2 + x_3 x_4 x_5^3.\]

If \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \) is the weight of a tableaux \( T \), then

\[x^T = x_1^{\mu_1} x_2^{\mu_2} \cdot x_n^{\mu_n}.\]

Note that the Schur symmetric function can also be computed by

\[s_\lambda = \sum_T x^T\]

where the summation is over all semi-standard tableaux of shape \( \lambda \) (see [2]).

Recall that \( K_{\lambda \mu} \) is the number of semi-standard tableaux of shape \( \lambda \) and weight \( \mu \). Macdonald [2] shows that the Schur functions can be written as sums of monomial symmetric functions in the following manner:

**Theorem 1.4.2.** The Schur functions can be expanded in terms of monomial symmetric functions by the following formula

\[s_\lambda = \sum_{\mu} K_{\lambda \mu} m_\mu.\]
Example 1.4.3. With \( n = 5 \) and \( \mu = (3, 1, 1, 0, 0) \) then

\[
\varsigma_{3, 1, 1, 0, 0} = m_{3, 1, 1, 0, 0} + m_{2, 2, 1, 0, 0} + 3m_{2, 1, 1, 1, 0} + 6m_{1, 1, 1, 1, 1}.
\]

Note that

\[
m_{3, 1, 1, 0, 0} = x_1^2 x_2 x_3 + x_1^3 x_2 x_4 + x_1^2 x_2 x_5 + x_1^3 x_3 x_4 + x_1^3 x_3 x_5 + x_1^3 x_4 x_5 + x_1^3 x_3 x_3 + x_1 x_2 x_4 + \]
\[
x_1 x_2^2 x_5 + x_2^3 x_3 x_4 + x_2^3 x_3 x_5 + x_2 x_4 x_5 + x_1 x_2 x_3^3 + x_1 x_3 x_3 x_4 + x_1 x_3 x_3 x_5 + x_2 x_3 x_4 + \]
\[
x_2 x_3^2 x_4 + x_2 x_3 x_5 + x_1 x_2 x_4 + x_1 x_3 x_4^3 + x_1 x_4 x_4 x_5 + x_2 x_3 x_4^3 + x_3 x_4 x_5 + x_1 x_2 x_5 + \]
\[
x_1 x_3 x_5^3 + x_1 x_4 x_5^3 + x_2 x_3 x_5^3 + x_2 x_4 x_5^3 + x_3 x_5^3.
\]

\[
m_{2, 2, 1, 0, 0} = x_1^2 x_2^2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_2 x_5 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4 + x_1^2 x_3 x_5 + x_1^2 x_4 x_5 + \]
\[
x_2^2 x_4 x_5 + x_2 x_4 x_5 + x_2 x_4 x_5 + x_2 x_4 x_5 + x_2 x_3 x_4 + x_2 x_3 x_5 + x_2 x_3 x_4 + \]
\[
x_2 x_3 x_4 + x_2 x_3 x_5 + x_1 x_2 x_4 + x_2 x_3 x_5 + x_2 x_4 x_5 + x_3 x_4 x_5.
\]

\[
3m_{2, 1, 1, 1, 0} = 3(x_1^2 x_2 x_3 x_4 + x_1^2 x_2 x_3 x_5 + x_1^2 x_2 x_4 x_5 + x_1^2 x_3 x_4 x_5 + x_1^2 x_3 x_5 + x_1^2 x_3 x_5 + \]
\[
x_1 x_2^2 x_4 x_5 + x_2^2 x_3 x_4 x_5 + x_2 x_3^2 x_4 x_5 + x_2 x_3^2 x_4 x_5 + x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 + \]
\[
x_1 x_2 x_3 x_4^2 + x_1 x_2 x_4 x_5 + x_1 x_3 x_4 x_5 + x_2 x_3 x_4^2 x_5 + x_1 x_2 x_3 x_4^2 + x_1 x_2 x_4 x_5.
\]

\[
6m_{1, 1, 1, 1, 1} = 6(x_1 x_2 x_3 x_4 x_5).
\]

This yields the same result as Example 1.4.1.

Note that the matrix \( (K_{\mu}) \) is unitriangular and thus \( \det(K_{\mu}) = 1 \) [2]. With

\[
\mathcal{G}_n^k = \{ \varsigma_{\mu} : \mu \vdash k, \text{length}(\mu) = n \},
\]

we have \( |\mathcal{G}_n^k| = |\mathcal{M}_n^k| = |\mathcal{C}_n^k| \) and Theorem 1.4.2 now yields

Corollary 1.4.4. The collection \( \mathcal{G}_n^k \) is a basis for \( \Lambda_n^k \).
1.5 Complete Symmetric Functions

The complete symmetric function, $h_a$ can be defined by

$$h_a = \sum_{a_1 + a_2 + \cdots + a_n = a; a_i \geq 0} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

with $h_0 = 1$. Then, with $\mu = (\mu_1, \cdots, \mu_n) \vdash k, k \leq n - 1$, we set

$$h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_n}.$$  

Define

$$S^k_n = \{ h_\mu : \mu \vdash k, \text{length}(\mu) = n \}.$$

**Example 1.5.1.** For $n=3$ the complete symmetric function $h_2$ is

$$h_2 = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3$$

and

$$h_{2,1} = h_2 h_1 = (x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3) = x_1^3 + 2x_1 x_2^2 + 2x_1 x_3^2 + 2x_1^2 x_2 + 2x_1 x_2 x_3 + 3x_1 x_2 x_3 + x_3^3 + 2x_2 x_3^2 + 2x_2^2 x_3 + x_3^3.$$

Macdonald [2] also establishes a relationship between the complete symmetric functions and the monomial symmetric functions.

**Theorem 1.5.2.** The complete symmetric functions can be expanded in terms of monomial symmetric functions by

$$h_\lambda = \sum_\mu \omega_{\lambda \mu} m_\mu$$

and the matrix $\Omega_{\lambda \mu} = (\omega_{\lambda \mu}) = (K'_{\lambda \mu}) K_{\lambda \mu}$, where $K_{\lambda \mu}$ is the Kostka matrix and $K'_{\lambda \mu}$ denotes its transpose.
Since $K_{\lambda\mu}$ is unitriangular, $det(K_{\lambda\mu}) = 1$. We therefore have,

$$det(e_{\lambda\mu}) = det(K'_{\lambda\mu}K_{\lambda\mu}) = det(K'_{\lambda\mu}) det(K_{\lambda\mu}) = 1.$$ 

It is also important to note that $|\Sigma^k_n| = |\mathcal{M}^k_n| = |\mathcal{E}^k_n|$ and we have the following Corollary.

**Corollary 1.5.3.** The collection $\mathcal{F}^k_n$ is a basis for $\Lambda^k_n$. 
Chapter 2: Symmetric Functions as Quotients

2.1 Determinantal Definitions

With $\mu = (\mu_1, \cdots, \mu_n) \vdash k, k \leq n - 1$, recall the elementary symmetric function $e_{\mu}$ is defined by

$$e_{\mu} = e_{\mu_1}e_{\mu_2}\cdots e_{\mu_n}$$

with

$$e_a = \sum_{1 \leq j_1 < j_2 < \cdots < j_a} x_{j_1}x_{j_2}\cdots x_{j_a}$$

where $e_0 = 1$. Our goal in this section is to define matrices $B_{\mu}$ and $C_{\mu}$ so that

$$e_{\mu} = \pm \frac{\det(B_{\mu})}{a_{\delta}} = \pm \frac{\det(C_{\mu})}{a_{\delta}}$$

where $a_{\delta}$ is the Vandermonde determinant.

Let

$$e_a^{(i)} = e_a|_{x_i=0}, \quad (2.1)$$

where $|_{x_i=0}$ signifies that we set $x_i = 0$ in $e_a$; $e_a^{(i)}$ is the $a^{th}$ elementary symmetric function in the variables $\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}$.

Example 2.1.1. Recall that if $n=5$, then the elementary symmetric function $e_2$ is

$$e_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5.$$ 

Setting $x_3 = 0$ yields

$$e_2^{(3)} = x_1x_2 + x_1x_4 + x_1x_5 + x_2x_4 + x_2x_5 + x_4x_5.$$
Let \( B_\mu = (B_{i,j})_{1 \leq i,j \leq n} \) be the \( n \times n \) matrix where

\[
B_{i,j} = e_{n-j}^{(i)} e_{\mu_j}^{(i)}.
\] (2.2)

Additionally, set \( C_\mu = (C_{i,j})_{1 \leq i,j \leq n} \), where

\[
C_{i,j} = x_i^{j-1} e_{\mu_j}^{(i)}.
\] (2.3)

It is important to note the connections between the definition of the matrix \( A_\mu \) in Equation (1.1) that was used in the definition of the Schur functions \( s_\mu \) and the definition of the matrix \( B_\mu \) that was given in Equation (2.2); we have substituted \( e_{\alpha} \) for \( x_\alpha \). One of the main theorems of this thesis is the following:

**Theorem 2.1.2.** Let \( \mu \vdash k \leq n-1 \), \( B_\mu = (B_{i,j})_{1 \leq i,j \leq n} \) with \( B_{i,j} = e_{n-j}^{(i)} e_{\mu_j}^{(i)} \) and \( C_\mu = (C_{i,j})_{1 \leq i,j \leq n} \), with \( C_{i,j} = x_i^{j-1} e_{\mu_j}^{(i)} \). Then,

\[
det(B_\mu) = det(C_\mu) = \pm e_\mu a_\delta
\] (2.4)

where \( a_\delta \) is the Vandermonde determinant and \( e_\mu \) is the elementary symmetric function corresponding to shape \( \mu \). Particularly,

\[
e_\mu = \pm \frac{\det(B_\mu)}{a_\delta} = \pm \frac{\det(C_\mu)}{a_\delta}.
\] (2.5)

Similarly, we also want to define matrices \( D_\mu \) and \( E_\mu \) whose quotients with the Vandermonde determinants \( a_\delta \) yield the complete symmetric functions \( h_{\mu} \). First, we let

\[
h_{a}^{(i)} = h_{a} |_{x_i = 0},
\] (2.6)

where \( |_{x_i = 0} \) signifies that we set \( x_i = 0 \) in \( h_{a} \); \( h_{a}^{(i)} \) is the \( a^{th} \) complete symmetric function in the variables \( \{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\} \).
Example 2.1.3. Recall that if \( n=3 \) and \( k=2 \), then
\[
h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3.
\]

Thus,
\[
h_2^{(2)} = x_1^2 + x_3^2 + x_1x_3.
\]

Set \( D_\mu \) to be the \( n \times n \) matrix
\[
D_{i,j} = e^{(i)}_{n-j} h_{\mu,j}^{(i)}.
\]

Additionally, let \( E_\mu = (E_{i,j})_{1 \leq i,j \leq n} \), where
\[
E_{i,j} = x_{j-1}^i h_{\mu,j}^{(i)}.
\]

Analogously to Theorem 2.1.2, it is important to note the similarities between the definitions of \( A_\mu, D_\mu, \) and \( E_\mu \). Now we have substituted \( h_{\mu,a}^{(i)} \) for \( x_i^a \), which yields the remaining main theorem of this thesis:

**Theorem 2.1.4.** Let \( \mu \vdash k \leq n-1 \), \( D_\mu = (D_{i,j})_{1 \leq i,j \leq n} \), with \( D_{i,j} = e^{(i)}_{n-j} h_{\mu,j}^{(i)} \) and \( E_\mu = (E_{i,j})_{1 \leq i,j \leq n} \), with \( E_{i,j} = x_{j-1}^i h_{\mu,j}^{(i)} \). Then,
\[
det(D_\mu) = det(E_\mu) = \pm h_\mu a_\delta,
\]
where \( a_\delta \) is the Vandermonde determinant and \( h_\mu \) is the complete symmetric function corresponding to shape \( \mu \). Particularly,
\[
h_\mu = \pm \frac{det(D_\mu)}{a_\delta} = \pm \frac{det(E_\mu)}{a_\delta}.
\]

### 2.2 Important Lemmas

In order to prove Theorem 2.1.2 and Theorem 2.1.4, we need to prove some important lemmas. Define
\[
\Delta(P(X)) = \sum_{\sigma \in S_n} sgn(\sigma) \sigma(P(X))
\]
where \( P(X) \in \mathbb{Z}[x_1, \ldots, x_n] \). Let \((a, b)\) be the permutation of \( S_n \) that only interchanges \( a \) and \( b \); such permutations are normally referred to as transpositions. Suppose

\[(a, b) \ P(X) = P(X).\]

However, \( \{\epsilon, (a, b)\} \) is a subgroup of \( S_n \) where \( \epsilon \) is the identity permutation. Let \( \tau \) be the set of coset representatives of \( \{\epsilon, (a, b)\} \subset S_n \) \([1]\), then

\[
\sum_{\sigma \in S_n} \text{sgn}(\sigma) \ \sigma(P(X)) = \sum_{\gamma \in \tau} \text{sgn}(\gamma) \ \gamma \ (\epsilon - (a, b)) \ (P(X)) = 0,
\]

where we interpret \( \epsilon - (a, b) \) as living in the group algebra \( \mathbb{Z}[S_n] \).

Note that

**Example 2.2.1.** If \( P(X) = x_2x_3 \), then \((2, 3)\) is the transposition \( \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \) and

\[
(2, 3) \ P(X) = P(X).
\]

In \( S_3 \), the subgroup \( \{\epsilon, (2, 3)\} \) has coset representatives

\[
\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}.
\]

Thus, we have shown:

**Lemma 2.2.2.** If

\[(a, b) \ P(X) = P(X)\]

for any \((a, b) \in S_n\) then

\[
\Delta(P(X)) = 0.
\]
\[ \det(B_{\mu}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \begin{pmatrix} e_{n-1}^{(1)} e_{n-2}^{(2)} \cdots e_0^{(n)} e_{\mu_1}^{(1)} e_{\mu_2}^{(2)} \cdots e_{\mu_n}^{(n)} \end{pmatrix}. \] (2.11)

Since \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \vdash k = n - p \), some integer \( p \geq 1 \),

\[ e_{n-1}^{(1)} e_{n-2}^{(2)} \cdots e_0^{(n)} e_{\mu_1}^{(1)} e_{\mu_2}^{(2)} \cdots e_{\mu_n}^{(n)} \]

has polynomial degree

\[ 0 + 1 + 2 + \cdots + (n - 1) + (n - p) = \frac{1}{2} n(n + 1) - p. \]

Thus,

**Lemma 2.2.3.** If \( \mu \vdash k = n - p \), \( p \) a positive integer, and if \( \det(B_{\mu}) \neq 0 \) then the degree of \( \det(B_{\mu}) \) as a polynomial is at most \( \frac{1}{2} n(n + 1) - p \).

Similarly, note that

\[ \det(D_{\mu}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \begin{pmatrix} e_{n-1}^{(1)} e_{n-2}^{(2)} \cdots e_0^{(n)} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} \end{pmatrix}. \]

Then since \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \vdash k = n - p \), some integer \( p \geq 1 \),

\[ e_{n-1}^{(1)} e_{n-2}^{(2)} \cdots e_0^{(n)} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} \]

has polynomial degree

\[ 0 + 1 + 2 + \cdots + (n - 1) + (n - p) = \frac{1}{2} n(n + 1) - p. \]

This implies an analogous result to Lemma 2.2.3:

**Lemma 2.2.4.** If \( \mu \vdash k = n - p \), \( p \) a positive integer, and if \( \det(D_{\mu}) \neq 0 \) then the degree of \( \det(D_{\mu}) \) as a polynomial is at most \( \frac{1}{2} n(n + 1) - p \).
Define the matrix $M_n = (M_{i,j})_{1 \leq i,j \leq n}$ by $M_{i,j} = e_{n-j}^{(i)}$. Macdonald ([2], pages 41, 42) showed the following:

**Lemma 2.2.5.**

$$\det(M_n) = (-1)^{\frac{n(n-1)}{2}} \det(V_n) = (-1)^{\frac{n(n-1)}{2}} a_\delta$$

where $a_\delta$ is the Vandermonde determinant.

### 2.3 Elementary Functions as Quotients of Determinants

Let $[n] = \{1, 2, \cdots, n\}$ and $Q_i = \{1, 2, \cdots, i-1, i+1, \cdots, n\}$. With $\xi \subseteq Q_i$ and $x_\xi = \prod_{j \in \xi} x_j$, note that

$$e_a^{(i)} = \sum_{\xi \subseteq Q_i \atop |\xi| = a} x_\xi.$$

Then using Equation (2.11), $\det(B_\mu)$ can be expanded as follows:

$$\det(B_\mu) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma \left( e_{n-1}^{(1)} e_{n-2}^{(2)} \cdots e_0^{(n)} e_{\mu_1}^{(1)} e_{\mu_2}^{(2)} \cdots e_{\mu_n}^{(n)} \right)$$

(2.12)

$$= \sum_{\sigma \in S_n} \sum_{\xi_1, \xi_2, \cdots, \xi_n \atop \xi_i \subseteq Q_i \atop |\xi_i| = n-i} \text{sgn}(\sigma) (x_{\xi_1} x_{\xi_2} \cdots x_{\xi_n} e_{\mu_1}^{(1)} \cdots e_{\mu_n}^{(n)})$$

(2.13)

$$= \sum_{\xi_1, \xi_2, \cdots, \xi_n \atop \xi_i \subseteq Q_i \atop |\xi_i| = n-i} \sum_{\sigma \in S_n} \text{sgn}(\sigma) (x_{\xi_1} x_{\xi_2} \cdots x_{\xi_n} e_{\mu_1}^{(1)} \cdots e_{\mu_n}^{(n)})$$

(2.14)

$$= \sum_{\xi_1, \xi_2, \cdots, \xi_n \atop \xi_i \subseteq Q_i \atop |\xi_i| = n-i} \Delta(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} e_{\mu_1}^{(1)} e_{\mu_2}^{(2)} \cdots e_{\mu_n}^{(n)})$$

(2.15)

where

$$x_1^{a_1} \cdots x_n^{a_n} = x_{\xi_1} x_{\xi_2} \cdots x_{\xi_n}.$$
We will represent $\Delta(x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \mu_1^{(1)} \cdots \mu_n^{(n)})$ by putting the integer $i \in [1, n]$ in the cell $(x^a, e_{\mu}(j))$ of the Elementary Expansion Diagram (EED) $Z$ given in (2.16):

$$Z =\begin{array}{cccccc}
\varepsilon^{(1)}_{0} & \varepsilon^{(1)}_{1} & \varepsilon^{(1)}_{2} & \cdots & \varepsilon^{(1)}_{n-1} & \varepsilon^{(1)}_{n} \\
\varepsilon^{(2)}_{0} & \varepsilon^{(2)}_{1} & \varepsilon^{(2)}_{2} & \cdots & \varepsilon^{(2)}_{n-1} & \varepsilon^{(2)}_{n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\varepsilon^{(n)}_{0} & \varepsilon^{(n)}_{1} & \varepsilon^{(n)}_{2} & \cdots & \varepsilon^{(n)}_{n-1} & \varepsilon^{(n)}_{n} \\
\end{array}$$

(2.16)

The collection

$$G = \left\{(x_1^{a_1}, e_{\mu_1}^{(1)}), (x_2^{a_2}, e_{\mu_2}^{(2)}), \ldots, (x_n^{a_n}, e_{\mu_n}^{(n)})\right\}$$

are the coordinates of the polynomial $\Delta(x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \mu_1^{(1)} \cdots \mu_n^{(n)})$ and we set

$$\Delta(Z) = \Delta(G) = \Delta(x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \mu_1^{(1)} \cdots \mu_n^{(n)}).$$

Example 2.3.1.

$$\Delta(x_1^2 x_2^2 x_3^2 x_4^4 x_5^3 x_6^4 e_3^{(1)} e_2^{(2)} e_1^{(3)} e_0^{(4)} e_0^{(5)} e_0^{(6)} e_0^{(7)})$$

corresponds to the coordinates

$$G = \left\{(x_1^2, e_3^{(1)}), (x_2^2, e_2^{(2)}), (x_3^2, e_1^{(3)}), (x_4^2, e_1^{(4)}), (x_5^3, e_0^{(5)}), (x_6^4, e_0^{(6)}), (x_7^4, e_0^{(7)})\right\}$$

and

$$Z = \begin{array}{cccccc}
\varepsilon^{(1)}_{0} & \varepsilon^{(1)}_{1} & \varepsilon^{(1)}_{2} & \cdots & \varepsilon^{(1)}_{n-1} & \varepsilon^{(1)}_{n} \\
\varepsilon^{(2)}_{0} & \varepsilon^{(2)}_{1} & \varepsilon^{(2)}_{2} & \cdots & \varepsilon^{(2)}_{n-1} & \varepsilon^{(2)}_{n} \\
\varepsilon^{(3)}_{0} & \varepsilon^{(3)}_{1} & \varepsilon^{(3)}_{2} & \cdots & \varepsilon^{(3)}_{n-1} & \varepsilon^{(3)}_{n} \\
\varepsilon^{(4)}_{0} & \varepsilon^{(4)}_{1} & \varepsilon^{(4)}_{2} & \cdots & \varepsilon^{(4)}_{n-1} & \varepsilon^{(4)}_{n} \\
\varepsilon^{(5)}_{0} & \varepsilon^{(5)}_{1} & \varepsilon^{(5)}_{2} & \cdots & \varepsilon^{(5)}_{n-1} & \varepsilon^{(5)}_{n} \\
\varepsilon^{(6)}_{0} & \varepsilon^{(6)}_{1} & \varepsilon^{(6)}_{2} & \cdots & \varepsilon^{(6)}_{n-1} & \varepsilon^{(6)}_{n} \\
\varepsilon^{(7)}_{0} & \varepsilon^{(7)}_{1} & \varepsilon^{(7)}_{2} & \cdots & \varepsilon^{(7)}_{n-1} & \varepsilon^{(7)}_{n} \\
\end{array}$$

(2.17)

Now, since $Z$ has 3 and 4 in the same cell, we have

$$(3, 4)(x_1^2 x_2^2 x_3^2 x_4^2 x_5^3 x_6^4 x_7^4 e_3^{(1)} e_2^{(2)} e_1^{(3)} e_0^{(4)} e_0^{(5)} e_0^{(6)} e_0^{(7)})$$

$$= (x_1^2 x_2^2 x_3^2 x_4^2 x_5^3 x_6^4 x_7^4 e_3^{(1)} e_2^{(2)} e_1^{(3)} e_0^{(4)} e_0^{(5)} e_0^{(6)} e_0^{(7)})$$
where $(3,4)$ is the permutation that interchanges 3 and 4. Thus by Lemma 2.2.2 we have
\[
\Delta(x_1^2 x_2^2 x_3^2 x_4^2 x_5^4 x_6^4 x_7^4 \epsilon_3^{(1)} \epsilon_2^{(2)} \epsilon_1^{(3)} \epsilon_1^{(4)} \epsilon_0^{(5)} \epsilon_0^{(6)} \epsilon_0^{(7)}) = 0. \qed
\]

It is not hard to see that
\[
e_{b+1} = x_i \epsilon_b + \epsilon_{b+1}^{(i)}.
\]

In particular, this gives the following result.

**Lemma 2.3.2.**
\[
x_i^a \epsilon_b^{(i)} = x_i^{a-1} \epsilon_{b+1} - x_i^{a-1} \epsilon_{b+1}^{(i)}.
\]

(2.18)

Lemma 2.3.2 implies an algorithm for moving entries west (left) in our Elementary Expansion Diagram. Suppose
\[
G = \left\{ (x_1^{a_1}, \epsilon_b^{(1)}), \ldots, (x_i^{a_{i-1}}, \epsilon_b^{(i-1)}), (x_i^{a_i}, \epsilon_b^{(i)}), (x_i^{a_{i+1}}, \epsilon_b^{(i+1)}), \ldots, (x_n^{a_n}, \epsilon_b^{(n)}) \right\},
\]

(2.19)

\[
G_1 = \left( G - \left\{ (x_i^{a_i}, \epsilon_b^{(i)}) \right\} \right) \cup \left\{ (x_i^{a_i-1}, \epsilon_0^{(i)}) \right\}
\]

\[
= \left\{ (x_1^{a_1}, \epsilon_b^{(1)}), \ldots, (x_i^{a_{i-1}}, \epsilon_b^{(i-1)}), (x_i^{a_i-1}, \epsilon_0^{(i)}), (x_i^{a_{i+1}}, \epsilon_b^{(i+1)}), \ldots, (x_n^{a_n}, \epsilon_b^{(n)}) \right\},
\]

(2.20)

and
\[
G_2 = \left( G - \left\{ (x_i^{a_i}, \epsilon_b^{(i)}) \right\} \right) \cup \left\{ (x_i^{a_i-1}, \epsilon_b^{(i+1)}) \right\}
\]

\[
= \left\{ (x_1^{a_1}, \epsilon_b^{(1)}), \ldots, (x_i^{a_{i-1}}, \epsilon_b^{(i-1)}), (x_i^{a_i-1}, \epsilon_b^{(i)}), (x_i^{a_{i+1}}, \epsilon_b^{(i+1)}), \ldots, (x_n^{a_n}, \epsilon_b^{(n)}) \right\}.
\]

(2.21)

Now, since $\epsilon_b \in \Lambda_n, \sigma(\epsilon_b) = \epsilon_b$ for all $\sigma \in S_n$,
\[
\Delta(\epsilon_b P(X)) = \epsilon_b \Delta(P(X)),
\]

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and
\[ \Delta(G) = e_5 \Delta(G_1) - \Delta(G_2). \]  

(2.22)

**Example 2.3.3.** From Example 2.3.1, we have
\[ G = \left\{ (x_1^3, e_2^{(1)}), (x_2^2, e_2^{(2)}), (x_3^2, e_1^{(3)}), (x_4^2, e_1^{(4)}), (x_5^2, e_0^{(5)}), (x_6^3, e_0^{(6)}), (x_7^4, e_0^{(7)}) \right\}. \]

Using Lemma 2.3.2, we can create two new sets \( G_1 \) and \( G_2 \) by replacing \( (x_1^3, e_2^{(1)}) \) with \( (x_1^2, e_0^{(1)}) \) and \( (x_1^2, e_3^{(1)}) \), respectively, yielding
\[ G_1 = \left\{ (x_1^2, e_0^{(1)}), (x_2^2, e_2^{(2)}), (x_3^2, e_1^{(3)}), (x_4^2, e_1^{(4)}), (x_5^2, e_0^{(5)}), (x_6^3, e_0^{(6)}), (x_7^4, e_0^{(7)}) \right\} \]
and
\[ G_2 = \left\{ (x_1^2, e_3^{(1)}), (x_2^2, e_2^{(2)}), (x_3^2, e_1^{(3)}), (x_4^2, e_1^{(4)}), (x_5^2, e_0^{(5)}), (x_6^3, e_0^{(6)}), (x_7^4, e_0^{(7)}) \right\} \]
with
\[ \Delta(G) = e_3 \Delta(G_1) - \Delta(G_2). \]

In terms of our EED, \( G, G_1 \) and \( G_2 \) correspond to the following diagrams \( Z, Z_1, \) and \( Z_2 \), respectively:
Note that “1” was found in the $x^3$ column in $Z$ but is found in the $x^2$ column in both $Z_1$ and $Z_2$, respectively. It is worthwhile to note that it is possible to move any of the integers west (left), one column at a time in this manner. □

The monomials of $e^{(i)}$ can be represented by fillings of the following diagram:

\[ e^{(3)} = x_1 x_2 + x_1 x_4 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_4 x_5. \]

Example 2.3.4. With $X = \{x_1, x_2, x_3, x_4, x_5\}$, the fillings corresponding to $e^{(3)}$ in the diagram in (2.23) are

These fillings correspond to the monomials $x_1 x_2, x_1 x_4, x_1 x_5, x_2 x_4, x_2 x_5$ and $x_4 x_5$, respectively. Recall from Example 2.1.1 that

\[ e^{(3)} = x_1 x_2 + x_1 x_4 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_4 x_5. \]
Monomial terms of the polynomial $c^{(1)}_{n-1}c^{(2)}_{n-2}\cdots c^{(n)}_{0}c^{(1)}_{\mu_1}c^{(2)}_{\mu_2}\cdots c^{(n)}_{\mu_n}$ (in Equation (2.11)) can be thought of as particular fillings $F$ of the cells of the following Monomial Product Diagram (MPD):

As before, $\uparrow b$ indicates that we must fill $b$ cells in that particular column; the “XX” indicates a cell that cannot be filled. In referring to the columns of the MPD, we will refer to the column with $\uparrow b$ as column $b$ and the column with $\uparrow \mu_i$ as column $\mu_i$. We will refer to fillings $F$ cells over columns $n-1, n-2, \cdots, 1, 0$ as a left-hand side filling (LHS). For a LHS filling $F$ of a MPD, let $L_i(F)$ denote the number of $\bullet$’s in row $i$ above the columns $n-1, n-2, \cdots, 2,1$ of the MPD. For a filling $F$, define

$$X_F = \prod_{i=1}^{n} x^{L_i(F)}_{\mu_i}. \quad (2.24)$$

We can rewrite Equation (2.11) as

$$B_{\mu} = \sum_{\text{fillings } F \text{ of the MPD}} \Delta(X_F). \quad (2.25)$$

**Example 2.3.5.** Consider the following LHS filling $F$ of our MPD that corresponds to a term in the expansion of the polynomial

$$c^{(1)}_{6}c^{(2)}_{5}c^{(3)}_{4}c^{(4)}_{3}c^{(5)}_{2}c^{(6)}_{1}c^{(7)}_{0}c^{(1)}_{0}c^{(2)}_{1}c^{(3)}_{1}c^{(4)}_{0}c^{(5)}_{0}c^{(6)}_{0}c^{(7)}_{0}.$$
We have \( L_1(F) = 3 \) since there are three dots in row 1, \( L_2(F) = 2 \), etc. Thus this filling \( F \) corresponds to the term

\[
X_F = x_1^3 x_2^2 x_3^2 x_4 x_5 x_6 x_7 e_2^{(1)} e_1^{(2)} e_4^{(3)} e_5^{(4)} e_6^{(5)} e_0^{(6)} e_0^{(7)}.
\]

The coordinates of \( \Delta(X_F) \) are

\[
G = \left\{ (x_1^3, e_2^{(1)}), (x_2^2, e_1^{(2)}), (x_3^2, e_4^{(3)}), (x_4^2, e_5^{(4)}), (x_5^3, e_6^{(5)}), (x_6^0, e_0^{(6)}), (x_7^0, e_0^{(7)}) \right\}.
\]

Correspondingly,

\[
Z = \begin{array}{cccccccc}
\hline
1^0 & 2^0 & 3^0 & 4^0 & 5^0 & 6^0 & 7^0 \\
\hline
2^1 & 3^1 & 4^1 & 5^1 & 6^1 & 7^1 & 8^1 \\
3^2 & 4^2 & 5^2 & 6^2 & 7^2 & 8^2 & 9^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
9^5 & 10^5 & 11^5 & 12^5 & 13^5 & 14^5 & 15^5 \\
\hline
\end{array}
\]

Since there are cells with more than a single entry, \( \Delta(X_F) = \Delta(Z) = 0 \). □

In the \( n - 1 \) column of the MPD there are exactly \( n - 1 \) open cells. Furthermore, no cells can be filled in the 0 column. Thus, we may assume that the summation in Equation (2.25) is over LHS fillings \( F \) of the following MPD:

Thus, for \( 2 \leq j \leq n \), we have that \( L_j(F) \geq 1 \).

Now, suppose that \( L_1(F) \geq 1 \) and \( \Delta(X_F) \neq 0 \). Since \( L_j(F) \geq 1 \), for \( 2 \leq j \leq n \), the polynomial degree of \( X_F \) would need to be at least

\[
1 + 2 + \cdots + n = \frac{1}{2} n(n + 1).
\]
$det(B_\mu)$, however, can have degree at most $\frac{1}{2}n(n+1) - 1$ (Lemma 2.2.3). So, $L_1 = 0$ and the cells in the top left-hand side of row 1 must all be empty. Note that there are now only $n - 2$ empty cells in the $n - 2$ column; so each empty cell in the $n - 2$ column without an “XX” must be filled with a •. Thus, we may assume that the sum in Equation (2.25) is over LHS fillings $F$ of the following MPD:

\[
\begin{array}{cccccccc}
1 & XX & XX & XX & XX & XX & XX & XX \\
2 & • & XX & XX & XX & XX & XX & XX \\
3 & • & • & XX & XX & XX & XX & XX \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 n-1 & • & • & • & • & • & • & • \\
 n & • & • & • & • & • & • & • \\
 \end{array}
\]

$$\text{(2.26)}$$

Now, suppose that $L_2(F) \geq 2$. The MPD diagram in (2.26) indicates that $L_j(F) \geq 2$ for $3 \leq j \leq n$. In the corresponding EED, we have a 1 in cell $(x^0, \mathcal{e}^{(j)}_{\mu_1})$ as well as entries $i$ in cells $(x^{a_i}, \mathcal{e}^{(j)}_{\mu_i})$ where $a_i \geq 2$ for $2 \leq i \leq n$. Note that this implies that there are $n - 1$ entries in the columns of $x^2, x^3, \cdots, x^{n-1}$ in the EED. Using the Elementary Expansion Algorithm, we can shift all the entries that start in columns $x^2, x^3, \cdots, x^{n-1}$ into the $x^2$ column. For these entries in the cells of the $x^2$ column to be distinct, so that $\Delta(X_F) \neq 0$, we need a polynomial of degree at least

$$\mu_1 + (2 + 0) + (2 + 1) + \cdots + (2 + (n - 2)) = \mu_1 + \frac{1}{2}n(n+1) - 1$$

$$\geq \frac{1}{2}n(n+1).$$

We only have, however, a polynomial of degree at most $\frac{1}{2}n(n+1) - 1$ (Lemma 2.2.3). Thus, we must have $L_2 = 1$ and we can assume that the sum in Equation (2.25) is
over fillings of the MPD of the following form:

\[
\begin{array}{cccccccc}
1 & XX & XX & XX & \cdots & XX & XX & XX \\
2 & \cdot & XX & XX & \cdots & XX & XX & XX \\
3 & \cdot & \cdot & XX & \cdots & XX & XX & XX \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
n-2 & \cdot & \cdot & \cdot & XX & \cdots & XX & XX \\
n-1 & \cdot & \cdot & \cdot & XX & \cdots & XX & XX \\
n & \cdot & \cdot & \cdot & XX & \cdots & XX & XX \\
\hline
1 & 0 & \mu_1 & \mu_2 & \cdots & \mu_{n-2} & \mu_{n-1} & \mu_n
\end{array}
\]

Recursively, we can continue this argument. We say that a filling \( F \) of a column \( b \) in the MPD is \textit{south-justified} if the filled cells are exactly the cells in the rows \( n - b + 1, n - b + 2, \ldots, n \). Similarly, a row with \( b \) filled cells of \( F \) is said to be \textit{west-justified} if the filled cells are in columns \( n - 1, n - 2, \ldots, n - b \). Note that if a MPD is south justified for columns \( n - 1 \) through \( n - q + 1 \), then rows \( j \), for \( 1 \leq j \leq q - 1 \) have at least \( j - 1 \) \( \cdot \)'s that are west justified.

Suppose that \( \Delta(X_F) \neq 0 \) and that columns \( n - 1 \) through \( n - q \) are all south justified. Furthermore, assume that row \( f \) is west-justified and \( L_f(F) = f - 1 \) for \( 1 \leq f \leq q - 1 \). This forces \( L_h(F) \geq q \) for \( q + 1 \leq h \leq n \). Assume that \( L_q(F) \geq q \).

Since \( L_j(F) \geq q \) for \( q + 1 \leq j \leq n \), we have that there will be at least \( n - q + 1 \) terms in columns \( x^q, x^{q+1}, \ldots, x^n \), in the EED. Using the Elementary Expansion Algorithm, we can expand these terms so that all \( n - q + 1 \) terms appear in the column \( x^n \). In order for \( \Delta(X_F) \neq 0 \), we need for these entries in the cells to be distinct by Lemma 2.2.2 which requires a polynomial of degree at least

\[
(q + 0) + (q + 1) + \cdots + (q + (n - q))
\]

\[
= q(n - q + 1) + \frac{1}{2}(n - q)(n - q + 1).
\]  

(2.27)

Additionally, by induction, we have entry \( i \) in the cells \((x^{i-1}, \epsilon_{ji}^{(j)})\) for \( 1 \leq i \leq q - 1 \).
which requires that the polynomial have an additional degree of at least

\[
(0 + \mu_1) + (1 + \mu_2) + \cdots + (q - 2 + \mu_{q-1}) \geq \frac{1}{2}(q - 1)(q - 2) + (q - p)
\]

\[
= \frac{1}{2}(q - 1)(q) - (p - 1) \quad (2.28)
\]

since

\[
\mu_1 + \mu_2 + \cdots + \mu_{q-1} \geq \min\{q - 1, n - p\} \geq q - p.
\]

Note that adding the degrees in Equations (2.27) and (2.28) yields

\[
q(n - q + 1) + \frac{1}{2}(n - q)(n - q + 1) + \frac{1}{2}(q)(q - 1) - (p - 1)
\]

\[
= \frac{1}{2}n(n + 1) - p + 1.
\]

But the degree of \( \det(B_\mu) \) is at most \( \frac{1}{2}n(n + 1) - p \) (Lemma 2.2.3). So, we must have

\( L_q(F) = q - 1 \). This forces row \( q \) to be west-justified and column \( n - q + 1 \) to be south-justified.

Inductively, the only fillings \( F \) of our MPD with the property that \( \Delta(X_F) \neq 0 \) must correspond to fillings of the following diagrams in Equation (2.25):

\[
\begin{array}{ccccccccccc}
1 & XX & XX & XX & \cdots & XX & XX & XX & XX & XX & XX \\
2 & \bullet & \bullet & \bullet & XX & \cdots & XX & XX & XX & XX & XX \\
3 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n - 2 & \bullet & \bullet & \bullet & \cdots & XX & \vdots & \vdots & \vdots & \vdots & \vdots \\
n - 1 & \bullet & \bullet & \bullet & \cdots & \bullet & XX & \vdots & \vdots & \vdots & \vdots \\
n & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \vdots & \vdots & \vdots & \vdots \\
n - 1 & n - 2 & n - 3 & \vdots & 2 & 1 & 0 & \mu_1 & \mu_2 & \mu_3 & \mu_{n - 2} & \mu_{n - 1} & \mu_n
\end{array}
\]

Thus, we have that

\[
\det(B_\mu) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma \left(x_1^0 x_2^1 \cdots x_{n-1}^{n-2} x_n^{n-1} \epsilon_{\mu_1}^{(1)} \epsilon_{\mu_2}^{(2)} \cdots \epsilon_{\mu_n}^{(n)} \right) \quad (2.29)
\]

\[
= \det(C_\mu) \quad (2.30)
\]
where $C_\mu$ is defined in Equation (2.3).

It is worthwhile to note in our proof of Theorem 2.1.2 that we never used properties of how we filled the $\mu_i$ columns of the MPD. All of our arguments concerned the degrees of the polynomial terms generated by the $\mu_i$ columns. Thus, we have in fact proved the following theorem.

**Theorem 2.3.6.** Let $W_\mu = (W_{i,j}) = (e_j^{(i)} f_{\mu_j}^{(i)})$ where $f_{\mu_j} \in \Lambda_n$ is a polynomial of degree $\mu_j$, $f_{\mu_j}^{(i)} = f_{\mu_j} |_{x_i=0}$ and $U_\mu = (U_{i,j}) = (x_j^{i-1} f_{\mu_j}^{(i)})$, then

$$\det(W_\mu) = \det(U_\mu).$$

We will now prove that

$$\det(C_\mu) = \pm \epsilon_\mu \det(M_n) = \pm \epsilon_\mu \det(V_n) = \pm \epsilon_\mu a_\delta,$$

where $a_\delta$ is the Vandermonde determinant. Before we give a proof, we give an example of the computation that will be used.

**Example 2.3.7.** Let $n=4$, $\mu = (2, 1, 0, 0)$ and $X = \{x_1, x_2, x_3, x_4\}$. Thus

$$C_\mu = \begin{bmatrix}
    e_2^{(1)} & e_1^{(1)} & e_0^{(1)} & e_1^{(1)} \\
    e_2^{(2)} & 1 & e_0^{(2)} & e_2^{(2)} \\
    e_2^{(3)} & e_1^{(3)} & e_0^{(3)} & e_3^{(3)} \\
    e_2^{(4)} & e_1^{(4)} & e_0^{(4)} & e_3^{(4)} 
\end{bmatrix}.$$  \hspace{1cm} (2.31)

$C_\mu$ corresponds to the EED

$$Z = \begin{array}{cccccccc}
    2 & 1 & 2 & 4 \\
    x^2 & x^3 & x^4 & x^5 \\
    1 & 2 & 4 & 0 \\
    x^0 & x^1 & x^2 & x^3 
\end{array}.$$

Let $Z_1$ and $Z_2$ be the diagrams formed by applying the Elementary Expansion Algorithm to the “4” in diagram $Z$, yielding:

$$Z_1 = \begin{array}{cccccccc}
    3 & 2 & 1 & 3 \\
    4 & 3 & 2 & 0 \\
    x^0 & x^1 & x^2 & x^3 
\end{array} \quad \text{and} \quad Z_2 = \begin{array}{cccccccc}
    3 & 2 & 1 & 3 \\
    4 & 3 & 2 & 0 \\
    x^0 & x^1 & x^2 & x^3 
\end{array}.$$
Since EED $Z_1$ has two entries in the same cell, $\Delta(Z_1) = 0$. Thus,

$$\det(C_\mu) = \epsilon_1 \Delta(Z_1) - \Delta(Z_2) = -\Delta(Z_2).$$

Applying the Elementary Expansion algorithm to the “3” in $Z_2$ yields

\[
Z_3 = \begin{array}{ccc|ccc}
3 & 4 & & 2 & 1 & \\
1 & 2 & & 3 & 4 & \\
0 & x^a & & x^b & x^c & x^d
\end{array}
\quad Z_4 = \begin{array}{ccc|ccc}
3 & 4 & & 2 & 1 & \\
1 & 2 & & 3 & 4 & \\
0 & x^a & & x^b & x^c & x^d
\end{array}
\]

with

$$\det(C_\mu) = -\Delta(Z_2) = - (\epsilon_1 \Delta(Z_3) - \Delta(Z_4)).$$

But, $\Delta(Z_4) = 0$, since $Z_4$ has two entries in the same cell and

$$\det(C_\mu) = -\epsilon_1 \Delta(Z_3).$$

Applying the Elementary Expansion Algorithm to the “4” in $Z_3$ gives

\[
Z_5 = \begin{array}{ccc|ccc}
4 & 2 & & 3 & 1 & \\
1 & 2 & & 3 & 4 & \\
0 & x^a & & x^b & x^c & x^d
\end{array}
\quad Z_6 = \begin{array}{ccc|ccc}
4 & 2 & & 3 & 1 & \\
1 & 2 & & 3 & 4 & \\
0 & x^a & & x^b & x^c & x^d
\end{array}
\]

where

$$\det(C_\mu) = -\epsilon_1 \Delta(Z_3) = -\epsilon_1 (\epsilon_2 \Delta(Z_5) - \Delta(Z_6)).$$

But, $\Delta(Z_5) = 0$ and

$$\det(C_\mu) = \epsilon_1 \Delta(Z_6).$$

Continuing this argument and setting $Z_7$ to be the EED

\[
Z_7 = \begin{array}{ccc|ccc}
2 & 3 & & 4 & 1 & \\
1 & 2 & & 3 & 4 & \\
0 & x^a & & x^b & x^c & x^d
\end{array}
\]

we note that

$$\Delta(Z_7) = (-1)^4 a_5$$

yielding

$$\det(C_\mu) = \pm \epsilon_1 \epsilon_2 a_5.\]
Now that we have shown an example of computation that will be used, we want to prove the following:

**Theorem 2.3.8.**

\[
\det(C_\mu) = \pm e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k} \det(V_n) = \pm e_{\mu} \det(V_n) \quad (2.32)
\]

**Proof.** Note that \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \vdash k \leq n - 1 \) with \( \mu_i \geq \mu_{i+1} \geq 0 \) implies that \( \mu_i \leq n - i \) and in particular \( \mu_n = 0 \) and either \( \mu_{n-1} = 1 \) or \( \mu_{n-1} = 0 \). In either case, applying the Elementary Expansion Algorithm to the “n” in the corresponding EED diagram \( Z \), yields a corresponding diagram \( Z^* \) with either of the following subdiagrams in the \( x^{n-2} \) and \( x^{n-1} \) columns:

(a) \[
\begin{array}{c|c|c|c|c|c}
 & 0 & n & \cdots & d & 1 \\
\hline
0 & n-1 & \cdots & d & 1 \\
1 & \cdots & d & 1 & 0 \\
\end{array}
\]

(b) \[
\begin{array}{c|c|c|c|c|c}
 & 0 & n & \cdots & d & 1 \\
\hline
0 & n-1 & \cdots & d & 1 & 0 \\
1 & \cdots & d & 1 & 0 \\
\end{array}
\]

With the (a) subdiagram, we have \( \det(C_\mu) = e_1 \Delta(Z^*) \) and in (b) we have \( \det(C_\mu) = -\Delta(Z^*) \), where \( Z^* \) is the EED found by substituting \( Z_{n-2,n-1} \) for columns \( x^{n-2} \) and \( x^{n-1} \) in \( Z \). Since, \( \mu_{n-2} \leq 2 \), inductively we may suppose that we have the following subdiagram \( \tilde{Z} \) corresponding to the entries in columns \( x^b \) and \( x^{b+1} \) of the EED \( Z \), where \( f + b = n \):

(a) \[
\begin{array}{c|c|c|c|c|c}
 & 0 & n & \cdots & d & 1 \\
\hline
0 & n-1 & \cdots & d & 1 & 0 \\
1 & \cdots & d & 1 & 0 \\
\end{array}
\]

(b) \[
\begin{array}{c|c|c|c|c|c}
 & 0 & n & \cdots & d & 1 \\
\hline
0 & n-1 & \cdots & d & 1 & 0 \\
1 & \cdots & d & 1 & 0 \\
\end{array}
\]
Now, let $Z_{b,b+1,1}$ and $Z_{b,b+1,2}$ be the diagrams associated with the Elementary Expansion Algorithm acting on the entry $i_d$. Specifically, we have

$$Z_{b,b+1,1} = \begin{array}{|c|c|} \hline n & \vdots \\ \vdots & \vdots \\ f & t_f \\ f-1 & t_{f-1} \\ \vdots & \vdots \\ d+1 & t_{d+1} \\ d & t_d \\ d-1 & t_{d-1} \\ 1 & t_1 \\ 0 & t_0 \\ \hline \end{array}$$

and

$$Z_{b,b+1,2} = \begin{array}{|c|c|} \hline n & \vdots \\ \vdots & \vdots \\ f & t_f \\ f-1 & t_{f-1} \\ \vdots & \vdots \\ d+1 & t_{d+1} \\ d & t_d \\ d-1 & t_{d-1} \\ 1 & t_1 \\ 0 & t_0 \\ \hline \end{array}$$

With $Z_1$ and $Z_2$ being the corresponding EED in which we substitute $Z_{b,b+1,1}$ and $Z_{b,b+1,2}$ for the columns $x^b$ and $x^{b+1}$, we have

$$\Delta(Z) = \epsilon_{d+1} \Delta(Z_1) - \Delta(Z_2).$$

But, $\Delta(Z_2) = 0$ since there are two entries in the same cell. So,

$$\Delta(Z) = \epsilon_{d+1} \Delta(Z_1).$$

Now, let $Z_{b,b+1,3}$ and $Z_{b,b+1,4}$ be the diagrams associated with using the Elementary Expansion Algorithm acting on the entry $i_0$ in $Z_{b,b+1,1}$ and let $Z_3$ and $Z_4$ be the diagrams with the corresponding column substitutions. Particularly, we have

$$Z_3 = \begin{array}{|c|c|} \hline n & \vdots \\ \vdots & \vdots \\ f & t_f \\ f-1 & t_{f-1} \\ \vdots & \vdots \\ d+1 & t_{d+1} \\ d & t_d \\ d-1 & t_{d-1} \\ 1 & t_1 \\ 0 & t_0 \\ \hline \end{array}$$

and

$$Z_4 = \begin{array}{|c|c|} \hline n & \vdots \\ \vdots & \vdots \\ f & t_f \\ f-1 & t_{f-1} \\ \vdots & \vdots \\ d+1 & t_{d+1} \\ d & t_d \\ d-1 & t_{d-1} \\ 1 & t_1 \\ 0 & t_0 \\ \hline \end{array}$$

with

$$\Delta(Z) = \epsilon_{d+1} ( \epsilon_1 \Delta(Z_3) - \Delta(Z_4) ).$$

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But, $\Delta(Z_3) = 0$ since there are two entries in the same cell. So,

$$\Delta(Z) = -\epsilon_{d+1} \Delta(Z_4).$$

Continuing this argument, we see that if we set $Z_{b,b+1,5}$ to be the diagram in which we substitute the following two columns for columns $x^b$ and $x^{b+1}$ in $Z$,

$$
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c |
Also, remember that

\[
det(D_\mu) = \sum_{\sigma \in S_n} sgn(\sigma) \sigma \left( c_{n-1}^{(1)} c_{n-2}^{(2)} \cdots c_0^{(n)} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} \right) \tag{2.33}
\]

and

\[
det(E_\mu) = \sum_{\sigma \in S_n} sgn(\sigma) \left( x_1^{0} x_2^{1} \cdots x_n^{n-1} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} \right) \tag{2.34}
\]

where \( c_{b}^{(i)} \) is given in Equation (2.1), and \( h_{b}^{(i)} \) is given in Equation (2.6).

By Theorem 2.3.6 we have the following:

**Theorem 2.4.1.** Suppose that \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \vdash k = n - p \) where \( p \) is a positive integer, then

\[
det(D_\mu) = \sum_{\sigma \in S_n} sgn(\sigma) \sigma \left( c_{n-1}^{(1)} c_{n-2}^{(2)} \cdots c_0^{(n)} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} \right) \tag{2.35}
\]

\[
= \sum_{\sigma \in S_n} sgn(\sigma) \left( x_1^{0} x_2^{1} \cdots x_n^{n-1} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} \right) \tag{2.36}
\]

\[
= \Delta \left( x_1^{0} x_2^{1} \cdots x_n^{n-1} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} \right) \tag{2.36}
\]

\[
= det(E_\mu)
\]

where \( c_{b}^{(i)} \) is given in Equation (2.1), and \( h_{b}^{(i)} \) is given in Equation (2.6).

We will represent \( \Delta(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)} ) \) by putting the integer \( i, 1 \leq i \leq n, \) in the cell \((x^{\alpha_i}, h_{\mu_i}^{(j)})\) of the Complete Expansion Diagram (CED) \( Y \) given in (2.37):

\[
Y = \begin{array}{cccccc}
\hline
h_{\mu_1}^{(j)} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\hline
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\hline
\hline
h_{\mu_2}^{(j)} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\hline
h_{\mu_3}^{(j)} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\hline
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\hline
\hline
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\hline
\hline
x^{\alpha_1} & x^{\alpha_2} & \cdots & x^{\alpha_{n-1}} & x^{\alpha_n} & \cdots \\
\hline
\end{array}
\tag{2.37}
\]

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Note that this differs from the *Elementary Expansion Diagram* given by Equation (2.16). The collection

$$J = \left\{ (x_{a_1}^{a_1}, h_{\mu_1}^{(1)}), (x_{a_2}^{a_2}, h_{\mu_2}^{(2)}), \ldots, (x_{a_n}^{a_n}, h_{\mu_n}^{(n)}) \right\}$$

are the *coordinates* of the polynomial $\Delta(x_{a_1}^{a_1} x_{a_2}^{a_2} \cdots x_{a_n}^{a_n} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)})$ and we set

$$\Delta(Y) = \Delta(J) = \Delta(x_{a_1}^{a_1} x_{a_2}^{a_2} \cdots x_{a_n}^{a_n} h_{\mu_1}^{(1)} h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)}).$$

**Example 2.4.2.**

$$\Delta(x_0^0 x_1^1 x_2^2 x_3^3 x_4^4 x_5^5 x_6^6 x_7^7 h_3^{(1)} h_2^{(2)} h_3^{(3)} h_2^{(4)} h_1^{(5)} h_0^{(6)} h_0^{(7)})$$

has coordinates

$$J = \left\{ (x_1^0, h_3^{(1)}), (x_2^1, h_3^{(2)}), (x_3^2, h_2^{(4)}), (x_4^3, h_1^{(5)}), (x_5^4, h_0^{(6)}), (x_6^5, h_0^{(7)}) \right\}$$

and

$$Y = \begin{array}{cccccccc}
    h_3^{(1)} & & & & & & & \\
    h_3^{(2)} & & & & & & & \\
    h_3^{(3)} & 1 & 2 & & & & & \\
    h_2^{(4)} & & 3, 4 & & & & & \\
    h_1^{(5)} & & & 5 & & & & \\
    h_0^{(6)} & & & & 6, 7 & & & \\
    x_0^0 & x_1^1 & x_2^2 & x_3^3 & x_4^4 & x_5^5 & x_6^6 & x_7^7
\end{array} \quad (2.38)$$

Now, $\Delta(Y) = \Delta(J) = 0$ since $Y$ has two entries in the same cell in (2.38).\Box

It is not difficult to see that $h_b = x_i h_{b-1} + h_b^{(i)}$ and in particular gives the following result

**Lemma 2.4.3.**

$$x_i^a h_b^{(i)} = x_i^a h_b - x_i^{a+1} h_{b-1}.$$
Note the difference between Lemma 2.4.3 and Lemma 2.3.2. This result, Lemma 2.4.3, implies an algorithm for moving entries south (down) in our Complete Expansion Diagram 2.37 instead of moving west (left) in the Elementary Expansion Diagram 2.4. Furthermore, in Example 2.4.2, Example 2.4.4.

From Lemma 2.4.3, we can define two sets $J_1$ and $J_2$ by acting on the “5” by replacing $(x_5, b_1^{(5)})$ with $(x_5, b_0^{(5)})$ and $(x_5, b_0^{(5)})$, respectively, yielding

$$J_1 = \left\{ (x_1^0, b_0^{(1)}), (x_2^1, b_0^{(2)}), (x_3^2, b_1^{(3)}), (x_4^3, b_2^{(4)}), (x_5^4, b_0^{(5)}), (x_6^5, b_0^{(6)}), (x_7^6, b_0^{(7)}) \right\}$$
and

\[ J_2 = \left\{ (x_1^0, b_3^{(1)}), (x_2^1, b_2^{(2)}), (x_3^2, b_1^{(3)}), (x_4^3, b_3^{(4)}), (x_5^4, b_6^{(5)}), (x_6^5, b_6^{(6)}), (x_7^6, b_0^{(7)}) \right\} \]

with

\[ \Delta(J) = b_1 \Delta(J_1) - \Delta(J_2) \]

In terms of our CED, \( J, J_1 \) and \( J_2 \) correspond to the following diagrams \( Y, Y_1, \) and \( Y_2 \) respectively:

\[ Y = \begin{array}{cccccccc}
 b_3^{(1)} & & & & & & & \\
 b_2^{(2)} & & & & & & & \\
 b_1^{(3)} & 1 & 2 & & & & & \\
 b_3^{(4)} & 3 & 4 & & & & & \\
 b_6^{(5)} & & & & & & & \\
 b_6^{(6)} & & & 5 & & & & \\
 b_0^{(7)} & x_0^0 & x_1^1 & x_2^2 & x_3^3 & x_4^4 & x_5^5 & x_6^6 \\
\end{array} \]

\[ Y_1 = \begin{array}{cccccccc}
 b_3^{(1)} & & & & & & & \\
 b_2^{(2)} & & & & & & & \\
 b_1^{(3)} & 1 & 2 & & & & & \\
 b_3^{(4)} & 3 & 4 & & & & & \\
 b_6^{(5)} & & & & & & & \\
 b_6^{(6)} & & & 5 & & & & \\
 b_0^{(7)} & x_0^0 & x_1^1 & x_2^2 & x_3^3 & x_4^4 & x_5^5 & x_6^6 \\
\end{array} \]

and

\[ Y_2 = \begin{array}{cccccccc}
 b_3^{(1)} & & & & & & & \\
 b_2^{(2)} & & & & & & & \\
 b_1^{(3)} & 1 & 2 & & & & & \\
 b_3^{(4)} & 3 & 4 & & & & & \\
 b_6^{(5)} & & & & & & & \\
 b_6^{(6)} & & & 5 & 6 & 7 & & \\
 b_0^{(7)} & x_0^0 & x_1^1 & x_2^2 & x_3^3 & x_4^4 & x_5^5 & x_6^6 \\
\end{array} \]

Note that “5” was in the \( x^3 \) column in \( Y \) and now is found in the \( x^3 \) column of \( Y_1 \) and the \( x^4 \) column in \( Y_2 \). It is worthwhile to note that it is possible to move any of the integers in this fashion. \( \square \)

We will prove that

\[ \text{det}(E_\mu) = b_\mu \text{det}(M_n) = b_\mu a_\delta, \]
where $M_n$ is the matrix defined by $M_{i,j} = c_{n-j}^{(i)}$. Before we give a proof, we give an example of the computation that we used.

**Example 2.4.5.** Let $\mu = (2, 1, 0, 0)$ and $X = \{x_1, x_2, x_3, x_4\}$. Set

$$E_\mu = \begin{bmatrix} b_2^{(1)} & b_1^{(1)} x_1 & b_0^{(1)} x_2 & b_0^{(1)} x_3 \\ b_2^{(2)} & b_1^{(2)} x_2 & b_0^{(2)} x_3 & b_0^{(2)} x_4 \\ b_2^{(3)} & b_1^{(3)} x_3 & b_0^{(3)} x_4 & b_0^{(3)} x_5 \\ b_2^{(4)} & b_1^{(4)} x_4 & b_0^{(4)} x_5 & b_0^{(4)} x_6 \end{bmatrix}.$$  

(2.40)

$E_\mu$ corresponds to the CED

$$Y = \begin{array}{cccc} b_2^{(1)} & b_1^{(1)} & b_0^{(1)} & b_0^{(1)} \\ b_2^{(2)} & b_1^{(2)} & b_0^{(2)} & b_0^{(2)} \\ b_2^{(3)} & b_1^{(3)} & b_0^{(3)} & b_0^{(3)} \\ b_2^{(4)} & b_1^{(4)} & b_0^{(4)} & b_0^{(4)} \end{array}.$$  

Let $Y_1$ and $Y_2$ be the diagrams formed by applying the Complete Expansion Algorithm acting on the "2" to the diagram $Y$, yielding:

$$Y_1 = \begin{array}{cccc} b_2^{(1)} & b_1^{(1)} & b_0^{(1)} & b_0^{(1)} \\ b_2^{(2)} & b_1^{(2)} & b_0^{(2)} & b_0^{(2)} \\ b_2^{(3)} & b_1^{(3)} & b_0^{(3)} & b_0^{(3)} \\ b_2^{(4)} & b_1^{(4)} & b_0^{(4)} & b_0^{(4)} \end{array} \quad \text{and} \quad Y_2 = \begin{array}{cccc} b_2^{(1)} & b_1^{(1)} & b_0^{(1)} & b_0^{(1)} \\ b_2^{(2)} & b_1^{(2)} & b_0^{(2)} & b_0^{(2)} \\ b_2^{(3)} & b_1^{(3)} & b_0^{(3)} & b_0^{(3)} \\ b_2^{(4)} & b_1^{(4)} & b_0^{(4)} & b_0^{(4)} \end{array}.$$  

Since CED $Y_2$ has two entries in the same cell, $\Delta(Y_2) = 0$. Thus,

$$\det(E_\mu) = \h_1 \Delta(Y_1) - \Delta(Y_2) = \h_1 \Delta(Y_1).$$

Applying the Complete Expansion algorithm to the "1" in $Y_1$ yields

$$Y_3 = \begin{array}{cccc} b_2^{(1)} & b_1^{(1)} & b_0^{(1)} & b_0^{(1)} \\ b_2^{(2)} & b_1^{(2)} & b_0^{(2)} & b_0^{(2)} \\ b_2^{(3)} & b_1^{(3)} & b_0^{(3)} & b_0^{(3)} \\ b_2^{(4)} & b_1^{(4)} & b_0^{(4)} & b_0^{(4)} \end{array} \quad \text{and} \quad Y_4 = \begin{array}{cccc} b_2^{(1)} & b_1^{(1)} & b_0^{(1)} & b_0^{(1)} \\ b_2^{(2)} & b_1^{(2)} & b_0^{(2)} & b_0^{(2)} \\ b_2^{(3)} & b_1^{(3)} & b_0^{(3)} & b_0^{(3)} \\ b_2^{(4)} & b_1^{(4)} & b_0^{(4)} & b_0^{(4)} \end{array}.$$  

with

$$\det(E_\mu) = \h_1 \Delta(Y_1) = \h_1 (\h_2 \Delta(Y_3) - \Delta(Y_4)) .$$

But, $\Delta(Y_4) = 0$, since $Y_4$ has two entries in the same cell and

$$\det(E_\mu) = \h_1 \h_2 \Delta(Y_3) \Box.$$
Now that we have given an example for the computation that will be used, we will prove the following:

**Theorem 2.4.6.** Let $\mu \vdash n - p$ where $1 \leq p \leq n$. Then,

$$\det(E_\mu) = \pm h_\mu a_\delta. \quad (2.41)$$

**Proof.** Note that $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \vdash k \leq n - 1$ with $\mu_i \geq \mu_{i+1} \geq 0$ implies that $\mu_i \leq n - i$ and in particular $\mu_n = 0$ and either $\mu_{n-1} = 1$ or $\mu_{n-1} = 0$. The coordinates of $\Delta(x_1^{\mu_1}x_2^{\mu_2} \cdots x_n^{\mu_n} h_{\mu_1}^{(1)}h_{\mu_2}^{(2)} \cdots h_{\mu_n}^{(n)})$ include $(x_n^{(n-1)}, h_0^{(n)})$ and either $(x_{n-1}^{(n-2)}, h_1^{(n-1)})$ or $(x_{n-1}^{(n-2)}, h_0^{(n-1)})$. These two situations correspond to the following diagrams, respectively.

(a) $Y = \begin{array}{cccccccc}
\h_0^{(1)} & & & & & & & \\
\h_0^{(2)} & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\h_0^{(n)} & & & & & & & \\
\h_0^{(n-1)} & & & & & & & \\
\h_0^{(n-2)} & & & & & & & \\
\h_0^{(n-3)} & & & & & & & \\
\h_0^{(n-4)} & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\h_0^{(2)} & & & & & & & \\
\h_0^{(1)} & & & & & & & \\
\h_0^{(0)} & & & & & & & \\
\end{array}$

(b) $Y = \begin{array}{cccccccc}
\h_0^{(1)} & & & & & & & \\
\h_0^{(2)} & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\h_0^{(n)} & & & & & & & \\
\h_0^{(n-1)} & & & & & & & \\
\h_0^{(n-2)} & & & & & & & \\
\h_0^{(n-3)} & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\h_0^{(n)} & & & & & & & \\
\h_0^{(n-1)} & & & & & & & \\
\h_0^{(n-2)} & & & & & & & \\
\end{array}$

Applying the Complete Expansion Algorithm to the “n-1” in the corresponding CED diagram (a) $Y$, yields a corresponding diagram $Y^*$ with either of the following sub-
diagrams in the $x^{n-1}$ and $x^{n-2}$ columns:

\[(c) \quad Y^* = \]

\[
\begin{array}{cccccccc}
\hline
b_{j-1}^{(f)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
b_{j-2}^{(f)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
b_{j-3}^{(f)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}
\]

\[(d) \quad Y^* = \]

\[
\begin{array}{cccccccc}
\hline
b_{j-1}^{(f)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
b_{j-2}^{(f)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
b_{j-3}^{(f)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}
\]

With the (c) subdiagram, we have $\det(D_\mu) = h_1 \Delta(Y^*)$ and in (d) we have $\det(E_\mu) = 0$, by Lemma (2.2.2). Inductively we may suppose that we have the following subdiagram $\bar{Y}$ corresponding to the entries in columns $x^{f-1}$ and $x^f$ of the CED $Y$.

\[
\bar{Y}_f = \]

\[
\begin{array}{cccccccc}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}
\]

(2.42)

Now, let $Y_{f,1}$ and $Y_{f,2}$ be the diagrams associated with the Complete Expansion
Algorithm acting on the entry $f$. Specifically, we have

\[
Y_{f,1} = \begin{bmatrix}
(h(j))_n^{-1} & (h(j))_{n-2} & \cdots & (h(j))_{f+1} & (h(j))_{f} & \cdots & (h(j))_{1} & (h(j))_0 \\
(h(j))_{f} & (h(j))_{f-1} & \cdots & (h(j))_{1} & (h(j))_0 & \cdots & (h(j))_{f+1} & (h(j))_{f+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(h(j))_1 & (h(j))_0 & \cdots & (h(j))_{f} & (h(j))_{f+1} & \cdots & (h(j))_{n-2} & (h(j))_{n-1} \\
\end{bmatrix}
\]

\[
Y_{f,2} = \begin{bmatrix}
(h(j))_n^{-1} & (h(j))_{n-2} & \cdots & (h(j))_{f+1} & (h(j))_{f} & \cdots & (h(j))_{1} & (h(j))_0 \\
(h(j))_{f} & (h(j))_{f-1} & \cdots & (h(j))_{1} & (h(j))_0 & \cdots & (h(j))_{f+1} & (h(j))_{f+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(h(j))_1 & (h(j))_0 & \cdots & (h(j))_{f} & (h(j))_{f+1} & \cdots & (h(j))_{n-2} & (h(j))_{n-1} \\
\end{bmatrix}
\]

With $Y_1$ and $Y_2$ being the corresponding CED in which we substitute $Y_{f,1}$ and $Y_{f,2}$ for the columns $x^f$ and $x^{f-1}$, we have

\[
\Delta(Y) = \mu_{f} \Delta(Y_{f,1}) - \Delta(Y_{f,2}).
\]

But, $\Delta(Y_{f,2}) = 0$ since there are two entries in the same cell. So,

\[
\Delta(Y) = \mu_{f} \Delta(Y_{f,1}).
\]

Continuing this argument, we see that if we set $Y_3$ to be the diagram in which we substitute the following two columns for columns $x^b$ and $x^{b+1}$ in $Y$, we have

\[
\Delta(Y) = \mu_{d+1} \Delta(Y_3).
\]

Hence, we have that

\[
\text{det}(E_{\mu}) = \mu \Delta(\hat{Y})
\]
where \(h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_n}\) and

\[
\hat{Y} = \begin{pmatrix}
h^{(j)}_{\mu_1} & & & \\
h^{(j)}_{\mu_2} & & & \\
\vdots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & n-1 & n
\end{pmatrix}
\]

Recall that Lemma 2.2.5 yields that \(\Delta(\hat{Y}) = a_\delta\). Therefore, by induction, we have

\[
det(E_\mu) = h_\mu a_\delta. \quad \Box
\]

Note: the proofs of our main theorems, Theorem 2.1.2 and Theorem 2.1.4, can be extended from partitions \(\mu\) to certain types of compositions \(\eta = (\eta_1, \eta_2, \cdots, \eta_n)\) where \(\eta_1 + \eta_2 + \cdots + \eta_n = k\) and \(\eta_i \geq 0\) for \(1 \leq i \leq n\). Specifically, all that the proofs require is that \(\sum_{j=1}^{j} \eta_j \geq n - j\) and \(\eta_j \leq n - j\) for all \(1 \leq j \leq n\).
Appendix A:

A = Matrix used as a quotient of determinants to create Schur functions
B = Matrix used to define analogue of Vandermonde in terms of elementary symmetric functions
C = Matrix used to define analogue of Vandermonde in terms of elementary symmetric functions
D = Matrix used to define analogue of Vandermonde in terms of complete symmetric functions
E = Matrix used to define analogue of Vandermonde in terms of complete symmetric functions
F = Fillings in diagrams
G = Coordinates of the polynomial for the elementary symmetric functions
J = Coordinates of the polynomial for the complete symmetric functions
K = Kosta matrix
L = Denotes the number of \( \cdot \)'s in diagrams
P = Polynomial
p = \( \mu \vdash n-p \)
S = Symmetric group
T = Tableaux
V = Vandermonde matrix
X = \{x_1, \cdots, x_n\}
Y = Diagrams used for the Complete symmetric functions
Z = Diagrams used for the Elementary symmetric functions
c_\mu = Elementary symmetric functions
h_\mu = Complete symmetric functions
\( m_\mu \) = Monomial symmetric functions

\( s_\mu \) = Schur functions
Bibliography


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• Wake Forest University News Center: “Computers in the middle”
  http://news.wfu.edu/2012/09/04/computers-in-the-middle

**Computer Skills:**

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