EXPLORATION OF THE THREE-PLAYER PARTIZAN GAME OF RHOMBINATION

BY

KATHERINE A. GREENE

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Approved By:

Sarah K. Mason, Ph.D., Advisor

Edward E. Allen, Ph.D., Chair

R. Jason Parsley, Ph.D.

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Table of Contents

List of Figures

List of Tables

Abstract

We introduce a new combinatorial three-player partizan game called Rhombination and expand some ideas from two-player game theory into three-player game theory.

Cincotti's three-dimensional Domineering is distinct from Rhombination. We prove the Fundamental Theorem for three-player games, given the player preference set by Li. We set a bound on the number of games born on day n. Comparisons are made with Cincotti's partial ordering and a generalization of two-player partial order is defined for three-player games. We make observations on outcome classes of games and find effects of rotation and reflection on outcome. We give many ideas for future theorems to prove or directions to take, including a potential method for defining three-player number games.

Chapter 1: Introduction to Combinatorial Game Theory

Who will win? How do we win? This is what we all ask when we play a game of any sort. In combinatorial game theory, we develop methods to find the optimal strategies and winners of given game positions.

1.1 Two-Player Games

We begin with two-player games, for which much of the following is standard. See [1] for details on two-player game theorems and for references on the given two-player games. In this section, all games are between two players. We begin by defining **combinatorial games**. There are two players, called Left (Player 1) and Right (Player 2), who take turns alternating moves until one no longer can play. Combinatorial games do not involve chance, meaning there is no random element such as dice rolling or card shuffling. These games have perfect information, meaning both players can see all possible moves. Combinatorial games must have a finite number of moves, contain no loops leading to previous positions, and under **normal play** the last player to move wins.

A game is **partizan** when different players cannot necessarily make the same moves and an **impartial** game is one in which the options are the same for each player. We will be considering partizan games.

Examples of combinatorial games include *Nim, Subtraction(L|R), Amazons, Clobber, Hackenbush, Push/Shove,* and *Toppling Dominoes.* See [1] for descriptions.

Subtraction(L|R) is a game in which we have a pile of counters and each player is given certain amounts they can remove. For example, Subtraction(1,4|2,3) means that Left can take either 1 or 4 counters out of the pile on their turn and Right may take either 2 or 3 on their turn. An example game sequence would be: $16 \rightarrow_L 15 \rightarrow_R 12 \rightarrow_L 11 \rightarrow_R 8 \rightarrow_L 7 \rightarrow_R 5 \rightarrow_L 1$ now Right cannot move, so Left wins.

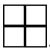
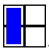
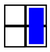
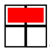
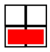
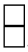
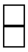
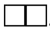
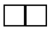
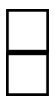
The following are common games that do not quite fit all the requirements for combinatorial games, but can be analyzed similarly.

- In *Dots and Boxes*, players do not always alternate as they get an extra move when they bound a box.
- *Battleship* does not have perfect information, as the players cannot see both boards.
- *Chess* contains draws, so there is not always a winner.
- *Go* determines the winner by who has the most pieces rather than who goes last.
- *Chomp* is a combinatorial game, but since the last player to move loses, it is not normal play, but **miseré play**.

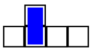
Some games that do not satisfy two or more of the requirements for combinatorial games are *Poker, Risk, Monopoly, Blackbox, Scrabble, etc.* These games are far from combinatorial games as some have chance, multiple players, imperfect information, and determine the winners in different ways.

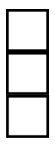
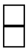
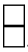
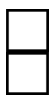
We will use the game ***Domineering*** for our two-player examples, defined as follows. We start with a board of squares in which players take turns placing 2x1 domino tiles on the board with no overlap in the tiles. Left can only place dominos oriented vertically and Right can only place dominos oriented horizontally.

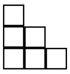
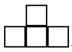
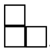
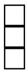
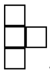
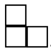
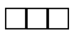
A game G is classified by a set of **game options**. Left has a set of possible moves, denoted G^L, and Right has a set of possible moves, denoted G^R, so the game $G = \{G^L | G^R\}$. The **zero game** is the game with no options: $0 = \{|\}$.

Example: Let G = . Then G = { ,  | ,  }. Notice here that once a section of the board is covered with a tile, it is the same as the position with the covered squares removed, so G = { ,  | ,  }. Since we have duplicate options here, and a player will choose only one, we can ignore the extra copies and G = {  |  }. Because  = {0|} and  = {|0}, we have G = {{0|}|{|0}}. These collections of sets within sets can be confusing though, so when showing the sets of game options within games, we use double bars to separate the options in the superset and single bars to separate the options in the subsets. Thus G = {{0|∅}|{∅|0}} can be written as G = {0|∅||∅|0}.

Notice that some moves in which we remove the squares covered by tiles will leave disconnected squares. These isolated squares can be ignored since no moves can be made on them. They are of no consequence to the game position.

Example:  =   =  .

We can create what we call a **game tree** which charts all the possible moves throughout the game. For full game trees, we include both Left and Right options because games can be subsets of other games and so we may move on a game in an order that is not the given player order. Two games, G and H, are called **isomorphic**, denoted by $G \cong H$, if their game trees are the same. If we have two options for a player that are isomorphic, we only need to consider one of those options, so if $G^L =$ {  ,  }, since  $\cong$ , then $G^L =$ {  }.

Example: The game tree for  = { , ,  | , ,  } is displayed in Figure 1.1.

The solitary boxes on the leaves of the game tree represent 0 games, which are where the game sequences end since there are no more possible moves for either player.

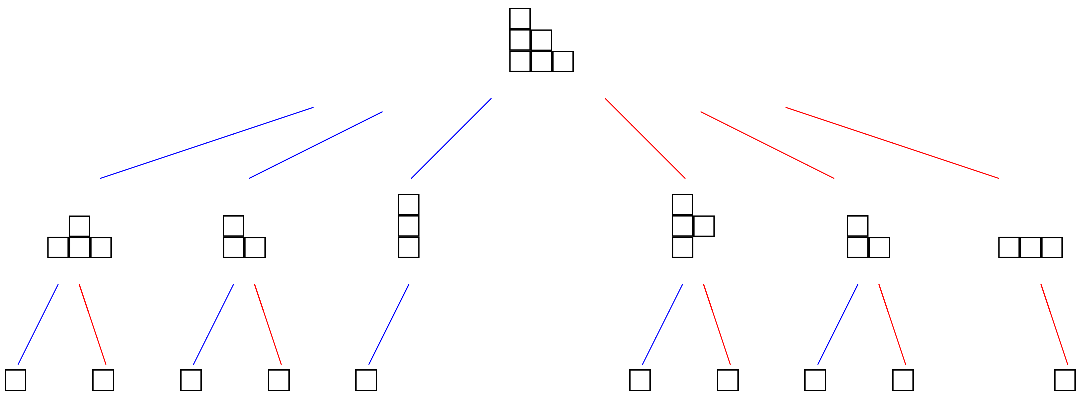


Figure 1.1: Two-Player Game Tree

Outcome classes		When Right moves first	
		Right wins	Left wins
When Left moves first	Left wins	$\mathcal{N}$	$\mathcal{L}$
	Right wins	$\mathcal{R}$	$\mathcal{P}$

Table I: Two-Player Outcome Classes

The *Fundamental Theorem of Combinatorial Game Theory* states that given any game with a certain player moving first, there is one player who can force a win, meaning there is a predetermined player with a winning strategy no matter what the other player does. Because of this, we develop **outcome classes**, as shown in Table I, which show for each game who has a winning strategy depending on who goes first.

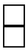
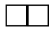
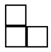
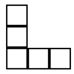
Here, $\mathcal{N}$ represents the outcome class where the $\mathcal{N}$ext player to move wins. $\mathcal{L}$ represents the class where $\mathcal{L}$eft wins, regardless of who goes first. Likewise, $\mathcal{R}$ is the class where $\mathcal{R}$ight wins, regardless of who moves first. Lastly, $\mathcal{P}$ is when the $\mathcal{P}$revious player, meaning the second player to move, wins.

Consider what kind of Left and Right options a game $G \in \mathcal{L}$ would have. In order for Left to win moving first, there must be a winning position for Left in G^L. Left must choose a game in $\mathcal{L}$ or $\mathcal{P}$ for Right to move on. As long as there exists an option in $\mathcal{L} \cup \mathcal{P}$ in G^L, Left can win moving first. Now for Left to win moving second, Right must have no losing positions to give Left in G^R. Losing positions for Left would be

	some $G^R \in \mathcal{R} \cup \mathcal{P}$	all $G^R \in \mathcal{L} \cup \mathcal{N}$
some $G^L \in \mathcal{L} \cup \mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$
all $G^L \in \mathcal{R} \cup \mathcal{N}$	$\mathcal{R}$	$\mathcal{P}$

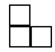
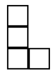
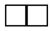
Table II: Two-Player Effects of Options on Outcome

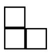
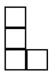
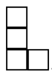
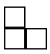
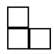
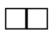
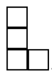
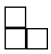
in $\mathcal{R}$ or $\mathcal{P}$. Since there are no Right options in $\mathcal{R} \cup \mathcal{P}$, all Right options must be in $\mathcal{L} \cup \mathcal{N}$. Now with similar observations on the other outcomes, we find the result in Table II.

Examples:  $\in \mathcal{L}$: Left wins no matter who goes first;  $\in \mathcal{R}$: Right wins no matter who goes first;  $\in \mathcal{N}$: whoever moves first wins; and  $\in \mathcal{P}$: the second player to move on this wins.

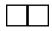
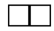
The **birthday** of a game is the number of layers beneath that position in the full game tree. It can also be recursively defined as one plus the maximum birthday of any of the options of the game. For the base case, the game $0 = \{|\}$ has birthday 0. Birthdays are very helpful because they give us the ability to induct on games, as all combinatorial games have a finite birthday. Notice the game in Figure 1.1 has birthday 2.

Definition 1. *For two-player combinatorial games G and H, $G + \emptyset = \emptyset$ and* $\mathbf{G} + \mathbf{H}$ $= \{G^L + H, G + H^L | G^R + H, G + H^R\}$.

Example: For $G =$  $= \{0|0\}$ and H $=$  $= \{$ 0,  |  $\}$:

$G + H =$  $+$  $= \{$ $0 +$ ,  $+ 0$,  $+$  | $0 +$ ,  $+$  $\}$.

Definition 2. *For games G and H,* $\mathbf{G} = \mathbf{H}$ *means that for all games X, $G + X$ has the same outcome as $H + X$.*

Example:  $+$  $=$  because for any combinatorial game X,  $+$  $+$ X has the same outcome as  $+$ X.

Theorem 1.1. *Addition is commutative and associative. That is,*

$$G + H = H + G$$

$$(G + H) + J = G + (H + J).$$

Lemma 1. $G + 0 = G$.

Proof. We will induct on the birthday of G.

Base Case: If G has birthday 0, $G = 0$, so

$$\begin{aligned} G + 0 &= \{|\} + \{|\} \\ &= \{\emptyset + 0, 0 + \emptyset | \emptyset + 0, 0 + \emptyset\} \\ &= \{\emptyset, \emptyset | \emptyset, \emptyset\} = \{|\} = 0. \end{aligned}$$

Inductive Step: Assume $G + 0 = G$ for G with birthday less than k. By Definition 1, $G + \emptyset = \emptyset$, so for G with birthday k,

$$\begin{aligned} G + 0 &= \{G^L | G^R\} + \{|\} \\ &= \{G^L + 0, G + \emptyset | G^R + 0, G + \emptyset\} \\ &= \{G^L, \emptyset | G^R, \emptyset\} \\ &= \{G^L | G^R\} = G. \end{aligned}$$

□

Theorem 1.2. *A game $G = 0$ if and only if $G \in \mathcal{P}$.*

Proof. ($\Rightarrow$) If $G = 0$, then for $X = 0$, $G + 0 = G$ and $0 + 0 = 0$ have the same outcome, so $G \in \mathcal{P}$.

($\Leftarrow$) For $G \in \mathcal{P}$, fix any game X.

Now if Left can win by moving second on X, then Left will win playing second on $G + X$ because whichever board Right plays on, Left will play the winning strategy immediately following Right.

If Left can win moving first on X, then moving first on $G + X$, Left will play first on X, then follow the previous player winning strategy moving on whichever board Right moves on.

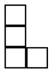
Symmetrically, if Right wins moving first or second on X, then they can use the same strategy to make the same outcome on $G + X$. Thus $G + X$ and $0 + X$ have the same outcome and $G = 0$. □

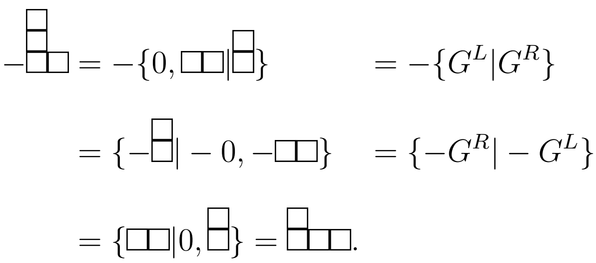
Now we have a set with a binary operation which is closed, associative, commutative, and has identity. This leads us to look at the algebraic structure of games.

Definition 3. *An* ***abelian group*** *is a set paired with a binary operation* $(S, \star)$ *which has the properties:*
Closure: $a \star b \in S$, $\forall a, b \in S$,
Associativity: $(a \star b) \star c = a \star (b \star c)$, $\forall a, b, c \in S$,
Identity: $\exists e \in S \ni e \star a = a = a \star e$, $\forall a \in S$,
Inverses: $\forall a \in S, \exists a^{-1} \in S \ni a \star a^{-1} = e = a^{-1} \star a$,
and Commutativity: $a \star b = b \star a$, $\forall a, b \in S$.

Example: The integers under addition, $(\mathbb{Z}, +)$ form an abelian group. Closure, associativity, and commutativity hold. Our identity is 0 and the inverse of any integer a is $-a$.

Games form an abelian group under game addition. The identity game is $0 = \{|\}$ and the inverse of game G is recursively given to be $-G = \{-G^R | - G^L\}$.

Example: The inverse of  is:



Notice that this inverse is a reflection along the diagonal. It turns out in Domineering that all game positions are inverses to their diagonal reflections.

Theorem 1.3. *For any two-player combinatorial game G, $G + (-G) = 0$.*

Proof. We will induct on the birthday.

Base Case: G = 0, so $0 + (-0) = 0 + 0 = 0$.

Assume this statement is true for games with birthday less than k. Let G have birthday k.

If Left moves first, Left can move either on G or $-G$.

Case 1: Left moves on G; then we have $G^L + (-G)$, and to force a win, Right will move on $-G$ to give us the game $G^L + (-G^L)$ which is $0 \in \mathcal{P}$ by the inductive hypothesis, so Right wins.

Case 2: Left moves on $-G$; then we have $G + (-G^R)$, and to force a win, Right will move on G to give us the game $G^R + (-G^R)$ which is $0 \in \mathcal{P}$ by the inductive hypothesis, so Right wins. The argument for Right moving first similarly has Left win in all cases.

Since we proved that regardless what the first player does, the second player can respond and force a win, we know the game is in $\mathcal{P}$, so it must be equal to 0. $\square$

Definition 4. *A **partially ordered set**, or **poset**, is a set paired with a partial*

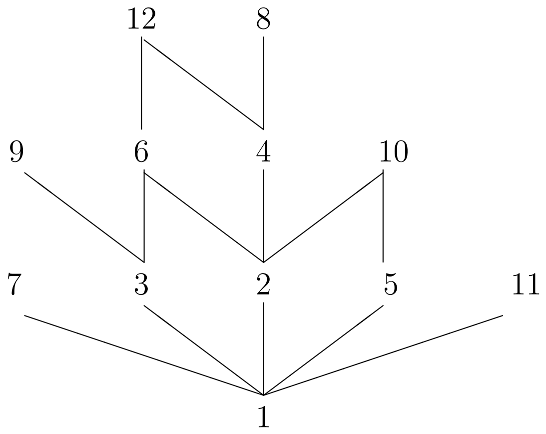


Figure 1.2: Poset Chart Example

ordering relation $(S, \geq)$ *where the partial ordering relation is reflexive, antisymmetric, and transitive.*

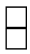
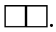
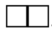
Reflexive: For $x \in S$*,* $x \geq x$*.*

Antisymmetric: For $x, y \in S$*, if* $x \geq y$ *and* $y \geq x$*, then* $x = y$*.*

Transitive: For $x, y, z \in S$*, if* $x \geq y$ *and* $y \geq z$*, then* $x \geq z$*.*

Example: Consider the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ with the ordering: $m \geq n$ iff $n|m$. This is a poset with Hasse Diagram [2] shown in Figure 1.2.

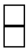
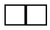
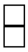
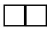
Definition 5. G ≥ H, *means for all games X, Left wins* $G + X$ *whenever Left wins* $H + X$*.*

Example:  $\geq$ . For any game X, if Left wins X + , then Left will win X +  because they play their winning strategy on X. Now Right has one less move, so there is no disadvantage to Left.

Games form a partially ordered set under the relation in Definition 5. It is reflexive because given game G, then for all X, if Left wins $G + X$ then Left will win $G + X$, so $G \geq G$. It is antisymmetric because if $G \geq H$ and $H \geq G$, then $\forall X$ Left wins $H + X$ implies Left wins $G + X$ and Left wins $G + X$ implies Left wins $H + X$. Thus

$G + X$ and $H + X$ have the same outcome, so $G = H$. Transitivity follows because if $G \geq H$ and $H \geq K$, then $\forall X$ Left wins $K + X$ implies Left wins $H + X$ which implies Left wins $G + X$, so $G \geq K$.

We can create what we call **values** of games, which are rational numbers that measure how favorable the games are for each player. The games that can be assigned a value are called **number games**. For any number game N, if N is negative, $N \in \mathcal{R}$, if N is positive, $N \in \mathcal{L}$, and if N is 0, $N \in \mathcal{P}$. No game in $\mathcal{N}$ can be a number game as it is neither better for Left nor Right and is not equal to 0. This creates a well-ordered subset of games in $\mathcal{L}, \mathcal{R}$, or $\mathcal{P}$.

Example:  is the number game 1 and  is the number game -1. As we would expect,  +  $= 1 - 1 = 0$.

For the sake of future chapters, we will say $\geq_L$ is the same as $\geq$, since larger numbers are better for Left and we will define $\geq_R$ to be the same as $\leq$, since smaller numbers are better for Right.

With two-player games, we can also trim the game trees, creating an equivalent game tree in reduced form, which is called **canonical form**. Given just two trimming moves, we can cut down the game trees so that all games that are equal have isomorphic game trees in canonical form.

We will glance over the three-player generalization for canonical form in Chapter 4. The following theorems are taken from [1].

Theorem 1.4. *If* $G = \{A, B, C, \ldots | H, I, J, \ldots\}$ *and* $B \geq_L A$*, then* $G = G'$ *where* $G' = \{B, C, \ldots | H, I, J, \ldots\}$.

Similarly for Right, if $H \geq_R I$, then option I can be removed. Here options A and I are called **dominated options**. The theorem allows us to remove these.

Theorem 1.5. *Fix a game* $G = \{A, B, C, \ldots | H, I, J, \ldots\}$ *and suppose that for*

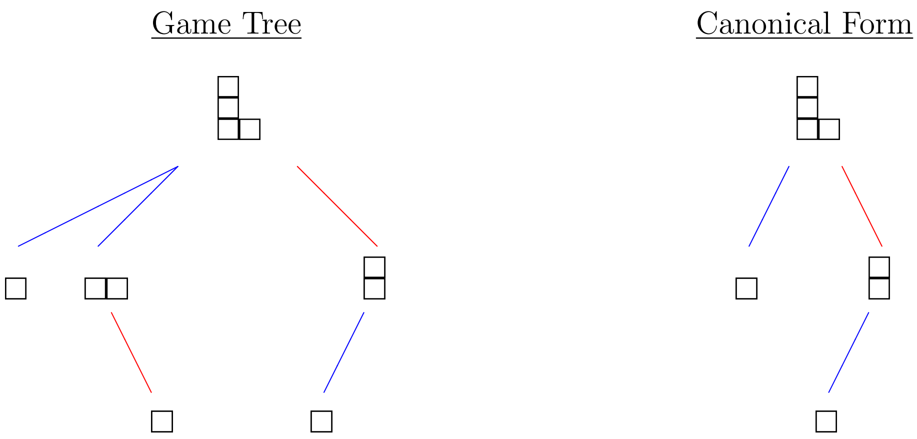


Figure 1.3: Game Tree Trimming

some Right option of A, call it A^R, $G \geq_L A^R$. If we denote the Left options of A^R by $\{W, X, Y, \ldots\}$ then $A^R = \{W, X, Y, \ldots | \ldots\}$, we can define the new game $G' = \{W, X, Y, \ldots, B, C, \ldots | H, I, J, \ldots\}$, then $G = G'$.

For the situations in which the theorem applies, we say A is a **reversible option**, meaning that the move reverses through A, so we can bypass it.

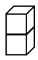
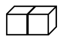
Example: Let G = $\begin{smallmatrix}\square & \\ \square & \\ \square & \square\end{smallmatrix}$ = $\{\ 0,\ \square\square \mid \begin{smallmatrix}\square\\ \square\end{smallmatrix}\ \}$. Here, $\square\square \in G^L$ is both a dominated option and a reversible option. The full game tree and canonical form are displayed in Figure 1.3. Note that G $\in \mathcal{L}$.

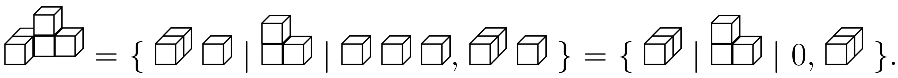
To show that $\square\square$ is dominated, notice that $0 \geq_L \square\square$, so by Theorem 1.4, we have G = $\{\ 0 \mid \begin{smallmatrix}\square\\ \square\end{smallmatrix}\ \}$.

To show that $\square\square$ is also reversible, we notice that the Right option of $\square\square$ is 0. Since $\square\square^R = 0 \leq_L G$, by Theorem 1.5, we can substitute the Right options of $\square\square$ as a Left option, giving us G = $\{\ 0,\ 0 \mid \begin{smallmatrix}\square\\ \square\end{smallmatrix}\ \}$. Now we can remove the extra 0 by isomorphic copies or by dominated options, giving us G = $\{\ 0 \mid \begin{smallmatrix}\square\\ \square\end{smallmatrix}\ \}$.

1.2 Three-Player Games

Three-player games require an additional player. We will call the new player Center. There is no general convention for three-player partizan games, so we will begin by looking at the research that has been done. In all cases, we play cyclically: (L,C,R,L,C,R...). We define **game options** similarly to two-player games, adding G^C, the Center options of a game $G = \{G^L|G^C|G^R\}$.

Cincotti introduces a three-dimensional *Domineering* game [3]. We will use it for our examples. This game is played on a finite three-dimensional grid with edges parallel to the x, y, and z axes, where players move by placing 2x1x1 dominoes. Left places dominoes parallel to the z-axis, Center places dominoes parallel to the x-axis, and Right places dominoes parallel to the y-axis. So  $= \{0||\}$ is a left tile,  $=$ $\{|0|\}$ is a center tile, and  $= \{||0\}$ is a right tile.

Example: 

Definition 6. *For three-player combinatorial games G and H,* $\mathbf{G} + \mathbf{H} = \{G^L + H, G + H^L | G^C + H, G + H^C | G^R + H, G + H^R\}$.

Proposition 1. *Game addition is commutative and associative for three-player games. Specifically, for games G, H, and K,*

$$G + H = H + G, \tag{1.1}$$

$$(G + H) + K = G + (H + K). \tag{1.2}$$

Proof. We will prove commutativity by induction on the sum of the birthdays of G and H.

Base Case: G and H both have birthday 0. Then $G + H = 0 + 0 = H + G$.

Assume commutativity for games whose birthdays sum to less than k. Now for G and H with birthdays that sum to k:

$$G + H = \{G^L + H, G + H^L | G^C + H, G + H^C | G^R + H, G + H^R\}$$
$$= \{H^L + G, H + G^L | H^C + G, H + G^C | H^R + G, H + G^R\} = H + G.$$

The proof of (1.2) is by a similar strategy. □

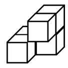
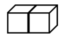
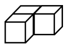
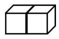
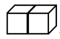
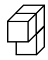
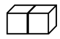
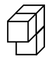
We now move to Propp's paper on three-player impartial games [4]. He gives us four outcome classes: $\mathcal{N}$ for a $\mathcal{N}$ext player win, $\mathcal{O}$ for an $\mathcal{O}$ther player (or second player) win, $\mathcal{P}$ for a $\mathcal{P}$revious player win, and $\mathcal{Q}$ for a $\mathcal{Q}$ueer game, where the winner cannot be determined.

Propp gives a recursive method to classify the outcome of an impartial game G:

1. $G \in \mathcal{N}$ if it has a $\mathcal{P}$ option.
2. $G \in \mathcal{O}$ if it has an option and all options are in $\mathcal{N}$.
3. $G \in \mathcal{P}$ if all its options are in $\mathcal{O}$.
4. $G \in \mathcal{Q}$ if none of these conditions are satisfied.

Propp also points out that only in two-player games do we have a strategy against an arbitrary adversary if and only if we have a winning strategy against a perfectly rational adversary. This means that in two-player games we can account for any adversary just by winning against the smartest possible opponent. However, when we add more players, winning against a perfect opponent will not guarantee a win against a random adversary.

In Li's paper about n-person Nim [5], he explains n-player impartial games. He introduces a ranking method in which everyone plays until one player cannot move. The game immediately stops and (for our three-player purposes) that person is given third place, the last person to move is given first place, and the second to last to move is given second place. This eliminates the $\mathcal{Q}$ueer outcome since it introduces a preference between players, so when a player is given a choice to give the game to one of the other two players, we know who will be chosen.

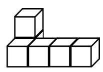
Example: Without player preference,  +  is $\mathcal{Q}$ueer because the Center options are  +  $\to_R$ , where Right wins or  +  $\to_R$  $\to_L$ , where Left wins. Player preference allows us to predict that Center will choose to let Right win so Center will come in second place.

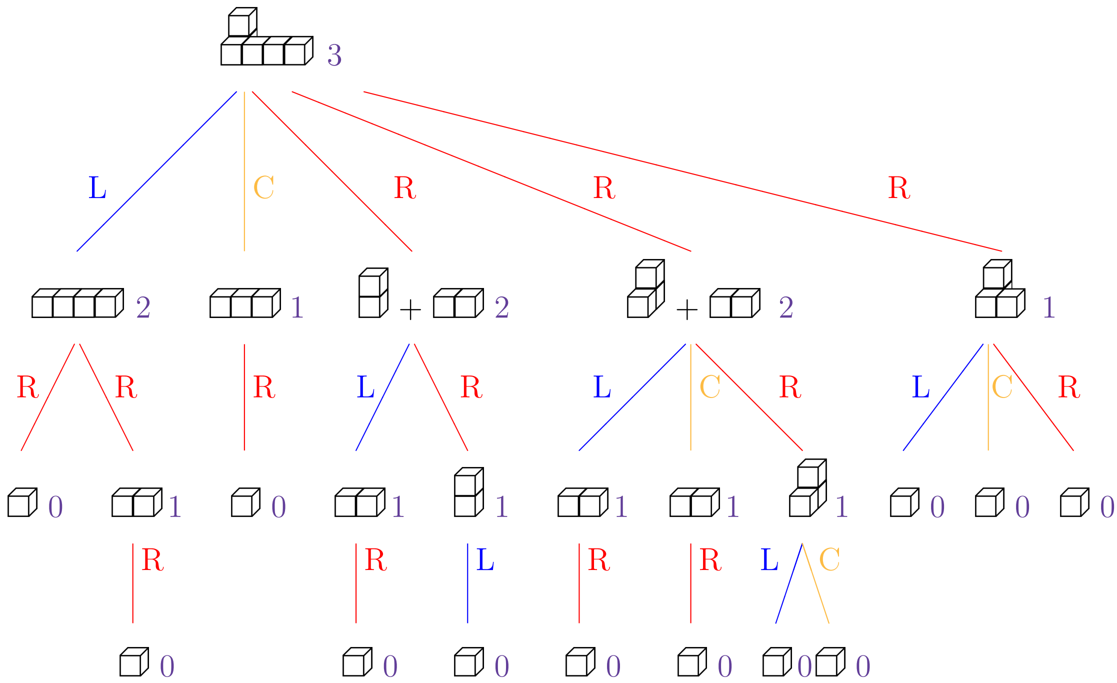
Cincotti proposes we go by elimination so that when a player no longer has a move, they are eliminated and the other players continue together as a two-player game [6]. Cincotti focuses on number games, which he recursively defines to have only numbers for options. He also defines inequalities and equalities with respect to each player, which we will look at in Chapter 3.

Induction is our biggest asset in three-player proofs, so we will use a similar approach to our two-player method.

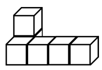
Definition 7. *We define the* ***birthday*** *for three-player games in the same way it is defined in two-player games: 0 has birthday 0 and the birthday of a game G is the maximum birthday of its options plus one. We say a game is born by day n if its birthday is less than or equal to n.*

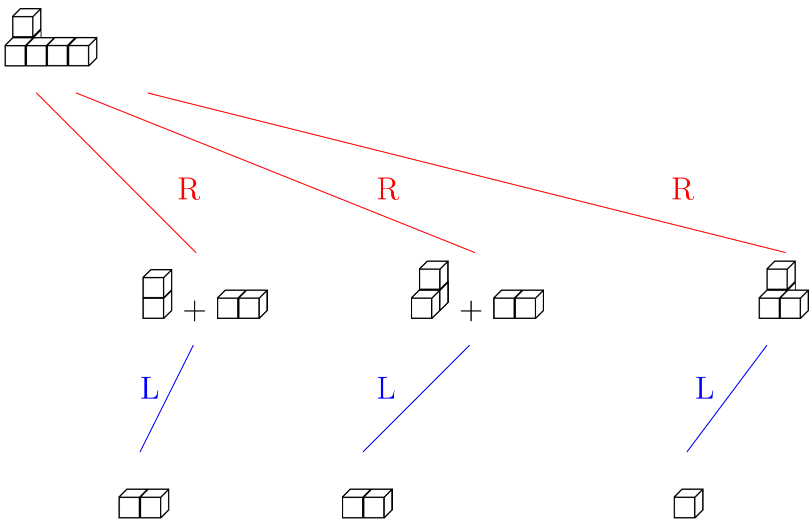
We also define three-player **game trees** similarly to two-player game trees, where we chart all possible moves in all possible sequences.

Example: Here we have the game tree for  with the game birthdays displayed in purple:



A **partial game tree** is a subtree of the full game tree where we only look at the game options for the given player order. Each level is based on only the options of the next player.

Example: The partial game tree of  where Right moves first:



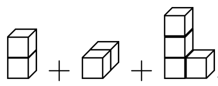
Chapter 2: Rhombination

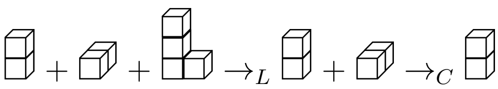
Now we solidify our rules. Left, Center, and Right will play cyclically until one player can not move on their turn. At that point, the game is over and the last player to move wins with the second to last player to move coming in second, and the player who was not able to move comes in last, as defined by Li. [5]

The rankings depend on who moves last:

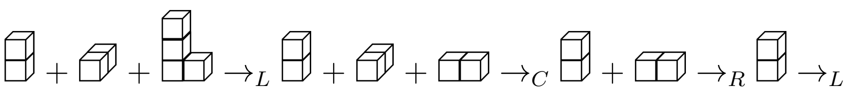
	1^{st} Place	2^{nd} Place	3^{rd} Place
L moves last	Left	Right	Center
C moves last	Center	Left	Right
R moves last	Right	Center	Left

For this reason, each player prefers the player immediately following them to win; namely, Left prefers Center, Center prefers Right, and Right prefers Left. However, if the player immediately after the given player runs out of moves before the other two then the given player wins. Thus the player they prefer to sabotage is also the player they prefer to win if there is no way to force a win for themselves, which creates an interesting dynamic between players. In two-player games, it is always advantageous to sabotage another player, but here we want to sabotage the next player and leave the player before us with enough moves to make it to our turn.

It is not advantageous for Left to sabotage Right in the game: .
If Left chooses to sabotage Right, the game sequence would be:

 and Center wins.

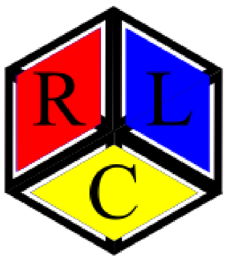
But if Left helps Right, the sequence is:

 0 and Left wins.

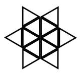
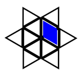
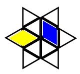
We now introduce the game on which we focus this research.

Rhombination is a three-player partizan game with perfect information and no chance. The winner is determined by who moves last, which happens in a finite number of turns.

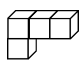
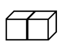
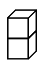
We begin with a finite subset of a grid of equilateral triangles. The players place rhombus shaped tiles on the board such that each tile covers exactly two of the triangles on the grid. Left is only allowed to place their tile with the acute vertex tilted $\frac{\pi}{6}$ radians to the left from the vertical axis, Center is only allowed to place a tile oriented horizontally, and Right is only allowed to place the tile tilted $\frac{\pi}{6}$ radians to the right of the vertical axis. Tiles are not allowed to overlap. When one of the players cannot place their tile, the game is over and the last player that moved wins. The player before that comes in second, and the player who could not move on their turn loses.



Orientation of the respective tiles.

Example game sequence:  $\rightarrow_L$  $\rightarrow_C$  $\rightarrow_R$  $\rightarrow_L$  and Left wins. (Note that this is not the optimal strategy, but an example of what players could do.)

Proposition 2. *Three-dimensional Domineering is distinct from Rhombination.*

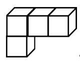
Proof. Consider the 3D *Domineering* position:  $= \{$  $\|\; 0,$  $\}$.

We want to show that this position can not be isomorphic to any *Rhombination* position, where a three-player isomorphism is similarly defined as having the same game trees, or isomorphic options, so we are looking for G = {  || 0, }.

Since Right can take the whole board with one move, we can only have one nontrivial component. Isolated triangles can be ignored in *Rhombination* because they add nothing to the game, which we call **trivial**. Trivial components are ◁ and ▷. Also, triangles are not considered connected through corners, but through sides. For instance,  is two components, namely  and ◁.

Since Right can take the board in one move, the isomorphic *Rhombination* position would be a Right tile with single triangles on the edges, which is a subset of . Notice that we cannot have two nontrivial components or Right would not have a 0 option, so the game must be a single connected component with only trivial pieces connected to the Right tile that can end the game.

As there are no center options, the side triangles cannot be included and the subsets we are left with are:  = {||0},  ≅  = {0||0}, and  = {  | | 0 }.

Thus there are no *Rhombination* games isomorphic to .

Therefore 3D *Domineering* and *Rhombination* are distinct. □

We will now prove the fundamental theorem in a context of three players. Notice that we name the players so we do not have to prove this is true in the three cases with Left, Center, and Right moving first. Alex can be Left, Center, or Right, so Beth and Charlie will be the naturally following players.

Theorem 2.1. *(Fundamental Theorem of Combinatorial Games for Three Players) Fix a game G played between Alex, Beth, and Charlie, with Alex moving first and Beth moving second. When everyone plays in their best interest, exactly one of these players can force a win with Alex moving first.*

Proof. We will proceed by induction on the birthday of G.
Base Case: If $G = 0$, then when Alex plays first, he does not have a move on G, so Charlie wins.

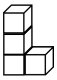
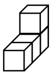
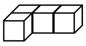
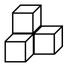
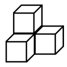
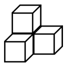
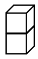
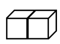
Assume the theorem is true for games born before day k. For G with birthday k, note that all of Alex's options allow someone to force a win with Beth moving first. If Alex has an option where he can force a win when Beth moves first on that option, he will pick that. If he has no options where he wins, but an option where Beth wins moving first, he will take that so he gets second place. And if all his options allow Charlie to force a win, then Charlie wins. $\square$

We make a strong assumption giving each player perfect strategy, but through this we can always determine the winner of a game.

Now that we know the winner can always be determined, we can sort games into **outcome classes**. We will notate these outcome classes by $\alpha\beta\gamma$ where α is the player that wins when Left moves first, β is the player who wins when Center moves first, and γ is the player who wins when Right moves first. This produces Table I. See Appendix A for examples in *Rhombination*.

Some outcome classes to take special notice of are: LLL, CCC, RRR, LCR, CRL, and RLC. LLL will be called $\mathcal{L}$ since it is a $\mathcal{L}$eft win. Similarly, CCC := $\mathcal{C}$ and RRR := $\mathcal{R}$. LCR is a $\mathcal{N}$ext player win, so LCR := $\mathcal{N}$. RLC is a $\mathcal{P}$revious player win, so RLC := $\mathcal{P}$. And lastly, CRL is a second player win, or an $\mathcal{O}$ther player win, so CRL := $\mathcal{O}$.

Examples:

 $\in \mathcal{L}$.  $\in \mathcal{C}$.  $\in \mathcal{R}$.  $\in \mathcal{N}$.  +  $\in \mathcal{O}$.  +  +  $\in \mathcal{P}$.

By making observations on how options affect outcome classes, we generate Table II, a three-player version of Table II from Chapter 1.

L wins when C goes first		L first		
		L wins	C wins	R wins
R first	L wins	LLL	CLL	RLL
	C wins	LLC	CLC	RLC
	R wins	LLR	CLR	RLR

C wins when C goes first		L first		
		L wins	C wins	R wins
R first	L wins	LCL	CCL	RCL
	C wins	LCC	CCC	RCC
	R wins	LCR	CCR	RCR

R wins when C goes first		L first		
		L wins	C wins	R wins
R first	L wins	LRL	CRL	RRL
	C wins	LRC	CRC	RRC
	R wins	LRR	CRR	RRR

Table I: Three-Player Outcome Classes

some $G^C \in \_\_C$	some $G^R \in R\_\_$	some $G^R \in L\_\_$, no $G^R \in R\_\_$	all $G^R \in C\_\_$
some $G^L \in \_L\_$	LCR	LCL	LCC
some $G^L \in \_C\_$, no $G^L \in \_L\_$	CCR	CCL	CCC
all $G^L \in \_R\_$	RCR	RCL	RCC

some $G^C \in \_\_R$, no $G^C \in \_\_C$	some $G^R \in R\_\_$	some $G^R \in L\_\_$, no $G^R \in R\_\_$	all $G^R \in C\_\_$
some $G^L \in \_L\_$	LRR	LRL	LRC
some $G^L \in \_C\_$, no $G^L \in \_L\_$	CRR	CRL	CRC
all $G^L \in \_R\_$	RRR	RRL	RRC

all $G^C \in \_\_L$	some $G^R \in R\_\_$	some $G^R \in L\_\_$, no $G^R \in R\_\_$	all $G^R \in C\_\_$
some $G^L \in \_L\_$	LLR	LLL	LLC
some $G^L \in \_C\_$, no $G^L \in \_L\_$	CLR	CLL	CLC
all $G^L \in \_R\_$	RLR	RLL	RLC

Table II: Three-Player Effects of Options on Outcome

For example, let $G \in LLL$. Since Left wins moving first, there must be a position Left can move on where Center moving next will allow Left to win, so there is a Left option in $\_L\_$. Similarly, when Center moves first, there must be no option where Center or Right wins, because Center prefers Left least, thus all the Center options are in $\_\_L$. Right must have an option where Left wins, but could also have options where Center wins, because they will choose Left to win, so there must be something in $L\_\_$ and there can be other options in $C\_\_$.

We will list the games of birthday 1 in *Rhombination*: There are seven possible three-player games with birthday 1, namely:

$\{0||\} =$ , $\{|0|\} =$ , $\{||0\} =$ ,

$\{|0|0\} =$ , $\{0||0\} =$ , $\{0|0|\} =$ ,

$\{0|0|0\} =$ .

For convenience, we will label four of these:

$1_L := \{0||\} =$  $\in LLC$, $1_C := \{|0|\} =$  $\in RCC$,

$1_R := \{||0\} =$  $\in RLR$, $\star := \{0|0|0\} =$  $\in LCR$.

It will be useful later to notate the sums of tiles such that for some number n, $n_L = 1_L + 1_L + \cdots + 1_L$ (n times). We similarly define sums of other tiles, so for example, $3_C = 1_C + 1_C + 1_C$.

Proposition 3. *We can set a recursive bound on the number of games with birthdays of any size. If G_n is the set of all Rhombination games born by day n and $g_n = |G_n|$, then $g_n \leq 2^{3g_{n-1}}$ and $g_0 = 1$.*

Proof. Since 0 is the only game with birthday 0, $g_0 = 1$. Now for any $G \in G_n$, the sets of options G^L, G^C, G^R are contained in G_{n-1}. Since the number of subsets of an m element set is 2^m, we have $g_n \leq 2^{g_{n-1}} 2^{g_{n-1}} 2^{g_{n-1}} = 2^{3g_{n-1}}$. □

We can set lower bounds for the birthdays of games in each outcome class. Due to symmetry, which we will see in Chapter 3, we only have to establish the bounds on LCR, CRL, RLC, LLL, LLC, LLR, LCL, LRL, CLL, RLL, and LRC. (We will skip LRC and its neighboring classes and look at them in Chapter 4.)

Since there is only one game of birthday 0, RLC is the only outcome with 0 as a lower bound on birthdays. Similarly, since we know all seven games of birthday 1, we can set minimal birthdays of 1 for outcomes LCR, LLC, and LLR. Then by examples in Appendix A, CRL, LLL, LCL, and LRL can be games with birthday at least 2.

For CLL, we have an example of a game with birthday 3, but we don't know for certain if there is a game with birthday 2 until we make an observation: When Left moves first, there must be at least two moves available (a Left move and then a Center move) in order for Center to win. Since a move for Center exists, there must be at least three moves when Center moves first (Center, Right, then Left) for Left to win. Thus the game has to have birthday at least 3. Thus the example in Appendix A is minimal.

We can make a similar argument for RLL, since Right moving first requires at least two moves (a Right and a Left move) for Left to win. When Left moves first, there are either no options or at least three moves. Since Left must have an option, the birthday must be at least 3. Now our example has birthday 4, so we know the minimal birthday for a game in RLL is either 3 or 4.

Because we cannot find examples of LRC, CLR, and RCL, we will examine those outcomes later. This gives us the minimal birthdays in Table III.

We will begin exploring properties of *Rhombination*, building off the ideas in two-

0	1	2	3	3 or 4	$\geq$ 4 (if they exist)
RLC	LCR	CRL	CLL	RLL	LRC
	LLC	LLL	CRC	CLC	CLR
	RCC	CCC	RRL	RRC	RCL
	RLR	RRR			
	LLR	LCL			
	LCC	CCR			
	RCR	LRR			
		LRL			
		CCL			
		CRR			

Table III: Minimal Birthdays by Outcome

player combinatorial game theory and finding the relationships in rotation, equivalence to zero, and properties of various outcome classes.

Chapter 3: Comparisons, Posets, and Symmetry

In two-player games, all games in the outcome class $\mathcal{P}$ are equal to the zero game. It would be nice to find a similar result in the three-player $\mathcal{P}$revious outcome class, RLC. However, by Cincotti's definitions, this is not true.

3.1 Cincotti's Approach to Partial Order

Consider the game $1_L + 1_C + 1_R \in$ RLC. We would expect this game to equal 0 since any sum gives each player one extra move and should not affect the outcome of a game.

In [6], Cincotti defines $\geq$ with respect to each of the players with the antisymmetric property. For example, $G =_L H$ iff ($G \geq_L H$ and $G \leq_L H$). General equality holds if games are equal with respect to all three players, meaning $G = H$ iff $G =_L H, G =_C H$, and $G =_R H$. Following Cincotti, we have the following definition.

Definition 8. *For games G and H, $G \geq_L H$ iff [($H \geq_L$ no G^C) and ($H \geq_L$ no G^R) and (no $H^L \geq_L G$)].*

Using this and the fact that $0 = \{||\}$, we simplify the definition to compare to 0:

Definition 9. *For game G, $G \geq_L 0$ iff ($0 \geq_L$ no G^C and $0 \geq_L$ no G^R). $0 \geq_L G$ iff (no $G^L \geq_L 0$).*

We will compute the Left comparison to 0 for games $1_L, 1_C$, and 1_R. 1_L has no Center or Right options, but the Left option is 0, so $1_L >_L 0$. 1_C has a 0 Center option, so $1_C \not\geq_L 0$. 1_C no Left options, so $0 \geq_L 1_C$. Thus $0 >_L 1_C$. Similarly, $0 >_L 1_R$.

Proposition 4. *By Cincotti's definition, $1_L + 1_C + 1_R \in RLC = \mathcal{P}$ is not equal to 0.*

Proof. Let $G = 1_L{+}1_C{+}1_R = \{1_C{+}1_R|1_L{+}1_R|1_L{+}1_C\} = \{\emptyset|1_R|1_C||1_R|\emptyset|1_L||1_C|1_L|\emptyset\}$.

Now $0 \geq_L G \Leftrightarrow$ no $G^L \geq_L 0 \Leftrightarrow \{|1_R|1_C\} \ngeq_L 0 \Leftrightarrow (0 \geq_L 1_R$ or $0 \geq_L 1_C)$ which is true. Thus $0 \geq_L G$.

Now $G \geq_L 0 \Leftrightarrow (0 \geq_L$ no G^C and $0 \geq_L$ no $G^R) \Leftrightarrow (0 \ngeq_L \{1_R||1_L\}$ and $0 \ngeq_L \{1_C|1_L|\}) \Leftrightarrow (1_R \geq_L 0$ and $1_C \geq_L 0)$ which is false, so $G \ngeq_L 0$.

Therefore $0{>_L}G$, so $G \neq 0$. □

This is less than ideal, so we will use the definitions of equality and $\geq$ with respect to different players in a generalization of the two-player definition.

3.2 Our Approach

We establish an equivalence for games. This parallels the two-player equivalence, looking at outcomes, instead of comparing options recursively as Cincotti defines.

Definition 10. *For games G and H,* $\mathbf{G} = \mathbf{H}$ *means for all games X, $G + X$ has the same outcome as $H + X$.*

Proposition 5. $=$ *is an equivalence relation.*

Proof. Fix a game X.

Reflexive: $G + X$ has the same outcome class at $G + X$, so $G = G$.

Symmetric: If $G = H$, then $G + X$ and $H + X$ have the same outcome, thus $H = G$.

Transitive: Let $G = H$ and $H = K$. $G + X$ has the same outcome as $H + X$, which has the same outcome as $K + X$, so $G + X$ has the same outcome as $K + X$. Thus $G = K$. □

Lemma 2. *For a game G, $G + 0 = G$.*

Proof. We will induct on the birthday of G.

Base Case: $G = 0$. Then $0+0 = \{||\}+\{||\} = \{\emptyset+0, 0+\emptyset|\emptyset+0, 0+\emptyset|\emptyset+0, 0+\emptyset\} = \{\emptyset, \emptyset|\emptyset, \emptyset|\emptyset, \emptyset\} = \{||\} = 0$.

Assume $G + 0 = G$ for G born before day k. Now for G with birthday k:

$$\begin{aligned} G + 0 &= \{G^L|G^C|G^R\} + \{||\} \\ &= \{G^L + 0, G + \emptyset|G^C + 0, G + \emptyset|G^R + 0, G + \emptyset\} \\ &= \{G^L + 0|G^C + 0|G^R + 0\} = \{G^L|G^C|G^R\} = G. \end{aligned}$$

Thus, by induction, $G + 0 = G$. □

Theorem 3.1. *For a game G,* $G = 0 \Rightarrow G \in \mathcal{P}$.

Proof. Let $G = 0$. This means $\forall X$, $G+X$ and $0+X$ have the same outcome. Choose $X = 0$. Now $G + 0 = G$ and $0 + 0 = 0 \in \mathcal{P}$ have the same outcome, so $G \in \mathcal{P}$. □

We conjecture that the converse is true in Chapter 4. Next we solidify our definition of game inequality, which we will use from here on.

Definition 11. *For games G and H,* $\mathbf{G \geq_L H}$ *means for all games X, Left wins G+X whenever Left wins* $H + X$. *(Meaning Left wins* $H + X \Rightarrow$ *Left wins* $G + X$.*)*

Proof of well-definedness: Let G and H be games such that $G \geq_L H$. Suppose G' and H' are games such that $G = G'$ and $H = H'$. Then $\forall X$, Left wins $H' + X$ means Left wins $H + X$, which implies Left wins $G + X$, which means Left wins $G' + X$. Thus $G' \geq_L H'$. □

The relation $\geq_L$ is also reflexive, transitive, and antisymmetric, which gives us a partially ordered set.

Proposition 6. *The combination of all partial orders:* $G \geq_L H$, $H \geq_L G$, $G \geq_C H$, $H \geq_C G$, $G \geq_R H$, *and* $H \geq_R G$ *implies that* $G = H$.

$\geq_L$	0	1_L	1_C	1_R
0	✓	×	✓	×
1_L	✓	✓	✓	×
1_C	×	×	✓	×
1_R	✓	×	✓	✓

$\geq_C$	0	1_C	1_R	1_L
0	✓	×	✓	×
1_C	✓	✓	✓	×
1_R	×	×	✓	×
1_L	✓	×	✓	✓

$\geq_R$	0	1_R	1_L	1_C
0	✓	×	✓	×
1_R	✓	✓	✓	×
1_L	×	×	✓	×
1_C	✓	×	✓	✓

Left Poset: $\geq_L$ Center Poset: $\geq_C$ Right Poset: $\geq_R$

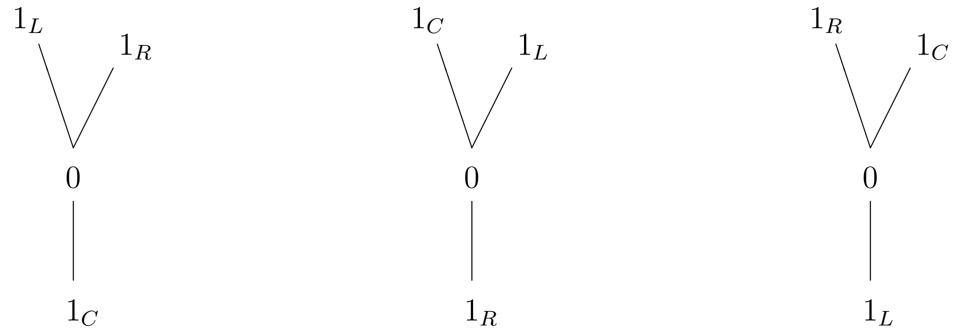


Figure 3.1: The Posets of $1_L, 1_C, 1_R$, and 0

Proof. Fix a game X. Now we look at the outcomes of $G+X$ and $H+X$. By $G \geq_L H$, we know Left wins on $G + X$ whenever Left wins on $H + X$. Also by $H \geq_L G$, we know Left wins on $H + X$ whenever Left wins on $G + X$. So Left must win on $G + X$ if and only if Left wins on $H + X$. By a similar argument, Center and Right must win on $G + X$ under the exact same conditions as $H + X$. Thus $G + X$ and $H + X$ must have the same outcomes. □

By these definitions, we develop Figure 3.1.

Proofs and examples for Figure 3.1:

- $1_L \geq_L 0$: If Left wins X, then Left wins $X + 1_L$, because Center will still run out of moves first.
- $1_C \not\geq_L 0$: Left wins going third on 0, but not on 1_C.
- $1_R \geq_L 0$: If Left wins X, Center runs out of moves first, so in $X + 1_R$, Center will still run out of moves before Right or Left.

- $0 \ngeq_L 1_L$: Left wins going first on 1_L, but not on 0.

- $1_C \ngeq_L 1_L$: Left wins going first on 1_L, but not on 1_C.

- $1_R \ngeq_L 1_L$: Left wins going first on 1_L, but not on 1_R.

- $0 \geq_L 1_C$: If Left wins $X + 1_C$, then Center runs out of moves first in $X + 1_C$, so Center will still run out of moves first in X.

- $1_L \geq_L 1_C$: Transitively, since $1_L \geq_L 0$ and $0 \geq_L 1_C$.

- $1_R \geq_L 1_C$: Transitively, since $1_R \geq_L 0$ and $0 \geq_L 1_C$.

- $0 \ngeq_L 1_R$: Left wins going second on $1_L + 1_R$, but not on 1_L.

- $1_L \ngeq_L 1_R$: Left wins going second on $1_L + 1_R$, but not on $1_L + 1_L$.

- $1_C \ngeq_L 1_R$: Left wins moving third on 1_R, but not on 1_C.

Proposition 7. *For any player α, and for games G, H, and K, if $G \geq_\alpha H$, then $G + K \geq_\alpha H + K$.*

Proof. Let X be any game. If α wins $(H+K)+X = H+(K+X)$, then since $K+X$ is a game, α must also win $G+(K+X) = (G+K)+X$. Therefore $G+K \geq_\alpha H+K$. $\square$

3.3 Rotations and Reflections

Definition 12. *For a game G, we define* $\mathbf{G^{60}}$ *as the* ***counter-clockwise rotation*** *of G by $\frac{\pi}{3}$ radians.*

Examples: ^{60} = , ^{60} = , and ^{60} = .

Theorem 3.2. *Let G be a game. Then* $G^{60} = \{(G^R)^{60} | (G^L)^{60} | (G^C)^{60}\}$ *and if the outcome class of G is* $\alpha\beta\gamma$ *then the outcome of* G^{60} *is* $(\gamma+1)(\alpha+1)(\beta+1)$*, where +1 is the function permuting the players:* $L \mapsto C$*,* $C \mapsto R$*, and* $R \mapsto L$*.*

Example:  $\in LCL$ and ^{60} $=$  $\in (L+1)(L+1)(C+1) = CCR$.

Proof. This rotation turns Left moves into Center moves; Center moves to Right moves; and Right moves to Left moves. So in G^{60}, Left's options are Right's options rotated by $\frac{\pi}{3}$. Likewise, Center's options are the rotated Left options and Right's options are the rotated Center options. Thus $G^{60} = \{(G^R)^{60} | (G^L)^{60} | (G^C)^{60}\}$.

Now we will prove the change of outcome class by inducting on the birthday of G. Notice that G and G^{60} have the same birthdays.

Base Case: If G has birthday 0, then $G = 0$ and $G^{60} = 0^{60} = 0$. $0 \in$ RLC, so here $\alpha = R, \beta = L, \gamma = C$, so $(\gamma+1)(\alpha+1)(\beta+1)$ is $(C+1)(R+1)(L+1)$ is RLC, which is the outcome class of 0^{60}. Thus the claim holds for birthday 0.

Assume the claim is true for games born before day k. Let $G \in \alpha\beta\gamma$ have birthday k. We know α wins G^L when Center moves first, β wins G^C when Right moves first, and γ wins G^R when Left moves first.

Specifically, notice that the fact α wins when Center moves first on G^L, puts G^L in the outcome class $\_\alpha\_$. Now by the inductive hypothesis, rotating G^L (which is born before day k) by $\frac{\pi}{3}$ radians gives us the outcome $(G^L)^{60} \in \_\_(\alpha+1)$. We can use this argument on all the options, giving us the implication, by the inductive hypothesis:

$$G^L \in \_\alpha\_ \Rightarrow (G^L)^{60} \in \_\_(\alpha+1),$$

$$G^C \in \_\_\beta \Rightarrow (G^C)^{60} \in (\beta+1)\_\_,$$

$$G^R \in \gamma\_\_ \Rightarrow (G^R)^{60} \in \_(\gamma+1)\_.$$

Since $(\alpha+1)$ wins on $(G^L)^{60}$ when Right moves first, then when Center moves first on G^{60}, $(\alpha+1)$ will win. By similar logic, we can determine the outcome of G^{60}.

Therefore $G^{60} = \{(G^R)^{60}|(G^L)^{60}|(G^C)^{60}\} \in (\gamma+1)(\alpha+1)(\beta+1)$. $\square$

A game G is **rotationally symmetric** when G^{60} is the exact same as G. For example, ^{60} is the same as .

Corollary 1. *If $G \in \alpha\beta\gamma$ is rotationally symmetric, then $G \in LCR, RLC$, or CRL.*

Proof. Let $G \in \alpha\beta\gamma$ be rotationally symmetric. Then G^{60} will be in the same outcome class as G, so $\alpha = (\gamma+1), \beta = (\alpha+1), \gamma = (\beta+1)$.
Case 1: $\alpha = L$. Then $\beta = L+1 = C$ and $\gamma = C+1 = R$, so $G \in LCR$.
Case 2: $\alpha = C$. Then $\beta = C+1 = R$ and $\gamma = R+1 = L$, so $G \in CRL$.
Case 3: $\alpha = R$. Then $\beta = R+1 = L$ and $\gamma = L+1 = C$, so $G \in RLC$. $\square$

Notice that the converse is not true. For example,  $\in$ CRL, but is clearly not rotationally symmetric.

Corollary 2. *G^{180} is isomorphic to G.*

Proof. First notice:

$$G = \{G^L|G^C|G^R\},$$

$$G^{60} = \{(G^R)^{60}|(G^L)^{60}|(G^C)^{60}\},$$

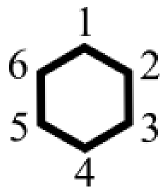
$$(G^{60})^{60} = G^{120} = \{(G^C)^{120}|(G^R)^{120}|(G^L)^{120}\},$$

$$((G^{60})^{60})^{60} = G^{180} = \{(G^L)^{180}|(G^C)^{180}|(G^R)^{180}\}.$$

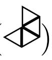
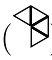
We will induct on the birthday of G. Base Case: $0 \cong 0^{180}$.

Assume this is true for games born before day k. Now let G have birthday k.
So $G = \{G^L|G^C|G^R\} \cong \{(G^L)^{180}|(G^C)^{180}|(G^R)^{180}\} = G^{180}$. $\square$

Definition 13. *The **dihedral motions** of a hexagon are the cyclic permutation representations of the rigid motions of a regular hexagon, with corners labeled below.*



Notice that the numbers refer to positions, so we read the permutation (123) as position 1 goes to position 2, which goes to position 3, which goes to position 1. (This is not a dihedral motion on a hexagon, but it illustrates the purpose.) If we were to permute the letters ABC by (123), we would have CAB. A was in position 1, but is now in position 2. Likewise, the permutation (12)(36)(45) on ABCDEF would be BAFEDC, as $A: 1 \to 2, B: 2 \to 1, C: 3 \to 6, D: 4 \to 5, E: 5 \to 4$, and $F: 6 \to 3$.

Example: Take the game . We will rotate it counter-clockwise by $\frac{\pi}{3}$, which is the permutation (165432), and then flip it vertically, which is the permutation (26)(35), giving us: $(26)(35) \circ (165432)($$) = (26)(35)($$) =$ .

Definition 14. *We will define a* ***vertical reflection*** *of a game G with respect to Center as the hexagonal dihedral motion* $v_C(G) = (26)(35)(G)$ *and a* ***horizontal reflection*** *with respect to Center as the dihedral motion:* $h_C(G) = (14)(23)(56)(G)$.

We similarly define vertical and horizontal reflections with respect to Left and Right. Vertical reflections have lines of reflection perpendicular to the player's tile and horizontal reflections run parallel with the player's tile.

$$v_L = (13)(46), \quad h_L = (16)(25)(34),$$

$$v_R = (15)(24), \quad h_R = (12)(36)(45).$$

Proposition 8. *For a game G, the vertical reflection of G with respect to any player is isomorphic to the horizontal reflection of G with respect to the same player.*

Proof. The proofs are similar, so we will consider reflections with respect to Center, since those are the most intuitive.

Geometrically consider what reflections do to the options of G. Both reflections preserve Center's moves and switch the moves of Left and Right, giving us:

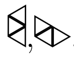
$v_C(G) = \{v_C(G^R)|v_C(G^C)|v_C(G^L)\}$ and $h_C(G) = \{h_C(G^R)|h_C(G^C)|h_C(G^L)\}$.

Now we will induct on the birthday of G:

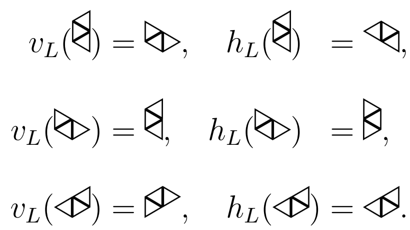
Base Case: $G = 0$. Then $v_C(0) = 0 = h_C(0)$.

Assume the theorem is true for games born before day k, and now for G with birthday k:

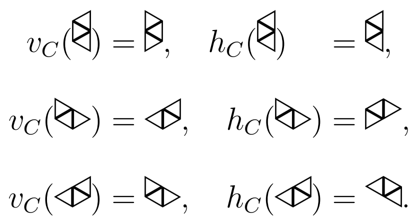
$v_C(G) = \{v_C(G^R)|v_C(G^C)|v_C(G^L)\} \cong \{h_C(G^R)|h_C(G^C)|h_C(G^L)\} = h_C(G)$. □

Consider the games , and . We will look at the vertical and horizontal reflections with respect to each player.

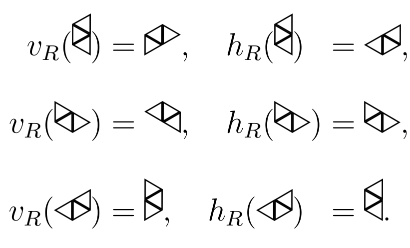
With respect to Left:



With respect to Center:



With respect to Right:



Chapter 4: Future Directions

4.1 Additional Concepts to Prove

Continuing to build on two-player game theory, we have a few unproven conjectures. We would naturally like to find the set of games that do not change the outcome of other games, leading to the first conjecture:

Conjecture 1 (Converse of Theorem 3.1)**.** *If* $G \in \mathcal{P}$ *then* $G = 0$*.*

Currently the set of *Rhombination* game positions under addition is an abelian monoid since we have a closed, associative, and commutative operation with an identity. However our identity is very limited, being only the game 0. If we can prove Conjecture 1, we would have a more generalized identity we can use more easily in proofs which could also help us find inverses.

We do not have a proof for Conjecture 1; however, we have looked into showing the particular game $1_L + 1_C + 1_R \in \mathcal{P}$ is equal to 0 through induction, though we have only figured out some of the cases.

In order to show $1_L + 1_C + 1_R = 0$, we need to show for any game H, that $H + 1_L + 1_C + 1_R$ has the same outcome as H. We will attempt strong induction on the birthday of H:

Base Case: $H = 0$. Then by Lemma 2, $H + 1_L + 1_C + 1_R = 1_L + 1_C + 1_R \in \mathcal{P}$ and $H \in \mathcal{P}$, so it holds.

Now we assume $H + 1_L + 1_C + 1_R$ has the same outcome as H for any H born before day k. For each player moving first, the argument is similar, so suppose Left is moving first on this position.

Case 1: Left moves on H. Then we have $H^L + 1_L + 1_C + 1_R$ which has the same outcome as H^L. By the inductive hypothesis, $1_L + 1_C + 1_R$ does not change the outcome.

Case 2: Left moves on 1_L. Then we are left with $H + 1_C + 1_R$. This is where it gets complicated. If Center and Right both move on 1_C and 1_R respectively, then after one cycle we have Left moving first on H and nothing is changed.

However, if Center and Right do not follow and we end up with a position like $H^C + 1_C$, then we do not know what happens to the outcome since we only know who wins H^C when Right moves first. Thus our conjecture remains unproven, even for a simple position like $1_L + 1_C + 1_R$.

Conjecture 2. *There are no Rhombination games in LRC, CLR, or RCL.*

So far, we have not found or been able to create a game that falls in any of these outcome classes, so we wonder if they may be empty.

Proposition 9. *A Rhombination game in LRC, if it exists, must have birthday at least 4.*

Proof. The fact that Left wins moving first implies we have to have birthday at least 1. Right wins when Center moves first means we need birthday at least 2 since Center and Right need successive moves. Center wins when Right moves first means that Right has either no options or there are at least 3 moves. Right has to have an option since they win when Center moves first, so there is a Right move. Thus, there has to be a Right move followed by a Left and Center move, making the game have birthday at least 3.

Left cannot win in one move because that would be isomorphic to a subset of  (see Proposition 2 for reasoning), which has birthday no greater than 2, so it cannot be in LRC. Thus Left's option must have a Left, Center, Right, and Left move for Left to win, giving us a birthday of at least 4. □

Since games in LRC, CLR, and RCL are equivalent up to rotation, the same result applies to CLR and RCL.

Conjecture 3. *Assuming Conjecture 1, for a game G,* $G + G^{60} + G^{120} \in \mathcal{P}$*, which gives us the inverse:* $-G = G^{60} + (G^{60})^{60}$.

If the proposed identity from Conjecture 1 and inverses from Conjecture 3 can be proven to work properly, that would make the set of Rhombination games an abelian group under game addition.

It would be convenient to have some form of trimming moves like we have in creating canonical form for two-player games. We want to show we can remove dominated options, which means for $A, B \in G^L$, if $B \geq_L A$, then $G^L = G^L - A$. The interesting problem we run into with three players is we do not necessarily know that a player will always pick the option that dominates with respect to themselves. If a winning strategy does not exist, they will aim to help the player after them.

For example, if Left had options like 1_C and $1_C + 1_R$, then even though $1_C + 1_R \geq_L 1_C$, Left is going to choose 1_C because then Left will come in second place instead of third. (Now this set of options may not be possible, but the concept is still something we have to consider.)

Reversible options are even more tricky to determine. The idea behind them in two-player games is that if one player can respond to the move of another with a game that is more favorable than the original, then we can bypass that option. However, since a player has to respond to the moves of two other players, this is difficult to imagine an analog to.

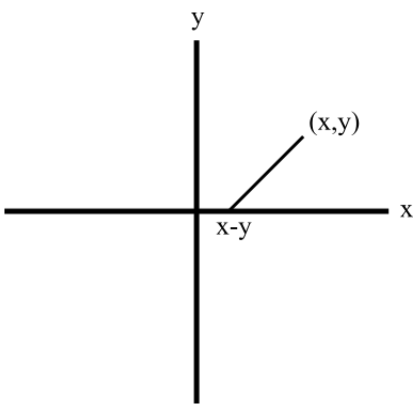
In this paper, we assume perfect strategy from all three players, but we think it is sufficient to only have the first and second place winners play optimally.

4.2 Number Games

It would be useful to quantify the benefit of a position for a certain player. This property in two-player games are called **number games**. When a two-player game is a number, we can immediately determine the outcome class along with total ordering with respect to any player compared to any other number games. Since the number line is not three-directional, it is difficult to extend this system to three players.

In two-player games, our numbers are one-dimensional, where positive numbers help Left and negatives help Right. We cannot add a third player to this system, so we will expand the two-player number games to the plane in $\mathbb{R}^2$, where Left's advantage is in the x-direction and Right's is in the y-direction. Notice that when we consider sums of number games such as $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\square\square$, we create a line of equivalence classes which form a bijection with the x-axis: $\mathbb{R}$. For instance, $3(\begin{smallmatrix}\square\\\square\end{smallmatrix}) + 2(\square\square) = (\mathbf{3}, \mathbf{2})$ in these coordinates, which belongs to the equivalence class with x-intercept 1, so $(\mathbf{3}, \mathbf{2}) = (\mathbf{2}, \mathbf{1}) = (\mathbf{1}, \mathbf{0})$.

The equivalence classes are all slope 1 lines. We see below a coordinate in the first quadrant being related to a number on the real line. Note that we can relate any number on the x-axis to a position in the first quadrant. For a number $(\mathbf{x}, \mathbf{y})$, if $x = y$, then $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$; if $x > y$, then $(\mathbf{x}, \mathbf{y}) \in \mathcal{L}$; and if $x < y$, then $(\mathbf{x}, \mathbf{y}) \in \mathcal{R}$.



This provides the intuition for our approach to three-player number games.

In defining this, perhaps if we could quantify the amount a game benefits a specific player, we could find three-dimensional coordinates in the first quadrant for a game. We will build these number games exclusively on sums of tiles, i.e. $a(\lhd) + b(\lhd\!\rhd) + c(\rhd)$. We define an (x, y, z)-coordinate value by first assigning the value $(\mathbf{1}, \mathbf{0}, \mathbf{0})$ to the smallest LLL game, $\lhd + \rhd$. Similarly, we will assign $(\mathbf{0}, \mathbf{1}, \mathbf{0})$ to $\lhd + \lhd\!\rhd$ and $(\mathbf{0}, \mathbf{0}, \mathbf{1})$ to $\lhd\!\rhd + \rhd$.

Example: A number $(\mathbf{1}, \mathbf{2}, \mathbf{3})$ would be: $1(\lhd + \rhd) + 2(\lhd + \lhd\!\rhd) + 3(\lhd\!\rhd + \rhd) = 3(\lhd) + 5(\lhd\!\rhd) + 4(\rhd) = 3_L + 5_C + 4_R \in RRR$.

A quick look at some basic sums gives us:

$$(\mathbf{0}, \mathbf{0}, \mathbf{0}) \in RLC \qquad (\mathbf{1}, \mathbf{1}, \mathbf{1}) \in RLC$$

$$(\mathbf{1}, \mathbf{0}, \mathbf{0}) + (\mathbf{1}, \mathbf{0}, \mathbf{0}) = (\mathbf{2}, \mathbf{0}, \mathbf{0}) \in LLL \quad (\mathbf{1}, \mathbf{0}, \mathbf{0}) + (\mathbf{0}, \mathbf{1}, \mathbf{0}) = (\mathbf{1}, \mathbf{1}, \mathbf{0}) \in LLC$$

$$(\mathbf{0}, \mathbf{1}, \mathbf{0}) + (\mathbf{0}, \mathbf{1}, \mathbf{0}) = (\mathbf{0}, \mathbf{2}, \mathbf{0}) \in CCC \quad (\mathbf{0}, \mathbf{1}, \mathbf{0}) + (\mathbf{0}, \mathbf{0}, \mathbf{1}) = (\mathbf{0}, \mathbf{1}, \mathbf{1}) \in RCC$$

$$(\mathbf{0}, \mathbf{0}, \mathbf{1}) + (\mathbf{0}, \mathbf{0}, \mathbf{1}) = (\mathbf{0}, \mathbf{0}, \mathbf{2}) \in RRR \quad (\mathbf{1}, \mathbf{0}, \mathbf{0}) + (\mathbf{0}, \mathbf{0}, \mathbf{1}) = (\mathbf{1}, \mathbf{0}, \mathbf{1}) \in RLR$$

The outcome of these games depends solely on who runs out of tiles first, as $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x + y)\lhd + (y + z)\lhd\!\rhd + (x + z)\rhd$. Once a player runs out of tiles, the player before them wins.

Conjecture 4. *Number games form an equivalence class where* $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ *if and only if* $x - u = y - v = z - w$.

This seems intuitive enough, but since we have not been able to prove that $1_L + 1_C + 1_R = 0$, we cannot show this in general yet.

How many outcomes can we produce with numbers and what can we generalize about them? We observe the following properties:

1. If $x = y = z = 0$, then $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in RLC$.

This is given by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (0)\lhd + (0)\lhd\!\rhd + (0)\rhd = 0 \in RLC$.

2. If $x > y \geq 0, z = 0$, then $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in LLL$.

This is given by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x + y)^{\triangleleft} + (y)\Diamond + (x)^{\triangleright}$, so Center will run out of moves first and Left will always win.

3. If $x = y > 0, z = 0$, then $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in LLC$.

This is given by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2x)^{\triangleleft} + (x)\Diamond + (x)^{\triangleright}$, which means Center will run out of moves first unless Right moves first.

4. If $y > x > 0, z = 0$, then $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in CCC$.

This is given by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x + y)^{\triangleleft} + (y)\Diamond + (x)^{\triangleright}$, so Right will run out of moves first and Center will always win.

Properties 1-4 are sufficient to classify the outcomes of all number games for two reasons:

Firstly, if $a = min\{x, y, z\}$, then $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ belongs to the same outcome class as $(\mathbf{x} - \mathbf{a}, \mathbf{y} - \mathbf{a}, \mathbf{z} - \mathbf{a})$. This is because subtracting a from all three coordinates is the same as removing $2a$ rounds of turns, which do not affect the outcome of the game. Thus the cases where one of x, y, z equal 0 are enough to determine the outcome.

Secondly, rotations permute our numbers, so since $(\mathbf{1}, \mathbf{0}, \mathbf{0})^{60} = (\mathbf{0}, \mathbf{1}, \mathbf{0})$, we can similarly argue that $(\mathbf{x}, \mathbf{y}, \mathbf{z})^{60} = (\mathbf{z}, \mathbf{x}, \mathbf{y})$. Thus the outcomes in all other cases can be found by rotation.

We are saying that given any number game $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, we can reduce at least one of the coordinates to zero without changing the outcome and then rotate to make $z = 0$, which has a well-defined effect on the outcome, and then the resulting game will follow one of the four given properties.

Thus by reduction and rotation, we can only produce the outcomes RLC, LLL, CCC, RRR, LLC, RCC, and RLR through these number games.

Now that we have number games and understand them fairly well, the questions are how do we generalize them and what do they mean? We can see definite benefits to each player given different sums, so it would be interesting to better understand how we might generalize numbers in order to apply them to more complicated game positions.

This thesis provides the foundation for this research by solidifying game rules and definitions in Rhombination and other three-player partizan games. We have made some discoveries such as the effect of rotation on outcome and created an abelian monoid with an identity we wish to generalize. The most important next step would be to prove a generalized identity for game addition along with finding inverses to games to give us an abelian group. Another important idea to look for is the existence or nonexistence of positions in the outcome classes LRC, CLR, and RCL.

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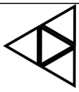
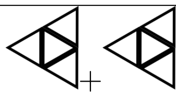
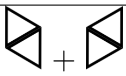
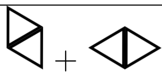
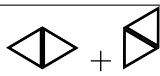
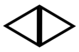
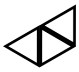
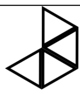
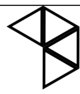
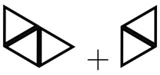
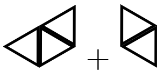
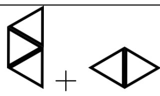
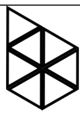
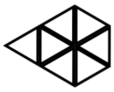
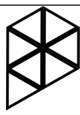
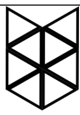
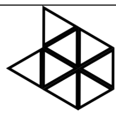
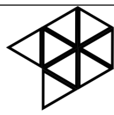
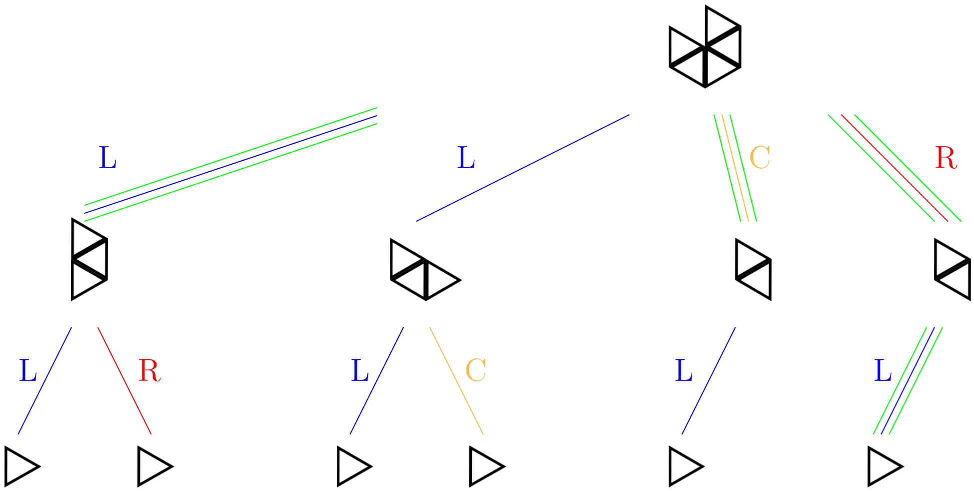
Appendix A: Rhombination Positions for the Outcome Classes

 ∈ LCR	 ∈ CRL	 ∈ RLC
 ∈ LLL	 ∈ CCC	 ∈ RRR
 ∈ LLC	 ∈ RCC	 ∈ RLR
 ∈ LLR	 ∈ LCC	 ∈ RCR
 ∈ LCL	 ∈ CCR	 ∈ LRR
 ∈ LRL	 ∈ CCL	 ∈ CRR
 ∈ CLL	 ∈ CRC	 ∈ RRL
 ∈ RLL	 ∈ CLC	 ∈ RRC
? ∈ LRC	? ∈ CLR	? ∈ RCL

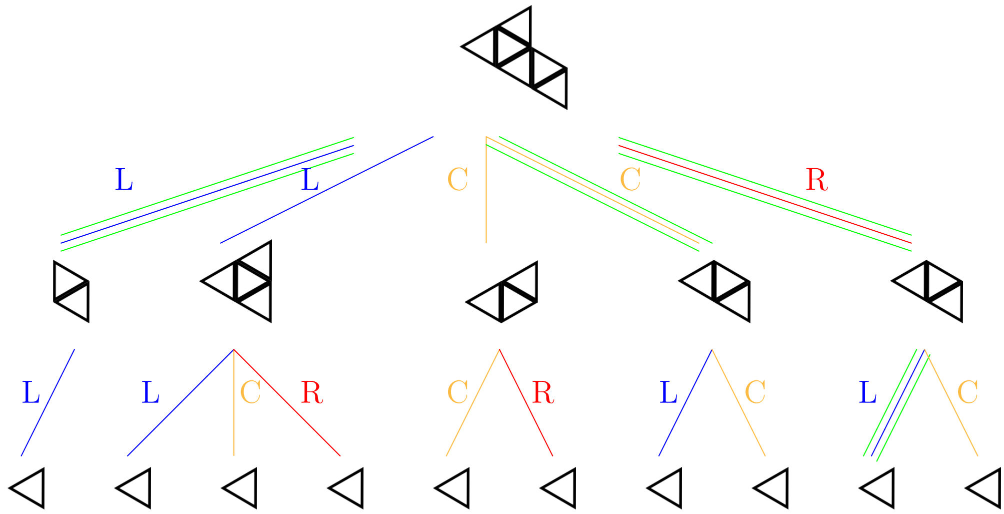
These game examples have minimal birthdays except for RLC, RLL, CLC, and RRC. We do not have minimal examples for RLL, CLC, and RRC, but the minimal RLC game is 0.

Appendix B: Sample Game Trees for Rhombination

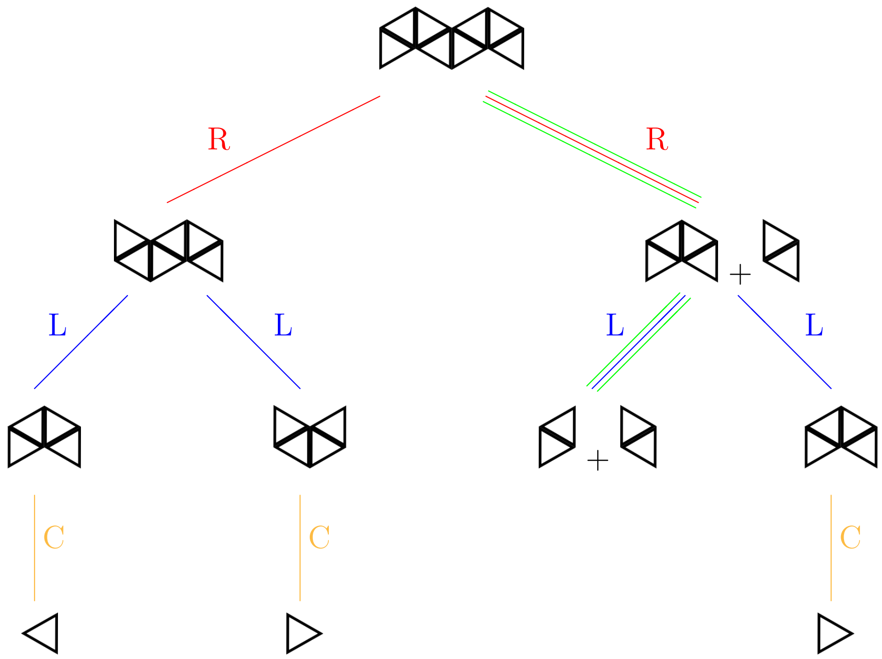
We will highlight the optimal game sequences in green.



This game is in LCL.



This game is also in LCL.



This is a partial game tree in RLL where we look at the sequence with Right moving first.

Curriculum Vitae : Katie Greene

Education:

Wake Forest University

Master of Arts in Mathematics

May 2017

Brigham Young University

Bachelor of Science in Mathematics

December 2014

Professional Experience:

Site Director at AoPS Academy (March 2017-Present)

Graduate Teaching Assistant at Wake Forest (August 2015-May 2017)

Grader for Measurement Incorporated (February 2015-July 2015)

Discrete Math Grader for Brigham Young University (January 2013-April 2014)

Math Circles Assistant for Brigham Young University (September 2013- April 2014)

PALS Math Teacher for Timpanogos Elementary (December 2013 - April 2014)

MathCounts Coach for Lakeridge Junior High (December 2013 - March 2014)

Math Tutor for Forsyth Tech (February 2011 - December 2011)

Performance Experience

Independent Performer with the Tricky Trio (February 2015 - May 2016)

Actress/Choreographer for Showtime Performers (July 1998 - August 2015)

Acrobat for the Carolina Renaissance Festival (October 2006 - November 2009)