TENSOR CATEGORIES, $\mathbb{Z}^+$-RING, AND GROTHENDIECK RINGS

BY

WOO HYUNG LEE

A Thesis Submitted to the Graduate Faculty of

WAKE FOREST UNIVERSITY GRADUATE SCHOOL OF ARTS AND SCIENCES

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF ARTS

Mathematics & Statistics

May, 2018

Winston-Salem, North Carolina

Approved By:

Frank Moore, Ph.D., Advisor

Ellen Kirkman, Ph.D., Chair

Hugh Howards, Ph.D.

Acknowledgements

The goal of this expository thesis is to provide a detailed explanation on the role of Grothendieck Rings in understanding tensor categories, and is heavily based on the monograph "Tensor Category" by Etingof et al.

The work would not have been possible without the guidance and supervision of my advisor William F. Moore, Ellen Kirkman, and Hugh Howards. I have my utmost gratitude for the help received in writing this thesis, and also, many thanks to my fellow graduate students for supporting me in this endeavor.

Table of Contents

Abstract

Woo Hyung Lee

In this paper, we will define and prove various properties of abelian categories, monoidal categories and tensor categories. Then, we will show how to define a ring that represents a tensor category, which will be called the Grothendieck Ring, and prove that it constitutes a very specific type of ring called a $\mathbb{Z}^+$-ring.

Chapter 1: Preliminary Materials

1.1 Abelian Categories and Exact Sequences

1.1.1 Additive Categories

What is a category? Informally one can think of a category simply as an algebraic structure comprising of objects with arrows in between them. At first, it may seem that categories are redundant as we already have well-defined notions of sets and functions in between them. However, unlike sets that come with the inherent problem regarding the set of all sets, categories deal with arbitrary objects that may be larger than sets. In fact, when the objects of a category and the set of maps between them are precisely sets, we denote such category to be small, and large if otherwise.

Now that we have a basic notion of what categories are, let us define them formally.

Definition 1.1.1. *A* ***category*** $\mathcal{C}$ consists of the following:

- A class $Ob(\mathcal{C})$ of objects
- A class $Hom(\mathcal{C})$ of morphisms between objects
- For any three objects A, B, C in $\mathcal{C}$, there exists a binary operation

$$\circ : Hom(A, B) \times Hom(B, C) \to Hom(A, C),$$

  denoted the composition of morphisms.

and satisfies the following axioms:

- For every object A in $\mathcal{C}$, there exists a morphism $id_A : A \to A$, denoted the identity morphism of A, and for any object X, Y in $\mathcal{C}$, and for any morphism $f : X \to A$, $g : A \to Y$, $id_A \circ f = f$ and $g \circ id_A = g$.

- For any four objects A, B, C, and D in $\mathcal{C}$, if $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Some elementary, but very important examples of categories are the categories, *Ab*, *Set*. *Ab* is the category of abelian groups, where its objects are abelian groups, and the morphisms are group homomorphisms. Recall that any abelian group can be thought of as a module over the ring of integers, and any $\mathbb{Z}$-module is an abelian group with respect to its additive operation. Hence it can be useful to think of *Ab* as the category of modules over $\mathbb{Z}$.

Similarly, the category *Set*, is a category with sets as objects and morphisms as functions between sets. From the definition one can easily see that if we forget the abelian group structure in the category *Ab*, any object of *Ab* becomes a set and any group homomorphism becomes a function between sets. What has been described here is an example of what is called a monoidal functor, and this particular functor is called the forgetful functor, as we are "forgetting" some structure on our objects and morphisms.

There is an object of particular importance when exploring categories, called the zero object. A zero object 0 in a category $\mathcal{C}$ is an object that is both an initial object and a terminal object. An initial object of $\mathcal{C}$ is an object I such that for every object $A \in \mathcal{C}$, there exists a unique morphism $i : I \to A$. Similarly, a terminal object of $\mathcal{C}$ is an object T such that for every object $A \in \mathcal{C}$, there exists a unique morphism $t : A \to T$. To give you an example of a category where there are no zero objects, consider **Set**, the category of sets. The morphisms in this category are precisely functions between sets, and we can see that the only initial object of **Set** is the empty set, whereas every singleton set is a terminal object.

Now, let us think about the category of vector spaces over a field $\mathcal{K}$. In this category, the objects are vector spaces, and the morphisms are vector space homomorphisms. Let V and W be two such vector spaces, and let $\phi, \psi : V \to$ be two vector space homomorphisms between V and W. Then, $\phi + \psi$ is another vector space homomorphism between V and W, and it is easy to see that the set of all vector space homomorphisms between V and W is an abelian group under addition of homomorphisms. Hence, we can say that $Hom(V, W)$ is an object in Ab, and this is precisely what it means to be an enriched category over Ab.

Definition 1.1.2. *Let $\mathcal{D}$ be a category. We say that a category $\mathcal{C}$ is enriched over $\mathcal{D}$ if for each pair of objects X, Y in $ob(\mathcal{C})$, $Hom(X, Y) \in \mathcal{D}$.*

For this paper, we will primarily be focusing on Ab-enriched categories. But now, let us formally explore the notion of functors that was briefly mentioned above.

Definition 1.1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is a map with the following properties:*

- *For each object $X \in \mathcal{C}$, there exists an object $Y \in \mathcal{D}$ such that $F(X) = Y$.*
- *For every morphism $f : X \to Y$ in $Hom_{\mathcal{C}}(X, Y)$, there exists a morphism $g : F(X) \to F(Y)$ in $Hom_{\mathcal{D}}(F(X), F(Y))$, such that $F(f) = g$ with the following properties:*
  - *(a) For every object $X \in \mathcal{C}$, $F(id_X) = id_{F(X)}$.*
  - *(b) For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $Hom(\mathcal{C})$, $F(g \circ f) = F(g) \circ F(f)$.*

The above definition of functors technically defines what are called **covariant functors**. We also have **contravariant functors** which are of similar construction with just the arrows reversed. For example, if $\mathcal{C}$ is a locally small category (i.e. $Hom_{\mathcal{C}}(X, Y)$ are sets), then $Hom(A, -) : \mathcal{C} \to$**Set** is a covariant functor, whereas $Hom(-, A) : \mathcal{C} \to$**Set** is a contravariant functor.

We have thus far explored the basic notions of categories and functors, so let us go ahead and define additive categories.

Definition 1.1.4. *An **additive category** $\mathcal{C}$* is a category that satisfies the following:

- Every set $Hom_{\mathcal{C}}(X, Y)$ has the structure of an abelian group such that the composition of morphisms is biadditive with respect to this structure. (Ab-enriched) Note that $Hom(-,-)$ is actually a set here, as it must be an object in Ab.

- There exists a zero object 0 in $\mathcal{C}$.

- For any two objects $X, Y \in \mathcal{C}$, there exists an object $Z \in \mathcal{C}$ and morphisms $i_X : X \to Z$, $i_Y : Y \to Z$, $p_X : Z \to X$, and $p_Y : Z \to Y$, such that $p_X \circ i_X = id_X$, $p_Y \circ i_Y = id_Y$, and $(i_X \circ p_X) + (i_Y \circ p_Y) = id_Z$. (Direct Sums)

A small Ab-enriched category is called a ringoid, which is a set with two binary operators $+$ and $\times$, such that $a(b+c) = ab+ac$, and $(b+c)a = ba+ca$. Hence the set $Hom(\mathcal{C})$ in a small additive category $\mathcal{C}$, with respect to its abelian group structures, if we take function compositions to be the multiplicative binary operator, becomes a ringoid. It is also easy to see that an Ab-enriched category $\mathcal{R}$ with precisely one object is a ring.

Lastly, notice that the third property of an additive category precisely refers to the direct sum. Since there must exist an object Z as defined above in $\mathcal{C}$ for every two objects $X, Y \in \mathcal{C}$, we have that any additive category $\mathcal{C}$ must be equipped with a bifunctor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and we denote $Z = X \oplus Y$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between two additive categories $\mathcal{C}$ and $\mathcal{D}$ is called additive if the maps

$$Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X),F(Y)), \quad X,Y \in \mathcal{C}$$

are homomorphisms of abelian groups.

Hence for any two morphisms $f,g : X \to Y$, $F(f+g) = F(f) + F(g)$. Now, from this definition and the definition of direct sums, we can easily obtain the following proposition.

Proposition 1.1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. Then, for any additive functor $F : \mathcal{C} \to \mathcal{D}$, there exists a natural isomorphism $F(X) \oplus F(Y) \simeq F(X \oplus Y)$.*

We are almost ready to define abelian categories, but before that, we must define the kernel and cokernel of a morphism in an additive category.

Definition 1.1.6. *Let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then, the kernel of f, denoted $Ker(f)$ is an object K with a morphism $k : K \to X$ such that $fk = 0$, and if $k' : K' \to X$ such that $fk' = 0$ exists, then there exists a unique morphism $l : K' \to K$ such that $kl = k'$.*

Similarly, the cokernel of f, denoted $Coker(f)$ is and object C with a morphism $c : Y \to C$ such that $cf = 0$, and if $c' : Y \to C'$ such that $c'f = 0$ exists, then there exists a unique morphism $r : C \to C'$ such that $rc = c'$.

If $Ker(f)$ and $Coker(f)$ exists, they are unique up to a unique isomorphism.

Now we are ready to define an abelian category.

1.1.2 Abelian Categories

Definition 1.1.7. *An **abelian category** is an additive category $\mathcal{C}$ such that for every morphism $\phi : X \to Y$, there exists a sequence*

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$$

where:

(a) $j \circ i = \phi$.

(b) $(K, k) = Ker(\phi)$, and $(C, c) = Coker(\phi)$.

(c) $(I, i) = Coker(k)$, and $(I, j) = Ker(c)$.

To illustrate the sequence in the above more concretely, let V and W be vector spaces over a field $\mathcal{K}$, and let $\phi : V \to W$ be a linear transformation. Then, the above sequence becomes,

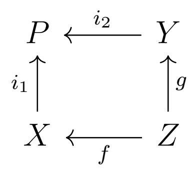
$$ker(\phi) \overset{i_{ker(\phi)}}{\to} V \xrightarrow{\overline{\phi}} im(\phi) \overset{i_{im(\phi)}}{\to} W \xrightarrow{c} W/im(\phi)$$

where $i_{ker(\phi)}$ is the embedding of $ker(\phi)$ in V, $\overline{\phi}$ is the a surjective linear transformation from V to $im(\phi)$ induced by ϕ, $i_{im(\phi)}$ is the embedding of $im(\phi)$ in W, and and c is a surjective map taking $w \in W$ to $w + im(\phi)$ (called a quotient map).

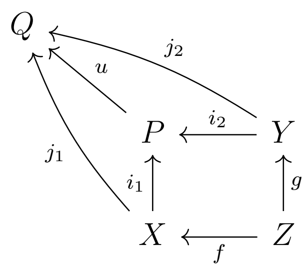
A more traditional way to define an abelian category is to define it as an additive category that has all kernels and cokernels, where all monomorphisms (injections) and epimorphisms (surjections) are normal, i.e. every monomorphism is a kernel of some morphism, and every epimorphism is a cokernel of some morphism. Notice that Definition 6 encodes this information via the illustrated sequence, which is called the **canonical decomposition** of a morphism.

Another important concept in abelian categories is the notion of pushouts and pullbacks.

Definition 1.1.8. *Let $\mathcal{C}$ be an abelian category, and let X, Y, Z be objects in $\mathcal{C}$. Let $f : Z \to X$ and $g : Z \to Y$ be two morphisms. Then, the pushout of f and g is an object P and two morphisms $i_1 : X \to P$, and $i_2 : Y \to P$ such that the following diagram*



commutes and for any other triple (Q, j_1, j_2) satisfying a similar diagram, then there exists a unique $u : P \to Q$ such that the following diagram:



commutes. If such a (P, i_1, i_2) exists, it is unique up to a unique isomorphism.

A pullback (P, i_1, i_2) is similarly defined by reversing the arrows in the above diagrams.

The following theorem gives us a more intuitive way of thinking about abelian categories.

Theorem 1. *(Mitchell) Every abelian category is equivalent, as an additive category, to a full subcategory of left modules over an associative unital ring A.*

A full subcategory of a category $\mathcal{C}$ is defined as:

Definition 1.1.9. *Let $\mathcal{C}$ be a category. Then, a full subcategory $\mathcal{S}$ of $\mathcal{C}$ is given by*

(a) A subcollection of $Ob(\mathcal{C})$, denoted $Ob(\mathcal{S})$.

(b) A subcollection of $Hom(\mathcal{C})$, denoted $Hom(\mathcal{S})$.

(c) For every $X \in \mathcal{S}$, $id_X \in Hom(\mathcal{S})$.

(d) For every $f : X \to Y$ in $Hom(\mathcal{S})$, X, Y are in $Ob(\mathcal{S})$.

(e) For every pair of morphisms $f, g \in Hom(\mathcal{S})$, if $f \circ g$ is defined, then $f \circ g \in Hom(\mathcal{S})$.

(f) For every pair of objects $X, Y \in \mathcal{S}$, $Hom_{\mathcal{S}}(X, Y) = Hom_{\mathcal{C}}(X, Y)$.

1.1.3 Exact Sequences and the Grothendieck Group

Definition 1.1.10. *Consider a sequence of morphisms, S, in an abelian category:*

$$\cdots \to X_{i-1} \stackrel{f_{i-1}}{\to} X_i \stackrel{f_i}{\to} X_{i+1} \to \cdots$$

- *We say S is exact in degree i if $Im(f_{i-1}) = Ker(f_i)$.*
- *We say S is exact if it is exact in every degree.*
- *An exact sequence of the form $0 \to X \stackrel{f}{\to} Y \stackrel{g}{\to} Z \to 0$ is called a short exact sequence.*

In the above short exact sequence, notice that f must be an injective morphism, and g must be a surjective morphism. Hence it is helpful to think of X as a subobject of Y, f as an embedding of X into Y, and similarly Z as the corresponding quotient object Y/X, i.e. $Coker(f)$.

We've looked at short exact sequences, and as one might expect, there are long exact sequences, defined as an exact sequence of more than 3 (often infinite) non-zero terms. A natural question that arises then is if we have a short exact sequence, how do we construct a long exact sequence? In order to answer this question, we will begin by defining Ext.

Let R be a ring, and let M, N be left R-modules. Then, a projective resolution of M is an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is a projective R-module. Applying the left-exact contravariant functor $Hom(-, N)$, we get the complex

$$0 \to Hom(P_0, N) \to Hom(P_1, N) \to \cdots .$$

Let d_i be the map $Hom(P_{i-1}, N) \to Hom(P_i, N)$ in the above complex, and let $H^i = Ker(d_{i+1}/Im(d_i)$ be the i^{th} cohomology. Then, we define $Ext^i(M, N) = H^i$.

Proposition 1.1.11. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Then, there is a long exact sequence given by*

$$0 \to Hom(C, N) \to Hom(B, N) \to Hom(A, N) \to$$

$$Ext^1(C, N) \to Ext^1(B, N) \to Ext^1(A, N) \to Ext^2(C, N) \to \cdots .$$

It is easy too see that $Ext^0(M, N) = Hom(M, N)$. Since $Hom(-, N)$ is a left exact functor, if we take the projective resolution of M, then $0 \to Hom(M, N) \xrightarrow{\alpha} Hom(P_0, N) \xrightarrow{\beta} Hom(P_1, N)$ is exact. This implies that $Ker(\beta) = Ext^0(M, N)$, is equal to $Im(\alpha)$, but α is injective. Hence we have that $Im(\alpha) \simeq Hom(M, N)$, and we have the claim.

Also, another interesting fact is that $Ext^i(M, N) = 0$ for every $i > 0$ if M is projective. This is because, if M is projective, M admits a finite projective resolution of the form $0 \to M \xrightarrow{id_M} M \to 0$, and we see that applying $Hom(-, N)$ to the resolution yields $H^i = 0$ for all $i > 0$. There is a similar result for injective modules as well.

Definition 1.1.12. *Let $S : 0 \to X \to Z \to Y \to 0$, and $S' : 0 \to X \to Z' \to Y \to 0$ be short exact sequences. The sequence S is called an extension of Y by X. A morphism f from S to S' is a morphism $f : Z \to Z'$ that restricts to id_X, and induces id_Y. The set of exact sequences $0 \to X \to Z \to Y \to 0$ is denoted $Ext(Y, X)$, and is called the set of extensions of Y by X.*

Proposition 1.1.13. *There exists a $1 - 1$ correspondence between the isomorphism classes of extensions of X by Y and $Ext^1(X, Y)$.*

Proof. Let $0 \to N \xrightarrow{\alpha} P \to X \to 0$ be a short exact sequence such that P is projective. By applying the functor $Hom(-, Y)$ and truncating, we get the exact sequence

$$Hom(P, Y) \to Hom(N, Y) \xrightarrow{\delta} Ext^1(X, Y) \to Ext^1(P, Y).$$

Since P is projective, we have that $Ext^1(P, Y) = 0$, and hence we obtain the exact sequence

$$Hom(P, Y) \to Hom(N, Y) \xrightarrow{\delta} Ext^1(X, Y) \to 0.$$

Because the sequence is exact, notice that δ must be surjective. Now, let $\sigma \in Ext^1(X, Y)$. Then, as δ is surjective, there must be a $\beta \in Hom(N, Y)$ such that $\delta(\beta) = \sigma$.

Take Z to be the pushout of α and β. In an abelian category, all pushouts exist, and to be more precise, Z is the cokernel of the map $N \to P \oplus Y$, defined by $n \mapsto \alpha(n) - \beta(n)$.

Hence we get the commuting diagram below, where the map $Z \to X$ is induced by the map $P \to X$, and the 0-map $Y \to X$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id_X} & & \\ 0 & \longrightarrow & Y & \xrightarrow{j} & Z & \xrightarrow{\epsilon} & X & \longrightarrow & 0 \end{array} \qquad (S)$$

Specifically, we will define the induced map $\epsilon : Z \to X$ by choosing an element $p + y \in P \oplus Y$ that maps to $z \in Z$, and setting $\epsilon(z) = \pi(p)$. If we have $p + y$ and $p' + y'$ that both map to z, then $p - p' + y - y'$ is in the image of the map $N \to P \oplus Y$, and this implies that $\pi(p) = \pi(p')$, and hence ϵ is well-defined. If z is in $Ker(\epsilon)$, then $z = p + y$ where $\pi(p) = 0_X$. Hence $p = \alpha(n)$ for some $n \in N$, and by how we defined Z, $p + y = p + y - (\alpha(n) - \beta(n)) = y + \beta(n) = y'$, and y' is clearly an element of $Im(j)$. Hence $Ker(\epsilon)$ is contained in $Im(j)$. On the other hand, because the map $Y \to X$ is the 0-map, $Im(j)$ must be in the kernel of the map $Z \to X$. Hence we have that S is exact at Z. Since Z is the cokernel of the map $N \to P \oplus Y$, $Ker(j)$ must be trivial, and hence S is exact at Y. Lastly, the map $P \to X$ is a surjection, and as the diagram commutes, the map $Z \to X$ must also be a surjection. Hence S is exact at X, and we have that S is a short exact sequence.

Now, let ϕ be a map from the isomorphism classes of extensions of X by Y to $Ext^1(X, Y)$. By construction of S, we see that for any $\sigma \in Ext^1(X, Y)$ we can find a short exact sequence S, so we see that ϕ is onto.

Given an arbitrary exact sequence $0 \to Y \to Z \to X \to 0$ and an exact sequence $0 \to N \to P \to X \to 0$ where P is projective, we can define a map from $P \to Z$ by projectivity of P, and then define a corresponding map $N \to Y$ using the exactness property. Hence this shows that for any short exact sequence we can find a $\beta \in Hom(N, Y)$.

Finally, suppose that there exist $\beta \neq \beta'$ such that $\delta(\beta) = \delta(\beta') = \sigma \in Ext^1(X, Y)$. Then, since $\delta(\beta') - \delta(\beta) = \delta(\beta' - \beta) \in Ker(\delta)$, we have that there exists an $f \in Hom(P, Y)$, such that $f \circ \alpha = \beta' - \beta$. Consider the following diagrams.

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & P \\ \big\downarrow{\scriptstyle \beta + f\alpha} & & \big\downarrow{\scriptstyle i + jf} \\ Y & \xrightarrow{j} & Z \end{array}$$

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & P \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle i' - j'f} \\ Y & \xrightarrow{j'} & Z' \end{array}$$

The first diagram is obtained by adding $f\alpha$ to β and jf to i in the pushout diagram of Z, and the second is obtained by subtracting $f\alpha$ from β' and $j'f$ from i' in the pushout diagram of Z'. Applying the universal property of pushouts to these diagrams yields that $Z' \to Z \to Z'$ is $id_{Z'}$, and similar construction yields that $Z \to Z' \to Z$ is id_Z. Hence we have that $Z \simeq Z'$, and combining all of the previous parts, we obtain the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j'} & Z' & \xrightarrow{\epsilon'} & X & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\simeq} & & \downarrow{=} & & \\ 0 & \longrightarrow & Y & \xrightarrow{j} & Z & \xrightarrow{\epsilon} & X & \longrightarrow & 0 \end{array}$$

One can easily check that the upper left box commutes, and applying the universal property of pushouts, and the right box commutes since for any $x \in X$, we can find a $p \in P$ such that $i\epsilon(p) = i'\epsilon'(p) = x$, and the isomorphism we constructed precisely sends $\epsilon(p)$ to $\epsilon'(p)$.

□

There is an important example to look at for the purpose of this paper. Let G be a group and A be an abelian group with a G-action. Then, A is a G module, and we have the following definition.

Definition 1.1.14. *A* ***cohomology group*** of G with coefficients in A is defined as the group $Ext^i_G(\mathbb{Z}, A)$ in the category of G-modules where G acts on $\mathbb{Z}$ trivially, and is denoted $H^i(G, A)$.

Definition 1.1.15. *A bar resolution of $\mathbb{Z}$ in the category of G-modules is defined by*

$$\cdots \to P_i \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to \mathbb{Z} \to 0$$

with the following properties:

- $P_i := \mathbb{Z}G^{i+1}$ *where the group action is given by* $g(g_0, g_1, \cdots, g_i) = (gg_0, g_1, \cdots, g_i)$.

- *$\partial_i : P_i \to P_{i-1}$ is defined by*

$$\partial_i(g_0, \cdots, g_i) = (-1)^0(g_0g_1, g_2, \cdots, g_i)+\cdots+(-1)^{i-1}(g_0, g_1, \cdots, g_{i-1}g_i)+(-1)^i(g_0, \cdots, g_{i-1}).$$

If we apply the contravariant functor $Hom_G(-, A)$ to the bar resolution, the induced map $d_i : Hom_G(P_{i-1}, A) \to Hom_G(P_i, A)$ is given by

$$d_i(f)(g_1, \cdots, g_i) = (-1)^0 g_1 f(g_2, \cdots, g_i)+(-1)^1 f(g_1g_2, \cdots, g_i)+\cdots+(-1)^{i-1} f(g_1, \cdots, g_{i-1}g_i)+(-1)^i f(g_1, \cdots, g_{i-1}).$$

Definition 1.1.16. *For the above complex $0 \to Hom_G(P_0, A) \stackrel{d_1}{\to} Hom_G(P_1, A) \stackrel{d_2}{\to} \cdots$, the i^{th}-cocycle and the i^{th}-coboundary are given by $Ker(d_{i+1})$ and $Im(d_i)$, and are denoted $Z^i(G, A)$ and $B^i(G, A)$.*

By Definition 11, we now have an alternate definition of $H^i(G, A)$, given by

$$H^i(G, A) := Z^i(G, A)/B^i(G, A).$$

In this section, we will conclude with an overview of (local) finiteness of abelian categories.

Definition 1.1.17. *Let $\mathcal{K}$ be a field. Then, a $\mathcal{K}$-linear abelian category $\mathcal{C}$ is locally finite if*

(a) For every $X, Y \in \mathcal{C}$, $Hom_{\mathcal{C}}(X, Y)$ is finite dimensional.

(b) Every object X in $\mathcal{C}$ has finite length, i.e. there exists a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that each X_i/X_{i-1} has no non-trivial proper sub-objects (simple).

Here, we also define injectivity and surjectivity for additive functors.

Definition 1.1.18. *Let $\mathcal{C}$, $\mathcal{D}$ be locally finite abelian categories. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is injective if F is bijective on the sets of morphisms (often denoted fully faithful), and is surjective if every simple object of $\mathcal{D}$ is a subquotient of $F(X)$ for some $X \in \mathcal{C}$.*

In a locally finite abelian category, we have an important corollary given by Schur's Lemma.

Lemma 1.1.19. *(Schur's Lemma)*

Let $\mathcal{C}$ be an abelian category, and X, Y be simple. Then, any nonzero morphism $f : X \to Y$ is an isomorphism. If X is not isomorphic to Y, then $Hom_{\mathcal{C}}(X, Y) = 0$, and $Hom_{\mathcal{C}}(X, X)$ is a division algebra.

Corollary 1.1.20. *Suppose $\mathcal{K}$ is a algebraically closed field, and let $\mathcal{C}$ be a locally finite abelian category over $\mathcal{K}$. Then, for any simple $X \in \mathcal{C}$, $Hom_{\mathcal{C}}(X, X) = \mathcal{K}$.*

Proof. By Schur's Lemma, we have that $Hom_{\mathcal{C}}(X, X)$ is a division algebra over $\mathcal{K}$. Since $\mathcal{C}$ is locally finite, $H = Hom_{\mathcal{C}}(X, X)$ is finite dimensional. Let $h \in H$. Then, as H is finite dimensional, there must be a polynomial $f(h)$ over $\mathcal{K}$, such that $f(h) = 0$. Suppose that $f(h)$ is of the minimal degree of all such polynomials. Because $\mathcal{K}$ is algebraically closed, by the fundamental theorem of algebra, $f(h)$ has a root in $\mathcal{K}$, and in particular, $f(h) = (h - k)g(h)$ for some $k \in \mathcal{K}$, and $g(h)$. However, we supposed that f is of minimal degree, and hence $g(h) \neq 0$, and hence $h - k = 0$, which implies $h = k$. Hence $h \in \mathcal{K}$, and it must be that $Hom_{\mathcal{C}}(X, X) = \mathcal{K}$.

□

Definition 1.1.21. *A $\mathcal{K}$-linear abelian category $\mathcal{C}$ is finite if*

(a) $\mathcal{C}$ has finite dimensional spaces of morphisms.

(b) Every object in $\mathcal{C}$ has finite length.

(c) $\mathcal{C}$ *has enough projectives, meaning every simple object of* $\mathcal{C}$ *has a projective cover, where a projective cover of* X *is defined as an object* $P(X)$ *such that* $Hom_{\mathcal{C}}(P(X), -)$ *is exact, and an epimorphism* $p : P(X) \to X$ *suc h that if* $g : P \to X$ *is another epimorphism from a projective object* P *to* X*, then there exists an epimorphism* $h : P \to P(X)$ *with* $ph = g$*.*

(d) There are finitely many isomorphism classes of simple objects.

An alternate definition of a $\mathcal{K}$-linear abelian category given in "Tensor Category" is a category $\mathcal{C}$ equivalent to the category A-mod of finite dimensional modules over a finite dimensional $\mathcal{K}$-algebra A.

Definition 1.1.22. *A* $\mathcal{K}$*-linear algebra* A *is given by*

(a) $(x + y) \cdot z = xz + yz$

(b) $x \cdot (y + z) = xy + xz$

(c) $(ax) \cdot (by) = (ab)(xy)$

fore every $a, b \in \mathcal{K}$ *and* $x, y, z \in A$*.*

Proposition 1.1.23. *A projective generator* P *of* $\mathcal{C}$ *represents a functor* $F : \mathcal{C} \to Vec$*, from* $\mathcal{C}$ *to the category of finite dimensional* $\mathcal{K}$*-vector spaces by* $F(X) = Hom_{\mathcal{C}}(P, X)$*. Since* P *is projective,* $Hom_C(P, -)$ *is an exact functor, and hence* F *is exact and faithful.*

It is well-known that an exact faithful functor $F : \mathcal{C} \to Vec$ is represented by a unique projective generator P up to a unique isomorphism.

The dual category of A-mod is A^{op}-mod, where the arrows are reversed, and the duality functor between them is $F : M \to M^*$, where M is a finite dimensional A module, and $M^* = Hom(M, A)$.

Definition 1.1.24. *An additive $\mathcal{K}$-linear functor $F : A - mod \to B - mod$ is $(\otimes -)$ representable if there exists a (B, A)-bimodule V such that F is naturally isomorphic to $(V \otimes_A -)$.*

Proposition 1.1.25. *An additive $\mathcal{K}$-linear functor $F : A - mod \to B - mod$ is representable if and only if it is right exact.*

Proof. The backward direction is obvious as the tensor product functor is right exact. Hence we only need to show the forward direction. A is a left module of A, and the map $\psi_a : A \to A$ defined by $x \mapsto ax$ induces $F(\psi_a) : F(A) \to F(A)$. This gives $F(A)$ a A-right module structure via $F(A) \cdot a = F(\psi_a)F(A) = F(aA)$.

Now, if X is a free $A - module$, $X = \oplus A$, and hence $F(A) \otimes_A X = F(A) \otimes_A (\oplus A)$, and as tensor product is distributive over $\oplus$, it suffices to consider $F(A) \otimes A$.

But then, $F(A) \otimes A$ is clearly isomorphic to $F(A)$ because $F(A)$ is a right A module. Also, as F is an additive functor, $F(X) \simeq F(\bigoplus A) \simeq \bigoplus(F(A) \otimes A) \simeq F(A) \otimes (\bigoplus A) \simeq F(A) \otimes X$ and hence we see that $F(X) \simeq F(A) \otimes X$.

Now, let $P_1 \xrightarrow{\pi} P_0 \xrightarrow{\sigma} X \to 0$ be a free resolution of X. Then, P_1 and P_0 are free, and hence by the previous proof, $F(P_1)$ and $F(P_0)$ are identified with $F(A) \otimes P_1$ and $F(A) \otimes P_0$. Applying F to the free resolution above, by right exactness of F, we now have that $F(X)$ must be the cokernel of $F(\pi) : F(A) \otimes P_1 \to F(A) \otimes P_0$. But then, $(F(A) \otimes -)$ is a right exact functor, and applied to the projective presentation of X, we have that the cokernel of $F(\pi)$ must be $F(A) \otimes X$.

Hence we have that $F(X) \simeq F(A) \otimes X$. Therefore we conclude that $F(A)$ is precisely the (B, A)-bimodule we were looking for and F is isomorphic to $(F(A) \otimes -)$.

□

Corollary 1.1.26. *Let $\mathcal{C}$ be a finite abelian $\mathcal{K}$-linear category, and let $F : \mathcal{C} \to Vec$ be an additive $\mathcal{K}$-linear left exact functor. Then, $F = Hom_{\mathcal{C}}(V, -)$ for some $V \in \mathcal{C}$.*

Proof. Let $\mathcal{C} = A-mod$ for some finite dimensional algebra A, and let $X \in A-mod^*$. Then, if $F : \mathcal{C} \to Vec$ is an additive left exact functor, $F^* : X \mapsto F(X^*)^*$ is a right exact functor defined on $A - mod^*$. Hence by Proposition 5, $F(X^*)^* = X \otimes_A V$ for some $V \in A - mod$. Dualizing on both sides, we have that $F(X^*) = (X \otimes_A V)^* \simeq Hom_A(V, X^*)$, and we have the corollary.

□

Definition 1.1.27. *Let $\mathcal{C}$ be a $\mathcal{K}$-linear finite abelian category. Then, we define the* ***Grothendieck group*** *of $\mathcal{C}$, denoted $Gr(\mathcal{C})$, as the free abelian group generated by isomorphism classes of simple objects in $\mathcal{C}$. To every object X in $\mathcal{C}$, we associate its class $[X] \in Gr(\mathcal{C})$ by*

$$[X] = \sum_i [X : X_i] X_i,$$

where $[X : X_i]$ counts the number of copies of X_i's in the decomposition of X into direct sums of simple objects.

Definition 1.1.28. *Let $F_1, F_2 : \mathcal{C} \to Vec$ be two exact and faithful functors. Then,*

$$F_1 \otimes F_2 : \mathcal{C} \times \mathcal{C} \to Vec$$

is defined by

$$(X, Y) \mapsto F_1(X) \otimes F_2(Y).$$

Proposition 1.1.29. *There exists a canonical algebra isomorphism*

$$\alpha_{F_1,F_2} : End(F_1) \otimes End(F_2) \simeq End_{F_1 \otimes F_2}$$

defined by

$$\alpha_{F_1,F_2}(\eta_1 \otimes \eta_2)|_{F_1(X) \otimes F_2(Y)} := \eta_1|_{F_1(X)} \otimes \eta_2|_{F_2(Y)}$$

where $\eta_i \in End(F_i)$.

Proof. Consider the following diagram.

$$\begin{array}{ccc} End(F_1(X)) \times End(F_2(Y)) & \longrightarrow & End(F_1(X) \otimes F_2(Y)) \\ \downarrow & \nearrow & \\ End(F_1(X)) \otimes End(F_2(Y)) & & \end{array}$$

It is clear that the basis of $End(F_1(X)) \times End(F_2(Y))$ maps surjectively to the basis of $End(F_1(X) \otimes F_2(Y))$ via the following map:

$$\tau_1 \times \tau_2 \to \tau$$

$$(x_1 \mapsto x_2, y_1 \mapsto y_2) \mapsto (x_1 \otimes y_1 \mapsto x_2 \otimes y_2)$$

So the basis of $End(F_1(X)) \otimes End(F_2(Y))$ maps onto the basis of $End(F_1(X) \otimes F_2(Y))$. But

$$dim(End(F_1(X)) \otimes End(F_2(Y))) = (dim(F(X)))^2(dim(F_2(Y)))^2$$

$$= (dim(F_1(X))dim(F_2(Y)))^2 = dim(End(F_1(X) \otimes F_2(Y))).$$

Therefore, as $End(F_1(X)) \otimes End(F_2(Y)) \to End(F_1(X) \otimes F_2(Y))$ is linear and finite dimensional, we have that $End(F_1(X)) \otimes End(F_2(Y)) \simeq End(F_1(X) \otimes F_2(Y))$. Now,

$$\alpha_{F_1,F_2} : End(F_1) \otimes End(F_2) \to End(F_1 \otimes F_2)$$

$$\eta_1|_{F_1(X)} \otimes \eta_2|_{F_2(Y)} \mapsto \eta_1 \otimes \eta_2|_{F_1(X) \otimes F_2(Y)}$$

is the unique map satisfying the following diagram.

$$\begin{array}{ccc} End(F_1(X)) \times End(F_2(Y)) & \longrightarrow & End(F_1(X) \otimes F_2(Y)) \\ \downarrow & \overset{\alpha_{F_1,F_2}}{\nearrow} & \\ End(F_1(X)) \otimes End(F_2(Y)) & & \end{array}$$

But we know that F_1, F_2 are exact and faithful, which implies that they are uniquely represented up to isomorphism by projective generators P_1, P_2 of $\mathcal{C}$, and we have:

$$End(F_1) = End(P_1)^{op} = A_1$$

$$End(F_2) = End(P_2)^{op} = A_2$$

for some finite dimensional algebra A_1, A_2.

But now, $P_1 = \bigoplus_{i=1}^{m} P_{1,i}$, $P_2 = \bigoplus_{j=1}^{n} P_{2,j}$, where $P_{1,i}, P_{2,j}$ are projective covers of all simple objects of $\mathcal{C}$. It follows that $End(F_1) \otimes End(F_2) = End(P_1)^{op} \otimes End(P_2)^{op} = End(\bigoplus P_{1,i})^{op} \otimes End(\bigoplus P_{2,j})^{op}$ implying

$$dim(End(F_1) \otimes End(F_2)) = (\sum_i dim(P_{1,i}))(\sum_j dim(P_{2,j})).$$

Also we claim that $End(F_1 \otimes F_2) = End(P_1 \otimes P_2)^{op}$. To show this, recall that $F(X) = Hom(P, X)$.

Then, it follows from finite dimensionality of P_1, P_2 that

$$F_1(X) \otimes F_2 = Hom(P_1, X) \otimes Hom(P_2, Y)$$

$$= (P_1^* \otimes X) \otimes (P_2^* \otimes Y)$$

$$= (P_1^* \otimes P_2^*) \otimes (X \otimes Y)$$

$$= Hom(P_1 \otimes P_2, X \otimes Y)$$

and so we have $F_1 \otimes F_2((X, Y)) = Hom(P_1 \otimes P_2, X \otimes Y)$, and it follows that $End(F_1 \otimes F_2) = End(P_1 \otimes P_2)^{op}$.

From the above we derive $End(F_1 \otimes F_2) = End(P_1 \otimes P_2)^{op} = End(\bigoplus_{i,j}(P_{1,i} \otimes P_{2,j}))^{op}$ and conclude that

$$dim(End(F_1 \otimes F_2)) = \sum_{i,j}(dim(P_{1,i}))(dim(P_{2,j})).$$

Hence we have that $dim(End(F_1) \otimes End(F_2)) = dim(End(F_1 \otimes F_2))$.

But notice that α_{F_1,F_2} is given by $\sum_{X,Y} \alpha_{F_1,F_2}|_{X,Y}$ where each $\alpha_{F_1,F_2}|_{X,Y}$ is the component-wise isomorphism in the previous universal diagram. Hence α_{F_1,F_2} is a linear injection, and by finiteness, we have that α_{F_1,F_2} must be an isomorphism. $\square$

Chapter 2: Monoidal Categories

Now that we have gone over some preliminary materials about abelian categories and exact sequences, we are ready to discuss monoidal categories.

2.1 Monoidal Categories and Monoidal Functors

2.1.1 Monoidal Categories

Definition 2.1.1. *A monoidal category is a quintuple* $(\mathcal{C}, \otimes, a, 1, \iota)$ *where* $\mathcal{C}$ *is a category,* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *is the tensor product bifunctor,* $a : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ *is a natural isomorphism defined by*

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \;\; \forall X, Y, Z \in \mathcal{C}$$

called the associativity constraint, $1 \in \mathcal{C}$ *is an object, and* $\iota : 1 \otimes 1 \xrightarrow{\sim} 1$ *is an isomorphim, which satisfies the following axioms:*

(a) Pentagon Axiom

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{a_{W,X,Y} \otimes id_Z} & & \searrow^{a_{W \otimes X, Y, Z}} & \\ (W \otimes (X \otimes Y)) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow a_{W, X \otimes Y, Z} & & & & \downarrow a_{W, X, Y \otimes Z} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{id_W \otimes a_{X,Y,Z}} & & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

(b) Unit Axiom

The functors:

$$L_1 : X \to 1 \otimes X$$

$$R_1 : X \to X \otimes 1$$

are autoequivalences of $\mathcal{C}$.

Definition 2.1.2. $(1, \iota)$ *is called the unit object of $\mathcal{C}$.*

We can extend the notion of subcategories of categories to monoidal subcategories in the following way.

Definition 2.1.3. *A monoidal subcategory of $(\mathcal{C}, \otimes, a, 1, \iota)$ is a quintuple $(\mathcal{D}, \otimes, a, 1, \iota)$ where $\mathcal{D}$ is a subcategory of $\mathcal{C}$ closed under the tensor product of morphisms and objects, and contains $(1, \iota)$.*

We want monoidal categories to be the categorification of monoids, and a monoid has a unique multiplicative identity. Hence, we would like to have a unique (up to isomorphism) unit object in a monoidal category. Not surprisingly, we can derive that the unit object in a monoidal category is unique up to a unique isomorphism from the pentagon axiom and unit axiom given in the definition of a monoidal category.

Let $(\mathcal{C}, \otimes, a, 1, \iota)$ be a monoidal category. We will define two natural isomorphism $l_X : 1 \otimes X \to X$, and $r_X : X \otimes 1 \to X$ where $L_1(l_X)$ and $R_1(r_X)$ are equal to:

$$L_1(l_X) := 1 \otimes (1 \otimes X) \overset{a^{-1}_{1,1,X}}{\to} (1 \otimes 1) \otimes X \overset{\iota \otimes id_X}{\to} 1 \otimes X$$

$$R_1(r_X) := (X \otimes 1) \otimes 1 \overset{a_{X,1,1}}{\to} X \otimes (1 \otimes 1) \overset{id_X \otimes \iota}{\to} X \otimes 1$$

Definition 2.1.4. *We call l_X and r_X the left and right unit constraints.*

Because we defined l_X and r_X to be natural isomorphisms we have the following proposition.

Proposition 2.1.5. *For any $X \in \mathcal{C}$, $l_{1\otimes X} = id_1 \otimes l_X$ and $r_{x\otimes 1} = r_X \otimes id_1$*

Proof. By naturality of l_X, if we let $F := (1 \otimes -)$, then we have the following commuting diagram:

$$\begin{array}{ccc} F(1\otimes X) & \xrightarrow{F(l_X)} & F(X) \\ \downarrow{\scriptstyle l_{1\otimes X}} & & \downarrow{\scriptstyle l_X} \\ 1\otimes X & \xrightarrow{\quad l_X \quad} & X \end{array}$$

Because l_X is an isomorphism, together with the diagram, we have that $l_{1\otimes X} = F(X) = id_1 \otimes l_X$.

Similarly, if we let $G := (- \otimes 1)$, by naturality of r_X, we get the following commuting diagram:

$$\begin{array}{ccc} G(X\otimes 1) & \xrightarrow{G(r_X)} & G(X) \\ \downarrow{\scriptstyle r_{X\otimes 1}} & & \downarrow{\scriptstyle r_X} \\ X\otimes 1 & \xrightarrow{\quad r_X \quad} & X \end{array}$$

This implies that $r_{X\otimes 1} = r_X \otimes 1$ and we have the proposition.

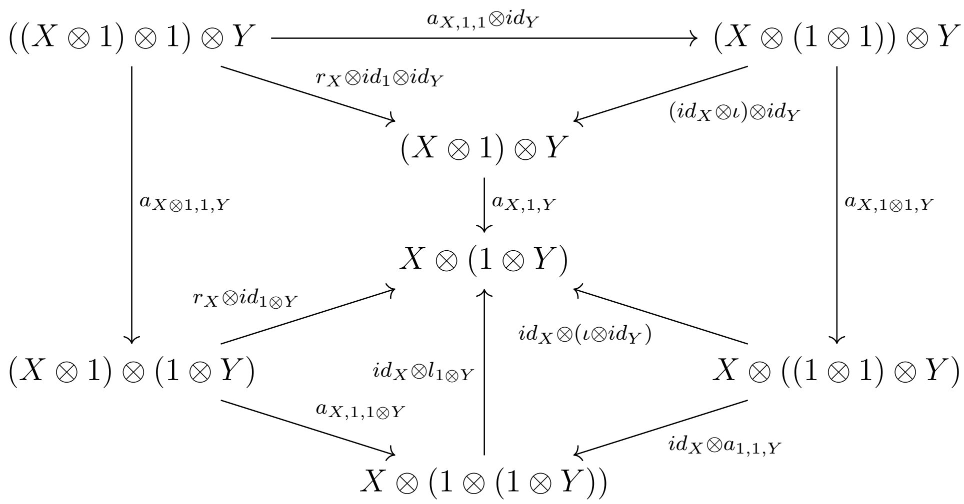
□

From the unit constraints and the associativity constraint, we get the **triangle diagram**:

$$\begin{array}{ccc} (X\otimes 1)\otimes Y & \xrightarrow{\quad a_{X,1,Y} \quad} & X\otimes(1\otimes Y) \\ & {\scriptstyle r_X\otimes id_Y} \searrow \quad \swarrow {\scriptstyle id_X\otimes l_Y} & \\ & X\otimes Y & \end{array}$$

Proposition 2.1.6. *The triangle diagram commutes.*

Proof. Consider the following diagram induced by the unit constraints and the pentagon axiom.



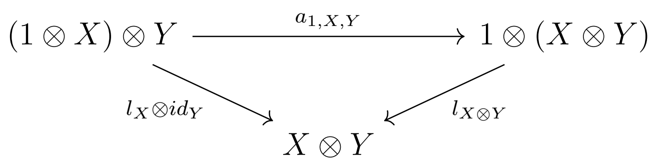
It is clear that the outside pentagon is induced by the pentagon axiom, and notice that if we let $Y' = 1 \otimes Y$, the bottom left triangle is precisely the triangle diagram that we want to show commutes.

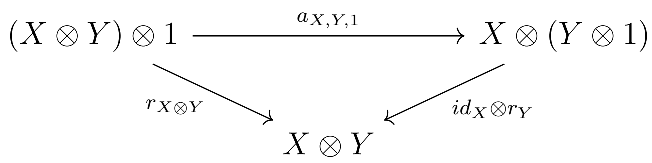
From how we defined r_X, $R_1(r_X) = r_X \otimes id_1 = (id_X \otimes \iota) \circ a_{X,1,1}$, and so it follows that the top triangle commutes. Similarly, by how we defined l_Y, $L_1(l_Y) = id_1 \otimes l_Y = l_{1 \otimes Y} = (\iota \otimes id_Y) \circ a^{-1}_{1,1,Y}$, and hence the bottom right triangle commutes. Lastly, by naturality of the associativity constraint $a_{X,Y,Z}$, the two trapezoids in the middle commute. Hence every component of the diagram outside of the bottom left triangle commutes, which forces the bottom left triangle to commute. Consequently we have that the triangle diagram commutes.

□

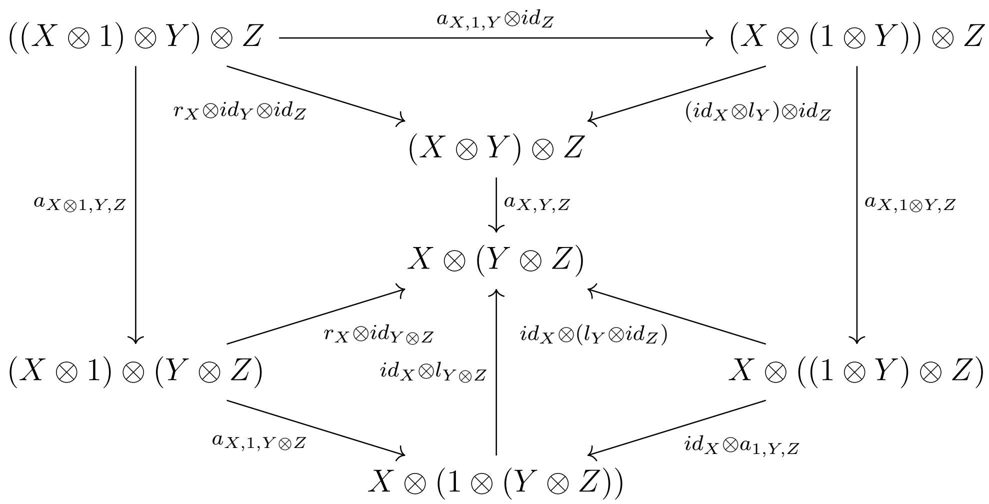
Now that we have the triangle diagram, we can use it to get the following proposition.

Proposition 2.1.7. *The following diagrams commute for every X, Y in $\mathcal{C}$.*



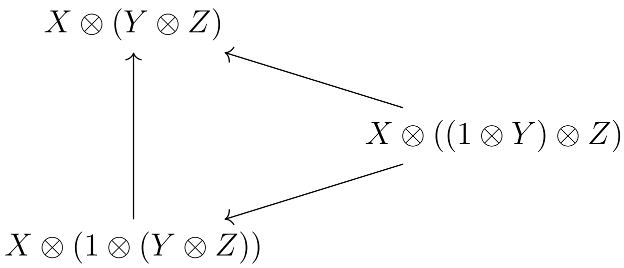


Proof. Consider the following diagram induced by the pentagon axiom and the unit constraints.

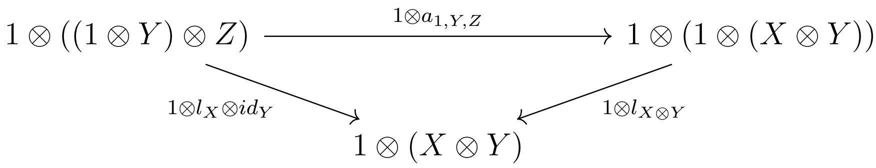


The naturality of $a_{X,Y,Z}$ and the commutativity of the triangle diagram gives us that the two left trapezoids and the bottom left triangle commutes, and combining them with the pentagon axiom, we can easily see that the above diagram must commute.

Now consider the bottom right triangle,

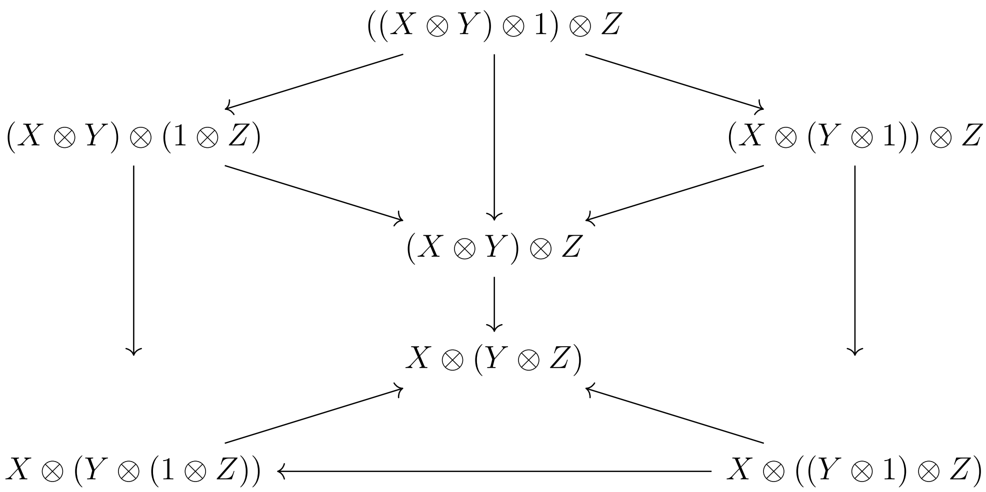


This diagram commutes for all X, Y, Z in $\mathcal{C}$, and hence if we let $X = 1$, we have



Applying $L_1^{-1} : 1 \otimes X \to X$ functor yields the desired result for the first triangle in the proposition.

Similarly it is easy to show using the following diagram to show that the second triangle in the proposition commutes.



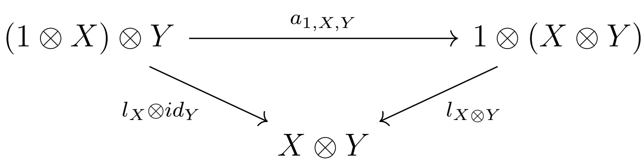
Notice that if we take $Z = 1$, then applying $R_1^{-1} : X \otimes 1 \to X$ functor to the top right triangle precisely gives us the second triangle in the proposition.

$\square$

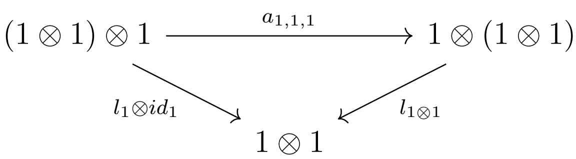
From the two triangles in the proposition above, we get an important corollary about the unit constraints.

Corollary 2.1.8. *In any monoidal category* $l_1 = r_1 = \iota$.

Proof. We have that the following diagram commutes for any X, Y in $\mathcal{C}$

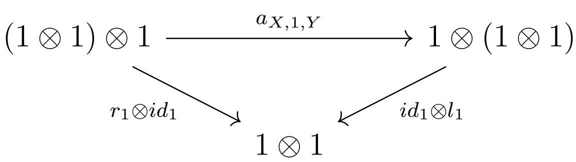


Set $X = Y = 1$ to obtain:



Hence we have that $l_1 \otimes id_1 = l_{1\otimes 1} \circ a_{1,1,1} = (id_1 \otimes l_1) \circ a_{1,1,1}$.

Also, from the triangle diagram, we can get the following diagram by setting $X = Y = 1$.



Hence $r_1 \otimes id_1 = (id_1 \otimes l_1) \circ a_{1,1,1}$. Recall that $L_1(l_1) = id_1 \otimes l_1 = (\iota \otimes id_1) \circ a_{1,1,1}^{-1}$. It follows that $(id_1 \otimes l_1) \circ a_{1,1,1} = \iota \otimes id_1$.

Combining the results gives us that $l_1 \otimes id_1 = r_1 \otimes id_1 = \iota \otimes id_1$, and this is equivalent to $R_1(l_1) = R_1(r_1) = R_1(\iota)$. Now, from the unit axiom, we have that R_1 is an autoequivalence on $\mathcal{C}$, hence we have that $l_1 = r_1 = \iota$.

□

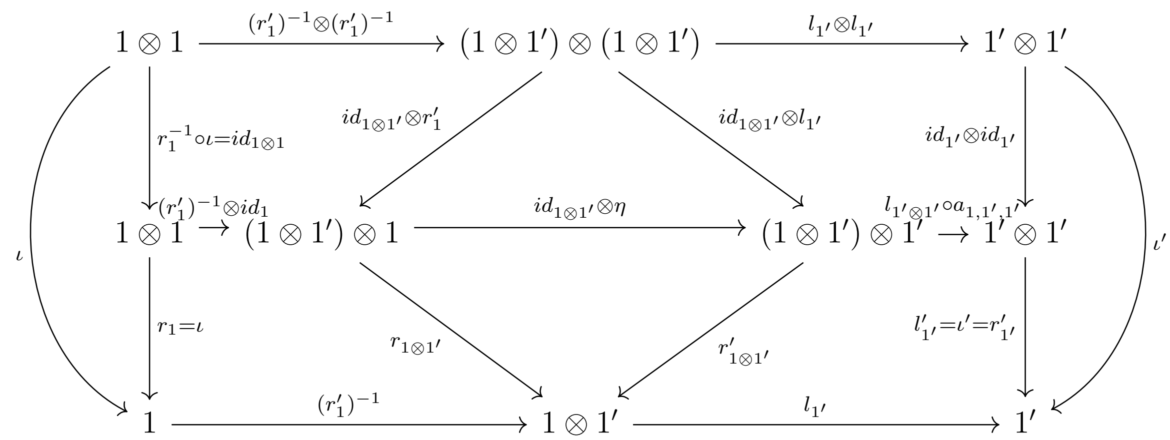
Finally, using all of the propositions and definitions we have discussed so far, we are ready to show the uniqueness of the unit.

Proposition 2.1.9. *The unit object in a monoidal category is unique up to a unique isomorphism.*

Proof. Let $(1, \iota), (1', \iota')$ be unit objects.

Then we have r, l and r', l' as our corresponding unit constraints. Now let $\eta := l_{1'} \circ r'_1 : 1 \xrightarrow{\sim} 1'$.

Consider the following diagram.



By directly composing, it is easy to see that the diagram commutes, and as r, l are natural isomorphisms, ι is an isomorphism, and we have that η is precisely the isomorphism taking ι to ι'. Hence we see that the unit object in a monoidal category is unique up to isomorphism, and the isomorphism is given by η.

To prove the prove the proposition, it remains that we show the isomorphism η is unique.

Let τ be any morphism from 1 to 1. Then,

$$\begin{array}{ccc} 1 \otimes 1 & \xrightarrow{\tau \otimes id_1} & 1 \otimes 1 \\ \downarrow{\iota} & & \downarrow{\iota} \\ 1 & \xrightarrow{\tau} & 1 \end{array}$$

commutes, by naturality of $\iota = r_1$.

Also notice that if $\sigma : 1 \xrightarrow{\sim} 1$ is an isomorphism from 1 to 1, then by naturality of $\iota = r_1$, we have the following commuting diagram.

$$\begin{array}{ccc} 1 \otimes 1 & \xrightarrow{\sigma \otimes \sigma} & 1 \otimes 1 \\ \downarrow{\iota} & & \downarrow{\iota} \\ 1 & \xrightarrow{\sigma} & 1 \end{array}$$

Setting $\tau = \sigma$, we have that $\sigma \otimes id_1 = \sigma \otimes \sigma$, and hence $\sigma = id_1$, and in particular, every isomorphism from 1 to 1 are equivalent to id_1. Hence $\eta = id_1$ must be unique, and we have the proposition.

□

Using a similar construction as the above we can deduce an interesting observation on monoidal categories.

Proposition 2.1.10. *Let $\mathcal{C}$ be a monoidal category. Then, $End_{\mathcal{C}}(1)$ is a commutative monoid under composition of morphisms. Also, $f \otimes g = \iota^{-1} \circ (f \circ g) \circ \iota$ for every $f, g \in End_{\mathcal{C}}(1)$.*

Proof. By naturality of $\iota = r_1$ we have the commutative diagram:

$$\begin{array}{ccc} 1 \otimes 1 & \xrightarrow{f \otimes id_1} & 1 \otimes 1 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ 1 & \xrightarrow{\quad f \quad} & 1 \end{array}$$

Hence $f \otimes id_1 = r_1^{-1} \circ f \circ r_1$, and similarly we have $id_1 \otimes g = l_1^{-1} \circ g \circ l$ by naturality of l_1. Since $\iota = l_1 = r_1$, we have that $f \otimes g = (f \otimes id_1) \circ (id_1 \otimes g) = \iota^{-1} \circ (f \circ g) \circ \iota$.

Repeating a similar process we get that $g \otimes f = \iota^{-1} \circ (f \circ g) \circ \iota$, and we conclude that $f \otimes g = g \otimes f$, and accordingly, $f \circ g = g \circ f$.

$\square$

The above proof can be thought of as an analogue of the definition of a monoid as a category with a single object. By the above proposition we can think of a commutative monoid as a monoidal category with a single object. It is quite surprising that the pentagon axiom and the unit axiom impose a commutative structure on the set of endomorphisms of the unit object.

Since we have examined the structure of monoidal categories, the natural progression would be to examine maps between monoidal categories, i.e. monoidal functors.

2.1.2 Monoidal Functors

Definition 2.1.11. *Let* $(\mathcal{C}, \otimes, 1, a, \iota)$ *and* $(\overline{\mathcal{C}}, \overline{\otimes}, \overline{1}, \overline{a}, \overline{\iota})$ *be monoidal categories.*

A monoidal functor from $\mathcal{C}$ *to* $\overline{\mathcal{C}}$ *is a pair* (F, J) *where* $F : \mathcal{C} \to \overline{\mathcal{C}}$ *is a functor, and* $J_{X,Y} : F(X) \overline{\otimes} F(Y) \xrightarrow{\sim} F(X \otimes Y)$ *is a natural isomorphism such that* $F(1) \simeq \overline{1}$ *and the following diagram commutes for every* X, Y, Z *in* $\mathcal{C}$*.*

$$\begin{array}{ccc}
(F(X)\overline{\otimes}F(Y))\overline{\otimes}F(Z) & \xrightarrow{\overline{a}_{F(X),F(Y),F(Z)}} & F(X)\overline{\otimes}(F(Y)\overline{\otimes}F(Z)) \\
{\scriptstyle J_{X,Y}\overline{\otimes}id_{F(Z)}}\downarrow & & \downarrow{\scriptstyle id_{F(Z)}\overline{\otimes}J_{Y,Z}} \\
F(X\otimes Y)\overline{\otimes}F(Z) & & F(X)\overline{\otimes}(F(Y\otimes Z)) \\
{\scriptstyle J_{X\otimes Y,Z}}\downarrow & & \downarrow{\scriptstyle J_{X,Y\otimes Z}} \\
F((X\otimes Y)\otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X\otimes(Y\otimes Z))
\end{array}$$

This diagram in called the monoidal structure axiom, and if F is an equivalence of $\mathcal{C}$ and $\overline{\mathcal{C}}$ as ordinary categories, then F is denoted an equivalence of monoidal categories.

Notice that the canonical natural isomorphism $\phi : \overline{1} \xrightarrow{\sim} F(1)$ is defined by the following diagram.

$$\begin{array}{ccc}
\overline{1}\overline{\otimes}F(1) & \xrightarrow{\bar{l}_{F(1)}} & F(1) \\
{\scriptstyle \phi\overline{\otimes}id_{F(1)}}\downarrow & & \downarrow{\scriptstyle F(l_1^{-1})} \\
F(1)\overline{\otimes}F(1) & \xrightarrow{J_{1,1}} & F(1\otimes 1)
\end{array}$$

where (r, l) and $(\overline{r}, \bar{l})$ are the unit constraints of $\mathcal{C}$ and $\overline{\mathcal{C}}$ in respective order.

Using the diagram to define ϕ, we get the following proposition.

Proposition 2.1.12. *For any monoidal functor $(F, J) : \mathcal{C} \to \overline{\mathcal{C}}$, the following diagrams commute.*

$$\begin{array}{ccc}
\overline{1}\overline{\otimes}F(X) & \xrightarrow{\bar{l}_{F(X)}} & F(X) \\
{\scriptstyle \phi\overline{\otimes}id_{F(X)}}\downarrow & & \downarrow{\scriptstyle F(l_X^{-1})} \\
F(1)\overline{\otimes}F(X) & \xrightarrow{J_{1,X}} & F(1\otimes X)
\end{array}$$

$$\begin{array}{ccc}
F(X)\overline{\otimes}\overline{1} & \xrightarrow{\overline{r}_{F(X)}} & F(X) \\
{\scriptstyle id_{F(X)}\overline{\otimes}\phi}\big\downarrow & & \big\downarrow{\scriptstyle F(r_X^{-1})} \\
F(X)\overline{\otimes}F(1) & \xrightarrow{J_{X,1}} & F(X\otimes 1)
\end{array}$$

Proof. We have the commuting diagram

$$\begin{array}{ccc}
\overline{1}\overline{\otimes}F(1) & \xrightarrow{\bar{l}_{F(1)}} & F(1) \\
{\scriptstyle \phi\overline{\otimes}id_{F(1)}}\big\downarrow & & \big\downarrow{\scriptstyle F(l_1^{-1})} \\
F(1)\overline{\otimes}F(1) & \xrightarrow{J_{1,1}} & F(1\otimes 1)
\end{array}$$

We can apply the functor $(-\overline{\otimes}F(X))$ to the above commutative diagram to get the following commuting diagram, which we will call diagram (a).

$$\begin{array}{ccc}
(\overline{1}\overline{\otimes}F(1))\overline{\otimes}F(X) & \xrightarrow{\bar{l}_{F(1)}\overline{\otimes}id_{F(X)}} & F(1)\overline{\otimes}F(X) \\
{\scriptstyle \phi\overline{\otimes}id_{F(1)}\overline{\otimes}id_{F(X)}}\big\downarrow & & \big\downarrow{\scriptstyle F(l_1^{-1})\overline{\otimes}id_{F(X)}} \\
(F(1)\overline{\otimes}F(1))\overline{\otimes}F(X) & \xrightarrow{J_{1,1}\overline{\otimes}id_{F(X)}} & F(1\otimes 1)\overline{\otimes}F(X)
\end{array}$$

We will now build a series of commutative diagrams to prove the proposition. Consider the following.

$$\begin{array}{ccccc}
(\overline{1}\overline{\otimes}F(1))\overline{\otimes}F(X) & \xrightarrow{\overline{a}_{\overline{1},F(1),F(X)}} & \overline{1}\overline{\otimes}(F(1)\overline{\otimes}F(X)) & \xrightarrow{id_{\overline{1}}\overline{\otimes}J_{1,X}} & \overline{1}\overline{\otimes}(F(1\otimes X)) \\
{\scriptstyle (\phi\overline{\otimes}id_{F(1)})\overline{\otimes}F(X)}\big\downarrow & & \big\downarrow{\scriptstyle \phi\overline{\otimes}(id_{F(1)}\overline{\otimes}id_{F(X)})} & & \big\downarrow{\scriptstyle \phi\overline{\otimes}id_{F(1\otimes X)}} \\
(F(1)\overline{\otimes}F(1))\overline{\otimes}F(X) & \xrightarrow{\overline{a}_{F(1),F(1),F(X)}} & F(1)\overline{\otimes}(F(1)\overline{\otimes}F(X)) & \xrightarrow{id_{F(1)}\overline{\otimes}J_{1,X}} & F(1)\overline{\otimes}(F(1\otimes X))
\end{array}$$

The left square commutes because $\overline{a}$ is functorial, and the right square commutes by naturality of $J_{1,X}$. Note that the top and bottom arrows are isomorphisms, hence we can reverse the arrows.

Also, as l_X is a natural isomorphism we get the following commutative diagram.

$$\begin{array}{ccc} \overline{1}\overline{\otimes}(F(1\otimes X)) & \xrightarrow{id_{\overline{1}}\overline{\otimes}F(l_X)} & \overline{1}\overline{\otimes}F(X) \\ {\scriptstyle \phi\overline{\otimes}id_{F(1\otimes X)}}\downarrow & & \downarrow{\scriptstyle \phi\overline{\otimes}id_{F(X)}} \\ F(1)\overline{\otimes}(F(1\otimes X)) & \xrightarrow{id_{F(1)}\overline{\otimes}F(l_X)} & F(1)\overline{\otimes}F(X) \end{array}$$

Combining the above two diagrams and reversing the arrows on the top and bottom gives us the following commutative diagram, which we will call diagram (b).

$$\begin{array}{ccc} \overline{1}\overline{\otimes}F(X) & \xrightarrow{\overline{a}^{-1}_{\overline{1},F(1),F(X)}\circ id_{\overline{1}}\overline{\otimes}J^{-1}_{1,X}\circ id_{\overline{1}}\overline{\otimes}F(l_X)^{-1}} & (\overline{1}\overline{\otimes}F(1))\overline{\otimes}F(X) \\ {\scriptstyle \phi\overline{\otimes}id_{F(1)}\overline{\otimes}id_{F(X)}}\downarrow & & \downarrow{\scriptstyle \phi\overline{\otimes}id_{F(X)}} \\ F(1)\overline{\otimes}F(X) & \xrightarrow{\overline{a}^{-1}_{F(1),F(1),F(X)}\circ id_{F(1)}\overline{\otimes}J^{-1}_{1,X}\circ id_{F(1)}\overline{\otimes}F(l_X)^{-1}} & (F(1)\overline{\otimes}F(1))\overline{\otimes}F(X) \end{array}$$

Similarly, we have the following commutative diagram, denoted diagram (c).

$$\begin{array}{ccccc} F(1)\overline{\otimes}F(X) & \xrightarrow{J_{1,X}} & F(1\otimes X) & \xrightarrow{F(l_X)} & F(X) \\ {\scriptstyle F(l_1^{-1})\overline{\otimes}id_{F(X)}}\downarrow & & \downarrow{\scriptstyle F(l_1^{-1}\otimes id_X)} & & \downarrow{\scriptstyle F(l_X^{-1})} \\ F(1\otimes 1)\overline{\otimes}F(X) & \xrightarrow{J_{1\otimes 1,X}} & F((1\otimes 1)\otimes X) & \xrightarrow{F(l_1\otimes id_X)} & F(1\otimes X) \end{array}$$

The left square commutes by naturality of J, and the right square commutes since l is natural and $l_1 \otimes id_X = l_{1\otimes X}$. Combining diagrams (a), (b), and (c) in the order (b)→(a)→(c) yields the desired result.

□

Using ϕ we can identify $\overline{1}$ with $F(1)$, so it is notationally beneficial if we take $F(1) = \overline{1}$, and $\phi = id_{\overline{1}}$. Under this convention, we have that $J_{1,X} = J_{X,1} = id_X$, which is similar to how we took $l_X = r_X = id_X$.

Now that we have what monoidal functors are, we can now define morphisms of monoidal functors.

Definition 2.1.13. *Let* $(\mathcal{C}, \otimes, 1, a, \iota)$ *and* $(\overline{\mathcal{C}}, \overline{\otimes}, \overline{1}, \overline{a}, \overline{\iota})$ *be two monoidal categories and* $(F^1, J^1), (F^2, J^2) : \mathcal{C} \to \overline{\mathcal{C}}$ *be two monoidal functors. Then, a morphism of monoidal functors* $\eta : (F^1, J^1) \to (F^2, J^2)$ *is a natural transformation* $\eta : F^1 \to F^2$ *where* η_1 *is an isomorphism, and the diagram*

$$\begin{array}{ccc} F^1(X)\overline{\otimes}F^1(Y) & \xrightarrow{J^1_{X,Y}} & F^1(X \otimes Y) \\ {\scriptstyle \eta_X \overline{\otimes} \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_{X\otimes Y}} \\ F^2(X)\overline{\otimes}F^2(Y) & \xrightarrow{J^2_{X,Y}} & F^2(X \otimes Y) \end{array}$$

commutes for all $X, Y \in \mathcal{C}$.

Notice that the naturality of $\eta : F^1 \to F^2$ makes η coherent with the isomorphisms of the unit objects. That is, if $\phi_1 : \overline{1} \xrightarrow{\sim} F^1(1)$, and $\phi_2 : \overline{1} \xrightarrow{\sim} F^2(1)$, then $\eta_1 \circ \phi_1 = \phi_2$, and as we have adopted the convention that $\phi_1 = \phi_2 = id_{\overline{1}}$, we have that $\eta_1 = id_{\overline{1}}$.

We've thus far defined monoidal functors and morphisms between them, so a natural question one might have is, what are some examples? An important, and intuitive example is the *forgetful functors.* For instance, the forgetful functor defined by $Rep(G) \to Vec$ lets us remove the G-representation structure and just view these objects as vector spaces. The monoidal structure is obvious for both categories, and the natural isomorphism J as defined in the definition of monoidal functors is trivially defined for this example, since the tensor structure on both categories are equivalent.

Another example of monoidal functors is one between the category of bimodules of a unital algebra A over a field and the endofunctors of the category of left modules of A. Let $\mathcal{K}$ be a field, and A be a unital $\mathcal{K}$-algebra. Let $\mathcal{C}$ be the category of left A-modules. Then, we have a monoidal functor,

$$F : A - bimod \to End(\mathcal{C})$$

$$M \mapsto (M \otimes_A -)$$

In fact, we can say much more about this monoidal functor.

Proposition 2.1.14. *The functor $F : A - bimod \to End(\mathcal{C})$ takes values in the full subcategory (the subcategory contains all of the morphisms between any two objects in it) $End_{re}(\mathcal{C})$, the right exact endofunctors of $\mathcal{C}$, and defines an equivalence between monoidal categories A-bimod and $End_r(\mathcal{C})$.*

Proof. We know that the tensor product is right exact, hence by definition, F must take values in $End_{re}(\mathcal{C})$. Hence the first part of the proposition holds.

Now, for every $G \in End_{re}(\mathcal{C})$, define $F^{-1} : End_{re}(\mathcal{C}) \to A - bimod$ by $G \mapsto G(A)$. Clearly, $G(A)$ is a left A-module, and notice that $G(A)$ admits a right action of A^{op} by $G(A) \cdot a = G(aA)$ where $a : A \to A$ is defined by $\alpha \in A \mapsto a\alpha$. Notice that the action must be given by A^{op} as $(G(A) \cdot b) \cdot a = G(abA) = G(A) \cdot (ab)$. Hence we have that $G(A)$ is an A-bimodule.

We claim that F^{-1} is a quasi-inverse to F, i.e. $F \circ F^{-1} \simeq id_{End_{re}(\mathcal{C})}$, and $F^{-1} \circ F \simeq id_{A-bimod}$.

Let $G \in End_{re}(\mathcal{C})$. We want to show that $G(X)$ for any $X \in \mathcal{C}$ can be identified with $((F \circ F^{-1})(G))(X) = G(A) \otimes_A X$. To do so, let us first take a projective resolution of X, given by $\cdots \to P_1 \to P_0 \to X \to 0$. Then, P_0 and P_1 are free, hence $G(P_0)$ and $G(P_1)$ are identified with $G(A) \otimes_A P_0$ and $G(A) \otimes_A P_1$ in respective order. Now, G is right exact, hence $G(X)$ is the cokernel of $G(P_1) \to G(P_0)$, and by the above, this is equivalent to $Coker(G(A) \otimes_A P_1 \to G(A) \otimes_A P_0)$. But $Coker(G(A) \otimes_A P_1 \to G(A) \otimes_A P_0)$ is canonically $G(A) \otimes_A X$, and hence we have that $G(A) \simeq G(A) \otimes_A X$. It follows that $((F \circ F^{-1})(G))(X) \simeq G(X)$, and we conclude that $F \circ F^{-1} \simeq id_{End_{re}(\mathcal{C})}$.

For the other direction, let $M \in A - bimod$. Then, $(F^{-1} \circ F)(M) = M \otimes_A A$. Now, observe the following diagram.

$$\begin{array}{ccc} M \times A & \longrightarrow & M \\ \downarrow & \nearrow_{\exists!\ identity\ map} & \\ M & & \end{array}$$

By the universal property of tensors, we have that $M \otimes_A A \simeq M$, and hence $F^{-1} \circ F \simeq id_{A-bimod}$.

Hence F^{-1} is a quasi-inverse of F, and hence F is an equivalence between $A - bimod$ and $End_{re}(\mathcal{C})$.

□

2.1.3 The Category of G-graded Vector Spaces and Cocycles

In the previous section of abelian categories, we have talked about exact sequences and cohomologies in categories of modules. With our understanding of monoidal categories and monoidal functors, we will begin examining similar ideas on categories of graded vector spaces.

First, consider the following. Let G be a monoid, and A be an abelian group. Let $\mathcal{C}_G = \mathcal{C}_G(A)$ be the category whose objects are labeled by elements of G, denoted δ_g. The morphisms in this category satisfies $Hom_{\mathcal{C}_G}(\delta_{g_1}, \delta_{g_2}) = \emptyset$ if $g_1 \neq g_2$, and $Hom_{\mathcal{C}_G}(\delta_g, \delta_g) = A$. The tensor product functor $\otimes$ in $\mathcal{C}_G$ is defined on the objects by $\delta_g \otimes \delta_h = \delta_{gh}$, and on the morphisms by $a \otimes b = ab$. Then, $\mathcal{C}_G$ is a monoidal category, where the associativity isomorphism is the identity, and 1 is represented by the unit element of G. Note that if G is a non-commutative group, $X \otimes Y$ is not necessarily isomorphic to $Y \otimes X$.

The category of G graded vector spaces is precisely the linear version of the above. Let $\mathcal{K}$ be a field, and let Vec_G the category of G-graded vector spaces over $\mathcal{K}$, such that $V = \bigoplus_{g \in G} V_g$. The morphisms in Vec_G must be linear maps that preserve the grading, and the tensor product in this category is defined by $(V \otimes W)_g = \bigoplus_{xy=g} V_x \otimes W_y$. The unit object 1 must satisfy $1_e = \mathcal{K}$, and $1_g = 0$ for any $g \neq e$, where e is the identity element of G. Let the associativity isomorphism a be defined as usual for tensor products of vector spaces, and let the unit constraint $\iota : 1 \otimes 1 \xrightarrow{\sim} 1$ be defined by the identity map $\mathcal{K} \otimes \mathcal{K} \xrightarrow{\sim} \mathcal{K}$. This construction equips Vec_G with the structure of a monoidal category, and we have a similar construction for the finite dimensional case.

Let $\overline{Vec_G}$ be the category of finite dimensional G-graded vector spaces over $\mathcal{K}$. In this category, the smallest objects are the pairwise non-isomorphic 1 dimensional G-graded vector spaces $\delta_g = \bigoplus_{h \in G} (\delta_g)_h$ such that $(\delta_g)_g \simeq \mathcal{K}$, and $(\delta_g)_h = 0$ for every $h \neq g$. Then, we see that

$$\delta_g \otimes \delta_h = \bigoplus_{k \in G} (\delta_g \otimes \delta_h)_k = \bigoplus_{k \in G} (\bigoplus_{xy=k} (\delta_g)_x \otimes (\delta_h)_y) = (\delta_g)_g \otimes (\delta_h)_h = (\delta_g \otimes \delta_h)_{gh} \simeq \delta_{gh}.$$

Notice that with this tensor structure, these objects form the G-graded monoidal category $\mathcal{C}_G(\mathcal{K}^\times)$. Since we take the abelian group to be $\mathcal{K} \backslash 0$, under multiplication, the 0 morphisms are missing, and hence we have that $\mathcal{C}_G(\mathcal{K}^\times)$ is a non-full monoidal subcategory of $\overline{Vec_G}$. Also, any object in $\overline{Vec_G}$ must be given by a finite direct sum of these one-dimensional vector spaces. Hence, we can view $\mathcal{C}_G(\mathcal{K}^\times)$ as the basis of $\overline{Vec_G}$ whose objects are given by the direct sum of non-negative integer multiples of finitely many δ_g's. This point of view directly mirrors the idea of $\mathbb{Z}^+$-rings that we will consider later on.

We can generalize the construction of graded monoidal categories further by "twisting" the associativity isomorphisms. Let G be a group, and A be an abelian group.

Let $\omega : G \times G \times G \to A$ be a 3-cocycle of G taking values in A, i.e.

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2g_3, g_4)\omega(g_2, g_3, g_4)$$

for every $g_1, g_2, g_3, g_4 \in G$.

Notice that this is exactly the pentagon axiom for monoidal categories.

Now we will define the category $\mathcal{C}_G^{\omega}(A)$.

As a category, $\mathcal{C}_G^{\omega}(A)$ is equivalent to $\mathcal{C}_G(A)$, and $\otimes$, $(1, \iota)$ are defined equivalently. However, we will twist the associativity isomorphism. a^{ω} is defined by

$$a^{\omega}_{\delta_g,\delta_h,\delta_i} = \omega(g, h, i)id_{\delta_{ghi}} : (\delta_g \otimes \delta_h) \otimes \delta_i \to \delta_g \otimes (\delta_h \otimes \delta_i),$$

where $g, h, i \in G$. Then, $\omega(g, h, i)$ defines some element in A, which corresponds to elements of $Hom_{C_G}(\delta_g, \delta_g)$, $Hom_{C_G}(\delta_h, \delta_h)$, and $Hom_{C_G}(\delta_i, \delta_i)$. Then, because ω satisfies the pentagon axiom, $a^{\omega}_{\delta_g,\delta_h,\delta_i}$ satisfies the pentagon axiom, and so $\mathcal{C}_G^{\omega}(A)$ makes sense as a monoidal category.

As before, we can get a linear version of $\mathcal{C}_G^{\omega}(A)$ by letting A be the field $\mathcal{K}$ and taking δ_g to be 1 dimensional G-graded vector spaces. By taking arbitrary direct sums of non-negative integer multiples of them, we get the category of G-graded vector spaces Vec_G^{ω}.

We can now examine the monoidal functors between categories of graded vector spaces with non-trivial associativity isomorphisms.

Let G_1, G_2 be groups, let A be an abelian group and let $\omega_1, \omega_2 \in Z^3(G_i, A)$ be 3-cocycles, and suppose the action of G_1, G_2 on A are trivial. Let $\mathcal{C}_i = \mathcal{C}_{G_i}^{\omega_i}$ be the monoidal categories of graded vector spaces.

Any monoidal functor $F : \mathcal{C}_1 \to C_2$, when restricted to the simple objects defines a group homomorphism $f : G_1 \to G_2$ by $F(\delta_g \otimes \delta_h) = F(\delta_g) \otimes F(\delta_h)$. By the monoidal structure axiom, we have

$$J_{g,h} = \mu(g, h)id_{\delta_{f(gh)}} \xrightarrow{\sim} F(\delta_{gh}), \; g, h \in G_1$$

where the map $\mu : G_1 \times G_1 \to A$ satisfies

$$\omega_1(g,h,l)\mu(gh,l)\mu(g,h) = \mu(g,hl)\mu(h,l)\omega_2(f(g),f(h),f(l)) \; \forall g,h,l \in G_1.$$

and this condition on μ encodes the monoidal structure axiom.

Notice here, that the codomain of ω_i, μ is A, an abelian group, and hence we can rewrite the above condition additively:

$$\omega_1(g,h,l) - \omega_2(f(g),f(h),f(l)) = \mu(g,hl) + \mu(h,l) - \mu(gh,l) - \mu(g,h).$$

Recall that ω_i is a map in the kernel of the the following ∂^4:

$$\cdots \xrightarrow{\partial^3} \{G_i \times G_i \times G_i \to A\} \xrightarrow{\partial^4} \{G_i \times G_i \times G_i \times G_i \to A\} \longrightarrow \cdots$$

where $g_1|g_2|g_3 + g_1|g_2g_3|g_4 + g_2|g_3|g_4 - g_1|g_2|g_3g_4 - g_1g_2|g_3|g_4 = 0.$

Also, $\mu(gh,l) + \mu(g,h) - \mu(g,hl) - \mu(h,l)$ is given by $g|h|l \mapsto gh|l + g|h - g|hl - h|l$ and because we assumed that the action of G on A is trivial, this tells us that the difference of ω_1 and $f^*\omega_2$, (i.e. $\omega_1(g,h,l)$ and $\omega_2(f(g),f(h),f(l))$), is given by the boundary map

$$\cdots \longrightarrow \{G_1 \times G_1 \to A\} \xrightarrow{\partial^3} \{G_1 \times G_1 \times G_1 \to A\} \longrightarrow \cdots$$

and hence $\omega_1(g,h,l) - f^*\omega_2(g,h,l) = \partial^3(\mu)$. Therefore, ω_1 and $f^*\omega_2$ are cohomologous. rewriting this condition multiplicatively, we have the condition

$$\omega_1 = f^*\omega_2 \cdot \partial^3(\mu).$$

Conversely, given a homomorphism $f : G_1 \to G_2$ and any function μ satisfying $\omega_1 = f^*\omega_2 \cdot \partial^3(\mu)$, we get a monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defined by

$$F(\delta_g) = \delta_{f(gh)},$$

$$J_{g,h} = \mu(g,h) id_{\delta_{f(gh)}} : F(\delta_g) \otimes F(\delta_h) \xrightarrow{\sim} F(\delta_{gh}).$$

Note that if F is an equivalence of monoidal categories, f must be an isomorphism, and conversely, any isomorphism f must induce an equivalence of monoidal categories.

Now let $F = F_{f,\mu}$, $F' = F_{f',\mu'}$ be two functors $\mathcal{C}_1 \to \mathcal{C}_2$. We will determine the natural monoidal transformations between F and F'. Let η be one such transformation. Then,

$$\begin{array}{ccc} F(\delta_g) & \longrightarrow & F(\delta_h) \\ \downarrow{\scriptstyle \eta_g} & & \downarrow{\scriptstyle \eta_h} \\ F'(\delta_g) & \longrightarrow & F'(\delta_h) \end{array}$$

commutes by naturality of η. However, $F(\delta_g) = \delta_{f(g)}$, $F'(\delta_g) = \delta_{f'(g)}$, and if $f(g) \neq f'(g)$, $Hom(\delta_{f(g)}, \delta_{f'(g)}) = \emptyset$. Hence if such an η exists, it must be that $f(g) = f'(g)$ for every $g \in G_1$, and hence $f = f'$.

Also, δ_g is simple, and hence $\delta_{f(g)}$ must also be simple, and as $\delta_{f(g)} = \delta_{f'(g)}$, η_g must be an isomorphism. Hence we have that $\eta : F \to F'$ is an isomorphism.

Lastly, because η is a transformation between monoidal functors, each $\eta_g : \delta_{f(g)} \to \delta_{f(g)}$ must satisfy

$$\mu'(g,h)(\eta_g \otimes \eta_h) = \eta_{gh}\mu(g,h)$$

and hence $\mu = \mu' \cdot \partial^2(\eta)$, i.e. μ is cohomologous to μ'.

On the other hand, any $\eta : G_1 \to A$ that satisfies $\mu = \mu' \cdot \partial^2(\eta)$ gives us a morphism of monoidal functors $\eta : F_{f,\mu} \to F_{f',\mu'}$ for fixed f, f' and μ, μ'.

Notice that given any $\alpha \in H^1(G_1, A)$ such that $\alpha \neq 1$ and η such that $\mu = \mu' \cdot \partial^2(\eta)$, which we can rewrite additively as A is abelian, $\eta + \alpha$ gives us

$$\eta_{gh} + \alpha(gh) - \eta_g \otimes \eta_h - \alpha(g) \otimes \alpha(h)$$

and as $\alpha \in H^1(G_1, A)$, $\alpha(g) \otimes \alpha(h) = \alpha(gh)$. Therefore,

$$\eta_{gh} + \alpha(gh) - \eta_g \otimes \eta_h - \alpha(g) \otimes \alpha(h) = \eta_{gh} - \eta_g \otimes \eta_h$$

and we see that $\mu - \mu' = \partial^2(\eta)$ is invariant for the action of $H^1(G_1, A)$ on η. Therefore $\{\eta : F_{f,\mu} \to F_{f,\mu'}\}$ is a torsor over $H^1(G, A)$.

Similarly, for a fixed homomorphism $f : G_1 \to G_2$, the set of $\mu : G_1 \times G_1 \to A$ that satisfies

$$\omega_1(g,h,l)\mu(gh,l)\mu(g,h) = \mu(g,hl)\mu(h,l)\omega_2(f(g),f(h),f(l))$$

parametrizing the isomorphism classes of $F_{f,\mu}$ is a torsor over $H^2(G_1, A)$, as any α in the kernel of

$$\{G_1 \times G_1 \to A\} \longrightarrow \{G_1 \times G_1 \times G_1 \to A\}$$

satisfies $\alpha(gh,l) + \alpha(g,h) - \alpha(g,hl) - \alpha(h,l) = 0$.

Lastly, equivalence classes of monoidal categories $\mathcal{C}_G^\omega$ are parametrized by the set $H^3(G,A)/Aut(G)$. However, A is abelian, so $Inn(G)$ must be trivial.

Hence we have that the parametrization is given by $H^3(G,A)/Out(G)$.

All of the above properties we derived about $\mathcal{C}_G^\omega$ can be specialized to Vec_G^ω. Take $A = \mathcal{K}^\times$, and let our monoidal functors to be additive. Then, as any morphism η of monoidal functors has the property that $\eta_1 \neq 0$, and using $\mu'(g,g^{-1})(\eta_g \otimes \eta_{g^{-1}}) = \eta_1\mu(g,g^{-1})$, we have that all η_g must be nonzero. Hence any $\eta : F_{f,\mu} \to F_{f',\mu'}$ must be an isomorphism, and hence $f = f'$.

Definition 2.1.15. *Let G be a group, and $\mathcal{K}$ be a field. Let Vec_G^ω be the category of G graded vector spaces whose associativity isomorphism is given by the 3-cocycle ω. Then ω is called normalized if $\omega(g,1,1) = \omega(1,1,g) = 1 \in \mathcal{K}^\times$.*

Note that by the triangle axiom, $\omega(g,1,h) = \omega(g,1,1)\omega(1,1,h)$, and hence we can alternatively define normalized by $\omega(g,1,h) = 1$.

Proposition 2.1.16. *For any 3-cocycle ω, ω is cohomologous to normalized ω.*

Proof. Let μ be a 2-cochain satisfying $\mu(g,1) = \omega(g,1,1)$, and $\mu(1,h) = \omega(1,1,h)^{-1}$.

Then, $\partial^3(\mu)(g,1,h) = g \cdot \mu(1,h)\mu(g,h)^{-1}\mu(g,h)\mu(g,1)^{-1}$, and as G acts trivially on A, we have

$$\begin{aligned}\partial^3(\mu)(g,1,h) =&\mu(1,h)\mu(g,h)^{-1}\mu(g,h)\mu(g,1)^{-1}\\ =&\mu(1,h)\mu(g,1)^{-1}\\ =&\omega(1,1,h)^{-1}\omega(g,1,1)^{-1}\\ =&\omega(g,1,h)^{-1}\end{aligned}$$

and we see that $\omega \cdot \partial^3(\mu)$ is normalized. Hence, ω and its normalized cocycles are cohomologous.

□

Now consider the case where $G = \mathbb{Z}/n\mathbb{Z}$ where $n > 1$ is an integer, and let $\mathcal{K} = \mathbb{C}$. As usual, assume that the action of G on $\mathcal{K}$ is trivial. Let us consider the cohomology of $\mathbb{Z}/n\mathbb{Z}$.

Let $ZG = \mathbb{Z}[x]/ < x^n - 1 >$, and take the following resolution:

$$0 \longleftarrow ZG \overset{g-1}{\longleftarrow} ZG \overset{g^{n-1}+\cdots+1}{\longleftarrow} ZG \longleftarrow \cdots .$$

We can apply the functor $Hom_{ZG}(-,\mathbb{C})$ to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_{ZG}(ZG,\mathbb{C}) & \xrightarrow{g-1} & Hom_{ZG}(ZG,\mathbb{C}) & \xrightarrow{1+\cdots+g^{n-1}} & Hom_{ZG}(ZG,\mathbb{C}) & \xrightarrow{g-1} & \cdots \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ 0 & \longrightarrow & \mathbb{C} & \xrightarrow{0} & \mathbb{C} & \xrightarrow{\times n} & \mathbb{C} & \xrightarrow{0} & \cdots \end{array}$$

Clearly $Hom_{ZG}(ZG,\mathbb{C}) \to \mathbb{C}$ is an isomorphism, as the map is entirely determined by where we map $1 \in ZG$. Also, as the action of G on $\mathbb{C}$ is trivial, the bottom complex alternates between the 0 map and the map $\times n$, defined as multiplication by n.

Computing the cohomology, we have that

$$\begin{aligned} H^0 =& \mathbb{C} \\ H^1 =& (0 = ker(\times n)/im(0) = 0) = 0 \\ H^2 =& (\mathbb{C} = ker(0)/im(\times n) = \mathbb{C}) = 0 \\ & \vdots \end{aligned}$$

Hence for every $i > 0$, $H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}) = 0$.

Now, $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \longrightarrow 0$ is a short exact sequence, and we can construct the following long exact sequence.

$$\begin{array}{ccccc} 0 \longrightarrow Hom(ZG, \mathbb{Z}) & \longrightarrow & Hom(ZG, \mathbb{C}) & \longrightarrow & Hom(ZG, \mathbb{C}^\times) \\ & & & & \downarrow \\ H^1(ZG, \mathbb{C}^\times) & \longleftarrow & H^1(ZG, \mathbb{C}) = 0 & \longleftarrow & H^1(ZG, \mathbb{Z}) \\ \downarrow & & & & \\ H^2(ZG, \mathbb{Z}) & \longrightarrow & H^2(ZG, \mathbb{C}) = 0 & \longrightarrow & H^2(ZG, \mathbb{C}^\times) \longrightarrow \cdots \end{array}$$

Hence we get that for every $i > 0$,

$$0 \to H^i(ZG, \mathbb{C}^\times) \to H^{i+1}(ZG, \mathbb{Z}) \to 0$$

is exact, and so $H^i(ZG, \mathbb{C}^\times) \simeq H^{i+1}(ZG, \mathbb{Z})$.

Also, by applying the functor $Hom_{ZG}(-, \mathbb{Z})$ to

$$0 \longleftarrow ZG \overset{g-1}{\longleftarrow} ZG \overset{g^{n-1}+\cdots+1}{\longleftarrow} ZG \longleftarrow \cdots$$

it is easy to see that $H^{2i-1}(ZG, \mathbb{Z}) = 0$ and $H^{2i}(ZG, \mathbb{Z} = \mathbb{Z}/n\mathbb{Z}$ for $i > 0$, and $H^0(ZG, \mathbb{Z}) = \mathbb{Z}$.

Now, $H^*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ is the cohomology ring given by $\bigoplus_{i\in\mathbb{Z}^+} H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$, and the cup product $\smile$ is defined by

$$H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \smile H^j(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = H^{i+j}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}).$$

Take α to be the generator of $H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. It is well known that cup product by α induces an isomorphism $H^i(ZG, \mathbb{Z}) \xrightarrow{\sim} H^{i+2}(ZG, \mathbb{Z})$ for all $i \in \mathbb{Z}$. For proof of this result, see Cassels-Frohlich, Section 8, Theorem 5.

2.1.4 Group Actions on Categories

We have talked a lot about G-gradings on categories so far, but then one may ask, is it possible to have group actions on categories? In order to define such a concept let us first begin by defining a special category.

Definition 2.1.17. *Let G be a group. Then, we define $Cat(G)$ to be the monoidal category with objects as elements of G, with the only morphisms being the identity morphisms on the elements, and $\otimes$ as multiplication in G.*

In the language of G-graded categories, $Cat(G)$ is equivalent to $\mathcal{C}_G(1)$, where 1 is the trivial group.

Now that we have a monoidal category that is essentially a group, we can define group actions on categories.

Definition 2.1.18. *Let G be a group, and $Aut(\mathcal{C})$ be the category of auto-equivalences of $\mathcal{C}$, whose morphisms are isomorphisms of functors. If $\mathcal{C}$ is a monoidal category, then we write $Aut_{\otimes}(\mathcal{C})$ to denote the category of monoidal auto-equivalences of $\mathcal{C}$. Then,*

- *An action of G on a category $\mathcal{C}$ is a monoidal functor*

$$T : Cat(G) \longrightarrow Aut(\mathcal{C}).$$

- *An action of G on a monoidal category $\mathcal{C}$ is a monoidal functor*

$$T : Cat(G) \longrightarrow Aut_{\otimes}(\mathcal{C}).$$

If $\mathcal{C}$ admits such a functor, we say G acts on $\mathcal{C}$.

Notice that in the above definition $\mathcal{C}$ need not be a monoidal category for the monoidal functor T to be defined. Indeed if we have the isomorphisms in $Aut(\mathcal{C})$ that play nicely with composition T_g's, then we may have a monoidal structure on T.

Let G act on the category $\mathcal{C}$, and let $g \longrightarrow T_g$ be the corresponding action. For every $g, h \in G$, let $\gamma_{g,h} : T_g \circ T_h \xrightarrow{\sim} T_{gh}$ be the isomorphism that defines the monoidal structure on the functor $Cat(G) \to Aut(\mathcal{C})$.

Definition 2.1.19. *A G-equivariant object in $\mathcal{C}$ is a pair (X, u), with X and object in $\mathcal{C}$ and a family of isomorphisms $u = \{u_g : T_g(X) \xrightarrow{\sim} X,\ g \in G\}$ such that*

$$\begin{array}{ccc} T_g(T_h(X)) & \xrightarrow{T_g(u_h)} & T_g(X) \\ \downarrow{\scriptstyle \gamma_{g,h}(X)} & & \downarrow{\scriptstyle u_g} \\ T_{gh}(X) & \xrightarrow{\quad u_{gh} \quad} & X \end{array}$$

commutes for all $g, h \in G$.

The morphisms of equivariant objects are morphisms in $\mathcal{C}$ that commute with u_g for every $g \in G$.

We denote the category of equivariant objects of $\mathcal{C}$, or the G-equivariantization of $\mathcal{C}$ as $\mathcal{C}^G$.

Similarly as before, we can examine a linear case by letting G act on $\overline{Vec}$, the category of finite dimensional $\mathcal{K}$-linear vector spaces. In this case we have the following proposition.

Proposition 2.1.20. *The action of G on the category $\overline{Vec}$ as an abelian category correspond to elements of $H^2(G, \mathcal{K}^\times)$, whereas any action of G on $\overline{Vec}$ as a monoidal category is trivial.*

Proof. G acts on $\overline{Vec}$ by $T : Cat(G) \to Aut(\overline{Vec})$. Let us begin by viewing $\overline{Vec}$ as an abelian category.

First, notice that $Aut(\overline{Vec})$ is a subcategory of $End(\overline{Vec})$. It follows that associativity is given by the identity, and $\otimes$ is the composition of automorphisms.

As T is a monoidal functor from $Cat(G)$ to $Aut(\overline{Vec})$, it must satisfy the following monoidal structure axiom.

$$\begin{array}{ccc}
(T(g)\otimes T(h))\otimes T(l) & \xrightarrow{id} & T(g)\otimes (T(h)\otimes T(l)) \\
\downarrow{\scriptstyle (\mu(g,h)\cdot id_{T(gh)})\otimes id_{T(l)}} & & \downarrow{\scriptstyle id_{T(g)}\otimes(\mu(h,l)\cdot id_{T(hl)})} \\
T(gh)\otimes T(l) & & T(g)\otimes T(hl) \\
\downarrow{\scriptstyle \mu(gh,l)\cdot id_{T(ghl)}} & & \downarrow{\scriptstyle \mu(g,hl)\cdot id_{T(ghl)}} \\
T((gh)l) & \xrightarrow{id} & T(g(hl))
\end{array}$$

and this gives us

$$\mu(gh, l)\mu(g, h) = \mu(g, hl)\mu(h, l)$$

where $\mu : G \times G \to \mathcal{K}^\times$.

Hence we see that $\mu \in ker(\partial^3) = Z^2(G, \mathcal{K}^\times)$.

Notice that any monoidal functor T induces a μ, and any μ satisfying the 2-cocycle condition, together with a functor T, turns T into a monoidal functor.

Hence, given a functor $T : Cat(G) \to Aut(\overline{Vec})$, the set of μ parametrizes the isomorphism classes of monoidal functors T_μ.

But now, any monoidal functor satisfies the following diagram:

$$\begin{array}{ccc} T(g) \otimes T(h) & \xrightarrow{\mu(g,h)id_{gh}} & T(gh) \\ {\scriptstyle \eta_g \otimes \eta_h}\downarrow & & \downarrow{\scriptstyle \eta_{gh}} \\ T'(g) \otimes T'(h) & \xrightarrow{\mu'(g,h)id_{gh}} & T'(gh) \end{array}$$

The morphisms in $Aut(\overline{Vec})$ are given by multiplication by $k \in \mathcal{K}^\times$, so $\eta_g : T(g) \to T'(g)$ for every $g \in G$ induces $\eta : G \to \mathcal{K}^\times$, and the converse also holds. Hence, we have that $\mu = \mu' \cdot \partial^2(\eta)$, and this tells us that the two functors $T_\mu = \bigoplus T(g)$, and $T_{\mu'} = \bigoplus T'(g)$ are isomorphic.

Hence the set of μ, given by $H^2(G, \mathcal{K}^\times)$ parametrizes the classes of monoidal functors $T_\mu : Cat(G) \to Aut(\overline{Vec})$.

Now we will view $\overline{Vec}$ as a monoidal category. Taking $\overline{Vec}$ as a monoidal category, we have that $T_g : \overline{Vec} \to \overline{Vec}$ is a monoidal equivalence for all $g \in G$. $\overline{Vec}$ is the category of finite dimensional vector spaces over $\mathcal{K}$, so any simple object of $\overline{Vec}$ is 1 dimensional. It follows that any isomorphism of simple objects are entirely determined by $\phi : \mathcal{K} \overset{\times k}{\tilde{\to}} \mathcal{K}$. Hence for any monoidal functors T_g, T_h, $\mu^g_{X,Y}$ satisfying $J^g_{X,Y} = \mu^g_{X,Y} \cdot id_{X \otimes Y}$, and $\mu^h_{X,Y}$ satisfying similar conditions, satisfies the following commutative diagram.

$$\begin{array}{ccc} T_g(X) \otimes T_g(Y) & \xrightarrow{\mu^g_{X,Y} \cdot id_{X \otimes Y}} & T_g(X \otimes Y) \\ {\scriptstyle \eta_X \otimes \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_{X \otimes Y}} \\ T_h(X) \otimes T_h(Y) & \xrightarrow{\mu^h_{X,Y} \cdot id_{X \otimes Y}} & T_h(X \otimes Y) \end{array}$$

So T_g's are isomorphic pairwise for all $g \in G$. In particular, T_1 must be the identity functor for $1 \in G$. Therefore T is trivial. $\square$

2.1.5 Duals in Monoidal Categories

Now we are almost ready to talk about rigid monoidal categories, but in order to do so, we must talk about the notion of "dual".

Definition 2.1.21. *Let $(\mathcal{C}, \otimes, 1, a, \iota)$ be a monoidal category, and let X be an object of $\mathcal{C}$. Then, an object X^* in $\mathcal{C}$ is said to be a left dual of X if there exist morphisms $ev_X : X^* \otimes X \to 1$ and $coev_X : 1 \to X \otimes X^*$ called evaluation and coevaluation such that*

$$X = 1 \otimes X \stackrel{coev_X \otimes id_X}{\longrightarrow} (X \otimes X^*) \otimes X \stackrel{a_{X,X^*,X}}{\longrightarrow} X \otimes (X^* \otimes X) \stackrel{id_X \otimes ev_X}{\longrightarrow} X \otimes 1 = X$$

$$X^* = X^* \otimes 1 \stackrel{id_{X^*} \otimes coev_X}{\longrightarrow} X^* \otimes (X \otimes X^*) \stackrel{a^{-1}_{X^*,X,X^*}}{\longrightarrow} (X^* \otimes X) \otimes X^* \stackrel{ev_X \otimes id_{X^*}}{\longrightarrow} 1 \otimes X^* = X^*$$

are identities.

An object ${}^*X$ in $\mathcal{C}$ is called the right dual of X if there exist morphisms $ev'_X : X \otimes {}^*X \to 1$ and $coev'_X : 1 \to {}^*X \otimes X$ such that

$$X = X \otimes 1 \stackrel{id_X \otimes coev'_X}{\longrightarrow} X \otimes ({}^*X \otimes X) \stackrel{a^{-1}_{X,{}^*X,X}}{\longrightarrow} (X \otimes {}^*X) \otimes X \stackrel{ev'_X \otimes id_X}{\longrightarrow} 1 \otimes X = X$$

$${}^*X = 1 \otimes {}^*X \stackrel{coev'_X \otimes id_{{}^*X}}{\longrightarrow} ({}^*X \otimes X) \otimes {}^*X \stackrel{a^{-1}_{{}^*X,X,{}^*X}}{\longrightarrow} {}^*X \otimes (X \otimes {}^*X) \stackrel{id_{{}^*X} \otimes ev'_X}{\longrightarrow} {}^*X \otimes 1 = {}^*X$$

are identities.

Notice that if X^* is a left dual of X, then X is a right dual of X^* by taking $ev_X = ev'_{X^*}$, and $coev_X = coev'_{X^*}$. Hence ${}^*(X^*) \simeq X \simeq ({}^*X)^*$ for any X admitting left and right duals. Also, in any monoidal category, $1^* = {}^*1 = 1$ with respect to ι and ι^{-1}.

Now, there is a well-defined notion of adjoint, and we want to show that our notion of left and right duals coincides with the notion of adjoints. In order to do so, let us define what adjoint functors are.

Definition 2.1.22. *A functor $F : \mathcal{D} \to \mathcal{C}$ is a left adjoint functor if for each object X in $\mathcal{C}$, there exists a terminal morphism from F to X. If, for each object X in $\mathcal{C}$, we choose an object G_0X of $\mathcal{D}$ for which there is a terminal morphism $\epsilon_X : F(G_0X)X$ from F to X, then there is a unique functor $G : \mathcal{C} \to \mathcal{D}$ such that $G(X) = G_0X$ and $\epsilon_{X'} \circ FG(f) = f \circ \epsilon_X$ for $f : X \to X$ a morphism in $\mathcal{C}$; F is then called a left adjoint to G.*

The right adjoint of a functor is similarly defined.

It is much more common to define adjoint functors using $Hom(-,-)$ functors, but the above definition is much more useful in our case to show that our notion of duals coincide with the notion of adjoints.

Proof. Suppose $G : \mathcal{C} \to \mathcal{C}$ is an endofunctor of $\mathcal{C}$, and has a left dual, denoted F. Then, we have the natural isomorphism

$$ev_G : F \otimes G \xrightarrow{\sim} 1$$

but note that in the category of $End(\mathcal{C})$, $\otimes$ is given by composition.

Hence, for every $X \in \mathcal{C}$,

$$\epsilon_X = ev_G(X) := FG(X) = (F \otimes G)(X) \to 1(X) = X.$$

Also, if $f : X \to X'$ is a morphism in $\mathcal{C}$ for some $X, X' \in \mathcal{C}$, we have the following commuting diagram by definition of ev_G.

$$\begin{array}{ccc} FG(X) & \xrightarrow{FG(f)} & FG(X') \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_{X'}} \\ X & \xrightarrow{f} & X' \end{array}$$

Hence $\epsilon_{X'} \circ FG(f) = f \circ \epsilon_X$, and we have that F is the left adjoint of G.

We now want to show the opposite direction, so let us suppose that F is the left adjoint of G in $End(\mathcal{C})$. We have that for every $X \in \mathcal{C}$, $G(X)$ is an object with a terminal morphism $\epsilon_X : FG(X) \to X$, and $\epsilon_{X'} \circ FG(f) = f \circ \epsilon_X$ for any $f : X \to X'$ in $\mathcal{C}$.

In particular, $FG = F \otimes G$ in $End(\mathcal{C})$, so we take ev_G to be ϵ. Since $ev_G = \epsilon$ defines an isomorphism between $F \otimes G$ and 1, and similarly, one can define an isomorphism ϵ' between $G \otimes F$ and 1, as G is the right adjoint of F. Let $coev_G = \epsilon'^{-1}$.

Now, for any object $X \in \mathcal{C}$, $(id_G \otimes \epsilon)(G \otimes (F \otimes G))(X) = id_G(G)(X) = G(X)$, and similarly $(\epsilon' \otimes id_G)((G \otimes F) \otimes G)(X) = (1 \otimes id_G)(1 \otimes G)(X) = G(X)$, so clearly the following diagram commutes.

$$\begin{array}{ccc} G \otimes (F \otimes G) & \xrightarrow{id_G \otimes \epsilon} & G \otimes 1 = G \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle id_G} \\ (G \otimes F) \otimes G & \xrightarrow{\epsilon' \otimes id_G} & 1 \otimes G = G \end{array}$$

However, $End(\mathcal{C})$ is a strict monoidal category, which tells us that $a_{G,F,G}$ is not just an isomorphism, but an equality. Also, ϵ' is an isomorphism, so we can reverse the bottom arrow, and rewriting ev_G and $coev_G$ as we've defined before we have,

$$\begin{array}{ccc} G \otimes (F \otimes G) & \xrightarrow{id_G \otimes ev_G} & G \\ {\scriptstyle a_{G,F,G}}\big\uparrow & & \big\uparrow{\scriptstyle id_G} \\ (G \otimes F) \otimes G & \xleftarrow[coev_G \otimes id_G]{} & G \end{array}$$

Note that this diagram precisely defines the first condition of left duals.

Similarly, we get the following commuting diagram

$$\begin{array}{ccc}
(F \otimes G) \otimes F & \xrightarrow{ev_G \otimes id_F} & F \\
{\scriptstyle a^{-1}_{F,G,F}} \uparrow & & \uparrow {\scriptstyle id_F} \\
F \otimes (G \otimes F) & \xleftarrow[id_F \otimes coev_G]{} & F
\end{array}$$

and the two diagrams show that the left adjoint functors are precisely the left dual objects in $End(\mathcal{C})$. The proof is similar for the right adjoint functors and right dual objects.

□

If we have left/right dual objects, we want them to be unique, and fortunately, our notion of dual has such a property.

Proposition 2.1.23. *Let $\mathcal{C}$ be a monoidal category. If $X \in \mathcal{C}$ has a left/right dual object, then it is unique up to a unique isomorphism.*

Proof. Let X_1^*, X_2^* be two left duals of X. Let (e_1, c_1) and (e_2, c_2) be the *ev* and *coev* maps in respective order.

We now have a morphism $\alpha : X_1^* \to X_2^*$ defined by

$$\alpha := X_1^* \overset{id_{X_1^*} \otimes c_2}{\longrightarrow} X_1^* \otimes (X \otimes X_2^*) \overset{a^{-1}_{X_1^*, X, X_2^*}}{\longrightarrow} (X_1 \otimes X) \otimes X_2^* \overset{e_1 \otimes id_{X_2^*}}{\longrightarrow} X_2^*$$

and similarly we have a morphism $\beta : X_2^* \to X_1^*$ defined by

$$\beta := X_2^* \overset{id_{X_2^*} \otimes c_1}{\longrightarrow} X_2^* \otimes (X \otimes X_1^*) \overset{a^{-1}_{X_2^*, X, X_1^*}}{\longrightarrow} (X_2 \otimes X) \otimes X_1^* \overset{e_2 \otimes id_{X_1^*}}{\longrightarrow} X_1^*.$$

By suppressing the associativity law, (i.e. just by taking $(A \otimes B) \otimes C = A \otimes (B \otimes C)$) we get the following diagram:

$$\begin{array}{ccccc}
X_1^* & \xrightarrow{id_{X_1^*}\otimes c_1} & X_1^*\otimes X\otimes X_1^* & \overset{id}{\searrow} & \\
{\scriptstyle id\otimes c_2}\downarrow & & {\scriptstyle id\otimes c_2\otimes id}\downarrow & & \\
X_1^1\otimes X\otimes X_2^* & \xrightarrow{id\otimes c_1} & X_1^*\otimes X\otimes X_2^*\otimes X\otimes X_1^* & \xrightarrow{id\otimes e_2\otimes id} & X_1^*\otimes X\otimes X_1^* \\
{\scriptstyle e_1\otimes id}\downarrow & & {\scriptstyle e_1\otimes id}\downarrow & & \downarrow{\scriptstyle e_1\otimes id} \\
X_2^* & \xrightarrow{id\otimes c_1} & X_2^*\otimes X\otimes X_1^* & \xrightarrow{e_2\otimes id} & X_1^*
\end{array}$$

Note that the composition of maps on the left side is α, the composition of maps on the top is $id_{X_1^*}$, and the composition of maps on the bottom is β. Hence $\beta\circ\alpha = id_{X_1^*}$, and using similar construction we can see that $\alpha\circ\beta = id_{X_2^*}$.

□

The uniqueness of the isomorphism above has a very subtle nuance. It should rather be interpreted as being "unique up to a canonical isomorphism". The canonical isomorphism is precisely given by α, i.e. $(e_1\otimes id)\circ(id\otimes c_2)$ and indeed if we alter α to α', this would induce a different e_1' and c_2', thereby altering the dual structure.

Now that we have the duality of objects, let us talk about duality of morphisms.

Definition 2.1.24. *Let $\mathcal{C}$ be a monoidal category. If X, Y are objects in $\mathcal{C}$ that have left duals X^*, Y^*, and $f : X \to Y$ is a morphism in $\mathcal{C}$, one defines the left dual $f^* : Y^* \to X^*$ of f by*

$$f^* := Y^* \overset{id_{Y^*}\otimes coev_X}{\longrightarrow} Y^*\otimes(X\otimes X^*) \overset{a^{-1}_{Y^*,X,X^*}}{\longrightarrow} (Y^*\otimes X)\otimes X^* \overset{(id_{Y^*}\otimes f)\otimes id_{X^*}}{\longrightarrow} (Y^*\otimes Y)\otimes X^* \overset{ev_Y\otimes id_{X^*}}{\longrightarrow} X^*$$

Likewise, if X, Y have right duals ${}^*X, {}^*Y$, and $f : X \to Y$ is a morphism in $\mathcal{C}$, we define the right dual ${}^*f : {}^*Y \to {}^*X$ of f by

$${}^*f := {}^*Y \overset{coev'_X\otimes id_{{}^*Y}}{\longrightarrow} ({}^*X\otimes X)\otimes{}^*Y \overset{a_{{}^*X,X,{}^*Y}}{\longrightarrow} {}^*X\otimes(X\otimes{}^*Y) \overset{id_{{}^*X}\otimes(f\otimes id_{{}^*Y})}{\longrightarrow} {}^*X\otimes(Y\otimes{}^*Y) \overset{id_{{}^*X}\otimes ev'_Y}{\longrightarrow} {}^*X$$

We would also like for the notion of dual objects and morphisms to behave nicely with monoidal functors, and indeed, we will see that if F is a monoidal functor, $F(X)^* = F(X^*)$. For this proposition, we want to be rigorous in showing that dual objects behave nicely with monoidal functors, so we will not be using the convention $\overline{1} = F(1)$, and use the canonical isomorphism $\phi : \overline{1} \xrightarrow{\sim} F(1)$.

Proposition 2.1.25. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and let $X, X^*\mathcal{C}$ be objects such that X^* is the left dual of X. Let (F, J, ϕ) be a monoidal functor from $\mathcal{C}$ to $\mathcal{D}$. Then, $F(X^*)$ is the left dual of $F(X)$.*

Proof. To show this, we define $ev_{F(X)}, coev_{F(X)}$ to be:

$$ev_{F(X)} := F(X^*) \otimes F(X) \overset{J_{X^*,X}}{\to} F(X^* \otimes X) \overset{F(ev_X)}{\to} F(1) \overset{\phi^{-1}}{\to} 1$$

$$coev_{F(X)} := 1 \overset{\phi}{\to} F(1) \overset{F(coev_X)}{\to} F(X \otimes X^*) \overset{J^{-1}_{X,X^*}}{\to} F(X) \otimes F(X^*)$$

Now we want to show that $ev_{F(X)}$ and $coev_{F(X)}$ satisfy the identity map.

$$\begin{array}{ccccccccc}
F(X) & \longrightarrow & F(1) \otimes F(X) & \xrightarrow{F(coev_X) \otimes id_{F(X)}} & F(X \otimes X^*) \otimes F(X) & \overset{J^{-1}_{X,X^*} \otimes id}{\longrightarrow} & (F(X) \otimes F(X^*)) \otimes F(X) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle J_{1,X} = id} & & \downarrow{\scriptstyle J_{X \otimes X^*, X}} & & \downarrow{\scriptstyle J_{X \otimes X^*, X} \circ (J_{X,X^*} \otimes id)} & & \\
F(X) & \longrightarrow & F(1 \otimes X) & \xrightarrow{F(coev_X \otimes id_X)} & F((X \otimes X^*) \otimes X) & \xrightarrow{id} & F((X \otimes X^*) \otimes X) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\xrightarrow{\overline{a}_{F(X),F(X^*),F(X)}} & F(X) \otimes (F(X^*) \otimes F(X)) & \overset{id \otimes J_{X^*,X}}{\longrightarrow} & F(X) \otimes F(X^* \otimes X) & \xrightarrow{id \otimes F(ev_X)} & F(X) \otimes F(1) & \longrightarrow & F(X) \\
 & \downarrow{\scriptstyle J_{X,X^* \otimes X} \circ (id \otimes J_{X^*,X})} & & \downarrow{\scriptstyle J_{X,X^* \otimes X}} & & \downarrow{\scriptstyle J_{X,1} = id} & & \downarrow{\scriptstyle id} \\
\xrightarrow{F(a_{X,X^*,X})} & F(X \otimes (X^* \otimes X)) & \xrightarrow{id} & F(X \otimes (X^* \otimes X)) & \xrightarrow{F(id \otimes ev_X)} & F(X \otimes 1) & \longrightarrow & F(X)
\end{array}$$

It is easy to see that this diagram commutes, and represents the first identity condition for $ev_{F(X)}$ and $coev_{F(X)}$. The other identity conditions can be shown in similar ways.

□

In a similar manner, one can check that if $U, V, W \in \mathcal{C}$ with left/right duals, and if $f : V \to W$ and $g : U \to V$, then $fg)^* = g^* f^*$. Also, $U \otimes V$ has a left dual $(U \otimes V)^* = V^* \otimes U^*$ and a right dual is similarly given.

We have shown that the notion of duality behaves well with monoidal functors, but the notion has even nicer properties with the $Hom_{\mathcal{C}}(-,-)$ functor.

Proposition 2.1.26. *Let $\mathcal{C}$ be a monoidal category, and let U, V, W be an object in $\mathcal{C}$.*

- *If V has a left dual V^*, then there are natural adjunction isomorphisms:*

$$Hom_{\mathcal{C}}(U \otimes V, W) \xrightarrow{\sim} Hom_{\mathcal{C}}(U, W \otimes V^*)$$

$$Hom_{\mathcal{C}}(U, V \otimes W) \xrightarrow{\sim} Hom_{\mathcal{C}}(V^* \otimes U, W)$$

- *If V has a right dual ${}^*V$, then there are natural adjunction isomorphisms:*

$$Hom_{\mathcal{C}}(U \otimes {}^*V, W) \xrightarrow{\sim} Hom_{\mathcal{C}}(U, W \otimes V)$$

$$Hom_{\mathcal{C}}(U, {}^*V \otimes W) \xrightarrow{\sim} Hom_{\mathcal{C}}(V \otimes U, W)$$

Proof. We will prove the two adjunction isomorphisms for the left dual case.

Let $f : U \otimes V \to W$, and $g : U \to W \otimes V^*$.

The first adjunction is given by the maps

$$\phi : f \longrightarrow (f \otimes id_{V^*}) \circ (id_U \otimes coev_V)$$

$$\psi : g \longrightarrow (id_W \otimes ev_V) \circ (g \otimes id_V)$$

Notice that $(\psi \circ \phi)(f)$ is given by the following diagram:

$$
\begin{array}{ccccccc}
U \otimes V & \xrightarrow{id \otimes coev_V \otimes id} & U \otimes (V \otimes V^*) \otimes V & \xrightarrow{f \otimes id \otimes id} & W \otimes V^* \otimes V & \xrightarrow{id_W \otimes ev_V} & W \\
{\scriptstyle id}\uparrow & & {\scriptstyle id}\uparrow & & {\scriptstyle f \otimes id \otimes id}\uparrow & & {\scriptstyle f}\uparrow \\
U \otimes V & \xrightarrow{id \otimes coev_V \otimes id} & U \otimes (V \otimes V^*) \otimes V & \xrightarrow{id} & U \otimes V \otimes V^* \otimes V & \xrightarrow{id_{U \otimes V} \otimes ev_V} & U \otimes V
\end{array}
$$

The diagram obviously commutes, and the bottom row is just the identity, and hence $(\psi \circ \phi)(f) = f$. Using a similar construction, we can show that $\phi \circ \psi)(f) = f$, and hence we have that $\psi = \phi^{-1}$, and conclude that ϕ is an isomorphism.

Now let $f : U \otimes^* V \to W$, and $g : U \to W \otimes V$.

For the second adjunction isomorphism, we can take two maps:

$$\phi : f \longrightarrow (id_V \otimes f) \circ (coev_V \otimes id_u)$$

$$\psi : g \longrightarrow (ev_V \otimes id_W) \circ (id_{V^*} \otimes g)$$

As in the above case, we have the diagram of $(\psi \circ \phi)(f)$ given by:

$$
\begin{array}{ccccccc}
V^* \otimes U & \xrightarrow{id \otimes coev_V \otimes id} & V^* \otimes V \otimes V^* \otimes U & \xrightarrow{id \otimes id \otimes f} & V^* \otimes V \otimes W & \xrightarrow{ev_V \otimes id_W} & W \\
{\scriptstyle id}\uparrow & & {\scriptstyle id}\uparrow & & {\scriptstyle id \otimes id \otimes f}\uparrow & & {\scriptstyle f}\uparrow \\
V^* \otimes U & \xrightarrow{id \otimes coev_V \otimes id} & V^* \otimes V \otimes V^* \otimes U & \xrightarrow{id} & V^* \otimes V \otimes V^* \otimes U & \xrightarrow{ev_V \otimes id_{V^* \otimes W}} & V^* \otimes U
\end{array}
$$

Hence we have that $(\psi \circ \phi)(f) = f$, and by a similar diagram, $(\phi \circ \psi)(f) = f$. Therefore, $\psi = \phi^{-1}$, and ϕ is an isomorphism.

$\square$

One can notice that Proposition 18 precisely tells us that if V has a left dual, then the left adjoint functor of $(V \otimes -)$ is $(V^* \otimes -)$, and if V has a right dual, then the right adjoint functor of $(- \otimes V)$ is $(- \otimes V^*)$.

Also, Proposition 18 provides another proof of Proposition 16, but this requires the Yoneda Lemma. While the Yoneda Lemma is not required for the purposes of this thesis, it is a very significant result in category theory. The statement is that if $\mathcal{C}$ is a locally small category, then

$$Nat(Hom_{\mathcal{C}}(*, -), Hom_{\mathcal{C}}(*, -)) \simeq Hom_{\mathcal{C}}(*, *).$$

This result generalizes Cayley's Theorem, and in fact, we can retrieve Cayley's Theorem by taking $\mathcal{C}$ to be a category with a single object with all of its morphisms as isomorphisms. Then, $Hom_{\mathcal{C}}(*, *)$ forms a group G under composition, and any covariant functor from $\mathcal{C}$ to Set takes the unique object of $\mathcal{C}$ to some set X, and the group of morphisms G to the group of permutations of X, and this tells us that X is a G-set. But then, a natural transformation between such functors is the equivariant maps from a G-set to itself, and the group of such equivariant maps form a subgroup of the group of permutations of G. The statement of Yoneda Lemma then tells us that the group of equivariant maps from a G-set to itself is isomorphic to G, hence G is isomorphic to a subgroup of the group of permutations of G, which is precisely the statement of Cayley's Theorem.

Now, using Proposition 18 and the Yoneda Lemma, we have this alternate proof of Proposition 16:

Proof. Let $\mathcal{C}$ be a monoidal category. If $X \in \mathcal{C}$ has a left dual X^*, then

$$Hom_{\mathcal{C}}(X^* \otimes U, W) \xrightarrow{\sim} Hom_{\mathcal{C}}(U, X \otimes W).$$

If we set $U = 1$, then we have the natural isomorphism

$$Hom(X^*, W) \simeq Hom(1, X \otimes W)$$

so if X_1 and X_2 are left duals of X, we have the natural isomorphism $Hom(X_1, W) \simeq Hom(X_2, W)$. By the Yoneda Lemma, we then have the canonical isomorphism $X_1 \simeq X_2$, and we have the proposition.

□

Now that we've fully explored the notion of duality in a monoidal category, we are ready to define rigid monoidal categories.

Definition 2.1.27. *An object in a monoidal category is called rigid if it has left and right duals. A monoidal category $\mathcal{C}$ is rigid if every object of $\mathcal{C}$ is rigid.*

So what are some examples of rigid monoidal categories? Our notion of duality coincides with the notion of duality for vector spaces, and in fact $\mathcal{K}$-$\overline{Vec}$, the category of finite dimensional vector spaces over the field $\mathcal{K}$ is a rigid monoidal category. The dual of any object V is its dual space V^*, and the required maps are:

- $ev_V : V^* \otimes V \to \mathcal{K}$ being the contraction, and is defined by, if $f \in V^*$ and $v \in V$, $ev_V : f \otimes v \mapsto f(v) \in \mathcal{K}$.
- $coev_V : \mathcal{K} \to V \otimes V^*$ defined by, if $c \in \mathcal{K}$, $coev_V : c \mapsto \sum_i cv_i \otimes v_i^*$, where $\{v_1, \cdots, v_n\}$ is the basis of V, and $\{v_1^*, \cdots, v_n^*\}$ is the dual basis.

But note that $\mathcal{K}$-Vec, the category of all vector spaces (not necessarily finite) over $\mathcal{K}$ is not rigid, since coevaluation maps are not well-defined for infinite dimensional vector spaces.

Similarly, $\overline{Rep(G)}$, the category of finite dimensional representation of G over $\mathcal{K}$ is rigid. If we take ρ as usual to be the G-representation, $\rho_{V^*}(g) = (\rho_V(g^{-1}))^*$, and this is usually given by $(\rho_V(g^{-1}))^T$, which is called the contragradient representation. Recall that the dual space is the vector space of linear functionals from V to $\mathcal{K}$, so the action of a functional in the dual space on a vector in V can be represented by the matrix multiplication, $f^T v$, and hence to be consistent with this action, we define the evaluation and coevaluation maps to be:

$$ev_G : \rho_{V^*}(g) \otimes \rho_V(g) \mapsto \rho_V(g^{-1})\rho_V(g) = \rho_V(g^{-1}g) = 1$$

$$coev_G : 1 \mapsto \sum_{g \in G} \rho_{V^*}(g) \otimes \rho_V(g)$$

Another example of a rigid monoidal category is $\overline{Vec_G}$, the category of finite dimensional G-graded vector spaces over $\mathcal{K}$, but it is rigid if and only if G is a group. We've already discussed the tensor structure on $\overline{Vec_G}$, given by $\delta_g \otimes \delta_h = \delta_{gh}$, so if we define $\delta_g^* = \delta_{g^{-1}}$, $\delta_g^* \otimes \delta_g = \delta_{g^{-1}}\delta_g = \delta_1 = \mathcal{K}$, and this induces the evaluation and coevaluation maps in an obvious way, where the coevaluation map is given by the diagonal sum. Notice that if we have that $\delta_g^* \neq \delta_{g^{-1}}$, i.e. G is not a group but a monoid, $Hom(\delta_g^* \otimes \delta_g, \mathcal{K}) = Hom(\delta_{hg}, \delta_1) = \emptyset$, where $hg \neq 1$. Hence we cannot define a dual structure, as we already have a predetermined tensor structure on the category. Therefore $\overline{Vec_G}$ is not rigid if G is not a group.

We can also think about the duality notion as being a functor. In a monoidal category $\mathcal{C}$ with left and right duals, one can define a left duality functor by

$$X \to X^*,\ f \to f^*,\ \mathcal{C} \to \mathcal{C}.$$

Clearly $(-)^*$ reverses the order of arrows, so it must be a contravariant functor. Also, the order of tensors are reversed so we have

$$(X \otimes Y)^* = Y^* \otimes X^* = X^* \otimes_{op} Y^*$$

$$(f \circ g)^* = g^* \circ f^*$$

$$a_{X,Y,Z} \xrightarrow{(-)^*} a^{-1}_{Z^*,Y^*,X^*}$$

where

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$$

$$a^{-1}_{Z^*,Y^*,X^*} : Z^* \otimes (Y^* \otimes X^*) \longrightarrow (Z^* \otimes Y^*) \otimes X^*$$

Hence we have that $(-)^* : \mathcal{C}^* \to \mathcal{C}^{op}$, where $\mathcal{C}^*$ is the dual category, is a monoidal functor, and it is easy to see from Proposition 18 that $(-)^*$ and ${}^*(-)$ are quasi-inverses. This tells us that at least for rigid monoidal categories, the dual categories and opposite categories are equivalent. (In the sense that the duality functors serve as monoidal equivalences)

We also have a very surprising result for the natural transformations between monoidal functors in rigid monoidal categories.

Proposition 2.1.28. *If $\mathcal{C}$, $\mathcal{D}$ are rigid monoidal categories, F_1, F_2 are monoidal functors and $\eta : F_1 \to F_2$ is a natural transformation, then η is an isomorphism.*

Proof. Let us explicitly write out the map ${}^*\eta_{X^*} : {}^*F_2(X^*) \to {}^*F_1(X^*)$.

$$\begin{aligned} {}^*\eta_{X^*} := {}^*F_2(X^*) &\xrightarrow{coev_{F_1(X^*)'} \otimes id} ({}^*F_1(X^*) \otimes F_1(X^*)) \otimes {}^*F_2(X^*) \\ &\xrightarrow{a} {}^*F_1(X^*) \otimes (F_1(X^*) \otimes {}^*F_2(X^*)) \\ &\xrightarrow{id \otimes \eta \otimes id} {}^*F_1(X^*) \otimes (F_2(X^*) \otimes {}^*F_2(X^*)) \\ &\xrightarrow{id \otimes ev'_{F_2(X^*)}} {}^*F_1(X^*) \end{aligned}$$

Using the explicit write-out of η, by naturality of η we obtain the following diagram:

$$\begin{array}{ccccccc} F_1(X) & \xrightarrow{coev \otimes id} & (F_1(X) \otimes F_1(X^*)) \otimes F_1(X) & \xrightarrow{a} & F_1(X) \otimes (F_1(X^*) \otimes F_1(X)) & \xrightarrow{ev} & F_1(X) \\ \downarrow \eta & & \downarrow id \otimes id \otimes \eta & & \downarrow id \otimes \eta \otimes \eta & & \downarrow \phi \\ F_2(X) & \xrightarrow{coev} & (F_1(X) \otimes F_1(X^*)) \otimes F_2(X) & \xrightarrow{\eta \circ a} & F_1(X) \otimes (F_2(X^*) \otimes F_2(X)) & \xrightarrow{ev} & F_1(X) \\ \downarrow id & & \downarrow \eta \otimes \eta \otimes id & & \downarrow \eta \otimes id \otimes id & & \downarrow \eta \\ F_2(X) & \xrightarrow{coev} & (F_2(X) \otimes F_2(X^*)) \otimes F_2(X) & \xrightarrow{a} & F_2(X) \otimes (F_2(X^*) \otimes F_2(X)) & \xrightarrow{ev} & F_2(X) \end{array}$$

Notice that all of the boxes commute except for the top right box, and in order for the top right box to commute, $\phi = id_{F_1(X)}$. Also notice that the top row is the identity map, and the middle row is precisely ${}^*\eta_{X^*}$, and by the commutativity of the diagram we have that $\eta \circ {}^*\eta_{X^*} = id$. Similarly, the middle row and the last row imply that

${}^*\eta_{X^*} \circ \eta = id$. Therefore, we have that ${}^*\eta_{X^*}$ is the inverse of η, thereby showing that η is an isomorphism.

□

Lastly, we can find a correspondence between special left modules and rigid bimodules over an algebra A.

Proposition 2.1.29. *$M \in A$-bimod has a dual(left) if and only if it is finitely generated projective when considered as a left A-module.*

Proof. First, note that having a left/right dual implies that it has a finite dual basis. So let us suppose M has a left dual, i.e. has a finite dual basis.

If we take $(m_1, \cdots, m_k)$ to be the basis of M, then $(m_1^*, \cdots, m_k^*)$ is the dual basis.

Now we take a free module F with basis $(x_1, \cdots, x_k)$ and a surjective homomorphism $\phi : F \to M$ defined by $\phi : x_i \mapsto m_i$.

But since we have a bijective correspondence between each $m_i^* : M \to A$ and x_i, we have the induced map:

$$\psi : \sum_{i=1}^{k} (m \cdot m_i^*) m_i \mapsto \sum_{i=1}^{k} (x \cdot m_i^*) x_i$$

Clearly, $\phi \circ \psi = id_M$, and ϕ is surjective. Hence we have:

$$\begin{array}{ccc} & & F(projective) \\ & \overset{\phi}{\nearrow} & \downarrow \phi \\ M & \xrightarrow{id} & M \end{array}$$

and we see that M is projective.

Conversely, suppose M is a finitely generated projective module.

Let F be a free module with an epimorphism from F to M. Then, as M is projective, this epimorphism splits, and we have a monomorphism from M to F. By choosing a basis of M, which is in our case $\{m_1, \cdots, m_k\}$, we have a finitely generated free module M' with $\{m'_1, \cdots, m'_k\}$ as the basis with an epimorphism $\phi : M' \to M$ defined by $m'_i \mapsto m_i$.

As M' is a free left A-module, for any $m' \in M'$, there exists $a_1, \cdots, a_k \in A$ such that $m' = \sum_{i=1}^{k} a_i m'_i$. So we will define a map:

$$m^* : M' \to A$$

defined by $m' \mapsto \sum a_i$.

This induces the dual basis $\{m_1^*, \cdots, m_k^*\}$, where

$$\begin{aligned} m_i^* : & m'_i \mapsto a_i \\ & m'_j \mapsto 0 \ \forall j \neq i \end{aligned}$$

Now, setting the monomorphism we had from M to F as ψ, ψ induces a monomorphism from M to M', which we will call ψ'. It is clear that $\phi \circ \psi = id_M$ as M is projective.

Hence we have that the basis for the left dual of M is given by $\{m_i^* \psi\}$.

□

Chapter 3: Tensor Categories, Grothendieck Rings, and $\mathbb{Z}^+$ Rings

3.1 Tensor Categories and Grothendieck Rings

Now, we have all the ingredients to define what tensor categories are.

Definition 3.1.1. *Let $\mathcal{C}$ be a locally finite $\mathcal{K}$-linear abelian rigid monoidal category. We say $\mathcal{C}$ is a multitensor category over $\mathcal{K}$ if the tensor bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms. We say that $\mathcal{C}$ is indecomposable if $\mathcal{C}$ is not categorically equivalent to a direct sum of nonzero multitensor categories. Lastly, if $End_{\mathcal{C}}(1) \simeq \mathcal{K}$, then we call $\mathcal{C}$ a tensor category.*

A finite semisimple multitensor category is called a multifusion category, and a fusion category is a multifusion category with $End_{\mathcal{C}}(1) \simeq \mathcal{K}$. It can alternatively be defined as a finite semisimple category.

Recall that $\overline{Vec}$, the category of finite dimensional vector spaces over $\mathcal{K}$ is a rigid monoidal category. Since it is comprised of finite dimensional vector spaces, it is locally finite and it is clearly abelian. Also, by taking $\otimes$ to be the Kronecker product between linear transformation matrices, we see that $\otimes$ is bilinear on morphisms in $\overline{Vec}$. Also, any finite dimensional vector space can be decomposed into a direct sum of 1-dimension vector spaces over $\mathcal{K}$, by choice of a set of basis. We can also see that $\overline{Vec}$ is not just locally finite, but finite, since any vector space is necessarily a free module, and hence projective, and there is exactly 1-isomorphism class of simple objects in $\overline{Vec}$. Lastly, $End_{\mathcal{C}}(1)$ is clearly just $\mathcal{K}$, and hence we have that $\overline{Vec}$ is a fusion category.

Similarly, $\overline{Rep(G)}$, the category of finite dimensional representations of a group G over $\mathcal{K}$, is a tensor category. Now, Maschke's Theorem states that if $\mathcal{K}$ is of characteristic

0 or $char(\mathcal{K})$ is coprime to $|G|$, then every representation of G over $\mathcal{K}$ is a direct sum of irreducible representations, or equivalently, $\mathcal{K}G$, the group algebra of G over $\mathcal{K}$ is semisimple. Therefore, we have that $\overline{Rep(G)}$ over $\mathcal{K}$ whose characteristic doesn't divide $|G|$ is a finite semisimple tensor category, i.e. a fusion category.

Lastly, $\overline{Vec_G^\omega}$, much like $\overline{Vec}$ is a tensor category. However, if G is an infinite group, this implies infinitely many non-isomorphic simple objects corresponding to each $g \in G$, hence $\overline{Vec_G^\omega}$ is a fusion category if G is a finite group.

For a multitensor category, the bifunctor $\otimes$ has a very nice property, but in order to show that we need the following proposition.

Proposition 3.1.2. *Let F, G be additive functors where F is the left adjoint of G and G is the right adjoint of F. Then, F is right exact, and G is left exact.*

Proof. Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Since for any X, $Hom(X, -)$ is left exact, $0 \to Hom(F(X), A) \to Hom(F(X), B) \to Hom(F(X), C)$ is exact, and by the naturality of adjunction isomorhpisms, we have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Hom(F(X), A) & \longrightarrow & Hom(F(X), B) & \longrightarrow & Hom(F(X), C) \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & Hom(X, G(A)) & \longrightarrow & Hom(X, G(B)) & \longrightarrow & Hom(X, G(C)) \end{array}$$

By the Yoneda Lemma, the exactness of the bottom row implies that $0 \to G(A) \to G(B) \to G(C)$ is exact, which implies G is left exact. Applying $Hom(-, G(X))$ on the short exact sequence yields a similar diagram, and we can conclude that F is right exact.

□

With Proposition 21, we have the follwing proposition.

Proposition 3.1.3. *Let $\mathcal{C}$ be a multitensor category. Then, the bifunctor $\otimes$ is exact in both factors.*

Proof. In our discussion of rigid monoidal categories, we saw that the functors $(V\otimes -)$ and $(-\otimes V)$ have both left and right adjoint functors if V has left and right duals. Since we are in a multitensor category, V is rigid, and therefore the above functors have both left and right adjoints. Applying Proposition 21 yields the desired result.

□

Ultimately, the goal of this paper is to provide a context in which we can examine mathematical objects that are called Grothendieck Rings, and it is useful to define a category that has slightly weaker conditions than a tensor category.

Definition 3.1.4. *A multiring category over $\mathcal{K}$ is a locally finite $\mathcal{K}$-linear abelian monoidal category $\mathcal{C}$ with bilinear and biexact tensor product. If $End_{\mathcal{C}}(1) = \mathcal{K}$, then we call $\mathcal{C}$ a ring category.*

Notice that for multiring categories, we must impose a condition that tensor products are biexact. For multitensor categories, we saw from Proposition 22 that rigidity forces tensor products to be biexact, but as multiring categories do not require such condition, we must impose the restriction on tensor products separately.

An example of a ring category is $\overline{Vec_G}$, the category of finite dimensional G-graded vector spaces over $\mathcal{K}$, with G a monoid. If G is a group, we saw previously that the existence of inverses extends to the rigidity condition of tensor categories, but as monoids do not require inverses to exist, we have a ring category.

As before, since we've defined tensor categories, we also want to define functors between tensor categories that behave nicely with them.

Definition 3.1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be multiring categories over $\mathcal{K}$, and let $F : \mathcal{C} \to \mathcal{D}$ be an exact and faithful $\mathcal{K}$-linear functor.*

F is called a quasi-tensor functor if it has a functorial isomorphism $J : F(-) \otimes F(-) \to F(- \otimes -)$, and $F(1) = 1$.

A quasi-tensor functor (F, J) is called a tensor functor if it satisfies the monoidal structure axiom.

Finally, we have gotten to the primary object of interest of this paper. Here, we have the definition of Grothendieck rings.

Definition 3.1.6. *Let $\mathcal{C}$ be a multiring category over $\mathcal{K}$. Let X_i be representatives of the isomorphism classes of simple objects in $\mathcal{C}$. Then, the tensor product on $\mathcal{C}$ induces a natural multiplication on the Grothendieck group $Gr(\mathcal{C})$ defined by:*

$$X_i X_j := [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k] X_k, \;\; i, j \in I$$

With multiplication defined as above, and addition defined as previously, $Gr(\mathcal{C})$ is called the Grothendieck ring of $\mathcal{C}$.

The multiplication rule above is called the fusion rule of $\mathcal{C}$, and is of importance in physics, as well as when we are examining fusion categories. The detailed explanation on fusion categories, or applications of fusion rules in physics is beyond the scope of this paper.

Grothendieck rings are very important because they have the potential to give us insight about categories, which are immense mathematical objects, by allowing us to use ring theoretic tools, which are much easier to handle. The caveat is that the definition of Grothendieck rings isn't quite so useful, as it does not give us information about the structure of the ring unless we arduously compute the ring, which is often times an incredibly difficult task. However, there is a special case, where a Grothendieck ring becomes a very special type of ring called $\mathbb{Z}^+$-ring. However, to show this, we must know and understand the properties of $\mathbb{Z}^+$-rings, which happens to be the topic of our next discussion.

3.2 $\mathbb{Z}^+$-rings

Definition 3.2.1. *Let $\mathbb{Z}^+$ denote the semiring of non-negative integers.*

Let A be a ring which is free as a $\mathbb{Z}$-module with the following properties:

- *A $\mathbb{Z}^+$ basis of A is a basis $B = \{b_i\}_{i \in I}$ such that $b_i b_j = \sum_{k \in I} c_{ij}^k b_k$, where $c_{ij}^k \in \mathbb{Z}^+$.*
- *A $\mathbb{Z}^+$ ring is a ring with fixed $\mathbb{Z}^+$ basis and with identity 1, which is a non-negative linear combination of the basis elements.*
- *A unital $\mathbb{Z}^+$ ring is a $\mathbb{Z}^+$ ring such that 1 is a basis element.*

Note that every $\mathbb{Z}^+$ ring is assumed to have an identity, but notice that not all $\mathbb{Z}^+$ rings are unital. It is important when understanding $\mathbb{Z}^+$ rings to see that the coefficients have to be non-negative induces a limitation on how we can change the basis, i.e. we cannot just pick an arbitrary basis as we can do for vector spaces.

Let A be a $\mathbb{Z}^+$ ring and let I_0 be the set of $i \in I$ such that b_i occurs in the decomposition of 1.

Let $\tau : A \to \mathbb{Z}$ denote the group homomorphism defined by

$$\tau(b_i) = 1 \; if \; i \in I_0, \; and \; 0 \; if \; i \notin I_0$$

In our discussion of rigid monoidal categories and tensor categories, the notion of duality and adjointness played a big role. As was stated previously, the Grothendieck rings become $\mathbb{Z}^+$ rings under a special case, so we can expect an equivalent of dual structure on $\mathbb{Z}^+$ rings.

Definition 3.2.2. *A $\mathbb{Z}^+$ ring A with basis $\{b_i\}_{i \in I}$ is called a based ring if there exists an involution $i \mapsto i^*$ of I such that the induced map*

$$a = \sum_{i \in I} a_i b_i \mapsto a^* = \sum_{i \in I} a_i b_{i^*}, \; a_i \in \mathbb{Z}$$

is an anti-involution of the ring A, i.e.

$$i \mapsto i^* \mapsto i^{**} = id_I$$

$$a \mapsto a^* \mapsto a^{**} = id_A$$

and we have that $\tau(b_i b_j) = 1$ *if* $i = j^*$*, and* 0 *if* $i \neq j^*$*.*

Perhaps it is trivial from our definition of I_0, but it is nice to explicitly identify how the multiplicative unit is defined in a unital based ring.

Proposition 3.2.3. *In every based ring,* $1 = \sum_{i \in I_0} a_i b_i$*. In other words, if* $\sum_{i \in I_0} a_i b_i = 1$*, then* $a_i = 1$ *for every* i*.*

Proof. $\sum_{i \in I_0} a_i b_i = 1$, hence $\sum_{i \in I_0} a_i b_{i^*} = 1$, as the induced map from $i \mapsto i^*$ is an involution, i.e. an automorphism of the ring.

Note that for $j \in I_0$,

$$\tau(b_j \sum_{i \in I_0} a_i b_{i^*}) = \sum_{i \in I_0} a_i \tau(b_j b_{i^*})$$

but $i \mapsto i^*$ is an automorphism. Hence it must be that the only nonzero term in the sum is $a_j \tau(b_j b_{j^*})$, but as $\sum_{i \in I_0} a_i b_{i^*} = 1$, we have

$$a_j = a_j \tau(b_j b_{j^*}) = \tau(b_j) = 1$$

and so we have the proposition.

□

The basis elements that appear in the decomposition of the unit in a $\mathbb{Z}^+$ ring have some interesting properties that are similar to the norm of an orthonormal basis for a vector space.

Proposition 3.2.4. *In a based ring, for $i, j \in I_0$, $b_i^2 = b_i$ and $b_i b_j = 0$ for $i \neq j$.*

Proof. WLOG let $I_0 = \{1, \cdots, n\}$.

Suppose $b_i b_j$ has b_l for some $l \notin I_0$ in its decomposition. Since $c_i > 0$ for every c_i such that $\sum c_i b_i = 1$, and

$$1 = (\sum c_i b_i)^2 = \sum c_i^2 b_i^2 + \sum c_j c_k b_j b_k$$

so b_l is in the decomposition of 1, which is absurd.

Hence $\forall i, j \in I_0$, $b_i b_j$ must be a $\mathbb{Z}^+$-linear combination of $b_1, \cdots, b_n$.

But now,

$$1 = \tau(b_i) = \tau(b_i \sum c_j b_j) = \sum c_j \tau(b_i b_j) = 1$$

Hence there exists a unique $j \in I_0$ such that $b_i b_j = b_k$ for some $k \in I_0$ and $c_j = 1$. For any other $j' \neq j$, $b_i b_{j'}$ must be 0.

Also, by similar logic, for each $i \in I_0$, there exists a unique $j \in I_0$ such that $b_j b_i = b_k$ for some $k \in I_0$, $c_j = 1$, and $b_{j'} b_i$ is 0 for any $j' \neq j$.

It follows that $\forall i \in I_0$, there exists a unique $i' \in I_0$ such that $b_i b_{i'} \neq 0$ and as we run through $i \in I_0$, i' runs through all of I_0.

We also have that $1 = b_1 + \cdots + b_n$, i.e. the coefficients must be all 1.

Now suppose for some $i \in I_0$, $i' \neq i$. Then, we have that $b_i \sum b_j = (\sum b_j) b_i$, so $b_i b_{i'} = b_{i''} b_i \neq 0$. Hence

$$1 = \sum_{l, l' \neq i} b_l b_{l'} + b_i b_{i'} + b_{i''} b_i = \sum_{l, l' \neq i} b_l b_{l'} + 2 b_i b_{i'}$$

But each of $b_j b_{j'}$ correspond to a unique basis element, so we have that for some $s, t \in I_0$, $b_s + b_t = 2b_s$, which is absurd since b_s and b_t are linearly independent.

Hence it must be that $\forall i \neq j$, $b_i b_j = 0$.

Now suppose that $b_i^2 = b_j$ for some $j \neq i$. Then, $(\sum b_s^2)(\sum b_s) = \sum(\sum b_s^2)b_t$, but notice that every b_t has exactly one b_s^2 such that $b_s^2 b_t \neq 0$. But for b_j, that is impossible, since b_i^2 is the only possible element such that $b_i^2 b_j \neq 0$, since $b_i^2 = b_j$, but $b_i^2 b_j = b_i(b_i b_j) = b_i \cdot 0 = 0$, which is absurd.

Hence $b_i^2 = b_i$, and this concludes the proof.

□

As a corollary to Proposition 24, we get that any involution on a based ring must fix any i in I_0.

The unique restrictions on the involution of a based ring also give rise to a useful property regarding the coefficients in the decomposition of the product of two basis elements.

Proposition 3.2.5. *In any based ring, c_{ij}^{k*}, defined as the coefficient of b_{k^*} in the decomposition of $b_i b_j$, is invariant under the cyclic permutation of i, j, k.*

Proof. $b_i b_j = \sum c_{ij}^k b_k$, so if we take a triple product of the basis elements, we get

$$\tau(b_i b_j b_k) = \tau(c_{ij}^{k^*} b_{k^*} b_k) = c_{ij}^{k^*}$$

and from the definition of τ, it is easy to see that $\tau(xy) = \tau(yx)$. Hence we have that $c_{ij}^{k^*}$ is cyclically symmetric, which implies $c_{ij}^{k^*} = c_{ki}^{j^*}$.

□

Definition 3.2.6. *A multifusion ring is a based ring of finite rank, and a fusion ring is a unital based ring of finite rank.*

If we recall the definition of multifusion categories and fusion categories, one can see that the multifusion rings and fusion rings are analogously defined. Hence it should be that based rings are somehow corresponding to tensor categories, and we will see that once we've fully explored $\mathbb{Z}^+$ rings.

Proposition 3.2.7. *Let A be a multifusion ring with basis $\{b_i\}$. For any $x \in A$, the element $Z(x) := \sum b_i x b_{i^*}$ is central in A.*

Proof. For a fixed $x \in A$, define $\otimes$ on A to be $\sum a_i b_i \times \sum a_j b_j \overset{\otimes}{\mapsto} (\sum a_i b_i) x (\sum a_j b_j)$. Note that if we fix a k,

$$\sum_i b_k b_i = \sum_{r,i} c_{ki}^r b_r$$

and $c_{ki}^r = \tau(b_k b_i b_{r^*})$.

However, as $\sum_{r,i} c_{ki}^r b_r$ runs through all r, i,

$$\sum_{r,i} c_{ki}^r b_r = \sum_{r^*,i^*} c_{r^*k}^{i^*} b_r$$

and as c_{ki}^r is invariant under cyclic permutations, we have

$$\sum_{r,i} c_{r^*k}^{i^*} b_{i^*} = \sum_r b_{r^*} b_k$$

and by using the bilinearity of $\otimes$, we get

$$\begin{aligned} \sum_i b_k b_i \otimes b_{i^*} &= \sum_{r,i} c_{ki}^r b_r \otimes b_{i^*} \\ &= \sum_{r,i} c_{r^*k}^{i^*} b_r \otimes b_{i^*} \\ &= \sum_{r,i} b_r \otimes c_{r^*k}^{i^*} b_{i^*} \\ &= \sum_r b_r \otimes b_{r^*} b_k \end{aligned}$$

Hence we see that $Z(x) := \sum_i b_i x b_{i^*}$ is central in A.

□

The idea of based ring may seem quite novel, but in fact we have a very classical example of a unital based ring. The group ring $\mathbb{Z}G$ is a unital based ring, with the group elements as the basis and take $g^* = g^{-1}$. Notice that in this case, we have $I_0 = \{1_g\}$. If G is finite, then $\mathbb{Z}G$ is of finite rank, and hence is a fusion ring.

It may seem unrelated, but Frobenius-Perron dimension is actually a defining characteristic of based rings. Hence we will explore this notion, and to do so we will begin by stating and proving the Frobenius-Perron Theorem.

Theorem 3.2.8. *(Frobenius-Perron Theorem)*

Let B be a square matrix with non-negative real entries.

(a) B has a non-negative real eigenvalue. The largest non-negative real eigenvalue $\lambda(B)$ of B dominates the absolute values of all other eigenvalues μ of B, i.e. $|\mu| \leq \lambda(B)$. Moreover, there exists an eigenvector of B with non-negative entries and eigenvalue $\lambda(B)$.

(b) If B has strictly positive entries, then $\lambda(B)$ is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries. Also, $|\mu| < \lambda(B)$ for any other eigenvalue μ of B.

(c) If a matrix B with non-negative entries has an eigenvctor v with strictly positive entries, then the corresponding eigenvalue is $\lambda(B)$.

Proof. Let B be an $n \times n$ matrix with non-negative entries.

If B has an eigenvector v with non-negative entries and eigenvalue 0, we are done.

Otherwise, let Σ be the set of column vectors $x \in \mathbb{R}^n$ with non-negative entries x_i, and $s(x) = \sum_{i=1}^{n} x_i$ equal to 1. Under this definition, notice that Σ is a simplex.

Now we will define a continuous map:

$$f_B : \Sigma \to \Sigma$$

$$x \mapsto \frac{Bx}{s(Bx)}$$

Since we assumed that B does not have an eigenvector with non-negative entries with eigenvalue 0, $s(Bx) > 0$ for every $x \in \Sigma$.

By the Brower Fixed Point Theorem, this map has a fixed point, so for some fixed point p, $B_p = s(B_p)p$, where $s(B_p) > 0$. So $s(B_p), p$ are the desired eigenvalue and eigenvector.

Now let $\lambda = \lambda(B)$ be the maximal non-negative eigenvalue of B for which there is an eigenvector with non-negative entries. We will denote this eigenvector as

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Suppose B has strictly positive entries. Then, $Bf = \lambda f$ has strictly positive entries, hence f must also have strictly positive entries, and $\lambda > 0$.

If $d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ is another real eigenvector of B with eigenvalue λ, let z be the smallest of the numbers $\frac{d_i}{f_i}$. Then, $v = d - zf$ satisfies $Bv = \lambda v$, and so has at least one entry 0. But any nontrivial eigenvector of B corresponding to λ must have strictly positive entries, so $v = 0$. Hence the eigenspace of λ must be 1-dimensional, i.e. simple.

Now let $y = (y_1, \cdots, y_n) \in \mathbb{C}^n$ be a row vector.

We will define the norm $|y|$ to be $\sum |y_j| f_j$. Then,

$$|yB| = \sum_j |\sum_i y_i b_{ij}| f_j \leq \sum_{i,j} |y_i| b_{ij} f_j = \lambda |y|,$$

and equality holds if and only if every non-zero y_i have the form $y_i = c_i e^{\pi i \theta}$ for some fixed θ, and $c_i > 0$. i.e. y_i's have the same argument.

So if $yB = \mu y$ then $|\mu| \leq \lambda$, and if $|\mu| = \lambda$, we can normalize y to have non-negative entries. This implies $\mu = \lambda$. So this proves (b).

Let B_N be a sequence of matrices with strictly positive entries that converges to B. Since the spectral radius function ρ is continuous, $\rho(B_N) \longrightarrow \rho(B)$ as $N \to \infty$.

Also, $\rho(B_N) = \lambda(B_N)$ for every N. Hence $\rho(B)$ is an eigenvalue of B, and there exists an eigenvector of B with non-negative entries corresponding to $\rho(B)$, and this implies $\rho(B) = \lambda(B)$. Hence we have (a).

Now suppose B has a row eigenvector y with strictly positive entries and eigenvalue μ. Then, $\mu yf = yBf = \lambda yf$. This implies $\mu = \lambda$ since $yf \neq 0$.

Hence taking any eigenvector g with strictly positive entries, you see that the corresponding eigenvalue must be $\mu = \lambda$, by the chain of equalities of the form

$$(y\mu)g = y(B)g = y(\eta g)$$

which implies that $\eta = \mu$. Hence we have (c), and we are done.

□

Now that we have the Frobenius-Perron Theorem, we can define the Frobenius-Perron dimension, which we will denote FPdim. To do so, we have a bit more definitions and properties we need to examine.

Let A be a $\mathbb{Z}^+$ ring with $\mathbb{Z}^+$ basis I.

Definition 3.2.9. *We say A is transitive if $\forall X, Z \in I$, there exists $Y_1, Y_2 \in I$ such that XY_1 and Y_2X have Z in their decompositions with non-zero coefficients.*

Proposition 3.2.10. *Any unital based ring is transitive.*

Proof. Let $X, Z \in I$.

Note that $\tau(XX^*) = 1$, so XX^*Z must contain a Z in its decomposition.

But $XX^*Z = X(a_1b_1+\cdots+a_nb_n) = a_1Xb_1+\cdots+a_nXb_n$, so for some $i \in \{1,\cdots,n\}$, a_iXb_i contains Z in its decomposition.

Hence Xb_i contains Z in its decomposition, as otherwise Z is a nontrivial linear combination of b_i's which is absurd, as $Z \in I$.

So let us set $Y_1 = b$. Then, XY_1 clearly contains Z in its decomposition, and we can find a desired Y_2 in a similar way.

Hence A is transitive.

□

Now let us define FPdim.

Definition 3.2.11. *Let A be a transitive unital $\mathbb{Z}^+$ ring of finite rank.*

Define a group homomorphism $FPdim : A \longrightarrow \mathbb{C}$ as follows:

For $X \in I$, let $FPdim(X)$ be the maximal non-negative eigenvalue of the matrix of left multiplication by X.

$$\begin{pmatrix} a_1 & \cdots & \\ \vdots & & \\ a_n & \cdots & \end{pmatrix}$$

where the first column represents the decomposition of Xb_1, i.e. $Xb_1 = a_1b_1 + \cdots + a_nb_n$.

We call the above defined homomorphism FPdim as the Frobenius-Perron dimension.

Notice that because we are in a $\mathbb{Z}^+$, the matrix of left multiplication defined above has non-negative real entries, so by the Frobenius-Perron Theorem, our homomorphism FPdim is validated. Also, as I is a basis, the homomorphism FPdim $\mathbb{Z}$-linearly extends from I to A via addition.

By how we defined FPdim, some restrictions are imposed on the possible values of $FPdim(X)$, which play an important role in proving some very nice properties of FPdim.

Proposition 3.2.12. *Let $X \in I$.*

(a) *The number $\alpha = FPdim$ is an algebraic integer, and for any algebraic conjugate α' of α, we have $\alpha \geq |\alpha'|$, i.e. α is a root of some monic polynomial in $\mathbb{Z}[x]$, and for any other root α' of the minimal polynomial of α, denoted $m_{\alpha,\mathbb{Z}}$, $\alpha \geq |\alpha'|$.*

(b) *$FPdim(X) \geq 1$.*

Proof. Since α is a root of the characteristic polynomial of N_X, the matrix of left multiplication by X defined previously, α is clearly an algebraic integer.

Also, $m_{\alpha,\mathbb{Z}}|char_{\alpha,\mathbb{Z}}$, so any algebraic conjugate of α, α' is an eigenvalue of the left-multiplication matrix N_X, and by Frobenius-Perron Theorem, $|\alpha'| \leq \alpha$, as the spectral radius is α. This proves (a).

Let r be the number of algebraic conjugates of α. Then, $\alpha^r \geq |N(\alpha)|$ where $N(\alpha)$ is the norm of α given by $\prod \alpha_i^{m_i}$, where α_i are all roots of $m_{\alpha,\mathbb{Z}}$, and m_i are the multiplicities. Clearly $|N(\alpha)| \geq 1$, as α is an algebraic integer, so $\alpha^r \geq 1$, which implies $\alpha \geq 1$. Therefore, $FPdim \geq 1$, and we have (b).

□

By Proposition 28, we have that all Frobenius Perron dimensions of elements of A are algebraic integers.

We will now use Proposition 28 to prove the follwing proposition, which makes it evident why FPdim is a defining characteristic of a transitive unital based ring.

Proposition 3.2.13. (a) *The function $FPdim : A \longrightarrow \mathbb{C}$ is a ring homomorphism.*

(b) *Up to scaling, there exists a unique non-zero element* $R \in A_{\mathbb{C}} := A \otimes_{\mathbb{Z}} \mathbb{C}$ *such that* $XR = FPdim(X)R$ *for all* $X \in A$*, and it satisfies the equality* $RY = FPdim(Y)R$ *for all* $Y \in A$*. After appropriate normalization, this element has positive coefficients, and hence* $FPdim(R) > 0$*.*

(c) $FPdim$ *is the unique character of* A *that takes non-negative values n* I*, and these values are actually strictly positive.*

(d) *If* $X \in A$ *has non-negative coefficients with respect to the basis of* A*, then* $FPdim(X)$ *is the largest non-negative eigenvalue* $\lambda(N_X)$ *of the matrix* N_X *of multiplication by* X*.*

Remark: A *must be unital to admit a homomorphism to* $\mathbb{C}$*.*

Proof. First, note that the matrix M of right multiplication by $\sum_{X \in I} X$ in A in basis I must have strictly positive entries by the transitive property of A.

For every $b_i, b_j \in I$, there exists an $\overline{X} \in I$ such that $b_i X$ contains b_j, and since we are multiplying by $\sum X$, $\overline{X}$ certainly is a component of $\sum X$.

Hence, by the Frobenius-Perron Theorem, M has a single eigenvector (up to scaling) with strictly positive entries, and the eigenvalue $\lambda(M)$. We denote this eigenvector $R \in A_{\mathbb{C}} = A \otimes_{\mathbb{Z}} \mathbb{C}$.

Since R is unique, and since $\forall X \in A$, $XR(\sum X) = \lambda XR$, XR is also an eigenvector of M, and hence $XR = d(X)R$ for some map $d : X \mapsto d(X) \in \mathbb{C}$.

But this implies that R is an eigenvector of the matrix N_X of left multiplication by X, and hence by the Frobenius-Perron Theorem, $d(X) = FPdim(X)$.

Now, $R = \sum_{X \in I} a_X X$, where $a_X > 0$. Hence we can scale R such that $a_X \geq 1$ for every $X \in I$, and by transitivity, left multiplication by R yields the matrix N_R which is broken down to $N_R = T + N_{\sum X}$, where T is a matrix with non-negative entries, and $N_{\sum X}$ has strictly positive entries. Hence $FPdim(R) > 0$, so this proves (a), and the first part of (b).

Since for every $X \in A$, $XR = FPdim(X)R$, we have a system of linear equations whose solution is R. Notice that R is the unique solution, as $(\sum X)R = FPdim(\sum X)R$, and $N_{\sum X}$ is a matrix with strictly positive entries, and so R is a simple eigenvector.

As RY is also a solution to the above system of linear equations, we have that $RY = d'(Y)R$ by uniqueness of R, for some number $d'(Y)$. Now, $FPdim(RY) = FPdim(d'(Y)R) = (FPdim(R))d'(Y)$. But $FPdim$ is a ring homomorphism, so $FPdim(R)FPdim(Y) = FPdim(RY) = (FPdim(R))d'(Y)$. But this is in $\mathbb{C}$ which is a field, hence $d'(Y) = FPdim(Y)$, and this proves (b).

The above proof makes it clear that $FPdim$ is a character. We now need to show that $FPdim$ is the unique character, and so suppose χ is another character of A taking non-negative values on I.

Then, for any $Y \in I$, $\chi(\sum_{X \in I} X \cdot Y) = \chi(\sum X)\chi(Y) = (\sum \chi(X))\chi(Y)$, so the vector with entries $\chi(Y)$ for $Y \in I$ is an eigenvector of $N_{\sum X}$.

Note that as $N_{\sum X}$ has positive entries, the eigenvector of $\chi(Y)$ must also have positive entries by the Frobenius-Perron Theorem, and so we have that $\chi(Y) = \lambda(N_{\sum X})$ by another application of the Frobenius-Perron Theorem.

Hence the the eigenvector corresponding to $\chi(Y)$, denoted $v_{\chi(Y)}$ must be equal to cR for some $c \in \mathbb{C}$, but χ is a character and this implies that $\chi(1) = 1$. Therefore, $c = 1$, and we have that $\chi = FPdim$. This proves (c).

Lastly one can easily see that (d) is immediate from (b) and the Frobenius-Perron Theorem.

$\square$

The element $R \in A_{\mathbb{C}}$, as one can see, plays a crucial role, and so we call R the regular element of A.

Now that we have defined $FPdim$, one can ask how $FPdim$ behaves with respect to involutions, and it turns out that $FPdim$ is invariant under involutions!

Proposition 3.2.14. *Let A be as usual, and $* : I \longrightarrow I$ be a bijection given by $* : x \cdot y \mapsto (x \cdot y)^* = y^* \cdot x^*$. Then, $FPdim$ is invariant under $*$.*

Proof. Note that the left multiplication by X corresponds to right multiplication by X^*. Since scalars are preserved, (by the τ map) the matrix of right multiplication by X^* is exactly the same as N_X.

However, R is the eigenvector of $N_{\sum X} = N'_{\sum X^*}$ i.e. the matrix of right multiplication by $\sum X^*$, and $RX^* = FPdim(X^*)R$.

Hence R is the eigenvector of $N_{X^*} = N_X$ corresponding to $\lambda(N_{X^*})$, where N_{X^*} is the matrix of right multiplication by X^*. Hence we have that $RX^* = \lambda(N_{X^*})R = FPdim(X^*)R$. But then, $\lambda(N_{X^*}) = \lambda(N_X) = FPdim(X)$.

Therefore, we have that $FPdim(X^*) = FPdim(X)$.

□

Finally we can use the propositions we have so far to justify why we call R a regular element, and define the Frobenius-Perron dimension of A.

Proposition 3.2.15. *Let A be a fusion ring. Then the element*

$$R = \sum_{Y \in I} FPdim(Y)Y$$

is a regular element.

Proof. Let us directly multiply an arbitrary basis element X of A and R.

$$XR = \sum_Y FPdim(Y)XY = \sum_{Y,Z} FPdim(Y)c_{XY}^{Z}Z = \sum_{Y,Z} FPdim(Y)c_{Z^*X}^{Y^*}Z$$

$$= \sum_{Y,Z} FPdim(Y)c_{X^*Z}^{Y}Z = \sum_{Z} FPdim(X^*Z)Z$$

$$= FPdim(X^*)(\sum_Z FPdim(Z)Z) = FPdim(X)R$$

As $FPdim(X)$ is nonzero for any $X \in I$, we see that R is regular with respect to the basis elements of X.

□

The element R normalized as in Proposition 31 is called the canonical regular element of A, and now we have a definition of the Frobenius-Perron dimension of A.

Definition 3.2.16. *Let A be a transitive unital based ring, and R be its canonical regular element. Then,*

$$FPdim(R) = \sum_{X \in I} FPdim(X)^2$$

is called the Frobenius-Perron dimension of A, and is denoted $FPdim(A)$.

Now that we've explored the notion of $\mathbb{Z}^+$ rings and $FPdim$, we will state the final theorem of this paper, connecting the notion of Grothendieck rings of tensor categories and transitive unital based rings.

3.3 $Gr(\mathcal{C})$ and $\mathbb{Z}^+$-rings

Theorem 3.3.1. *If $\mathcal{C}$ is a ring category with left duals, then $Gr(\mathcal{C})$ is a transitive unital $\mathbb{Z}^+$ ring.*

Proof. Recall the fusion rule given by

$$X_iX_j := [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k] X_k$$

The fusion rule is defined as the natural multiplication on $Gr(\mathcal{C})$, and we will now show that it is associative.

By the exactness of tensor products,

$$[(X_i \otimes X_j) \otimes X_p : X_l] = \sum_k [X_i \otimes X_j : X_k][X_i \otimes X_k : X_l] \ \ \forall i, j, p, l.$$

Similarly, we can see that

$$[X_i \otimes (X_j \otimes X_p) : X_l] = \sum_k [X_j \otimes X_p : X_k][X_i \otimes X_k : X_l].$$

Lastly we know that

$$(X_i \otimes X_j) \otimes X_p \simeq X_i \otimes (X_j \otimes X_p)$$

by associativity of tensor products.

Combining all of the information above, we have that the product as defined above in $Gr(\mathcal{C})$ is associative.

Now, notice that the index above must always take on non-negative integer values, and so by choosing a set of representatives of isomorphism classes of simple objects in $\mathcal{C}$, we have that $Gr(\mathcal{C})$ is a unital based ring, as the identity object must be simple. We can see that by taking a simple subobject of 1, denoted X and taking a short exact sequence

$$0 \longrightarrow X \longrightarrow 1 \longrightarrow Y \longrightarrow 0.$$

Applying the left dualization functor, which is exact, (the proof of this fact will be given after) we get the exact sequence

$$0 \longrightarrow Y^* \longrightarrow 1 \longrightarrow X^* \longrightarrow 0,$$

and tensoring with sequence with X on the left, we get

$$0 \longrightarrow X \otimes Y^* \longrightarrow X \longrightarrow X \otimes X^* \longrightarrow 0.$$

As X is simple and $X \otimes X^*$ is nonzero, we have that $X \otimes X^* \simeq X$. Using this isomorphism and the coevaluation map, we now have a surjective morphism

$$1 \to X \otimes X^* \to X,$$

and since X is a subobject of 1, we have the embedding $X \to 1$. Composing these maps gives us a non-zero morphism $1 \to 1$, and as $End_{\mathcal{C}}(1) = \mathcal{K}$, this morphism must be multiplication by a non-zero element of $\mathcal{K}$, and hence $X = 1$.

Thus we have that 1 is a simple object, and hence $Gr(\mathcal{C})$ must be a unital based ring.

Now, due to the evaluation map, $X \otimes X^*$ contains 1 in its decomposition, and hence for any simple $X, Z \in \mathcal{C}$, $X \otimes X^* \otimes Z$ contains Z in its decomposition. Hence there must be a simple object Y_1 in the decomposition of $X^* \otimes Z$ such that $X \otimes Y_1$ contains Z in its decomposition.

Similarly, the object $Z \otimes X^* \otimes X$ contains Z in its decomposition, and by the evaluation map, $X^* \otimes X$ contains 1 in its decomposition. Therefore, we can find a Y_2 in the decomposition of $Z \otimes X^*$, such that Z appears in the decomposition of $Y_2 \otimes X$. Hence $Gr(\mathcal{C})$ is transitive.

□

Lastly, we will give a short justification of why we call R the regular element.

Proposition 3.3.2. *In $\overline{Rep(G)}$, the regular representation is the regular element of $Gr(\overline{Rep(G)})$ as the unital transitive based ring defined above.*

Proof. Consider $Rep(G)$, the category of finite dimensional representations of a group G over $\mathcal{K}$. Let $S_1, \cdots, S_n$ be simple, i.e. irreducible representations of G over $\mathcal{K}$.

Then, by the fusion rule and the definition of a regular element, if R is a regular element, it must be that $R \otimes_{\mathcal{K}} S_k \simeq (dim(S_k))R$.

Take $R = \mathcal{K}G$, the regular representation of G over $\mathcal{K}$. Now consider $\mathcal{K}G \otimes_{\mathcal{K}} S_k$ and $(\mathcal{K}G)^{dim(S_k)}$.

$Rep(G)$ is semisimple, hence irreducible implies projective. So we can apply tensor evaluation to get

$$Hom_{\mathcal{K}G}(S_i, \mathcal{K}G \otimes_{\mathcal{K}} S_k) \simeq Hom_{\mathcal{K}G}(S_i, \mathcal{K}G) \otimes_{\mathcal{K}} S_k$$

We know that the regular representation contains exactly $dim(S_i)$ many copies of S_i, and hence $Hom_{\mathcal{K}G}(S_i, \mathcal{K}G) \otimes_{\mathcal{K}} S_k$ has dimension precisely $dim(S_i)dim(S_k)$.

On the other hand, the dimension of $Hom_{\mathcal{K}G}(S_i, \mathcal{K}G^{dim(S_k)})$ is given by

$$dim(Hom_{\mathcal{K}G}(S_i, \mathcal{K}G))dim(S_k) = dim(S_i)dim(S_k).$$

Hence the dimensions match, which implies that there are exactly the same number of copies of S_i in the decomposition of $Hom_{\mathcal{K}G}(S_i, \mathcal{K}G \otimes_{\mathcal{K}} S_k)$ and $Hom_{\mathcal{K}G}(S_i, (\mathcal{K}G)^{dim(S_k)})$, and so we have

$$\mathcal{K}G \otimes_{\mathcal{K}} S_k \simeq (\mathcal{K}G)^{dim(S_k)}.$$

Since each irreducible representation appears positive-integer many times in the regular representation, we have that $\mathcal{K}G$ corresponds to a vector with strictly positive integer valued entries, and by Frobenius-Perron, we have that it must be the eigenvector corresponding to $FPdim$. Therefore, we have a justification of the term "regular".

□

Chapter 4: Conclusion

In this paper, we have shown that we can examine tensor categories as unital based rings by looking at the respective Grothendieck Rings. This notion is merely a precursor to a prospering area of research in mathematics called algebraic K-theory, and $Gr(\mathcal{C})$ is usually referred to as K_0. As one can infer from the subscript, K_0 is the most basic K-theory group, and there are higher K-groups, denoted K_i. However, the K_0 we have looked at in this paper is particularly nice even in the context of K-theory, as the tensor structure in tensor categories allows us to define a ring on K_0, which opens up to the use of more robust tools in ring theory, such as the Frobenius-Perron dimension which is, in our case, a defining invariant.

Vita

Woohyung Lee
1004 Aspen Trail, Winston-Salem, NC

27106 United States of America

(818)318-3155

leew16@wfu.edu

Education

Masters in Mathematics *Expected: May, 2018*

Wake Forest University

- Master's thesis in Tensor Categories
- Graduate Coursework: Abstract Algebra, Galois Theory, Representation Theory, Point Set Topology, Algebraic Topology, Real Analysis, Measure Theory, Advanced Linear Algebra, and Elliptic Curves
- Reading Sessions: Homological Algebra, Category Theory, and Cohomology

Bachelor of Arts in Mathematics and Economics *May, 2015*

Pomona College

- Undergraduate thesis on *n-Player Game Equilibria and Algebraic Varieties*
- Independent study with Dr. Karaali at Pomona College on Bruhat order, KazhdanLusztig polynomial, and Supercharacter theory

Research

Original

- Classification of symmetries of distinct embeddings of θ_4-graph in S^3 (In progress)
- Classifying homophonic quotient groups of various languages including Korean and Turkish
- Supercharacter theory of symmetric groups given by filtrations of S_n

Expository

- Expository master's thesis on tensor categories and fusion categories based on the monograph "Tensor Category" by Etingof et al (In progress)

Presentations

Pomona College

- Undergraduate mathematics thesis presentation on *n-Player Game Equilibria and Algebraic Varieties*
- Chosen class speaker for economics majors at Pomona College, on an undergraduate economics project regarding mathematical modeling of altruistic behaviors

Professional Experience

Teaching Assistant

Department of Mathematics, Wake Forest University *August 2016 - Present*

- Linear Algebra, Calculus 2, and Modern Algebra
- Tutor for various undergraduate courses, including calculus, linear algebra, etc

- Ran 4 hours of weekly study sessions for the above courses
- Weekly assignment grading

Department of Mathematics, Pomona College *Fall, 2014*

- Ran weekly study sessions for Calculus and Applied Mathematics for Science and Economics

Other Work Experiences

- Served as the Republic of Korea Navy liason/translator during ROK-US joint military trainings
- Taught mathematics, english, and economics to Korean high school students living in Qing-Dao, China in fall, 2010
- Paid vocalist at bars in Ap-gu-jeong, South Korea in summer, 2008