NEUMANN SOLUTIONS TO A TWO-PHASE ELLIPTIC FREE BOUNDARY PROBLEM IN $\mathbb{R}^2$

BY

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Abstract

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This thesis concerns the problem of minimizing

$$J[v] = \int_\Omega |\nabla v|^2 + q^2(x)\lambda^2(v)$$

over an appropriate class of admissible functions. We have $\lambda^2(v) = \lambda_1^2$ for $v < 0$, $\lambda^2(v) = \lambda_2^2$ for $v > 0$ and $q(x) \neq 0$. We require admissible functions to satisfy Dirichlet boundary conditions on $S \subset \partial\Omega$ and Neumann boundary conditions on $N = \partial\Omega \setminus S$. In this thesis we consider the behavior of minimizers in a neighborhood of some $x \in N$. The primary result is that minimizers are Lipschitz continuous up to the Neumann fixed boundary.

Chapter 1: Introduction

Many natural phenomena can be modeled using partial differential equations (PDEs). For example, the diffusion of heat, the behavior of water waves and the behavior of jet flows can all be modeled using partial differential equations. This thesis is concerned with a specific problem within the study of partial differential equations, specifically a particular free boundary problem. In a free boundary problem some portion of the domain is unknown and must be determined along with the solution of the equation. The primary focus of this thesis is the study of a particular elliptic free boundary problem in $\mathbb{R}^2$. Specifically, we consider functions u which minimize the functional

$$J[v] = \int_{\Omega} |\nabla v|^2 + q^2(x)\lambda^2(v) \tag{1.1}$$

on a bounded, convex domain $\Omega \subset \mathbb{R}^2$. The minimizer, u, is harmonic in its positive and negative phases and satisfies $|\nabla u(x)| = q(x)$, in a weak sense, along the free boundary. In this case the free boundary is $\Gamma := \partial\{\{u > 0\} \cup \{u < 0\}\} \cap \Omega$ where $\{u > 0\} := \{x \in \Omega : u(x) > 0\}$ and $\{u < 0\}$ is defined likewise.

This particular free boundary problem has a number of interesting applications, primarily in physics. Most commonly it is used in the modeling of jet flows as well as in some steady-state heating and cooling models. For example, consider a two-dimensional flow of an ideal fluid, that is, a fluid which is incompressible, irrotational and inviscid. Let u be the velocity potential of the fluid. Then, the velocity vector is ∇u. Incompressibility implies

$$0 = \nabla \cdot (\nabla u) = \Delta u$$

in $\Omega \cap \{u > 0\}$. Let v denote the stream function (the harmonic conjugate of u). Recall that v is conjugate to u in $\Omega \cap \{u > 0\}$ if and only if u and v satisfy the

Cauchy-Riemann equations

$$\partial_x u = \partial_y v$$
$$\partial_y u = -\partial_x v$$

in $\Omega \cap \{u > 0\}$. Then, v satisfies

$$\Delta v = 0 \text{ in } \Omega \cap \{u > 0\} \tag{1.2}$$

Further, let Γ denote the free boundary $\partial\{u > 0\}$. Since Γ is a streamline we have

$$v = 0 \text{ on } \Gamma \tag{1.3}$$

Recall Bernoulli's principle for an incompressible flow, which says that at any point on a streamline we have

$$\frac{1}{2}|\nabla u|^2 + gy + \frac{p}{\rho} = C$$

where g is the gravitational constant, y is the height, p is the pressure at the given point and ρ is the density of the fluid. See [14] for a more detailed discussion. The incompressibility of the fluid implies $\rho =$ constant. Since the pressure is continuous across the boundary of the fluid we have $p =$ constant on Γ. Using the Cauchy-Riemann equations we can substitute $|\nabla v|^2$ for $|\nabla u|^2$. It then follows that $\frac{1}{2}|\nabla v|^2 + gy = C$ and setting $Q^2 = C - 2gy$ gives

$$|\nabla v| = Q \text{ on } \Gamma. \tag{1.4}$$

Conditions (1.2) - (1.4) are satisfied by minimizers u of the functional

$$J[v] = \int_\Omega |\nabla v|^2 + \chi_{\{v>0\}} Q^2 \tag{1.5}$$

where the minimization is carried out over $K = \{v \in L^1_{\text{loc}}(\Omega) : \nabla v \in L^2(\Omega)$ and $v = u^0$ on $S \subset \partial\Omega\}$ where $u^0 \in L^1_{\text{loc}}(\Omega)$ and $\nabla u^0 \in L^2(\Omega)$. We will study this problem in more detail in Chapter 3.

The objective of this thesis is to examine the behavior of solutions near the intersection of the free boundary and the portion of the fixed boundary, $N \subset \partial\Omega$, on which $\partial_\nu u = 0$. In this thesis we will prove that in this region solutions are Lipschitz continuous, that is $|\nabla u(x)| \leq C$. This problem was first considered by Alt and Caffarelli in "Existence and Regularity for a Minimum Problem with Free Boundary". In that paper the authors considered the one-phase problem (i.e. they considered (1.5)) and proved that solutions exist and are Lipschitz continuous. They additionally attained some results about the regularity of the free boundary. These results will be discussed in Chapter 3. The one-phase problem with mixed (i.e., Dirichlet and Neumann) boundary conditions was studied by Raynor in "Regularity of Neumann Solutions to an Elliptic Free Boundary Problem". In that thesis it was shown that Neumann solutions to the one-phase problem are Lipschitz continuous up to the Neumann fixed boundary. The two-phase problem (minimizers u can assume both positive and negative values) was then considered by Alt, Caffarrelli and Friedman in "Variational Problems with Two Phases and their Free Boundaries". The authors demonstrated the existence and Lipschitz continuity of minimizers. They also established some results about the regularity of the free boundary, including showing that the free boundary is C^1 in $\mathbb{R}^2$.

Chapter 2 will review the mathematical background necessary to deal with this particular free boundary problem. The partial differential equation in our free boundary problem is Laplace's equation. Thus, in Section 1 of Chapter 2 we will first examine some fundamental properties of harmonic functions. As will be explained in more detail later Sobolev spaces provide a natural place to search for solutions to PDEs. So, in Section 2, we proceed to discuss Sobolev spaces and the properties of functions belonging to them. In Section 3, we consider the theory of general second-order elliptic PDEs. Then, in Section 4, we discuss the calculus of variations, which will provide us a useful way to reformulate PDEs and is a powerful tool in the analysis of partial differential equations. Finally, in Section 5, we will examine properties of Green's functions for equations with bounded and measurable coefficients which will be useful for forming representation formulas for solutions to our problem later in Chapter 5. Next, in Chapter 3 we will present some results about the one-phase problem (both the Dirichlet and Neumann problems). After this, in Chapter 4, we will examine the two-phase Dirichlet problem and present some relevant results. Finally, in Chapter 5, we will present some new results regarding the two-phase Neumann problem problem in $\mathbb{R}^2$. The primary new result to be presented is the Lipschitz continuity of solutions up to the Neumann fixed boundary. We will also derive a monotonicity formula for Neumann solutions in a neighborhood of the Neumann fixed boundary.

Chapter 2: Preliminaries

2.1 Properties of Harmonic Functions

Laplace's equation is one of the most important partial differential equations. Given that the objective of this thesis is the study of a free boundary problem for Laplace's equation it will be useful to first understand some properties of solutions to Laplace's equation on a fixed domain. Consequently, in this section we will survey a number of well-known results about such solutions. Proofs will be omitted, but can be found in many standard texts discussing elliptic PDEs (e.g. [3], [9]). Throughout this thesis we will let Ω denote an open, connected subset of $\mathbb{R}^n$. Occasionally, we will need to make some stronger assumptions about the domain Ω. However, at such times these assumptions will be explicitly stated.

Notation 2.1. $\partial^k_{x_i} u = \frac{\partial^k u}{\partial x_i^k}$. *So, if* $L = \sum_{i=1}^n \partial^{k_i}_{x_i}$, *then* $Lu = \sum_{i=1}^n \frac{\partial^{k_i} u}{\partial x_i^{k_i}}$

Definition 2.1. *Let* $\Omega \subset \mathbb{R}^n$. *Then,* $u \in C^2(\Omega)$ *is harmonic in* Ω *if it satisfies Laplace's equation* $-\Delta u = 0$ *in* Ω *where* $\Delta := \sum_{i=1}^n \partial^2_{x_i}$.

Definition 2.2. *The function* $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ *given by*

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \tag{2.1}$$

is called the fundamental solution of Laplace's equation. Here $\alpha(n)$ *denotes the volume of the unit ball in* $\mathbb{R}^n$.

The fundamental solution satisfies Laplace's equation at every point in $\mathbb{R}^n$ except at the origin. Further, the fundamental solution is a radial solution. That is, if $x, y \in \mathbb{R}^n$ with $|x| = |y|$ and $x, y \neq 0$, then we have $\Phi(x) = \Phi(y)$.

Notation 2.2. *Let $\Omega \subset \mathbb{R}^n$. We denote*

$$\fint_{\Omega} u := \frac{1}{|\Omega|} \int_{\Omega} u$$

$$\fint_{\partial\Omega} u := \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u$$

where $|\cdot|$ denotes the Lebesgue measure.

These integrals give the average of u over Ω and $\partial\Omega$ respectively.

Theorem 2.1 (Mean-Value Property and Converse)**.** *$u \in C^2(\Omega)$ is harmonic in Ω if and only if*

$$u(x) = \fint_{S_r(x)} u \; d\sigma = \fint_{B_r(x)} u \; dy \tag{2.2}$$

for each $B_r(x) \subset \Omega$. The forward implication is called the mean-value property.

The previous theorem tells us that if u is harmonic, then $u(x)$ is equal to the average of u over $S_r(x)$ as well as the average of u over $B_r(x)$. Since the statement in the preceding theorem is biconditional we can readily interchange harmonic functions and functions which satisfy the mean-value property. Thus, any property that is satisfied by functions satisfying the mean-value property is also satisfied by harmonic functions and vice versa.

Theorem 2.2 (Strong Maximum Principle)**.** *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω.*

(i) Then,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

(ii) If Ω is connected and there is a point $x^0 \in \Omega$ such that

$$u(x^0) = \max_{\overline{\Omega}} u$$

then u is constant in Ω.

(i) is referred to as the weak maximum principle. Notice also that (ii) implies (i). The same results hold if we consider the minimum instead of the maximum. This follows directly from the above as we can simply replace u with $-u$, which is still harmonic. The maximum principle can be used to establish the following two results.

Corollary 2.1. *If Ω is connected and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$-\Delta u = 0 \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

where $g \geq 0$. Then $u > 0$ in Ω if $g > 0$ somewhere on $\partial\Omega$.

Theorem 2.3. *Let $g \in C(\partial\Omega)$. Then, there is at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary-value problem*

$$-\Delta u = 0 \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

A natural question to ask about solutions to PDEs is how smooth they are. The following result tells us that harmonic functions are in fact infinitely continuously differentiable.

Theorem 2.4. *If $u \in C(\Omega)$ satisfies the mean-value property for each $B_r(x) \subset \Omega$, then $u \in C^\infty(\Omega)$.*

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ and assume u is harmonic in Ω. Then,*

$$|\nabla u(x^0)| \leq \frac{C(n)}{r^{n+1}} \|u\|_{L^1(B_r(x^0))}$$

The following theorem will strengthen an earlier theorem which established that harmonic functions are smooth.

Definition 2.3. *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Then, u is real-analytic in Ω if for any $x^0 \in \Omega$ u can be represented by a convergent power series in some neighborhood of x^0.*

Theorem 2.6. *If u is harmonic in Ω, then u is real-analytic in Ω.*

Notation 2.3. *We use $\Omega' \subset\subset \Omega$ to indicate that Ω' is compactly contained in Ω. That is, $\Omega' \subset \overline{\Omega'} \subset \Omega$ where $\overline{\Omega'}$ is compact. Note that if Ω were not open we would have $\Omega' \subset\subset \overline{\Omega'} \subset \Omega^{\circ}$.*

Theorem 2.7 (Harnack's Inequality)**.** *For each connected open set $\Omega' \subset\subset \Omega$, there is a positive constant $C = C(\Omega')$ such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

for all nonnegative harmonic functions u in Ω'.

In fact, this theorem implies that for all points $x, y \in \Omega'$ the following inequalities hold

$$C^{-1}u(y) \leq u(x) \leq Cu(y)$$

Thus, if $u : \Omega' \to \mathbb{R}$ is a nonnegative harmonic function, then all values of u in Ω' are proportional. So, if $|u(x^0)|$ is very large for some $x^0 \in \Omega'$, then $|u(x)|$ is very large for all $x \in \Omega'$. It should be noted that it is important the we consider only points in $\Omega' \subset\subset \Omega$ as the above does not hold for points not sufficiently far from the boundary. However, similar results, which hold up to the boundary, can be established.

Rather than simply solving a partial differential equation, we often want to solve a boundary value problem. That is, we want to solve the PDE while additionally requiring that the solution assumes certain prescribed values on the boundary of the domain. Earlier in this section we presented the fundamental solution of Laplace's equation (2.1). Recall, that the fundamental solution solves Laplace's equation in $\mathbb{R}^n$ (except at a single point), but given a domain $\Omega \subset \mathbb{R}^n$ it will not necessarily satisfy any boundary conditions that we wish to impose. We can however use the fundamental solution to derive a solution that will satisfy Laplace's equation everywhere in the domain and also satisfy prescribed boundary conditions. We do this by making use of a corrector function which is harmonic and equal to the fundamental solution on the boundary of the domain.

Definition 2.4. *Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary. Then, the Green's function for the region Ω is*

$$G(x, y) := \Phi(y - x) - \phi^x(y) \qquad (x, y \in \Omega, x \neq y)$$

where Φ is the fundamental solution of Laplace's equation and, for fixed $x \in \Omega$, $\phi^x(y)$ is a corrector function which satisfies

$$\begin{aligned} -\Delta\phi^x(y) &= 0 && \text{in } \Omega \\ \phi^x(y) &= \Phi(y - x) && \text{on } \partial\Omega \end{aligned}$$

We can now use Green's function to construct a formula to represent solutions to boundary value problems for Laplace's equation.

Theorem 2.8. *The Green's function satisfies the following: for all $x, y \in \Omega$ with $x \neq y$ we have $G(x, y) = G(y, x)$.*

Definition 2.5. *Poisson's equation is given by $-\Delta u = f$.*

Theorem 2.9 (Green's Function Representation Formula)**.** *If $u \in C^2(\overline{\Omega})$ solves*

$$-\Delta u = f \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

where $f \in C(\Omega)$ and $g \in C(\partial\Omega)$, then for $x \in \Omega$

$$u(x) = -\int_{\partial\Omega} g\partial_\nu G + \int_\Omega fG$$

where $\partial_\nu G$ denotes the outward normal derivative of G.

Remark 2.1. *Above we have used $\partial_\nu G$ to denote the outward normal derivative of G. Let ν be the outward pointing normal vector on $\partial\Omega$. Then, $\partial_\nu G$ is the directional derivative of G in the direction of ν, that is, $\partial_\nu G = \nabla G \cdot \nu$.*

Notice that this representation formula actually solves Poisson's equation. However, if we let $f \equiv 0$ we have a representation formula for solutions to Laplace's equation satisfying the prescribed boundary conditions. This representation formula will be very useful later when we wish to prove some results about solutions to boundary value problems for Laplace's equation.

Theorem 2.10 (Dirichlet's Principle)**.** *Consider the boundary value problem*

$$-\Delta u = f \ \text{in } \Omega$$

$$u = g \ \text{on } \partial\Omega$$

where $f \in C(\Omega)$, $g \in C(\partial\Omega)$ and Ω is open and bounded with C^1 boundary. Let

$$J[w] := \int_\Omega \frac{1}{2}|\nabla w|^2 - wf$$

and

$$K := \{w \in C^2(\overline{\Omega}) : w = g \ \text{on } \partial\Omega\}$$

If $u \in C^2(\overline{\Omega})$ solves the boundary-value problem, then

$$J[u] = \min_{w \in K} J[w]$$

Conversely, if $u \in K$ minimizes the energy functional, J, then u solves the boundary-value problem.

Later in this chapter, in the section on the calculus of variations, we will take an in-depth look at this approach to studying PDEs.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be convex with $\partial\Omega = \{y_2 > f(y_1)\}$ where f is a Lipschitz function with Lipschitz constant K, $f(0) = 0$ and $\min_{y\in\mathbb{R}} f(y) = 0$. Let u be a harmonic function on $B_4(0)\cap\Omega$, with $0 \leq u \leq A$, such that $\partial_\nu u = 0$ along $\partial\Omega\cap B_4(0)$. Then,*

$$|\nabla u| \leq C$$

on $B_2(0) \cap \Omega$ where the constant $C = C(K)A$.

2.2 Sobolev Spaces

2.2.1 Introduction

Sobolev spaces are of fundamental importance in the study of the theory of partial differential equations. Specifically, Sobolev spaces are a natural place to search for solutions to partial differential equations. Using the theory that we will develop in this chapter we can think of PDEs (e.g. $Lu = f$) as operators acting on a Sobolev space (e.g. $L : X \to Y$, where X and Y are Sobolev spaces). Given the importance of Sobolev spaces and the power of them as a tool for studying PDEs, it will be useful to understand some properties of functions that belong to various Sobolev spaces.

Notation 2.4. *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and let $t \in \mathbb{R}$. Then, we denote*

$$\{u > t\} \coloneqq \{x \in \Omega : u(x) > t\}$$

Definition 2.6. *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Then, u is a measurable function if for each $t \in \mathbb{R}$ the set $\{u > t\}$ is measurable.*

Recall that Ω is open. If Ω weren't an open set, then we would also need to impose the additional constraint that Ω be measurable.

Definition 2.7. *A measurable function u is* <u>*summable*</u> *if*

$$\int_\Omega |u| < \infty$$

Definition 2.8. *Let* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$*. Then,* $u \in L^p(\Omega)$ *if* u *is a measurable function and* u^p *is summable.*

Definition 2.9. $u \in L^p_{\text{loc}}(\Omega)$ *if* $u \in L^p(\Omega')$ *for every* $\Omega' \subset\subset \Omega$*.*

Definition 2.10. *A vector* $\alpha = (\alpha_1, ..., \alpha_n)$ *where* $\alpha_i \in \mathbb{N}$ *for each* $1 \leq i \leq n$ *is called a* <u>*multiindex*</u> *of order* $|\alpha| = \sum_{i=1}^n \alpha_i$*.*

2.2.2 Definition of Sobolev Spaces

Given that functions in Sobolev spaces may not be differentiable in the classical sense we will need to develop a new concept of differentiability to use when working in Sobolev spaces. We can use the integration by parts formula to define a "weak derivative" which can be applied to functions belonging to Sobolev spaces.

Definition 2.11. *Let* $u, v \in L^1_{loc}(\Omega)$ *and let* α *be a multiindex. Then,* v *is the* <u>α^{th}*-weak partial derivative*</u> *of* u*, denoted* $D^\alpha u = v$*, if*

$$\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega v\phi$$

for all $\phi \in C_c^\infty(\Omega)$*.*

The weak derivative satisfies several of the important properties that are satisfied by the classical derivative. For example, uniqueness, linearity and Leibniz's rule are all satisfied by the weak derivative.

Theorem 2.11 (Properties of Weak Derivatives)**.** *If* $u, v \in W^{k,p}(\Omega)$ *and* $|\alpha| \leq k$*, then* *(i)* $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ *and* $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ *for all multiindices* β *with* $|\alpha| + |\beta| \leq k$.

(ii) *For each* $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ *and* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.

(iii) *If* Ω' *is an open subset of* Ω, *then* $u \in W^{k,p}(\Omega')$.

(iv) *If* $\zeta \in C_c^\infty(\Omega)$, *then* $\zeta u \in W^{k,p}(\Omega)$ *and*

$$D^\alpha(\zeta u) = \sum_{|\beta|\leq|\alpha|} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \qquad \textit{(Leibniz's formula)}$$

Theorem 2.12. *Let* α *be a multiindex. Then,* $D^\alpha u$, *if it exists, is uniquely defined (up to a set of measure zero).*

Definition 2.12. *A function* u *is* <u>*locally summable*</u> *if* $u \in L^1_{\text{loc}}$ *and* <u>*locally p-summable*</u> *if* $u \in L^p_{\text{loc}}$.

Definition 2.13. *The Sobolev space* $W^{k,p}(\Omega)$ *consists of all locally p-summable functions* $u : \Omega \to \mathbb{R}$ *such that for each* $|\alpha| \leq k$, $D^\alpha u$ *exists (in the weak sense) and* $D^\alpha u \in L^p(\Omega)$.

Definition 2.14. *Let* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *be measurable. Then, the* <u>*essential supremum*</u> *of* u *is*

$$\operatorname*{ess\,sup}_\Omega u := \inf\{t \in \mathbb{R} : |\{u > t\}| = 0\}$$

It is important to note the difference between the essential supremum and the supremum. Let $u : \Omega \to \mathbb{R}$ be measurable. Then, we have

$$u(x) \leq \sup_\Omega u \text{ for all } x \in \Omega$$

However,

$$u(x) \leq \operatorname*{ess\,sup}_\Omega u \text{ for almost all } x \in \Omega$$

So, it may be the case that for some $x \in \Omega$ we have

$$u(x) > \operatorname*{ess\,sup}_{\Omega} u$$

The definition of essential supremum only requires that the Lebesgue measure of the set of such x be zero.

Definition 2.15. *If $u \in W^{k,p}(\Omega)$, then we define its norm as follows*

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} (\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u| & (p = \infty) \end{cases}$$

Definition 2.16. *$W_0^{k,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.*

Notation 2.5. *We will write $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ for $k \in \mathbb{N}$.*

Theorem 2.13. *$W^{k,p}(\Omega)$ is a Banach space for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$.*

Definition 2.17. *Let $\{u_m\} \subset W^{k,p}(\Omega)$ and $u \in W^{k,p}(\Omega)$. Then, $u_m \to u$ in $W_{\text{loc}}^{k,p}(\Omega)$ if $u_m \to u$ in $W^{k,p}(\Omega')$ for each $\Omega' \subset\subset \Omega$.*

2.2.3 Approximation by Smooth Functions

While the weak derivative is a powerful tool, it would be preferable to be able to work with functions which are smooth, so we can use the classical notion of differentiability and take advantage of other convenient properties of smooth functions. Fortunately, functions belonging to Sobolev spaces can be approximated by smooth, that is infinitely continuously differentiable, functions. The following theorems provide increasingly strong results regarding the use of smooth functions to approximate functions belonging to Sobolev spaces.

Definition 2.18. *Let $u : \Omega \to \mathbb{R}$, $\eta : \mathbb{R}^n \to \mathbb{R}$. Then, the convolution of η and u is given by*

$$\eta * u(x) := \int_\Omega \eta(x-y)u(y)dy \qquad (x \in \Omega)$$

Theorem 2.14. *Let $\eta \in C^\infty(\mathbb{R}^n)$. For each $\varepsilon > 0$, set $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}(\frac{x}{\varepsilon})$. Fix $k \in \mathbb{N}$ and assume $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Let $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and set*

$$u^\varepsilon = \eta_\varepsilon * u \text{ in } \Omega_\varepsilon$$

Then,

(i) $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$ *for each* $\varepsilon > 0$.

(ii) $u^\varepsilon \to u$ *in* $W^{k,p}_{\text{loc}}(\Omega)$ *as* $\varepsilon \to 0$.

Theorem 2.15. *Suppose Ω is bounded and $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then, there exists $\{u_m\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that*

$$u_m \to u \text{ in } W^{k,p}(\Omega) \text{ as } m \to \infty$$

Notice that the preceding approximation theorems made no assumptions about the smoothness of $\partial\Omega$. But, it should be noted that we only claimed that the approximating functions were in $C^\infty(\Omega)$. In the following theorem we will assert that functions in $W^{k,p}(\Omega)$ can be approximated using functions in $C^\infty(\overline{\Omega})$. However, this will only be possible if we make some assumptions about the regularity of $\partial\Omega$.

Theorem 2.16. *Assume Ω is bounded with C^∞ boundary. If $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$, then there exists $\{u_m\} \subset C^\infty(\overline{\Omega})$ such that*

$$u_m \to u \text{ in } W^{k,p}(\Omega)$$

2.2.4 Traces

Given that we wish to search for solutions to boundary value problems in Sobolev spaces it is important to be able to assign values to functions in $W^{k,p}(\Omega)$ along $\partial\Omega$. However, functions in Sobolev spaces need only be defined almost everywhere on Ω. Given that the n-dimensional Lebesgue measure of $\partial\Omega$ is zero we cannot naively restrict u to $\partial\Omega$. In fact, $u|_{\partial\Omega}$ does not have any readily apparent meaning in this context. The following theorem will provide a way for us to consider functions in $W^{k,p}(\Omega)$ restricted to $\partial\Omega$ and thus solve the problem of prescribing boundary conditions in the Sobolev space setting.

Theorem 2.17 (Trace Theorem)**.** *Suppose Ω is bounded and $\partial\Omega$ is C^1. Then, there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

(i) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$

(ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C(p, \Omega)\|u\|_{W^{1,p}(\Omega)}$

for each $u \in W^{1,p}(\Omega)$.

Definition 2.19. *Tu is called the trace of u on $\partial\Omega$.*

Earlier we defined $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The next theorem provides an alternative way of describing functions belonging to $W_0^{1,p}(\Omega)$ based on the trace of those functions on $\partial\Omega$.

Theorem 2.18. *If Ω is bounded, $\partial\Omega$ is C^1 and $u \in W^{1,p}(\Omega)$, then*

$$u \in W_0^{1,p}(\Omega) \text{ if and only if } Tu = 0 \text{ on } \partial\Omega$$

2.2.5 Sobolev Inequalities

Definition 2.20. *If $1 \leq p < n$, then the Sobolev conjugate of p is*

$$p^* := \frac{np}{n-p}$$

A natural question to ask about functions belonging to a Sobolev space is whether it belongs to other function spaces. The next several theorems will establish estimates that will allow us to conclude that functions in $W^{k,p}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ are, after possibly being redefined on a set of measure zero, Hölder continuous given certain conditions on k, p and n.

Theorem 2.19 (Gagliardo-Nirenberg-Sobolev Inequality)**.** *Assume* $1 \leq p < n$. *Then, there is a constant* $C = C(p,n)$ *such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 2.20. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded with* $\partial\Omega$ *being* C^1. *If* $1 \leq p < n$ *and* $u \in W^{1,p}(\Omega)$, *then there is a constant* $C = C(p,n,\Omega)$ *such that* $u \in L^{p^*}(\Omega)$ *with*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

Theorem 2.21. *If* $\Omega \subset \mathbb{R}^n$ *is bounded and open and* $u \in W_0^{1,p}(\Omega)$ *for some* $1 \leq p < n$, *then there is a constant* $C = C(p,q,n,\Omega)$ *such that for* $q \in [1, p^*]$ *we have*

$$\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$$

Corollary 2.2 (Poincaré's Inequality)**.** *Suppose* $\Omega \subset \mathbb{R}^n$ *is bounded and open. For all* $1 \leq p \leq \infty$, *if* $u \in W_0^{1,p}(\Omega)$, *then there exists a constant* $C = C(p,q,n,\Omega)$

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$$

Theorem 2.22. *If* $n < p \leq \infty$, *then there is a constant* $C = C(p,n)$ *such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$, *where* $\gamma := 1 - \frac{n}{p}$.

Definition 2.21. *Let* $u \in W^{k,p}(\Omega)$. *Then,* $u^* \in W^{k,p}(\Omega)$ *is a* *representative* *of* u *if*

$$u = u^* \text{ a.e. in } \Omega$$

Theorem 2.23. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega$ being C^1. If $n < p \le \infty$ and $u \in W^{1,p}(\Omega)$, then u has a representative $u^* \in C^{0,\gamma}(\overline{\Omega})$ where $\gamma = 1 - \frac{n}{p}$. Further, there is a constant $C = C(p, n, \Omega)$ such that*

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \le C\|u\|_{W^{1,p}(\Omega)}$$

Theorem 2.24 (General Sobolev Inequalities)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with a C^1 boundary. Suppose $u \in W^{k,p}(\Omega)$.*

(i) If $k < \frac{n}{p}$, then $u \in L^q(\Omega)$ where $\frac{1}{q} = \frac{1}{p} - \frac{n}{k}$. Further, we have the estimate

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{k,p}(\Omega)}$$

(ii) If $k > \frac{n}{p}$, then $u \in C^{k-\lfloor\frac{n}{p}\rfloor-1,\gamma}(\overline{\Omega})$ where

$$\gamma = \begin{cases} \lfloor\frac{n}{p}\rfloor + 1 - \frac{n}{p} & , \frac{n}{p} \notin \mathbb{Z} \\ \text{any positive number } < 1 & , \frac{n}{p} \in \mathbb{Z} \end{cases}$$

where the constant $C = C(k, p, n, \Omega)$.

In addition, there is a constant $C = C(k, p, n, \gamma, \Omega)$ such that

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})} \le C\|u\|_{W^{k,p}(\Omega)}$$

Definition 2.22. *Let X, Y be Banach spaces with $X \subset Y$. Then, X is precompact in Y, if for any sequence $\{u_k\} \subset X$, there is a subsequence $\{u_{k_j}\} \subset \{u_k\}$ and $u \in Y$ with*

$$\lim_{j\to\infty} \|u_{k_j} - u\|_Y = 0$$

Definition 2.23. *Let X, Y be Banach spaces with $X \subset Y$. Then, X is compactly embedded in Y, denoted $X \subset\subset Y$, if*

1. *For some constant C, $\|u\|_Y \le C\|u\|_X$*

2. Each bounded sequence in X is precompact in Y.

The Gagliardo-Nirenberg-Sobolev inequality gave an embedding of $W^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ for $1 \leq p < n$. The following theorem will establish that $W^{1,p}(\Omega)$ is actually compactly embedded in $L^q(\Omega)$ for $1 \leq q < p^*$ under the appropriate conditions.

Theorem 2.25 (Rellich-Kondrachov Compactness Theorem)**.** *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary and $1 \leq p < n$. Then, for each $1 \leq q < p^*$ we have*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

Theorem 2.26 (Poincaré's Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected with C^1 boundary. If $1 \leq p \leq \infty$, then there exists a constant $C = C(n, p, \Omega)$ such that for each $u \in W^{1,p}(\Omega)$ we have*

$$\left\| u - ⨍_{\Omega} u \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

2.2.6 Difference Quotients and Differentiability

At times it is necessary to approximate weak derivatives. In these cases difference quotients provide a nice way to do this. After defining a difference quotient we state a theorem which demonstrates that difference quotients indeed provide a good way to approximate weak derivatives.

Definition 2.24. *Assume $u : \Omega \to \mathbb{R}$ is locally summable and $\Omega' \subset\subset \Omega$.*

(i) The i^{th}-difference quotient of size $h \in \mathbb{R}$ is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} \qquad (i = 1, ..., n)$$

for $x \in \Omega'$ and $h \in \mathbb{R}$ with $0 < |h| < \mathrm{dist}(\Omega', \partial\Omega)$.

(ii) $D^h u := (D_1^h u, ..., D_n^h(u))$

where $\{e_i\}_{i=1}^n$ *is the standard basis for* $\mathbb{R}^n$*.*

Theorem 2.27. *(i) If* $1 \leq p < \infty$ *and* $u \in W^{1,p}(\Omega)$*, then for every* $\Omega' \subset\subset \Omega$

$$\|D^h u\|_{L^p(\Omega')} \leq C\|\nabla u\|_{L^p(\Omega)}$$

for some constant C *and each* $0 < |h| < \frac{1}{2}\text{dist}(\Omega', \partial\Omega)$*.*

(ii) If $1 < p < \infty$*,* $u \in L^p(\Omega')$ *and there is a constant* C *such that*

$$\|D^h u\|_{L^p(\Omega')} \leq C$$

for all $0 < |h| < \frac{1}{2}\text{dist}(\Omega', \partial\Omega)$*, then* $u \in W^{1,p}(\Omega')$ *with*

$$\|\nabla u\|_{L^p(\Omega')} \leq C$$

Recall that a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $x^0 \in \Omega$ if there exists $D \in \mathbb{R}^n$ such that

$$f(x) = f(x^0) + D \cdot (x - x^0) + o(|x - x^0|)$$

Recall that functions belonging to Sobolev spaces are not necessarily differentiable everywhere in their domain. In fact, they need not be defined everywhere in their domain. This motivated the definition of the weak derivative. A natural question to ask is whether these functions must be differentiable anywhere in their domain and if so how the derivative relates to the weak derivative. The following theorem will resolve these questions, thus providing a link between the classical derivative and the weak derivative.

Theorem 2.28 (Differentiability Almost Everywhere)**.** *If* $u \in W^{1,p}_{\text{loc}}(\Omega)$ *for some* $n < p \leq \infty$*, then* u *is differentiable a.e. in* Ω *and its gradient equals its weak gradient a.e. Note that we identify* u *with its continuous representative.*

2.2.7 Dual Spaces

Definition 2.25. *Let X be a Banach space. Then, the collection of all bounded linear functionals on X is the* <u>*dual space*</u> *of X and is denoted X^*.*

Definition 2.26. *$H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.*

So, $f \in H^{-1}(\Omega)$ if f is a bounded linear functional on $H_0^1(\Omega)$.

Definition 2.27. *The* <u>*dual pairing*</u> *between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is a bilinear map $< \cdot, \cdot >: H^{-1}(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$.*

Definition 2.28. *If $f \in H^{-1}(\Omega)$, then we define the norm*

$$\|f\|_{H^{-1}(\Omega)} := \sup\{< f, u >: u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1\}$$

Theorem 2.29 (Characterization of H^{-1})**.** *If $f \in H^{-1}(\Omega)$, then there exist functions $f^0, f^1, ..., f^n \in L^2(\Omega)$ such that*

(i)

$$< f, v >= \int_\Omega \left[f^0 v + \sum_{i=1}^{n} f^i v_{x_i} \right] \qquad (v \in H_0^1(\Omega))$$

(ii)

$$\|f\|_{H^{-1}(\Omega)} = \inf\{ \left(\int_\Omega \sum_{i=0}^{n} |f^i|^2 \right)^{\frac{1}{2}} : f \text{ satisfies (i) for } f^0, ..., f^n \in L^2(\Omega)\}$$

(iii)

$$< v, u >= (v, u)$$

for all $u \in H_0^1(\Omega)$, $v \in L^2(\Omega) \subset H^{-1}(\Omega)$ where $(\cdot, \cdot)$ denotes the inner product on $L^2(\Omega)$.

2.3 Second-Order Elliptic Equations

2.3.1 Introduction

Before we study an elliptic partial differential equation with free boundary conditions it will be helpful to establish some properties about elliptic PDEs with fixed boundary conditions.

Definition 2.29. *The partial differential operator* $L = \sum_{i,j=1}^{n} a^{ij}\partial_{x_i}\partial_{x_j}$ *is uniformly elliptic if there exists a constant* $\theta > 0$ *such that*

$$\sum_{i,j=1}^{n} a^{i,j}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for almost all $x \in \Omega$ *and all* $\xi \in \mathbb{R}^n$*.*

Definition 2.30. *A partial differential operator having the form*

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j}$$

is said to be in divergence form.

Definition 2.31. *The bilinear form* $B[\ ,\ ]$ *on* $H_0^1(\Omega)$ *associated with the divergence form elliptic operator* L *is*

$$B[u,v] := \int_{\Omega} \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j}$$

Recall that since functions belonging to Sobolev spaces are not classically differentiable we needed to define the weak derivative. However, our reason for examining the theory of Sobolev spaces was because they provide a natural place to look for solutions to partial differential equations. Issues could arise as a classical solution to a second-order PDE must be C^2. The next definition will allow us to overcome this problem by defining a new notion of the solution to a partial differential equation.

Definition 2.32. *$u \in H_0^1(\Omega)$ is a weak solution of the boundary-value problem*

$$Lu = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

if

$$B[u, v] = (f, v)$$

for all $v \in H_0^1(\Omega)$.

Definition 2.33. *Let $f^i \in L^2(\Omega)$ for each $i = 0, 1, ..., n$. That is, $f := f^0 - \sum_{i=1}^n f^i_{x_i}$ is in $H^{-1}(\Omega)$. We say that $u \in H_0^1(\Omega)$ is a weak solution of*

$$Lu = f^0 - \sum_{i=1}^n f^i_{x_i} \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

provided

$$B[u, v] =< f, v >$$

for all $v \in H_0^1(\Omega)$.

Problems with nonzero Dirichlet boundary conditions can be easily transformed into problems with zero boundary conditions. For example, let $u \in H^1(\Omega)$ be a weak solution of

$$Lu = f \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

where Ω has C^1 boundary and the prescribed boundary conditions are defined in the trace sense. Then, it must be the case that g is the trace of some function $w \in H^1(\Omega)$.

We can then define $\hat{u} := u - w$ and $\hat{f} := f - Lw$. It then follows that $\hat{u} \in H_0^1(\Omega)$, $\hat{f} \in H^{-1}$ and $\hat{u}$ solves the following boundary value problem in the weak sense

$$L\hat{u} = \hat{f} \text{ in } \Omega$$

$$\hat{u} = 0 \text{ on } \partial\Omega$$

Consequently, all of the results from this section will be presented for problems with zero boundary conditions in the trace sense.

2.3.2 Existence

A natural question to ask when studying boundary value problems is under what conditions one can guarantee the existence of a solution. The next several results address the question of the existence of weak solutions to second-order elliptic boundary value problems. Notice that many of these results not only guarantee the existence of solution to the elliptic boundary value problem, but they further guarantee that this solution is unique.

Theorem 2.30 (Lax-Milgram Theorem)**.** *Let H be a real Hilbert space with norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $<\cdot,\cdot>_{H^*\times H}$ denote the pairing of H with its dual space.*
Suppose that $B : H \times H \to \mathbb{R}$ is a bilinear mapping, for which there are constants $\alpha, \beta > 0$ such that

$$|B[u,v]| \leq \alpha\|u\|\|v\| \qquad (u, v \in H)$$

and

$$\beta\|u\|^2 \leq B[u,u] \qquad (u \in H)$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then, there exists a unique element $u \in H$ such that

$$B[u,v] = < f, v >$$

for all $v \in H$.

Theorem 2.31 (Energy Estimates)**.** *There exist constants* $\alpha, \beta > 0$ *and* $\gamma \geq 0$ *such that*

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

and

$$\beta \|u\|_{H_0^1(\Omega)}^2 \leq B[u,u] + \gamma \|u\|_{L^2(\Omega)}^2$$

for all $u, v \in H_0^1(\Omega)$.

Theorem 2.32 (First Existence Theorem for Weak Solutions)**.** *There is a number* $\gamma \geq 0$ *such that for each* $\mu \geq \gamma$ *and each function* $f \in L^2(\Omega)$ *there exists a unique weak solution* $u \in H_0^1(\Omega)$ *of the boundary value problem*

$$Lu + \mu u = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

2.3.3 Regularity

Notation 2.6. *We will occasionally use* $H^0(\Omega)$ *to denote* $L^2(\Omega)$.

Another fundamental question in the study elliptic partial differential equations (in fact, partial differential equations in general) relates to the smoothness of solutions. Given assumptions regarding the smoothness of the equation's coefficients and the smoothness of the source term we will be able to draw some conclusions about the smoothness of solutions to the partial differential equation. Additionally, we will be able to derive some estimates on the norm of solutions.

Theorem 2.33 (Interior Regularity)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded and let* m *be a nonnegative integer, and assume*

$$a^{ij} \in C^{m+1}(\Omega) \qquad (i, j = 1, ..., n)$$

and

$$f \in H^m(\Omega)$$

Suppose $u \in H^1(\Omega)$ *is a weak solution of the elliptic PDE*

$$Lu = f \qquad in\ \Omega$$

Then,

$$u \in H^{m+2}_{\text{loc}}(\Omega)$$

and for each $\Omega' \subset\subset \Omega$ *there is a constant* $C = C(m, \Omega', \Omega, a^{ij})$ *such that*

$$\|u\|_{H^{m+2}(\Omega')} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$$

Theorem 2.34 (Infinite Differentiability in the Interior)**.** *Assume*

$$a^{ij} \in C^\infty(\Omega) \qquad (i, j = 1, ..., n)$$

and

$$f \in C^\infty(\Omega)$$

Suppose $u \in H^1(\Omega)$ *is a weak solution of the elliptic PDE*

$$Lu = f \qquad in\ \Omega$$

Then,

$$u \in C^\infty(\Omega)$$

Theorem 2.35 (Boundary Regularity)**.** *Let* m *be a nonnegative integer, and assume*

$$a^{ij} \in C^{m+1}(\overline{\Omega}) \qquad (i, j = 1, ..., n)$$

and

$$f \in H^m(\Omega)$$

Suppose $u \in H_0^1(\Omega)$ *is a weak solution of the elliptic boundary-value problem*

$$Lu = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

Assume finally $\partial\Omega$ *is* C^{m+2}. *Then,* $u \in H^{m+2}(\Omega)$ *and there exists a constant* $C = C(m, \Omega, a^{ij})$ *such that*

$$\|u\|_{H^{m+2}(\Omega)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$$

Theorem 2.36 (Infinite Differentiability up to the Boundary)**.** *Assume*

$$a^{ij} \in C^\infty(\overline{\Omega}) \qquad (i, j = 1, ..., n)$$

and

$$f \in C^\infty(\overline{\Omega})$$

Suppose $u \in H_0^1(\Omega)$ *is a weak solution of the elliptic boundary-value problem*

$$Lu = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

Finally, suppose $\partial\Omega$ *is* C^∞. *Then,* $u \in C^\infty(\overline{\Omega})$.

2.3.4 Maximum Principle

Recall from the first section in this chapter that harmonic functions attain their maximum value on the boundary of their domain. Further, recall that if a harmonic function attains its maximum in the interior of their domain, then that function is constant. Loosely speaking elliptic partial differential operators generalize the Laplacian. Thus, we would like for general second-order elliptic partial differential equations to satisfy a similar property. The next several results will establish that second-order elliptic equations do indeed satisfy versions of the strong and weak maximum principles.

Theorem 2.37 (Weak Maximum Principle). *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

(i) If $Lu \leq 0$ in Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

(ii) If $Lu \geq 0$ in Ω, then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

A function u satisfying $Lu \leq 0$ is called a subsolution while a function u satisfying $Lu \geq 0$ is called a supersolution.

Lemma 2.2 (Hopf's Lemma). *Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Further, suppose that $Lu \leq 0$ in Ω and there is a point $x^0 \in \partial\Omega$ with $u(x^0) > u(x)$ for all $x \in \Omega$. Finally, assume that there is an open ball $B \subset \Omega$ with $x^0 \in \partial B$. Then,*

(i)

$$\frac{\partial u}{\partial \nu}(x^0) > 0$$

where ν is the outer unit normal vector to B at x^0

(ii) If $c \geq 0$ in Ω, then the result holds given $u(x^0) \geq 0$

Theorem 2.38 (Strong Maximum Principle). *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Additionally, suppose that Ω is connected, open and bounded. Then,*

(i) If $Lu \leq 0$ in Ω and u attains its maximum over $\overline{\Omega}$ at a point in the interior of Ω, then u is constant within Ω.

(ii) If $Lu \geq 0$ in Ω and u attains its minimum over $\overline{\Omega}$ at a point in the interior of Ω, then u is constant within Ω.

Another theorem that can be extended from harmonic functions to functions satisfying a general second-order elliptic partial differential equation is Harnack's inequality. The statement of that result is as follows.

Theorem 2.39 (Harnack's Inequality)**.** *Assume that* $u \geq 0$ *is a* C^2 *solution of* $Lu = 0$ *in* Ω *and* $\Omega' \subset\subset U$ *is connected. Then, there is a constant* $C = C(\Omega', a^{ij})$ *so that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

Note that this again implies that all values of u in Ω' are proportional. Thus, for all $x, y \in \Omega'$ the following holds

$$C^{-1}u(y) \leq u(x) \leq Cu(y)$$

2.3.5 Eigenvalues

Definition 2.34. *Consider the boundary-value problem*

$$Lu = \lambda u \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

λ *is an* *eigenvalue* *of* L *if there is a nontrivial solution of the given boundary-value problem. A solution of the boundary-value problem is called an* *eigenfunction.*

Theorem 2.40. *Let* L *be a symmetric elliptic partial differential operator. Then, we have*

(i) Each eigenvalue of L *is real.*

(ii) If we repeat each eigenvalue according to its finite multiplicity, then we have $\Sigma := \{\lambda_k\}_{k=1}^{\infty}$ *where* $0 < \lambda_1 \leq \lambda_2 \leq ...$ *and* $\lambda_k \to \infty$ *as* $k \to \infty$.

(iii) *There is an orthonormal basis* $\{w_k\}_{k=1}^{\infty}$ *of* $L^2(\Omega)$*, where* $w_k \in H_0^1(\Omega)$ *is an eigenfunction corresponding to* λ_k*. That is for* $k = 1, 2, ...,$

$$Lw_k = \lambda_k w_k \text{ in } \Omega$$

$$w_k = 0 \text{ on } \partial\Omega$$

Definition 2.35. $\lambda_1 > 0$ *is called the* *principal eigenvalue* *of* L*.*

Theorem 2.41. *Let* $\lambda_1 > 0$ *be the principal eigenvalue of* L*. Then, we have*

(i) $\lambda_1 = \min\{B[u,u] : u \in H_0^1(\Omega), \|u\|_{L^2} = 1\}$ *(Rayleigh's formula)*

(ii) *The above minimum is attained for a function* w_1*, positive in* Ω*, which solves*

$$Lw_1 = \lambda_1 w_1 \text{ in } \Omega$$

$$w_1 = 0 \text{ on } \partial\Omega$$

(iii) *If* $u \in H_0^1(\Omega)$ *is a weak solution of*

$$Lu = \lambda_1 u \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

then u *is a multiple of* w_1*.*

2.4 Calculus of Variations

2.4.1 Introduction

As we saw earlier in this chapter (recall Dirichlet's principle) it is at times possible to recast the solution to a boundary value problem as the minimizer of an appropriate functional (called the energy functional). This is useful, because it is sometimes easier to solve for the minimizer of the energy functional than it is to solve the original partial differential equation directly. In order to utilize these tools we will need to undertake a study of the calculus of variations.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded where $\partial\Omega$ is smooth and let $L : \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ be a smooth function (we will call L the Lagrangian).

Notation 2.7. *We will make occasionally make use of the notation*

$$L = L(p, z, x) = L(p_1, ..., p_n, z, x_1, ..., x_n)$$

So, p will denote $\nabla v(x)$ and z will denote $v(x)$. Further, set

$$\nabla_p L = (L_{p_1}, ..., L_{p_n})$$
$$\nabla_z L = L_z$$
$$\nabla_x L = (L_{x_1}, ..., L_{x_n})$$

The general energy functional takes the following form.

Notation 2.8. *Let*

$$J[v] \coloneqq \int_\Omega L(\nabla v(x), v(x), x)$$

Claim 2.1. *Let $K \coloneqq \{v \in C^\infty(\overline{\Omega}) : v = g \text{ on } \partial\Omega\}$. If $u \in K$ is a minimizer of $J[\cdot]$ over all functions in K, then u is a solution of the nonlinear partial differential equation*

$$-\sum_{i=1}^{n}(L_{p_i}(\nabla u, u, x))_{x_i} + L_z(\nabla u, u, x) = 0 \text{ in } \Omega$$

Definition 2.36. *The PDE satisfied by the minimizer u in Claim 2.1 is called the Euler-Lagrange equation associated with the energy functional $J[\cdot]$.*

Definition 2.37. *Fix $1 < q < \infty$ and suppose there are constants $\alpha > 0, \beta \geq 0$ such that $L(p, z, x) \geq \alpha|p|^q - \beta$ for all $p \in \mathbb{R}^n, z \in \mathbb{R}, x \in \Omega$. Then,*

$$J[v] \geq \delta\|\nabla v\|^q_{L^q(\Omega)} - \beta|\Omega|$$

for some constant $\delta > 0$. We call this a coercivity condition on $J[\cdot]$.

Remark 2.2. *An important step in searching for minimizers of functionals is to determine where we should search for these minimizers. In this section we will, unless otherwise specified, let*

$$K := \{v \in W^{1,q}(\Omega) : v|_{\partial\Omega} = g\}$$

be the class of admissible functions (i.e. the set of functions over which we will attempt to minimize the energy functional $J[\cdot]$). Notice that when choosing an appropriate admissible class we need to search in the proper Sobolev space, that is we need to ensure the required number of weak partial derivatives. Further, we need to impose the correct boundary conditions.

Definition 2.38. *We say that $I[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(\Omega)$, if*

$$J[u] \leq \liminf J[u_k]$$

whenever

$$u_k \rightharpoonup u \text{ weakly in } W^{1,q}(\Omega)$$

Theorem 2.42. *Assume that L is smooth, bounded below and convex in p. Then, $J[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(\Omega)$.*

2.4.2 Existence

Given that we are searching for functions in a certain admissible class which will minimize a given functional it is important to establish the conditions under which we can ensure that a minimizer actually exists. Further, given the existence of a minimizer we would like to know whether that minimizer will be the unique minimizer. The following two theorems will allow us to do this.

Theorem 2.43. *If L satisfies the coercivity inequality, L is convex in p and the admissible set K is nonempty, then there exists at least one function $u \in K$ solving*

$$J[u] = \min_{v \in K} J[v]$$

Theorem 2.44. *Suppose* $L = L(p, x)$ *and there exists* $\theta > 0$ *such that* $\sum_{i,j=1}^{n} L_{p_i p_j}(p, x)\xi_i \xi_j \geq \theta|\xi|^2$ $(p, \xi \in \mathbb{R}^n; x \in \Omega)$. *Then, any minimizer* $u \in K$ *of* $J[\cdot]$ *is unique.*

Remark 2.3. *For the remainder of this section we will assume the following growth conditions hold for all* $p \in \mathbb{R}^n, z \in \mathbb{R}, x \in \Omega$.

$$|L(p, z, x)| \leq C(|p|^q + |z|^q + 1)$$

$$|D_p L(p, z, x)| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$$

$$|D_z L(p, z, x)| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$$

where C *is a constant.*

Definition 2.39. $u \in K$ *is a weak solution of the boundary-value problem*

$$-\sum_{i=1}^{n}(L_{p_i}(\nabla u, u, x))_{x_i} + L_z(\nabla u, u, x) = 0 \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

for the Euler-Lagrange equation if

$$\int_{\Omega} \sum_{i=1}^{n} L_{p_i}(\nabla u, u, x))v_{x_i} + L_z(\nabla u, u, x)v = 0$$

for each $v \in W_0^{1,q}(\Omega)$.

Given a particular energy functional we would like to be able to determine what boundary value problem is solved by the minimizer. The following theorem will allow us to do this.

Theorem 2.45. *If* L *satisfies the aforementioned growth conditions and* $u \in K$ *satisfies*

$$J[u] = \min_{w \in K} J[w]$$

Then, u is a weak solution of

$$-\sum_{i=1}^{n}(L_{p_i}(\nabla u, u, x))_{x_i} + L_z(\nabla u, u, x) = 0 \ \textit{in}\ \Omega$$

$$u = g \ \textit{on}\ \partial\Omega$$

This theorem tells us that the minimizer u solves the Euler-Lagrange equation corresponding to the functional $J[\cdot]$ subject to the boundary conditions imposed on the functions in the admissible functions K.

Remark 2.4. *The Euler-Lagrange equation*

$$-\sum_{i=1}^{n}(L_{p_i}(\nabla u, u, x))_{x_i} + L_z(\nabla u, u, x) = 0 \ \textit{in}\ \Omega$$

$$u = g \ \textit{on}\ \partial\Omega$$

will generally have other solutions which are not minimizers of $J[\cdot]$. However, if the mapping $(p, z) \mapsto L(p, z, x)$ is convex for each x, then each weak solution is a minimizer.

2.4.3 Regularity

As before when we were considering solutions to second-order elliptic partial differential equations it is natural to ask what conclusions if any we can make about the smoothness of minimizers and what assumptions are necessary to draw those conclusions. The following remarks and theorem will accomplish this.

Remark 2.5. *For the following regularity theorem and remark we will assume that the functional $J[\cdot]$ is of the form*

$$J[w] := \int_{\Omega} L(\nabla w) - wf$$

for $f \in L^2(\Omega)$. We will further assume $q = 2$ and

$$|\nabla_p L(p)| \leq C(|p| + 1) \qquad (p \in \mathbb{R}^n)$$

Then, any minimizer $u \in K$ is a weak solution of the Euler-Lagrange equation

$$-\sum_{i=1}^{n} (L_{p_i}(\nabla u))_{x_i} = f \qquad \text{in } \Omega$$

So, we have

$$\int_\Omega \sum_{i=1}^{n} L_{p_i}(\nabla u) v_{x_i} = \int_\Omega f v$$

for all $v \in H_0^1(\Omega)$.

Theorem 2.46. *(i) Let $u \in H^1(\Omega)$ be a weak solution of the above Euler-Lagrange equation where L satisfies the following:*

$$|\nabla^2 L(p)| \leq C \qquad (p \in \mathbb{R}^n)$$

$$\sum_{i,j=1}^{n} L_{p_i p_j}(p) \xi_i \xi_j \geq \theta |\xi|^2 \qquad (p, \xi \in \mathbb{R}^n)$$

where $\theta > 0$ is a constant.

Then,

$$u \in H^2_{\text{loc}}(\Omega)$$

(ii) If we also have $u \in H_0^1(\Omega)$ and $\partial\Omega$ is C^2, then $u \in H^2(\Omega)$ with the estimate

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

Remark 2.6. *If L is infinitely differentiable, then u is as well.*

2.5 Green's Function for Elliptic Equations with Bounded and Measurable Coefficients

In this section we will examine some properties of Green's function for elliptic PDEs where the coefficients of the equation need only be bounded and measurable. We will present results from [11].

Earlier in this chapter we defined Green's function and used it to construct a representation formula for solutions to boundary value problems for Poisson's equation (and of course Laplace's equation). In chapter 5 of this thesis we will need to work with a PDE whose coefficients need only be bounded and measurable. In doing so we will make use of the Green's function for the equation. In order to do this we will need to understand some properties of the Green's function for such partial differential equations. In this section we will present the needed results regarding the properties of Green's function for PDEs with bounded and measurable coefficients.

Theorem 2.47. *Let* $L^\varepsilon u := -(a_\varepsilon^{ij} u_{x_i})_{x_i}$ *with*

$$a_\varepsilon^{ij}(x) := \int_{\mathbb{R}^n} a^{ij}(y)\eta_\varepsilon(x-y)dy$$

where $a_{ij} = \delta_{ij}$ *outside of* Ω *and* $\eta_\varepsilon(x)$ *is a family of mollifiers having the properties*

(i) $\eta_\varepsilon(x) \in C^\infty(\mathbb{R}^n)$

(ii) $\eta_\varepsilon(x) \geq 0$

(iii) $\eta_\varepsilon(x) = 0$ *for* $|x| > \frac{1}{\varepsilon}$

(iv) $\int_{\mathbb{R}^n} \eta_\varepsilon dx = 1$

For any measure μ *of bounded variation on* Ω *the weak solutions* u^ε *of* $L^\varepsilon u^\varepsilon = \mu$ *converges to the weak solution* u *of* $Lu = \mu$ *weakly in* $W_0^{1,p'}(\Omega)$ *for any* $p' < \frac{n}{n-1}$ *and thus strongly in* $L^q(\Omega)$ *for any* $q < \frac{n}{n-2}$*. [11]*

One important issue to address is whether the Green's function can be used to construct a representation formula for the solution to such an equation, that is an equation with bounded and measurable coefficients. The next theorem will address this question.

Theorem 2.48. *Let $G(x, y)$ be the Green's function for the uniformly elliptic operator L. For every measure μ of bounded variation the integral*

$$u(x) := \int G(x, y) d\mu(y)$$

exists and is finite almost everywhere, and is the weak solution vanishing on $\partial\Omega$ of the equation $Lu = \mu$. [11]

Theorem 2.49. *Let $G(x, y)$ and $\overline{G}(x, y)$ be the Green's functions for any uniformly elliptic operators L and $\overline{L}$ with ellipticity constant θ on a sphere Ω. Then, for any compact $\Omega' \subset \Omega$ there is a constant $C = C(\Omega', \Omega, \theta)$ we have*

$$C^{-1}\overline{G}(x, y) \leq G(x, y) \leq C\overline{G}(x, y) \qquad (x, y \in \Omega')$$

[11]

The previous theorem tells us that the Green's functions for any two uniformly elliptic operators with the same ellipticity constants are comparable. So, if we have an elliptic PDE whose coefficients are only bounded and measurable and an elliptic PDE with smooth coefficients with both PDEs having the same ellipticity constant we can produce estimates on the Green's function for the PDE whose coefficients are merely bounded and measurable using the Green's function for the PDE with smooth coefficients. This is helpful as we can then work with the Green's function for a PDE with smooth coefficients. It should be noted that this theorem can be readily extended to any simply connected domain Ω which can be smoothly mapped onto the sphere.

2.6 Mapping a Lipschitz Domain to a Ball

In this section we will consider a domain deformation and its effect on the coefficients of a uniformly elliptic PDE.

Remark 2.7. *Suppose that we have some $\Omega \subset \mathbb{R}^n$ which is a bounded, convex, Lipschitz domain. However, at times it may be helpful to work on a domain whose geometry we have more information about. To accomplish this we can construct a bilipschitz map that maps Ω to a half ball. Then, an even reflection can be used to map Ω to a ball. However, after this transformation our PDE will only have bounded, measurable coeffficients. Since Ω is a Lipschitz domain there is a bilipschitz map*

$$F : \Omega \to B_t^+(0)$$

with the Lipschitz constants of F and F^{-1} controlled by the Lipschitz constant of $\partial\Omega$. We then let

$$(a^{ij}(y)) = |\det(\nabla F^{-1}(y))|((\nabla F)^T \nabla F)(F^{-1}(y)),$$

where $y_2 \geq 0$ and

$$(a^{ij}(y_1, y_2)) = \begin{cases} a^{ij}(y_1, -y_2) & , i \neq 2, j \neq 2 \\ -a^{ij}(y_1, -y_2) & , i = 2 \text{ or } j = 2 \text{ but } i \neq j \\ a^{ij}(y_1, -y_2) & , i = j = 2 \end{cases}$$

In this case the coefficients a^{ij} are uniformly elliptic as well as bounded and measurable on all of $B_t(0)$ and the bounds on a^{ij} depend solely on the Lipschitz constant of $\partial\Omega$. For a more detailed discussion of this bilipschitz map and its effect on the equation's coefficients see [13]. The results regarding Green's function for equations with bounded and measurable coefficients given in Section 5 of Chapter 2 will be particularly useful when working with this transformed equation.

Chapter 3: The One-Phase Problem

3.1 Introduction

In this section we will consider a nonnegative function u which minimizes the functional

$$J[v] := \int_{\Omega} |\nabla v|^2 + Q^2(x)\chi_{\{v>0\}}$$

with $u = 0$ on $\Omega \cap \partial\{u > 0\}$ and certain boundary conditions on $\partial\Omega$. In parts of this section we will impose homogeneous Neumann conditions on part of $\partial\Omega$. We will assume that $\Omega \subset \mathbb{R}^n$ is a bounded, convex domain. The function u minimizing the above functional is harmonic in its positive phase and satisfies $|\nabla u(x)| = Q(x)$ weakly along the free boundary, $\partial\{u > 0\}$. We will present results from [13] and [1]. A detailed discussion of this one-phase problem and some extensions of this problem is contained in [5].

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$. Then, the* <u>*diameter*</u> *of Ω is given by*

$$\text{diam } \Omega := \inf\{\text{dist}(x, y) : x, y \in \Omega\}$$

Definition 3.2. *Let $\Omega \subset \mathbb{R}^n$ and define*

$$H_{n,\delta}(\Omega) := \inf\{\sum_{k=1}^{\infty} (\text{diam } B_k)^n : B_k \subset \mathbb{R}^n, \Omega \subset \bigcup_{k=1}^{\infty} B_k, \text{diam } B_k \leq \delta\}$$

Then, the <u>*n-dimensional Hausdorff measure*</u> *of Ω is given by*

$$\mathscr{H}^n(\Omega) := \lim_{\delta \to 0} H_{n,\delta}(\Omega)$$

3.2 Existence

Assume $\partial\Omega$ is locally a Lipschitz graph. Let $S \subset \partial\Omega$ be measurable with $\mathscr{H}^{n-1}(S) > 0$. The Dirichlet data on S are given by a nonnegative function $u^0 \in L^1_{\text{loc}}(\Omega)$ with $\nabla u^0 \in L^2(\Omega)$. The force function Q is nonnegative and measurable. Let

$$K \coloneqq \{v \in L^1_{\text{loc}}(\Omega) : \nabla v \in L^2(\Omega) \text{ and } v = u^0 \text{ on } S\}$$

As we have noted several times before when considering a boundary-value problem (in this case a free boundary problem) a natural starting point for analysis is whether solutions are guaranteed to exist and what conditions are necessary to guarantee existence. In this case we are formulating the boundary-value problem as a variational problem. As a result, the objective of this section is to establish under what conditions we can establish the existence of such an absolute minimum. Additionally, we must determine an appropriate admissible class of functions in which to search for minimizers. In this section we will search for an absolute minimum of the functional J in K.

Theorem 3.1 (Existence)**.** *If $J[u^0] < \infty$, then there exists $u \in K$ which is an absolute minimum of the functional J. [1]*

Remark 3.1. *If $Q \equiv 0$, then the solution u is harmonic in all of Ω (recall Theorem 2.10). Therefore we have a free boundary only in regions where Q is positive. Since we wish to study the free boundary we will assume that*

$$0 < \min_{\Omega} Q \leq Q \leq \max_{\Omega} Q < \infty$$

[1]

3.3 Local Minima

In this section we will examine some properties of local minima of the functional J. For the remainder of this chapter we will denote minimizers of J by u. We will present several lemmas, including that u is harmonic in its positive phase. This lemma will help establish that, for local minima, the free boundary condition is satisfied in a weak sense. That is,

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u>\varepsilon\}} |\nabla u|^2 \eta \cdot \nu = \lim_{\varepsilon \downarrow 0} \int_{\partial\{u>\varepsilon\}} Q^2 \eta \cdot \nu$$

for each $\eta \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$.

Definition 3.3. *We call $u \in K$ a <u>local minimum</u> of J, if for some small $\varepsilon > 0$ we have $J[u] \leq J[v]$ for every $v \in K$ with*

$$\|\nabla(v-u)\|_{L^2(\Omega)} + \|\chi_{\{v>0\}} - \chi_{\{u>0\}}\|_{L^1(\Omega)} \leq \varepsilon$$

Lemma 3.1. *If u is a local minimum, then u is subharmonic. We can thus assume that*

$$u(x) = \lim_{r \downarrow 0} \fint_{B_r(x)} u \qquad \textit{for } x \in \Omega$$

[1]

Lemma 3.2. *If u is a local minimum, then u is harmonic in the open set $\{u > 0\}$. [1]*

The following theorem establishes that the free boundary condition is weakly satisfied for local minima.

Theorem 3.2. *Let u be a local minimum and suppose that $Q^2 \in W^{1,1}(\Omega)$. Then,*

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u>\varepsilon\}} (|\nabla u|^2 - Q^2)\eta \cdot \nu = 0$$

for every $\eta \in W_0^{1,\infty}(\Omega;\mathbb{R}^n)$*. [1]*

3.4 Regularity and Nondegeneracy

In this section we will look at some results dealing with the growth of u, a local minimizer, away from the free boundary and we will determine several nondegeneracy conditions. As a corollary we will be able to establish that local minimizers u are Lipschitz continuous.

Lemma 3.3. *There is a constant* $C_{\max} = C(n)\max_\Omega Q$*, such that each absolute (local) minimum* u *has the following property for every (small) ball* $B_r \subset \Omega$*.*

$$\frac{1}{r}\fint_{S_r} u \geq C_{\max} \textit{ implies } u > 0 \textit{ in } B_r$$

[1]

Note that the above holds if and only if u is Lipschitz continuous. We can thus conclude that any minimizer, u, of J is Lipschitz continuous.

Corollary 3.1. u *is Lipschitz continuous in* Ω' *for any* $\Omega' \subset\subset \Omega$*. That is,* $u \in C^{0,1}_{\text{loc}}(\Omega)$*. [1]*

Lemma 3.4. *For* $\kappa < 1$ *there is a constant* $C_{\min} = C(n,\kappa)\min_\Omega Q$*, such that for each (local) minimum* u *and for (small) balls* $B_r \subset \Omega$ *the following conclusion holds:*

$$\frac{1}{r}\fint_{S_r} u \leq C_{\min} \textit{ implies } u = 0 \textit{ in } B_{\kappa r}$$

[1]

Lemma 3.5. *There is a constant* $c > 0$*, such that for (local) minima* u *and for (small) balls* $B_r \subset \Omega$ *with center in the free boundary*

$$c \leq \frac{|B_r \cap \{u > 0\}|}{|B_r|} \leq 1 - c$$

[1]

3.5 Neumann Solutions near the Neumann Fixed Boundary

This section is concerned with the behavior of solutions to the one-phase Neumann problem near the Neumann fixed boundary. W As before we will have $u = u^0$ on $S \subset \partial\Omega$ where $u^0 \in L^1_{\text{loc}}(\Omega)$, $\nabla u^0 \in L^2(\Omega)$ with $\mathscr{H}^{n-1}(S) > 0$. We will further have $\partial_\nu u = 0$ on $N \subset \Omega$ where $N = \partial\Omega \backslash S$. The Lipschitz continuity of Neumann solutions up to the Neumann fixed boundary will be established.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex domain. Let $r_0 > 0$. Let $x \in \partial\Omega$ with* $\text{dist}(x, \Gamma) < r_0$, *where Γ is the free boundary, and* $\text{dist}(x, S) \geq r_0$. *Then,*

$$|\nabla u(x)| \leq C$$

That is, u is Lipschitz continuous. [13]

Chapter 4: The Two-Phase Problem

4.1 Introduction

In this chapter we will consider the problem of minimizing

$$J[v] := \int_\Omega |\nabla v|^2 + q^2(x)\lambda^2(v), \qquad v \in K,$$

where $q^2(x) \neq 0$,

$$\lambda^2(v) = \begin{cases} \lambda_1^2 & , \text{ if } v < 0 \\ \lambda_2^2 & , \text{ if } v > 0 \end{cases}$$

and λ^2 is lower semicontinuous at $v = 0$. Let $\Lambda := \lambda_1^2 - \lambda_2^2 \neq 0$ and let

$$K := \{v \in L^1_{\text{loc}}(\Omega) : \nabla v \in L^2(\Omega) \text{ and } v = u^0 \text{ on } S\}$$

where S is a given open subset of $\partial\Omega$. Notice that in the previous chapter we only looked for minimizers in

$$K^+ := \{v \in K : v \geq 0 \text{ a.e.}\}$$

In this section we will present results from [2].

Notation 4.1. *Let $u : \Omega \to \mathbb{R}$. Then, $u^+ := \max\{u, 0\}$ and $u^- := -\min\{u, 0\}$. It should be noted that both u^+ and u^- are nonnegative functions, even though u^- is referred to as the negative part of u.*

4.2 Existence

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and let $S \subset \partial\Omega$ be open with $S \neq \emptyset$. Let $u^0 \in L^1_{\text{loc}}(\Omega)$ be a given function with $\nabla u^0 \in L^2(\Omega)$ and let q be a strictly positive uniformly Lipschitz continuous function in compact subsets of $\overline{\Omega}$. Let

$$\lambda(u) = \begin{cases} \lambda_1 & , \text{ if } u < 0 \\ \lambda_2 & , \text{ if } u > 0 \end{cases}$$

where $\lambda_1, \lambda_2 \geq 0$ and $\lambda(0)$ is defined so that

$$0 \leq \lambda(0) \leq \min\{\lambda_1, \lambda_2\}$$

Set $Q(u, x) \coloneqq q(x)\lambda(u)$. Then,

$$J[v] = \int_\Omega |\nabla v|^2 + Q^2(v, x), \qquad v \in K$$

So, the problem that we are concerned with in this chapter is to find $u \in K$ such that

$$J[u] = \min_{v \in K} J[v]$$

As we have already done several times in previous chapters we wish to establish conditions under which the existence of an absolute minimum is guaranteed.

Theorem 4.1. *If $J[u^0] < \infty$, then there exists $u \in K$ such that*

$$J[u] = \min_{v \in K} J[v]$$

[2]

4.3 Continuity, Subharmonicity and the Free Boundary Condition

For the remainder of this chapter we will denote minimizers of the functional J by u. In this section results about minimizers will be presented. Among them are that minimizers are harmonic in their positive and negative phases and that minimizers satisfy the free boundary condition in a weak sense. Additionally, it is shown that minimizers are either subharmonic or superharmonic (depending on whether Λ is positive or negative).

Theorem 4.2. *For any compact $D \subset \Omega$ there is a constant C such that*

$$|u(x) - u(y)| \leq C|x-y| \ln\left(\frac{1}{|x-y|}\right)$$

if $x, y \in D$, $|x-y| < \frac{1}{2}$. [2]

Theorem 4.3. *u is harmonic in $\{u \neq 0\}$. [2]*

Theorem 4.4. *If $\lambda(0) = \lambda_1$ and $\Lambda < 0$ ($\lambda(0) = \lambda_2$ and $\Lambda > 0$), then u is subharmonic (superharmonic) in Ω. [2]*

The following theorem shows that a minimizer u satisfies the free boundary condition weakly.

Theorem 4.5. *Suppose $|\{u = 0\}| = 0$. Then, for any $\eta \in C_0^1(\Omega; \mathbb{R}^n)$,*

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u<-\varepsilon\}} (|\nabla u|^2 - \lambda_1^2 q^2(x))\eta \cdot \nu + \lim_{\delta \downarrow 0} \int_{\partial\{u>\delta\}} (|\nabla u|^2 - \lambda_2^2 q^2(x))\eta \cdot \nu = 0$$

where ν is the outward normal vector. [2]

4.4 Nondegeneracy

Let $0 \leq q_1 \leq q(x) \leq q_2 < \infty$ and let $|\Lambda| \geq l_0 > 0$. In this section we present some results establishing that minimizers are not degenerate in their positive phase. With some additional assumptions the nondegeneracy of minimizers in their negative phase can be demonstrated as well.

Theorem 4.6. *Suppose $\Lambda < 0$. For any $0 < \kappa < 1$, if $B_r \subset \Omega$ and $\frac{1}{r}\int_{S_r} u^+ < c(\kappa, q_1^2 l_0)$, then $u^+ = 0$ in $B_{\kappa r}$. [2]*

Corollary 4.1. *Suppose $\Lambda < 0$. If $B_r \subset \Omega$ with center in the free boundary $\partial\{u > 0\}$, then $\frac{1}{r}\int_{S_r} u^+ \geq c(q_1^2 l_0) > 0$. [2]*

Remark 4.1. *If $\lambda^2(0) < \min\{\lambda_1^2, \lambda_2^2\}$, then Theorem 4.6 applies to the minimizer in its positive and negative phases (i.e. u^+ and u^-). As a result if $B_r \subset \Omega$ is centered in $\partial\{u > 0\}$ ($\partial\{u < 0\}$), then*

$$\frac{1}{r}\fint_{S_r} u^+ \geq c \qquad \left(\frac{1}{r}\fint_{S_r} u^- \geq c\right)$$

4.5 Upper Estimates on Averages

Let $\max\{\lambda_1^2, \lambda_2^2\} \leq l_1$. The primary result in this section is a result regarding an upper estimate on the average of minimizers. This theorem will require a lemma which establishes an estimate on the measure Δu. That Δu is a measure follows from the fact that minimizers are subharmonic (superharmonic). Notice that if u is subharmonic (superharmonic), then Δu is a positive (negative) measure.

Lemma 4.1. *If $\Lambda < 0$ and $B_r \subset \Omega$, then*

$$\Delta u(B_{\frac{r}{2}}) \leq C r^{n-1}$$

[2]

Theorem 4.7. *Suppose that $\lambda(0) = \min\{\lambda_1, \lambda_2\}$. Then, if $B_r \subset \Omega$ with center in $\{u = 0\}$, then there is a positive constant $C = C(q_2, l_1)$*

$$\frac{1}{r}\left|\fint_{\partial\ B_r} u\right| \leq C$$

[2]

4.6 Lipschitz Continuity

In this section we present several lemmas with the objective of establishing the Lipschitz continuity of minimizers. This includes an important result known as the monotonicity formula (Lemma 4.2). This lemma and the upper estimate on averages from the previous section will suffice to establish the Lipschitz continuity of a minimizer u.

Lemma 4.2. *Let $u \in C(B_R(x^0)) \cap W^{1,2}(B_R(x^0))$ where $u(x^0) = 0$ and u is harmonic in $B_R \setminus \{u = 0\}$. Set*

$$\phi(r) := \frac{1}{r^2} \int_{B_r} \rho^{2-n} |\nabla u^+|^2 \cdot \frac{1}{r^2} \int_{B_r} \rho^{2-n} |\nabla u^-|^2$$

where $\rho = |x - x^0|$. Then, $\phi(r) < \infty$ and ϕ is increasing in r for $r \in (0, R)$. [2]

Monotonicity formulas are a powerful tool for studying the regularity of minimizers. This tool was first introduced in [2]. Monotonicity formulas have been used in the study of other elliptic free boundary problems as well as parabolic free boundary problems.

Lemma 4.3. *Assume that $\lambda(0) = \min\{\lambda_1, \lambda_2\}$. Then, for any domain $D \subset\subset \Omega$ there is a constant $C > 0$ such that if $B_r \subset D$ has center in $\{u = 0\}$, then*

$$\frac{1}{r} \fint_{S_r} |u| \leq C$$

[2]

Theorem 4.8. *If $\lambda(0) = \min\{\lambda_1, \lambda_2\}$, then u is Lipschitz continuous in any compact subset of Ω. [2]*

Chapter 5: The Two-Phase Neumann Problem Near the Neumann Fixed Boundary

In this chapter we will consider the two-phase Neumann problem near the Neumann fixed boundary. That is, we will consider minimizers u of the functional

$$J[v] = \int_{\Omega} |\nabla v|^2 + q^2(x)\lambda^2(v)$$

where the minimization is carried out over the class of admissible functions $K = \{v \in L^1_{\text{loc}}(\Omega) : \nabla v \in L^2(\Omega) \text{ and } v = u^0 \text{ on } S \subset \partial\Omega\}$ with $u^0 \in L^1_{\text{loc}}(\Omega)$ and $\nabla u^0 \in L^2(\Omega)$. Further, $\Omega \subset \mathbb{R}^2$ is a bounded, convex domain (open and connected). In this case we have $S \subsetneq \partial\Omega$. On $N = \partial\Omega \setminus S$ we have $\partial_\nu u = 0$. We will consider the behavior of minimizers in $B_r(x) \cap \Omega$ where $r > 0$ and $x \in N$. That is we want to consider the behavior of minimizers near the Neumann fixed boundary. The primary result is that solutions are Lipschitz continuous up to the Neumann fixed boundary.

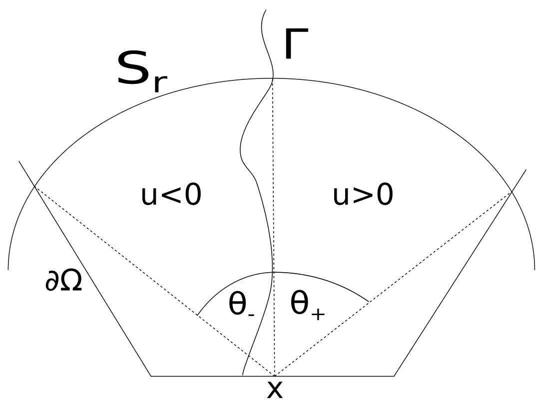

Figure 5.1: Domain

5.1 Monotonicity Formula

Lemma 5.1. *Let $r_0 > 0$ and let $x \in \partial\Omega$ with $\operatorname{dist}(x,\Gamma) < r_0$ where Γ is the free boundary. Let $u \in C(B_r \cap \Omega) \cap H^1(B_r \cap \Omega)$ with $\partial_\nu u = 0$ on $N \subset \partial\Omega$ and $\Delta u = 0$ in $B_r \cap \Omega \setminus \{u = 0\}$. Let $\operatorname{dist}(x, S) \geq r_0$ where $S \subset \partial\Omega$ and $u = u^0$ on S. Set*

$$\phi(r) := \frac{1}{r^4} \int_{B_r\cap\Omega} |\nabla u^+|^2 \cdot \int_{B_r\cap\Omega} |\nabla u^-|^2$$

Then, $\phi'(r) \geq 0$.

Proof. Let $x \in N$, let $B_r := B_r(x)$, $S_r := S_r(x)$ and let θ_+, θ_- be defined as in Figure 5.1.

To begin notice that $\theta_+ + \theta_- \leq \pi$ (see Figure 5.1). Consider

$$\frac{-1}{r^2}\frac{d^2 f_+}{d\theta^2} = \lambda_+ f_+, \;\; f_+(\theta_+) = f_+'(0) = 0$$

Then,

$$f_+(\theta) = C_1^+ \sin\sqrt{\lambda_+}r\theta + C_2^+ \cos\sqrt{\lambda_+}r\theta$$

Further, $f_+'(0) = 0$ implies $C_1^+ = 0$. So,

$$f_+(\theta) = C_2^+ \cos\sqrt{\lambda_+}r\theta$$

Further, $f_+(\theta_+) = 0$ gives $\cos\sqrt{\lambda_+}r\theta_+ = 0$. It then follows that

$$\sqrt{\lambda_+}r\theta_+ = \frac{\pi}{2}$$

Now consider

$$\frac{-1}{r^2}\frac{d^2 f_-}{d\theta^2} = \lambda_- f_-, \;\; f_-(\theta_+) = f_-'(\theta_+ + \theta_-) = 0$$

Equivalent boundary conditions are given by $f_-(\theta_-) = f_-'(0) = 0$. Thus, we have, as before,

$$\sqrt{\lambda_-}r\theta_- = \frac{\pi}{2}$$

By scaling we may assume $r = 1$. Consider

$$\theta_+^{-1} + \theta_-^{-1} \geq \theta_+ + (\pi - \theta_+)^{-1}$$

and

$$(\theta_+^{-1} + (\pi - \theta_+)^{-1}))' = -\theta_+^{-2} + (\pi - \theta_+)^{-1})^{-2}$$

So, $(\theta_+^{-1}+(\pi-\theta_+)^{-1})^{-1})' = 0$ implies $(\pi-\theta_+)^{-1})^{-2} = \theta_+^{-2}$ which yields $\pi^2 - 2\pi\theta_+ + \theta_+^2 = \theta_+^2$. Note that there is no maximum. Thus, $\theta_+ = \frac{\pi}{2}$ must be a local minimum. Hence, $\theta_+^{-1} + \theta_-^{-1} \geq \frac{4}{\pi}$. Therefore,

$$\sqrt{\lambda_+} + \sqrt{\lambda_-} = \frac{\pi}{2\theta_+} + \frac{\pi}{2\theta_-} \geq 2$$

Let U_+ and U_- be the support of u^+ and u^- respectively on $S_1 \cap \Omega$. Then, applying the Rayleigh quotient gives

$$\lambda_+ = \inf_{v \in H_0^1(U_+)} \frac{\int_{U_+} (\partial_\theta v)^2}{\int_{U_+} v^2} \leq \frac{\int_{S_1 \cap \Omega} (\partial_\theta u^+)^2}{\int_{S_1 \cap \Omega} (u^+)^2}$$

and

$$\lambda_- = \inf_{v \in H_0^1(U_-)} \frac{\int_{U_-} (\partial_\theta v)^2}{\int_{U_+} v^2} \leq \frac{\int_{S_1 \cap \Omega} (\partial_\theta u^-)^2}{\int_{S_1 \cap \Omega} (u^-)^2}$$

So,

$$\sqrt{\int_{S_1 \cap \Omega} (\partial_\theta u^+)^2} \geq \sqrt{\lambda_+} \sqrt{\int_{S_1 \cap \Omega} (u^+)^2}$$

and

$$\sqrt{\int_{S_1 \cap \Omega} (\partial_\theta u^-)^2} \geq \sqrt{\lambda_-} \sqrt{\int_{S_1 \cap \Omega} (u^-)^2}$$

It then follows that

$$\begin{aligned}\int_{S_1\cap\Omega}|\nabla u^+|^2 &= \int_{S_1\cap\Omega}(\partial_r u^+)^2+(\partial_\theta u^+)^2\\ &\geq 2\sqrt{\int_{S_1\cap\Omega}(\partial_r u^+)^2\cdot\int_{S_1\cap\Omega}(\partial_\theta u^+)^2}\\ &\geq 2\sqrt{\lambda_+}\sqrt{\int_{S_1\cap\Omega}(\partial_r u^+)^2\cdot\int_{S_1\cap\Omega}(u^+)^2}\\ &\geq 2\sqrt{\lambda_+}\int_{S_1\cap\Omega}|u^+\partial_r u^+|\end{aligned}$$

Similar computations show that we also have

$$\int_{S_1\cap\Omega}|\nabla u^-|^2\geq 2\sqrt{\lambda_-}\int_{S_1\cap\Omega}|u^-\partial_r u^-|$$

Recall that,

$$\phi(r)=\frac{1}{r^4}\int_{B_r\cap\Omega}|\nabla u^+|^2\cdot\int_{B_r\cap\Omega}|\nabla u^-|^2$$

Hence,

$$\begin{aligned}\phi'(r)=\frac{-4}{r^5}\int_{B_r\cap\Omega}|\nabla u^+|^2\cdot\int_{B_r\cap\Omega}|\nabla u^-|^2+\frac{1}{r^4}\left[\int_{S_r\cap\Omega}|\nabla u^+|^2\cdot\int_{B_r\cap\Omega}|\nabla u^-|^2\right]+\\ \frac{1}{r^4}\left[\int_{B_r\cap\Omega}|\nabla u^+|^2\cdot\int_{S_r\cap\Omega}|\nabla u^-|^2]\right]\end{aligned}$$

But,

$$\begin{aligned}\int_{B_r\cap\Omega}|\nabla u^\pm|^2 &= -\int_{B_r\cap\Omega}u^\pm\Delta u^\pm+\int_{S_r\cap\Omega}u^\pm\partial_\nu u^\pm+\int_{B_r\cap\partial\Omega}u^\pm\partial_\nu u^\pm\\ &= \int_{S_r\cap\Omega}u^\pm\partial_r u^\pm\end{aligned}$$

as in Ω either $u^\pm = 0$ or $\Delta u^\pm = 0$, $\partial_\nu u^\pm = 0$ on $\partial\Omega$ and $\partial_\nu u^\pm = \partial_r u^\pm$ on S_r. So,

$$\phi'(r) = \frac{-4}{r^5}\int_{S_r\cap\Omega} u^+\partial_r u^+ \cdot \int_{S_r\cap\Omega} u^-\partial_r u^- + \frac{1}{r^4}\left[\int_{S_r\cap\Omega} |\nabla u^+|^2 \cdot \int_{S_r\cap\Omega} u^-\partial_r u^-\right] +$$
$$\frac{1}{r^4}\left[\int_{S_r\cap\Omega} u^+\partial_r u^+ \cdot \int_{S_r\cap\Omega} |\nabla u^-|^2\right]$$

However, previous computations showed that

$$-4\int_{S_1\cap\Omega} |u^+\partial_r u^+| \cdot \int_{S_1\cap\Omega} |u^-\partial_r u^-| + \int_{S_1\cap\Omega} |\nabla u^+|^2 \cdot \int_{S_1\cap\Omega} u^-\partial_r u^- +$$
$$\int_{S_1\cap\Omega} u^+\partial_r u^+ \cdot \int_{S_1\cap\Omega} |\nabla u^-|^2$$
$$\geq -4\int_{S_1\cap\Omega} |u^+\partial_r u^+| \cdot \int_{S_1\cap\Omega} |u^-\partial_r u^-| + 2\sqrt{\lambda_+}\int_{S_1\cap\Omega} |u^+\partial_r u^+| \cdot \int_{S_1\cap\Omega} |u^-\partial_r u^-| +$$
$$2\sqrt{\lambda_-}\int_{S_1\cap\Omega} |u^+\partial_r u^+| \cdot \int_{S_1\cap\Omega} |u^-\partial_r u^-|$$
$$= [-4 + 2(\sqrt{\lambda_+} + \sqrt{\lambda_-})]\int_{S_1\cap\Omega} u^+\partial_r u^+ \cdot \int_{S_1\cap\Omega} u^-\partial_r u^-$$
$$\geq 0$$

as $\sqrt{\lambda_+} + \sqrt{\lambda_-} \geq 2$. Therefore, we can conclude, after rescaling, that

$$\phi'(r) \geq 0$$

□

5.2 Upper Estimates on the Averages

Lemma 5.2. *If* $\Lambda = \lambda_1^2 - \lambda_2^2 > 0$*, then* $\Delta u(B_{\frac{r}{2}} \cap \Omega) \leq Cr$.

Proof. Let v be the solution of

$$\Delta v = 0 \text{ in } B_r$$
$$v = u \text{ on } S_r$$

Then, since u is a minimizer we have

$$\begin{aligned}\int_{B_r\cap\Omega}|\nabla u|^2-\int_{B_r\cap\Omega}|\nabla v|^2 &\le \int_{B_r\cap\Omega}Q^2(v,x)-Q^2(u,x)\\ &=\int_{B_r\cap\Omega}(\lambda^2(v)-\lambda^2(u))q^2(x)\\ &\le|\lambda_1^2-\lambda_2^2|\int_{B_r\cap\Omega}q^2(x)\\ &\le|\lambda_1^2-\lambda_2^2|\|q^2\|_{L^\infty(B_r\cap\Omega)}|B_r\cap\Omega|\\ &\le Cr^2\end{aligned}$$

But,

$$\begin{aligned}\int_{B_r\cap\Omega}|\nabla u|^2-|\nabla v|^2&=\int_{B_r\cap\Omega}\nabla(u-v)\cdot\nabla(u+v)\\ &=\int_{B_r\cap\Omega}\nabla(u-v)\cdot\nabla(u-v)+\int_{B_r\cap\Omega}2\nabla(u-v)\cdot\nabla v\end{aligned}$$

Notice that

$$\begin{aligned}\int_{B_r\cap\Omega}2\nabla(u-v)\cdot\nabla v&=-\int_{B_r\cap\Omega}2(u-v)\Delta v+\int_{S_r\cap\Omega}2(u-v)\partial_\nu v+\int_{B_r\cap\partial\Omega}2(u-v)\partial_\nu v\\ &=0\end{aligned}$$

as $\Delta v=0$ in B_r, $u=v$ on S_r and $\partial_\nu v=0$ on $\partial\Omega$.

So,

$$
\begin{aligned}
\int_{B_r\cap\Omega} |\nabla u|^2 - |\nabla v|^2 &= \int_{B_r\cap\Omega} \nabla(u-v)\cdot\nabla(u-v) \\
&= \int_{B_r\cap\Omega} |\nabla u|^2 - \nabla u\cdot\nabla v - \nabla v\cdot\nabla u + |\nabla v|^2 \\
&= \int_{B_r\cap\Omega} |\nabla u|^2 - \nabla v\cdot\nabla u + \int_{B_r\cap\Omega} u\Delta v - v\Delta v + \int_{S_r\cap\Omega} v\partial_\nu v - u\partial_\nu v + \\
&\quad \int_{B_r\cap\partial\Omega} v\partial_\nu v - u\partial_\nu v \\
&= \int_{B_r\cap\Omega} |\nabla u|^2 - \nabla v\cdot\nabla u \\
&= \int_{B_r\cap\Omega} \nabla(u-v)\cdot\nabla u \\
&= \int_{B_r\cap\Omega} (v-u)\Delta u + \int_{S_r\cap\Omega} (v-u)\partial_\nu u + \int_{B_r\cap\partial\Omega} (v-u)\partial_\nu u \\
&= \int_{B_r\cap\Omega} (v-u)\Delta u \\
&= \int_{B_r\cap\Omega} v\Delta u
\end{aligned}
$$

as Δu is a measure supported on $\{u=0\}$.

Therefore, we conclude that

$$\int_{B_{\frac{r}{2}}\cap\Omega\cap\{u=0\}} v\Delta u \leq Cr^2$$

Since Ω is a Lipschitz domain there is a bilipschitz map

$$F: B_r\cap\Omega \to B_t^+$$

We can then use an even reflection to map our domain Ω to the ball B_t.

Then, by Theorem 2.49 there exists $\tilde{G_{x^0}}$ such that

$$\tilde{u}(x^0) = \tilde{v}(x^0) - \int_{B_t} \tilde{G_{x^0}} L\tilde{u}(y)$$

where

$$L\tilde{v} = 0 \text{ in } B_t$$

$$\tilde{v} = \tilde{u} \text{ on } S_t$$

where the boundary conditions hold in the trace sense.

Then, $u(x^0) = \tilde{u}(F(x^0))$. Further, note that on B_t^+ we have $\tilde{G}(F(x^0)) = G(x^0)$ for the Green's function on $B_r \cap \Omega$.

$$u(x^0) = \int_{S_t} \tilde{P_{x^0}}(y)\tilde{u}(y) - \int_{B_t} \tilde{G_{x^0}}(y) L\tilde{u}(y)$$

Let $x^0 \in \{\tilde{u} = 0\}$. So,

$$\tilde{v}(x^0) = \int_{B_t} \tilde{G_{x^0}}(y) L\tilde{u}(y)$$

Let $V = F(B_{\frac{r}{2}} \cap \Omega)$. Then, $V \subset B_t$ and F being bilipschitz imply

$$Cr^2 \geq \int_{B_{\frac{r}{2}} \cap \Omega} v\Delta u \geq \int_V \tilde{v} L\tilde{u} = c\int_V \left(\int_{B_t} \tilde{G_{x^0}}(y) L\tilde{u}(y) \right) L\tilde{u}(x)$$

Notice that $\tilde{G_{x^0}}(y) \geq c > 0$ for $x, y \in V$ by Theorem 2.49. It then follows that

$$\begin{aligned}
Cr^2 &\geq \int_V \left(\int_{B_t} \tilde{G_{x^0}}(y) L\tilde{u}(y) \right) L\tilde{u}(x) \\
&\geq c \int_V \left(\int_{B_t} L\tilde{u}(y) \right) L\tilde{u}(x) \\
&= c \int_V (L\tilde{u}(B_t)) L\tilde{u}(x) \\
&= c(L\tilde{u}(B_t)) \int_V L\tilde{u}(x) \\
&= cL\tilde{u}(B_t) L\tilde{u}(V) \\
&\geq c(L\tilde{u}(V))^2
\end{aligned}$$

Therefore, $L\tilde{u}(V) \leq Cr$. Since F is bilipschitz it follows that $\Delta u(B_{\frac{r}{2}} \cap \Omega) \leq Cr$. □

Lemma 5.3. *Let* $\max\{\lambda_1^2, \lambda_2^2\} \leq \ell_1$ *and assume that* $\lambda(0) = \min\{\lambda_1, \lambda_2\}$*. If* B_r *has center in* $\{u = 0\}$*, then there is a positive constant* $C = C(q_2, \ell_1)$ *such that*

$$\frac{1}{r} \left| ⨍_{S_r \cap \Omega} u \right| \leq C$$

where $0 \leq q_1 \leq q(x) \leq q_2 < \infty$.

Proof. Assume that $\Lambda < 0$ and that the center of B_r is the origin. Then,

$$\begin{aligned}
0 &= u(0) \\
&= \int_{B_r \cap \Omega} G_0(y) \Delta u(y) - v(0)
\end{aligned}$$

with v as before. Let

$$I := \int_{B_r \cap \Omega} G_0(y) \Delta u(y)$$

Then,

$$c \int_{B_t} \tilde{G}_0(y) L\tilde{u}(y) \leq I \leq C \int_{B_t} \tilde{G}_0(y) L\tilde{u}(y)$$

Then,

$$I = \int_{B_t} \tilde{G}_0(y) L\tilde{u}(y)$$

$$= \int_0^r \int_{B_s} \tilde{G}_0(s,\theta) L\tilde{u}(s,\theta)\ d\theta\ ds$$

$$= \int_0^r G_0(s) \int_{B_s} \frac{\tilde{G}_0(s,\theta)}{G_0(s)} L\tilde{u}(s,\theta)\ d\theta\ ds$$

where

$$\frac{\tilde{G}_0(s,\theta)}{G_0(s)} \leq C$$

by Theorem 2.49.

Let G_0 denote the standard Green's function on B_r. Let $G_0(s) = G_0(|s|, \theta) = -\log(\frac{s}{r})$ for $0 \leq \theta \leq 2\pi$ noting that G_0 is radially symmetric. Let

$$h(r) = r \int_{S_1} L\tilde{u}(r\xi)\ d\mathscr{H}^1(\xi)$$

Then,

$$
\begin{aligned}
I &= \int_0^r \tilde{G}(s\xi) L\tilde{u}(s\xi)\ d\mathscr{H}^1\ ds \\
&= \int_0^r sG_0(s) \int_{S_1} \frac{\tilde{G}(s\xi)}{G_0(s\xi)} L\tilde{u}(s\xi)\ d\mathscr{H}^1(\xi)\ ds \\
&= \int_0^r -s\log(\frac{s}{r}) \int_{S_1} \frac{\tilde{G}(s\xi)}{G_0(s\xi)} L\tilde{u}(s\xi)\ d\mathscr{H}^1(\xi)\ ds \\
&\leq C \int_0^r -s\log(\frac{s}{r}) \int_{S_1} L\tilde{u}(s\xi)\ d\mathscr{H}^1(\xi)\ ds \\
&= C \int_0^r -\log(\frac{s}{r}) h(s)\ ds \\
&= C \int_0^r -\log(\frac{s}{r}) \frac{d}{ds} \left(\int_0^s h(t)\ dt \right)\ ds \\
&= C \left[-\log(\frac{s}{r}) \int_0^s h(t)\ dt \right]_0^r - C \int_0^r -\frac{1}{s} \int_0^s h(t)\ dt\ ds \\
&= 0 + \lim_{s\to 0} C \left(\log(\frac{s}{r}) \int_0^s h(t)\ dt \right) + C \int_0^r \frac{1}{s} \int_0^s h(t)\ dt\ ds \\
&\leq C \int_0^r \frac{1}{s} \int_0^s r \int_{S_1} L\tilde{u}(r\xi)\ d\xi\ dr\ ds \\
&\leq C \int_0^r \frac{1}{s} Cs\ ds \\
&= C \int_0^r\ ds \\
&= Cr
\end{aligned}
$$

We can obtain a similar estimate for the measure Δu.

$$
\int_{B_r \cap \Omega} \Delta u \leq Cr
$$

Since u is subharmonic, v is harmonic in $B_r \cap \Omega$ and $v = u$ on $S_r \cap \Omega$ we have $v \geq u$

in $B_r \cap \Omega$. So, we have

$$\begin{aligned} \fint_{S_r\cap\Omega} u &= \int_{S_r\cap\Omega} P_0 u \\ &= \int_{B_r\cap\Omega} G_0 \Delta u \\ &= \int_{B_t} \tilde{G}_0 L\tilde{u} \\ &\leq Ct \\ &\leq Cr \end{aligned}$$

as F is a bilipschitz map.

Since u is subharmonic in $B_r \cap \Omega$ and $u(0) = 0$ it follows by the mean-value property for subharmonic functions that

$$\fint_{S_r\cap\Omega} u \geq 0$$

Therefore,

$$\begin{aligned} \frac{1}{r}\left|\fint_{S_r\cap\Omega} u\right| &= \frac{1}{r}\fint_{S_r\cap\Omega} u \\ &\leq C \end{aligned}$$

□

5.3 Regularity

Theorem 5.1. *Let $r_0 > 0$ and define $\Omega_{r_0} := \{x \in \Omega : \mathrm{dist}(x, S) > r_0\}$. Then, there is a constant C such that if u is a minimizer of the functional J then for almost every $x \in \Omega_{r_0}$ we have*

$$|\nabla u(x)| \leq C$$

Proof. Let $U_+ := \{u > 0\}$, $U_- := \{u < 0\}$ and let $\Gamma = \{\partial U_+ \cup U_-\} \cap \Omega$ be the free boundary. As in [13] we consider five (possibly overlapping) cases.

1. $x \in U_0 := \Omega \setminus \{U_+ \cup U_-\}$. Note that $|\Gamma| = 0$, so we need not consider $|\nabla u(x)|$ for $x \in \Gamma$. Since $u = 0$ in $U_0 \setminus \Gamma$ we have $|\nabla u(x)| = 0$ for $x \in U_0 \setminus \Gamma$.

2. $x \in U_+$ ($x \in U_-$) with $\text{dist}(x, \partial U_+) \geq 1$ ($\text{dist}(x, \partial U_-) \geq 1$). Notice that $x \notin U_+ \cap U_-$ as a result of the distance from ∂U_+ (∂U_-). The result follows immediately from Theorem 2.5.

3. $x \in U_+ \cup U_-$ with $\text{dist}(x, \partial\Omega) > \text{dist}(x, \Gamma)$. In this case Lipschitz continuity was show in ([2], Theorem 5.3).

4. $x \in U_+$ or $x \in U_-$ with $\text{dist}(x, \Gamma) \geq 1$. If $x \in U_+$ we can apply Lemma 2.1 to get the desired result. If $x \in U_-$ we can apply Lemma 2.1 to u^+ and u^- separately to get $|\nabla u^+(x)| \leq C_+$ and $|\nabla u^-(x)| \leq C_-$. The fact $\nabla u = \nabla u^+ - \nabla u^-$ then implies the desired result.

5. $x \in \{U_+ \cup U_-\} \cap \Omega_{r_0}$ with $\text{dist}(x, N) \leq \text{dist}(x, \Gamma) \leq 1$. We can apply the proof of case (5) in the proof of Theorem 2 in [13] to u^+ and u^- separately. This will allow us to derive estimates on $|\nabla u^+|$ and $|\nabla u^-|$. In particular, we will get $|\nabla u^+(x)| \leq C_+$ and $|\nabla u^-(x)| \leq C_-$. We can then use the fact that $\nabla u = \nabla u^+ - \nabla u^-$ to conclude that $|\nabla u(x)| \leq C$.

□

Chapter 6: Conclusion

In this thesis we have considered a two-phase elliptic free boundary problem formulated as a variational problem. In [1] the interior regularity (Lipschitz continuity) of minimizing solutions to the n-dimensional one-phase problem with Dirichlet data was established. These results were extended in [13] where Neumann boundary conditions were imposed on part of the fixed boundary and regularity (Lipschitz continuity) was extended up to the Neumann fixed boundary. The two-phase problem with Dirichlet data was considered in [2]. The authors established interior regularity (Lipschitz continuity). In this thesis we considered the two-phase problem in $\mathbb{R}^2$. As in [13] we imposed Dirichlet boundary conditions on part of the fixed boundary $S \subset \partial\Omega$ and Neumann conditions on the rest of the fixed boundary $N \subset \partial\Omega$. We then demonstrated that minimizing solutions were Lipschitz continuous up to the Neumann fixed boundary.

There are many areas of possible further inquiry related to this problem. One natural avenue of exploration would be to attempt to establish these results in $\mathbb{R}^n$. Another interesting area of inquiry would be to consider the properties of the free boundary. In [1] the authors demonstrated that if for some m we have $0 < m \leq Q$ where Q is smooth then the free boundary is a $C^{1,\alpha}$ surface for some α except at a set of $\mathscr{H}^{n-1}$-measure zero. Further, for $n = 2$ the free boundary is analytic if the force function Q is. In [2] it was shown that for $n = 2$ the free boundary is continuously differentiable. It would be natural to attempt to extend these results, first in the two-dimensional case, and examine the regularity of the free boundary near the Neumann fixed boundary. It would then be interesting to investigate what sort of regularity results hold in this case when $n > 2$. To establish this one would likely need to first establish that γ-flatness condition holds. That is, the free boundary in $B_\delta(x^0)$ lies in a strip with center x^0 and width 2γ.

An additional problem to consider is how the free boundary intersects the Neumann fixed boundary. In [8] the author demonstrated that the free boundary must approach a Dirichlet fixed boundary tangentially. Further, it has been conjectured, but not proven, that the free boundary should intersect a Neumann fixed boundary orthogonally. It would certainly be of interest to prove or disprove this conjecture. This conjecture also gives rise to questions of the stability of the free boundary with respect to Dirichlet boundary data. Specifically, as the Dirichlet data is perturbed the free boundary can be pushed towards a corner of the Neumann fixed boundary. How will the free boundary traverse the corner? If the conjecture holds and the corner angle is sufficiently small, then the free boundary cannot intersect the corner. This will force a jump (i.e. instability with respect to the Dirichlet data). How close will the free boundary come to the corner before the jump? Numerical experiments would be a very convenient and informative way to carry out a preliminary study of this question.

It would additionally be interesting to consider this problem with more complicated boundary conditions. For example, imposing inhomogeneous Neumann boundary conditions or even Robin boundary conditions and attempting to obtain regularity results for minimizers as well as the free boundary.

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