QUASISYMMETRIC FUNCTIONS AND PATTERN AVOIDANCE; SEEKING QUASISYMMETRIC SCHUR POSITIVITY

BY

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Abstract

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In Hamacker, Pawlowski, and Sagan (2018) the authors characterized which subsets of $\mathfrak{S}_n$ avoiding particular subsets of $\mathfrak{S}_3$ resulted in symmetric and Schur nonnegative functions for all n. They defined the pattern symmetric quasisymmetric function which takes permutations to quasisymmetric functions. A natural extension of this work is to ask for which subsets this pattern symmetric quasisymmetric function is quasisymmetric Schur nonnegative. We offer a list of two subsets of $\mathfrak{S}_3$ that result in quasisymmetric Schur nonnegative functions for all n. Additionally, we list thirteen additional subsets of $\mathfrak{S}_3$ which we conjecture will result in quasisymmetric Schur nonnegative functions.

Chapter 1: Introduction

Let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n] = \{1, 2, \cdots, n\}$. We denote a permutation π in $\mathfrak{S}_n$ as $\pi = \pi_1 \cdots \pi_n$, we call this notation "one-line notation". For example, in $\mathfrak{S}_4$, the permutation $\pi = 4231$ sends $1 \to 4, 2 \to 2, 3 \to 3$, and $4 \to 1$. We define the *standardization* of a subsequence γ of a permutation π in $\mathfrak{S}_n$ to be the function which sends the smallest element of γ to 1, the second smallest element to 2, and so on. We say a permutation π in $\mathfrak{S}_n$ *contains* a pattern $\sigma \in \mathfrak{S}_k$ (where $k \leq n$) if π contains a subsequence whose standardization is σ. If π does not contain the pattern σ we say that π *avoids* σ. For example, if $\pi = 31427856$ then π contains the pattern $\sigma = 23154$ since the subsequence $\gamma = 34286$ of π has standardization σ.

Given a set of permutations Π we denote, as in Hamacker, et al [3],

$$\mathfrak{S}_n(\Pi) := \{\pi \in S_n \mid \pi \text{ avoids every } \sigma \in \Pi\}.$$

For example, if $\Pi = \{123, 321\}$ then $\mathfrak{S}_4(\Pi) = \{2143, 2314, 2413, 3142, 3412\}$.

We have now built all the necessary notation and definitions necessary to understand pattern avoidance with permutations that will be used in later results. We will switch our focus to the types of functions that arise in the results of this thesis.

We define the formal power series ring

$$\mathbb{C}[[\mathbf{x}]] := \{\sum_{n \geq 0} a_n \mathbf{x}^\alpha \mid a_n \in \mathbb{C} \text{ for all } n \text{ and } \alpha \text{ is any composition of } n\},$$

where $\mathbf{x} = \{x_1, x_2, x_3 \cdots\}$. This is a ring under the standard operations of addition and multiplication.

We say a monomial $x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$ has degree n if the sum of the exponents of the monomial is n, that is if $\sum_i \lambda_i = n$. If $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ and every monomial in $f(\mathbf{x})$ has degree n then we say $f(\mathbf{x})$ is homogeneous of degree n. For example, the function $f(\mathbf{x}) = x_1x_2 + x_1x_3 + x_1x_4 + \cdots$ is homogeneous of degree 2.

On any $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ there is an action by the symmetric group such that $\pi \in \mathfrak{S}_n$ acts on $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ by permuting the variables. That is,

$$\pi \cdot f(x_1, x_2, x_3 \cdots) = f(x_{\pi 1}, x_{\pi 2}, x_{\pi 3}, \cdots),$$

where $\pi i = i$ when $i > n$. We say $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ is *invariant* under this action if the coefficient of any monomial in $f(\mathbf{x})$ is fixed under the action of $\pi \in \mathfrak{S}_n$ for all $\pi \in \mathfrak{S}_n$ and for all n [7]. For example, the function $f(\mathbf{x}) = \prod_{i=1}^{\infty} 1 + x_i$ is invariant under the action of the symmetric group. However, the function $f(\mathbf{x}) = x_1x_2^2 + x_1x_3^2 + x_1x_4^2$ is not, since it fails to be invariant under $\pi = 21$.

We say $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ is *shift invariant* if the coefficient of the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$ in $f(\mathbf{x})$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$ and any composition $(\alpha_1, \cdots, \alpha_k)$ [5]. For example, the function $f(\mathbf{x}) = x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_2^2x_3x_4$ is shift invariant in four variables.

Lastly, we say $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ is of bounded degree k if the degree of any monomial in $f(\mathbf{x})$ is bounded. E.g: The function $f(\mathbf{x}) = x_1^2x_2x_3^5x_4 + x_2x_3 + x_4^7x_5^3$ is of bounded degree 10.

These definitions provide the necessary background to define the ring of symmetric functions and the ring of quasisymmetric functions.

Let the ring of symmetric functions, denoted Λ or Sym, be the set of all formal power series $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ such that [7]:

1. $f(\mathbf{x})$ is invariant under the action of the symmetric group
2. $f(\mathbf{x})$ is of bounded degree

The function $f(\mathbf{x}) = \sum_{i_1<i_2<i_3} x_{i_1}^3 x_{i_2} x_{i_3} + x_{i_1} x_{i_2}^3 x_{i_3} + x_{i_1} x_{i_2} x_{i_3}^3$ is a symmetric function, since it is bounded of degree 5 and is invariant under the action of the symmetric group. Notice that not all functions invariant under the symmetric group are symmetric functions; in particular, the previous example $\prod_{i=1}^{\infty} 1 + x_i$ is invariant under the action, but is not of bounded degree.

Let Sym_n denote the vector space of homogeneous symmetric functions of degree n. The function $f(\mathbf{x}) = \sum_{i_1<i_2<i_3} x_{i_1}^3 x_{i_2} x_{i_3} + x_{i_1} x_{i_2}^3 x_{i_3} + x_{i_1} x_{i_2} x_{i_3}^3$ noted above is in Sym_5.

Let the ring of quasisymmetric functions [5], denoted QSym, be the set of all formal power series $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ such that:

1. $f(\mathbf{x})$ is shift invariant
2. $f(\mathbf{x})$ is of bounded degree

An example of a quasisymmetric function is $f(\mathbf{x}) = \sum_{i_1<i_2<i_3<i_4} x_{i_1}^4 x_{i_2} x_{i_3}^2 x_{i_4}$. This function is bounded of degree 8 and shift invariant.

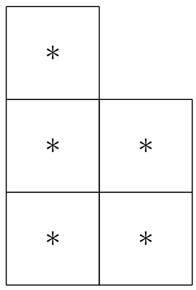
Notice that every symmetric function is a quasisymmetric function, since if a function is invariant under the action of the symmetric group it is also shift invariant. However, a quasisymmetric function is not necessarily symmetric. The function previously shown $f(\mathbf{x}) = \sum_{i_1<i_2<i_3<i_4} x_{i_1}^4 x_{i_2} x_{i_3}^2 x_{i_4}$ is a quasisymmetric function , but is not symmetric function since the monomial $x_1^4 x_2^2 x_3 x_4$ is in f, but the monomial $x_1^2 x_2^4 x_3 x_4$ is not, and it would need to be for this function to be invariant under the permutation $\pi = 21$.

Let QSym_n be the vector space of homogeneous quasisymmetric functions of degree n. The above function is a homogeneous quasisymmetric function of degree 8.

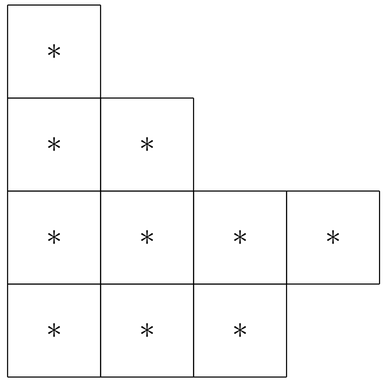
In the results shown later, Sym_n and QSym_n are of particular importance, since the functions we generate from the pattern avoidance discussed before lie in these spaces.

The next set of definitions introduce the combinatorial objects which we will be studying and which most of our results directly apply to. These objects are tableaux which will be transformed into quasisymmetric functions via a procedure we will discuss in Chapter 2.

Define $\lambda = (\lambda_1, \cdots, \lambda_k)$ to be a partition of n, such that $|\lambda| := \sum_i \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$ is a weakly decreasing sequence of positive integers. For example, $(2, 2, 1)$ is a partition of 5. We associate to a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ a *Young Diagram* consisting of k left justified rows with the ith row of length λ_i and with the first row at the bottom of the diagram and the other rows placed above this row. In the literature, this is called French notation, compared to the alternative English notation where the first row is at the top of the diagram and the other rows are placed below the first. Such a Young Diagram is said to have shape λ. The *conjugate* of $\lambda = (\lambda_1, \cdots, \lambda_k)$, denoted $\lambda' = (\lambda'_1, \cdots, \lambda'_m)$, is the partition where λ'_i is the length of the ith column of the Young Diagram with shape λ. Further, define a *composition* of n to be a sequence of positive integers $\alpha = (\alpha_1, ..., \alpha_k)$ such that $|\alpha| = n$. For example, the sequence $(3, 4, 2, 1)$ is a composition of 10, but is not a partition of 10 since the parts are not in weakly decreasing order. We can similarly associate a diagram to α called a composition diagram. We index the positions of these diagrams as (i, j), where i denotes the ith row and j denotes the jth column. The reader can refer to figure 1.1 for examples of these diagrams.



(a) a Young diagram of shape (2,2,1)



(b) a composition diagram of shape (3,4,2,1)

Figure 1.1: Examples of young diagram and composition diagram

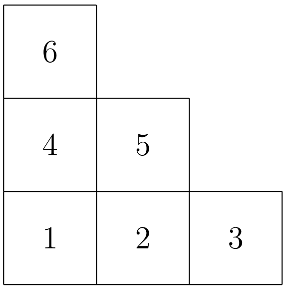
We now define tableaux which are needed to understand the objects which generate quasisymmetric and symmetric functions in later results.

Given a partition λ of n, a *standard Young tableau* (SYT) T is a Young Diagram of shape λ, denoted $Sh(T) = \lambda$, in which the boxes of T are filled with the numbers $\{1, \cdots, n\}$ with no repetitions such that the entries of the rows and columns of T are strictly increasing left to right and bottom to top.

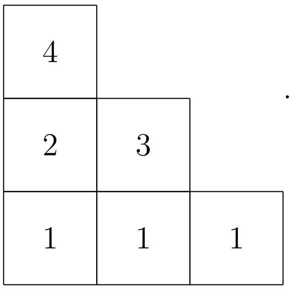
Further, define the descent set of a SYT T as:

$$Des(T) := \{i \mid i + 1 \text{ is in a higher row than } i \text{ in } T\} \subseteq [n - 1].$$

A *semistandard Young tableau* (SSYT) T is a Young Diagram of shape λ with the boxes of T filled so that the entries in the rows weakly increase from left to right and the entries in the columns strictly increase from bottom to top.



(a) a SYT with $Des(T) = \{3, 5\}$



(b) a SSYT

Figure 1.2: Examples of SYT and SSYT

We define the *Schur function* corresponding to a partition λ as

$$s_\lambda := \sum_T \prod_{i \in T} x_i$$

where T is a SSYT of shape λ. The Schur functions are an important basis for Sym_n. There are many results from algebraic geometry and representation theory associated with them, in particular, the coefficients of Schur functions count objects of interest in the respective fields.

Given a composition $\alpha = (\alpha_1, \cdots, \alpha_k)$ of n, a *Reverse Composition Tableau* (RCT) T is a composition diagram of shape α with the boxes of T filled so that they obey the following rules:

1. Row entries weakly decrease read from left to right
2. Entries in the left most column strictly increase from bottom to top
3. (Triple Rule) Let $m = max\{\alpha_l\}_{l=\{1,\ldots,k\}}$. Let $\hat{T}$ be the $n \times m$ rectangle diagram made by extending T with zero cells to create a rectangular shape. Then, for $1 \le i < j \le n$, $2 \le k \le m$ and $\hat{T}(j, k) \neq 0$ we must have if $\hat{T}(j, k) \le \hat{T}(i, k-1)$ then $\hat{T}(j, k) < \hat{T}(i, k)$

We provide this diagram to demonstrate the triple rule:

$\cdots$	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$
$\cdots$	b	c

If $a \leq b$ then $a < c$.

Lastly, a *Standard Reverse Composition Tableau* (SRCT) is a reverse composition tableau T filled with the numbers $\{1, \cdots, n\}$ with no repetitions.

To define the quasisymmetric Schur functions we first define the fundamental basis of QSym_n. Given a subset $S \subseteq [n-1]$, the associated fundamental quasisymmetric function is

$$F_S = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ i_s < i_{s+1} \text{ if } s \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For example, if $S = \{2, 3\} \subseteq [4]$ then we have the sum over $x_{i_1} x_{i_2} x_{i_3} x_{i_4}$ where $i_1 \leq i_2 < i_3 < i_4$ so, $F_{\{2,3\}} = x_1^2 x_2 x_3 + x_1 x_2 x_3 x_4 + x_2^2 x_3 x_4$.

Define the descent set of a SRCT T to be

$$Des(T) := \{i \mid i+1 \text{ appears weakly right of } i \text{ in } T\} \subseteq [n-1].$$

For example the SRCT,

$T =$

7	4	
6	5	3
2	1	

has $Des(T) = \{2, 4, 6\}$.

Lastly, the descent set of a permutation $\pi = \pi_1 \cdots \pi_n$ is:

$$Des(\pi) = \{i \mid \pi_i > \pi_{i+1} \subseteq [n-1].$$

For example, the permutation $\pi = 34251$ has $Des(\pi) = \{2, 4\}$.

We now define the quasisymmetric Schur Function corresponding to a composition α to be:

$$S_\alpha := \sum_T F_{Des(T)}$$

where T is a SRCT of shape α.

For example, the only SRCT of shape $\alpha = (2, 1, 3)$ are

$$T_1 = \begin{array}{|c|c|c|} \hline 6 & 5 & 2 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \quad T_2 = \begin{array}{|c|c|c|} \hline 6 & 5 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \quad T_3 = \begin{array}{|c|c|c|} \hline 6 & 5 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

We have $Des(T_1) = \{1, 3, 4\}, Des(T_2) = \{2, 4\}$, and $Des(T_3) = \{2, 3\}$. Thus, $S_{2,1,3} = F_{\{1,3,4\}} + F_{\{2,4\}} + F_{\{2,3\}}$.

We note the following relationship between Schur functions and quasisymmetric Schur functions:

$$s_\lambda = \sum_{\hat{\alpha}=\lambda} S_\alpha.$$

where $\hat{\alpha}$ is the rearrangement of α into parts of weakly decreasing order. [5]

For example, $s_{3,2} = S_{2,3} + S_{3,2}$

We now define, in the same fashion as [3], the pattern quasisymmetric function:

$$Q_n(\Pi) := \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{Des(\sigma)}$$

Prior results by Hamaker, et al [3], gave precisely the subsets of S_3 for which $Q_n(\Pi)$ was Schur nonnegative, that is, a linear combination of Schur functions with nonnegative coefficients. These results are shown in Figure 1.3. As stated previously we ask the question: When is $Q_n(\Pi)$ quasisymmetric Schur nonnegative for all n, that is when is $Q_n(\Pi)$ a sum of quasisymmetric Shur functions with nonnegative coefficients?

In Figure 1.4, we have a table of theorems and conjectures regarding this question. The first three rows of the table are subsets of $\mathfrak{S}_3$ that are proven to be quasisymmetric Schur nonnegative with formulas given in the right column for their quasisymmetric Schur expansion. Proofs of these theorems can be found in Section 3. The rest of the subsets in the table are conjectured to be nonnegative with conjectured formulas given.

Π	$Q_n(\Pi)$ for $n \geq 3$
$\emptyset$	$\sum_{\lambda} f^\lambda s_\lambda$
$\{123\}$	$\sum_{\lambda_1 < 3} f^\lambda s_\lambda$
$\{321\}$	$\sum_{\lambda_1^t < 3} f^\lambda s_\lambda$
$\{132, 213\}; \{132, 312\}; \{213, 231\}; \{231, 312\};$	$\sum_{\lambda \text{ a hook}} f^\lambda s_\lambda$
$\{123, 132, 312\}; \{123, 213, 231\}; \{123, 231, 312\};$	$s_{1^n} + s_{2,1^{n-2}}$
$\{132, 213, 321\}; \{132, 312, 321\}; \{213, 231, 321\};$	$s_n + s_{n-1,1}$
$\{132, 213, 231, 312\}$	$s_n + s_{1^n}$
$\{123, 132, 213, 231, 312\}$	s_{1^n}
$\{132, 213, 231, 312, 321\}$	s_n

Figure 1.3: Hamacker, et al theorems

Π	$Q_n(\Pi)$ for $n \geq 3$
$\{213, 231, 321, 312\}; \{132, 213, 312, 321\};$	$S_{n-1,1} + S_n$
$\{123, 213, 231, 312\}; \{123, 132, 213, 312\};$	$S_{1^n} + S_{2,1^{n-2}}$
$\{123, 132, 231, 312\}; \{123, 132, 213, 231\};$	$S_{1^n} + S_{1^{n-2},2}$
	Conjectured
$\{213, 312, 321\};$	$(n-1)S_{n-1,1} + S_n$
$\{132, 213, 312\}; \{213, 231, 312\};$	$S_{1^n} + S_{1^{n-2},2} + \cdots + S_n$
$\{123, 213, 312\}; \{123, 132, 231\};$	$S_{1^n} + (n-1)S_{2,1^{n-2}}$
$\{321, 312\};$	If n even: $S_{1,n-1} + S_{2,n-2} + \cdots + S_{\frac{n}{2},\frac{n}{2}} + 3S_{\frac{n}{2}+1,\frac{n}{2}-1}$ $+5S_{\frac{n}{2}+2,\frac{n}{2}-2} + \cdots + (n-1)S_{1,n-1} + S_n$ If n odd: $S_{1,n-1} + S_{2,n-2} + \cdots + S_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil} + 2S_{\lfloor\frac{n}{2}\rfloor+1,\lceil\frac{n}{2}\rceil-1}$ $+4S_{\lfloor\frac{n}{2}\rfloor+2,\lceil\frac{n}{2}\rceil-2} + \cdots + (n-1)S_{1,n-1} + S_n$
$\{213, 312\};$	$\binom{n-1}{n-1}S_{1^n} + \binom{n-1}{2}S_{2,1^{n-2}}+$ $\binom{n-1}{3}S_{3,1^{n-3}} + \cdots + \binom{n-1}{0}S_n$
$\{123, 312\};$	$S_{1^n} + S_{1^{n-2},2} + 2S_{1^{n-3},2,1}+$ $3S_{1^{n-4},2,1^2} + \cdots + (n-1)S_{2,1^{n-2}}$
$\{123, 231\};$	$S_{1^n} + (n-1)S_{1^{n-2},2} + (n-2)S_{1^{n-3},2,1}+$ $(n-3)S_{1^{n-4},2,1^2} + \cdots + S_{2,1^{n-2}}$

Figure 1.4: Table of Theorems and Conjectures

Chapter 2: A bijection from permutations to Standard Reverse Composition Tableaux

We begin by introducing an algorithm known as row insertion. This algorithm takes a permutation $\pi \in \mathfrak{S}_n$ and takes it to a pair of Standard Young Tableau. Under this algorithm, there is bijective correspondence between permutations and Standard Young Tableau, known as the Robinson-Schensted-Knuth correspondence. We will denote the algorithm as RSK for the remainder of the thesis.

Let $\sigma = a_1 a_2 \cdots a_n$ in S_n be a permutation.

1. Start with σ in two-line notation, that is $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$.

2. Set $P = \emptyset$ and $Q = \emptyset$.

3. We insert the entries of σ from left to right, the first row inserted into Q and the second into P as follows:

   (a) Inserting $k \in \{a_1, \cdots, a_n\}$: If k is at least as large as all the entries in the first row of P, add k to a new box at the end of the first row. If not, find the left-most entry in the first row that is strictly larger than k and put k in its place, call the replaced entry k now. Continue this process on the second row with the new k value. Keep going until a k value is placed at the end of a row or once it forms a new row.

   (b) Place the corresponding recording value coming from the first row of σ into Q in the same spot k was placed at the end of the previous step.

The reader can find an example of RSK in Figure 2.1.

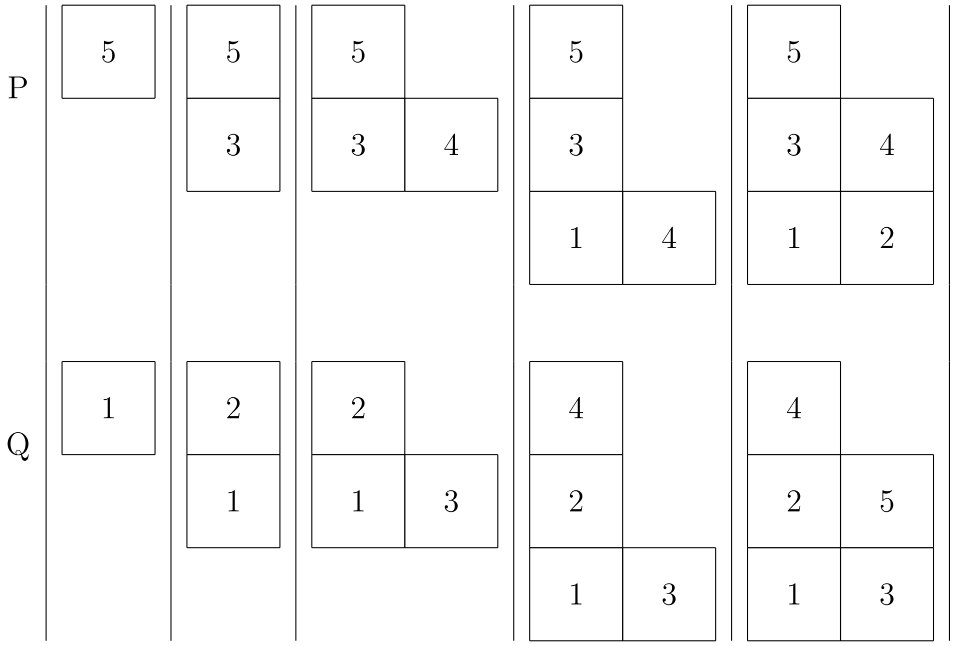


Figure 2.1: Insertion sequence of $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$ via RSK

The RSK algorithm has several nice properties, some of note are listed in the folowing proposition.

Theorem 2.1. [6]

1. If $\pi \xrightarrow{RSK} (P, Q)$ then $Des(\pi) = Des(Q)$
2. Let
   - $i(\pi)$ = length of the longest increasing subsequence of π
   - $d(\pi)$ = length of the longest decreasing subsequence of π.
   - If $\pi \xrightarrow{RSK} (P, Q)$ where $Sh(P) = \lambda = (\lambda_1, \cdots, \lambda_k)$ then $i(\pi) = \lambda_1$ and $d(\pi) = \lambda_1^t$

We describe an algorithm that takes a permutation $\pi \in \mathfrak{S}_n$ to a pair of SRCT (S, T). This algorithm was first introduced in [4] for semi-skyline augmented fillings and then extended in [2] for SRCTs. This is essentially an extension of RSK for a different type of tableaux. The correspondence between SRCTs and permutations is bijective. Note, the shape of the two tableau (S, T) produced are not necessarily equal, instead they are a row permutation of each other. We denote this algorithm cRSK for the remainder of the thesis.

Let $\sigma = a_1 a_2 \cdots a_n \in S_n$ be a permutation.

1. Starting with σ in two-line notation, reverse the top and bottom row of this array from right to left to obtain $\hat{\sigma}$

$$\text{If } \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \text{ then } \hat{\sigma} = \begin{pmatrix} n & n-1 & \cdots & 1 \\ a_n & a_{n-1} & \cdots & a_1 \end{pmatrix}$$

2. Set $S = \emptyset$ and $T = \emptyset$

3. Insert the first entry a_n of the bottom row of $\hat{\sigma}$, into a cell of the currently empty tableau S and place the corresponding entry n of the top row of $\hat{\sigma}$ into the tableau T.

4. To insert the remainder of the entries, one at a time, scan column to column, from bottom to top, right to left in our Composition Tableaux S and follow these rules:

   (a) Set $j :=$ length of the longest row $+1$

   (b) Let k be the current insertion entry

   (c) Start scanning at the bottom of column j

   (d) If the position at the end of the current row you are scanning is empty and k is smaller than the entry in position $j-1$ then insert k at the end of the row in position j. Else, if the jth position is nonempty and contains

$\hat{k} < k$ and k is less than the entry in the $j-1$ position then swap $\hat{k}$ and k. If neither of these two situations occur, continue onto the next step.

(e) With the possible new k value continue to the next higher position in column j and repeat the previous step. If you reach the end of the column, set $j := j-1$ and return to step (c). If $j = 1$ create a new row containing only k and place the row so that the first column is strictly increasing from bottom to top.

5. In the Recording Tableau T we insert the corresponding recording entry at the end of the bottom most row of length $j-1$.

6. Continue this process of insertion with each element of the permutation until it terminates

The reader can find an example of this algorithm in Figure 2.2.

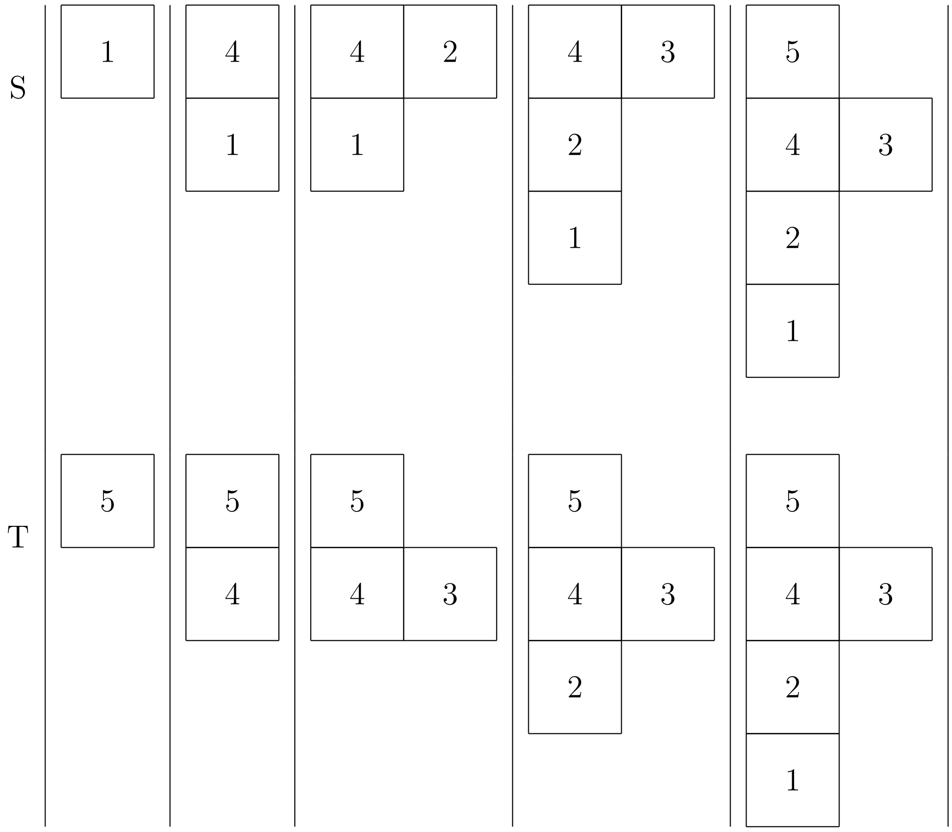


Figure 2.2: Insertion sequence of $\pi = 53241$ showing pairs (S, T)

A particular property of this algorithm is that $\pi \xrightarrow{cRSK} (S, T)$ then $Des(T) = Des(\pi)$. To prove this property, we utilize a bijection between SRCT of shape n and Standard Reverse Young Tableau of shape n described in [4]. These are tableau that are obtained by "reversing" the entries of a Standard Young Tableau of shape n; that is, we send the entry 1 to n, 2 to $n-1$, and so on. In particular, the columns of a Standard Reverse Young Tableau increase from bottom to top and the rows decrease from left to right. The map ρ taking a Standard Reverse Composition Tableau T to a Standard Reverse Young Tableau Q is as follows (its inverse can be found in [4]):

1. First reverse the first column of T, so that this left most column is increasing from top to bottom.

2. Insert the subsequent columns of T by taking each entry from the column of T and placing it into the corresponding column of Q in order from largest to smallest

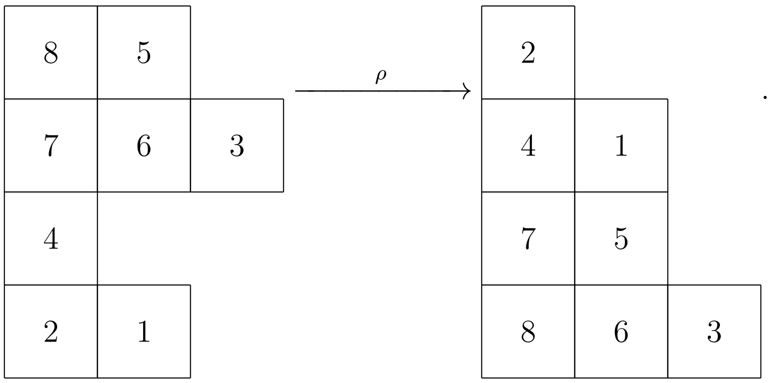


Figure 2.3: The map ρ

Note that this map commutes with RSK and cRSK, in the sense that if $\pi \xrightarrow{cRSK} (S, T)$ then $\pi \xrightarrow{RSK} (\rho(S), \rho(T))$ [4].

Standard Reverse Young Tableau have descent set given by,

$$Des(T) = \{i \mid i+1 \text{ is in a lower row than } i \text{ in } T\}.$$

It is known that under RSK, if $\pi \in \mathfrak{S}_n \xrightarrow{RSK} (P, Q)$ where P and Q are Standard Reverse Young Tableau then $Des(\pi) = Des(Q)$. We also note that the map ρ commutes with RSK and cRSK, in the sense that if $\pi \xrightarrow{cRSK} (S, T)$ then $\pi \xrightarrow{RSK} (\rho(S), \rho(T))$ [4]. So it suffices to show that under ρ descents are preserved.

Lemma 2.1. The map ρ preserves descent sets between a SRCT T and Standard Reverse Young Tableau Q. That is, if $T \xrightarrow{\rho} Q$ then $Des(Q) = Des(T)$.

Proof. Let T be a RCT and let $i \in Des(T)$; that is $i+1$ appears weakly right of i in T. We must show that ρ sends $i+1$ to a lower row than i.

If i is in the first column in T then $i+1$ is also in the first column of T and $i+1$ must lie above i in T since T is a Standard Reverse Composition Tableau. Therefore, the first column of T must increase from bottom to top. Hence, by ρ reversing the entries of the first column, we have that $i+1$ lies above i.

If i is not in the first column of T and $i+1$ lies in the same column as i, then $i+1$ will be in the same column as i in Q. Since ρ orders entries in columns by largest to smallest from top to bottom, then $i+1$ will be in a lower row than i.

If i is not in the first column of T and $i+1$ lies in a column strictly right then we note that $i+1$ could not map to the row i is in or a row higher than i in Q, since this would violate that the columns of Q strictly increase.

Thus, in all cases i is in $Des(Q)$.

Now we show that if $i \notin Des(T)$ then $i \notin Des(Q)$.

If $i \notin Des(T)$ then $i+1$ lies strictly left of i in T. So, in Q the only way we could have $i \in Des(Q)$, would be if $i+1$ was in a column to the left of i, but below i. However, the columns of Q decrease from bottom to top, so this would imply that any element that lies above $i+1$ is smaller than i, hence the element in the same row as i, but left of i would be smaller than i, contradicting the rows of Q decreasing.

Thus, $i \notin Des(T)$.

Hence, $Des(T) = Des(Q)$. ■

We have shown that $Des(\pi) = Des(T)$. Recall that the pattern quasisymmetric function is defined as follows:

$$Q_n(\Pi) := \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{Des(\sigma)}.$$

Since we now know $Des(\pi) = Des(T)$, we have:

$$Q_n(\Pi) := \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{Des(T)},$$

where $\sigma \xrightarrow{cRSK} (S, T)$. Thus property allows us to view the T tableau as the correct tableau that determines our quasisymmetric Schur functions.

Chapter 3: cRSK and Tableau Shapes

In this chapter, we will prove several lemmas that determine which permutation patterns correspond to Standard Reverse Composition Tableaux shapes. The particular shapes studied arose from Sage calculations that showed conjectured formulas for the pattern quasisymmetric function. The interested reader can find the Sage code in Appendix A. The proofs are all extremely similar in technique. For the reader mainly interested in the results from Figure 1.3, those main theorems will be proven in the next chapter.

Lemma 3.1. Let $\pi \xrightarrow[cRSK]{} (S, T)$ and $Sh(T) = (n-1, 1)$. Then π can only contain the length three patterns $231, 132,$ and 123.

Proof. Assume $\pi = a_1 a_2 \cdots a_n \xrightarrow[cRSK]{} (S, T)$ and $Sh(T) = (n-1, 1)$. We then have that T has shape shown in Figure 3.1(a). We now look for where we can place the entry n in T. Since T is a Standard Reverse Composition tableau the columns of T must increase from bottom to top and rows must decrease from left to right. Hence, the only position for n is $(2, 1)$ since it is the largest element. Therefore, we obtain the filling in Figure 3.1(b). Next, to fill the rest of the entries we follow the same rules. Thus, the rest of the filling of T must go in descending order term by term resulting in the full filling of T shown in Figure 3.1(c).

$*$			
$*$	$*$	$\cdots$	$*$

(a) Shape of T

n			
$*$	$*$	$\cdots$	$*$

(b) Filling n

n			
$n-1$	$n-2$	$\cdots$	1

.

(c) Filling in the rest of T

Figure 3.1: Process for Filling T

This shape of T and the current filling shows us that at step 2 of insertion, when we insert a_{n-1}, it must lie above a_n, since this is the only way to produce the second row starting with $n-1$ in T. Thus, we have the following shape at step 2 of insertion:

$$S_2 = \begin{array}{|c|} \hline a_{n-1} \\ \hline a_n \\ \hline \end{array}.$$

To produce the first row in T, we have two scenarios, Case 1: either all other entries are larger than a_n, but successively decrease, that is, $a_{n-1} > a_{n-2} > \cdots > a_1$.

Case 2: Only a_{n-1} is larger than a_n and every other element is smaller than a_n and successively decreasing.

If there was no successive decreasing in either of these two scenarios, then some element would be larger than a previous element along the row, which would cause an additional bump and not produce the desired shape of T.

$$S = \begin{array}{|c|c|c|c|}\hline a_{n-1} & a_{n-2} & \cdots & a_1 \\ \hline a_n \\ \cline{1-1}\end{array} \qquad S = \begin{array}{|c|c|c|c|}\hline a_{n-1} \\ \hline a_n & a_{n-2} & \cdots & a_1 \\ \hline\end{array}$$

Figure 3.2: Case 1 and Case 2 shown

In the permutation $a_1 \cdots a_{n-1}a_n$ we obtain a 231 from $a_{n-1} > a_{n-2} > a_n$, this will arise from either of Case 1 or Case 2. Additionally, in only Case 2, we obtain the pattern 132 from $a_{n-1} > a_n > a_1$.

From any row in S we will only obtain the pattern 123 in the permutation. It is clear that no other interaction between elements in the tableau occur that would result in a new pattern. ■

Lemma 3.2. Let $\pi \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (1, n-1)$ then π can only contain the length three patterns $132, 312, 213$, and 123.

Proof. Let $\pi = a_1 a_2 \cdots a_n \xrightarrow[cRSK]{} T$ and let T have shape $(1,(n-1))$.

Thus, $T =$

$*$	$*$	$*$	$\cdots$	$*$
$*$				

.

Recall, that in a reverse composition tableau, the entries in the first column must increase from bottom to top and the entries in any row must decrease from left to right. Therefore, the entry n must appear in position $(2,1)$ in T for T to be a valid Reverse Composition Tableau. Further, some entry j must go into the position (1,1) in T and thus T is determined as in the figure below by the above rules:

n	$n-1$	$n-2$	$\cdots$	$j+1$	$j-1$	$\cdots$	1
j							

where $j \in \{n-1, \cdots, 1\}$

Thus, the corresponding shape S at insertion step $j-1$ is of the form,

a_n	a_{n-1}	$\cdots$	a_{j+1}

.

Given the shape of T, we know that some element either already placed in S, or the element a_j, must be placed in a lower column and this happens precisely at insertion step j.

We now have two options: (1) the entry $a_j > a_n$ which forces a_j to be placed in the lower column or (2) $a_n > a_j$. Notice, we cannot have $a_k > a_j$ for $k \in \{n-1, ..., j+1\}$

since we know some element must end up in a lower column and this would force a_j to sit at the end of the row in S, thus producing an undesired shape for T. In fact, the only element that can be smaller than a_j is a_{j+1} since otherwise we would break the triple rule. To see this suppose there exists a_k with $k \in \{n-1, ..., j+1\}$ and $a_k < a_j$. Choose a_k to be the first such occurrence of such an element, so that we have $a_{k-1} > a_j$. Then, we know at the jth step of insertion, a_j replaced a_k and a_k was moved to a lower column. So the shape of S at the jth step is:

a_n	a_{n-1}	$\cdots$	a_{k+1}	a_j	a_{k-1}	$\cdots$	a_{j+1}
a_k							

.

Then, $a_{k-1} < a_k$, but by the triple rule we must have $a_{k-1} < 0$ which is nonsensical.

We now see our resulting shapes:

Option (1): If $a_j > a_n$ then the shape of S at step k must be

a_j			
a_n	a_{n-1}	$\cdots$	a_{j+1}

and the shape of T implies the positions of the last $j-1$ entries must be at the end of the first row. Thus, we obtain pattern 312 from $a_j > a_n > a_{j+1}$ and 132 from $a_j > a_n > a_1$ (Notice, 132 is not obtained if $a_j = a_1$). From any row decrease we obtain 123. By inspection, we see these are the only patterns possible.

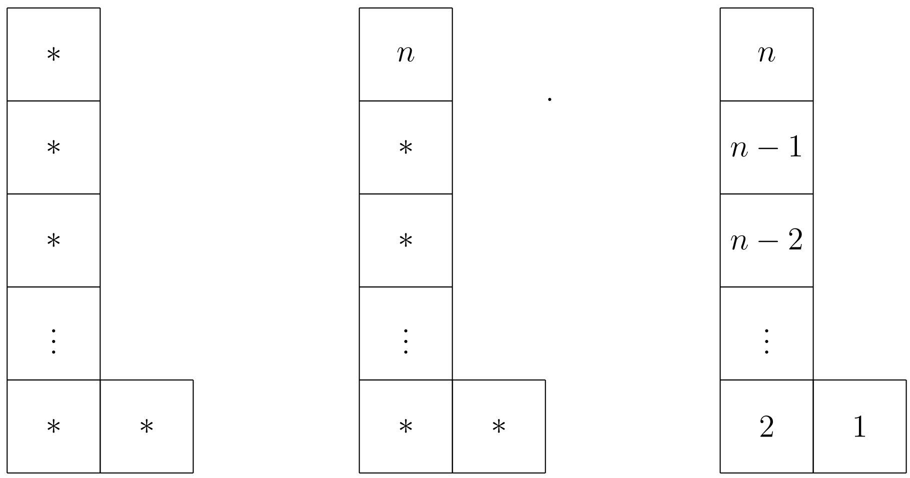
Option (2): If $a_n > a_j$ then we know $a_{j+1} < a_j$. Then, we know at the jth step of insertion we have that a_j replaced a_{j+1} and a_{j+1} was moved to a lower column. So the shape of S at the jth step is:

a_n	a_{n-1}	$\cdots$	a_{j+2}	a_j
a_{j+1}				

Again, our shape of T determines the positions of the last $k-1$ entries at the end of the longest row. Again, by the triple rule, these remaining entries are also larger than a_{j+1}. Thus, we obtain pattern 213 from $a_{j+2} > a_j > a_{j+1}$ and from any row decrease we obtain 123. ■

Lemma 3.3. Let $\pi \xrightarrow[cRSK]{} (S, T)$ and $Sh(T) = (2, 1^{n-2})$ then π can only contain the length three patterns $231, 132$, and 321.

Proof. Let $\pi = a_1 a_2 \cdots a_n$. Let (S, T) be the pair of reverse composition tableau resulting from cRSK. Let T have shape $(2, 1^{n-2})$ so that T looks like Figure 1.



(a) Figure 1 (b) Figure 2 (c) Figure 3

Since T is a reverse composition tableau, its left most column entries must increase from bottom to top and its row entries must decrease from left to right. Thus, the recording entry n must be in the position $(n-1, 1)$. Otherwise there would be a decrease in the left most column since some $i \in \{1, \cdots, n-1\}$ would be above n or an increase in the bottom row if n is in position $(1, 2)$.

Thus, we fill in the shape of T to match Figure 2.

Now by the same logic, we must have $n-1$ in position $(n-2, 1)$ and continuing this process for $i \in \{1, 2, ..., n-2\}$ we obtain that T is determined as Figure 3.

In particular, this implies that after insertion step $n-1$ we have that T looks like Figure 4. This reduces our only option for S to be as in Figure 5. This is because a single column in T implies the shape of S is a single column as well. Now, the only way for S to be a valid reverse composition tableau *and* be a single column would be if when inserting π (from right to left, as defined previously), we have $a_n < a_{n-1} < \cdots < a_3 < a_2$.

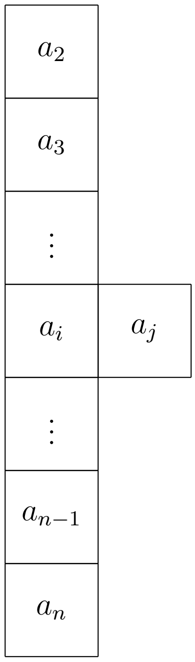
$$T = \begin{array}{|c|} \hline n \\ \hline n-1 \\ \hline n-2 \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}.$$

(a) Figure 4

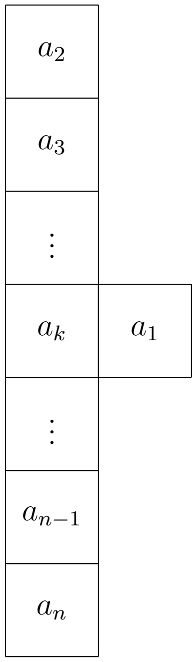
$$S = \begin{array}{|c|} \hline a_2 \\ \hline a_3 \\ \hline \vdots \\ \hline a_{n-1} \\ \hline a_n \\ \hline \end{array}.$$

(b) Figure 5

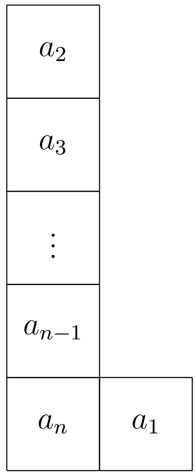
If this were not the case, that is to say we had one increase, so for $a_i, a_j \in \{a_n, a_{n-1}, \cdots, a_2\}$ we have $a_i > a_j$ with $i < j$. Then during insertion of a_j we would have a_j inserted into the row with a_i and would create the diagram in Figure 6, which is no longer a single column.



(a) Figure 6



(b) Figure 7



(c) Figure 8

In the following step, we will insert a_1 into S. We notice that when inserting a_1, in the recording tableau we would always be forced to insert the recording entry 1 next to our recording entry 2, because we always insert the recording entry into the lowest row of greatest length, which in this case, all rows are of length 1 in T. Thus, a_1 can be inserted anywhere along the second column of S and would result in the correct T shape. We also note that a_1 cannot be larger than every element that has already been inserted, since this would imply S is a straight column and thus T would also be a straight column, contradicting the assumed shape of T.

Next, we observe that if a_1 were inserted into any row other than the first row in S, a_1 must be larger than anything below it, since otherwise we would break the triple rule and S would not be a valid reverse composition tableau. For example, we provide the diagram if a_1 were inserted in row k next to a_k in Figure 7. In this case, if $a_1 < a_k$ implies $0 > a_1$ which is a contradiction, since $a_1 \in \{1, \cdots, n\}$.

From the first possible shape of S, demonstrated in Figure 7, we obtain the pattern $a_n < a_1 < a_k$ or 231 in our original permutation.

Alternatively, in the case that a_1 is not larger than any elements already inserted, we would have a_1 at the end of the bottom-most row (to avoid breaking the triple rule as discussed previously). Hence, we would result in the shape of S in Figure 8.

From the second shape of S, pictured in Figure 8, we obtain the pattern $a_1 < a_n < a_{n-1}$ or 132 in our original permutation.

The only other pattern possible comes from interactions where the elements ascend in a column, thus we only obtain 321 from these interactions. ■

Lemma 3.4. Let $\pi \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (1^{n-2-k}, 2, 1^k)$. Then π can only contain the length three patterns $231, 132, 312, 213$ and 321.

Proof. Assume $\pi = a_1 a_2 \cdots a_n \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (1^{n-2-k}, 2, 1^k)$.

So, we know

$$T = \begin{array}{|c|c} \hline n & \\ \hline \vdots & \\ \hline k & \multicolumn{1}{|c|}{k-1} \\ \hline \vdots & \multicolumn{1}{|c}{} \\ \cline{1-1} 1 & \\ \hline \end{array} .$$

We know n must be in the top-most position since T is a reverse composition tableau. The entry 1 must be in the lowest position, since having it in the position the element $k-1$ is occupying would break the triple rule, and having any other element other than $k-1$ in its current position would break the triple rule.

Thus at step k of insertion we have, $T_k = \begin{array}{|c|} \hline n \\ \hline \vdots \\ \hline k \\ \hline \end{array}$ and so $S_k = \begin{array}{|c|} \hline a_k \\ \hline \vdots \\ \hline a_n \\ \hline \end{array}$ to obtain the appropriate column shape of T_k.

We know $a_{k-1} < a_k$ since otherwise we would continue the column shape in T. The rest of S is determined by the following cases:

Case 1: $a_{k-1} > a_j$ for $k+1 \le j \le n$

We must have,

$$S_{k-1} = \begin{array}{|c|c|} \hline a_k & a_{k-1} \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Else, if a_{k-1} were in any other row position, we would violate the weakly decreasing rows condition and S would not be a reverse composition tableau.

Now, we note that if $a_k > a_{k-2} > a_{k-1}$ then a_{k-2} will replace a_{k-1} and a_{k-1} will be placed in column 1 in the position below a_k as shown in the figure below.

$$S_{k-2} = \begin{array}{|c|c|} \hline a_k & a_{k-2} \\ \hline a_{k-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

This scenario can continue similarly with any a_i with $1 \leq i \leq k-3$. If the process continues for all such a_i we will have:

$$S = \begin{array}{|c|c|} \hline a_k & a_1 \\ \hline a_2 & \multicolumn{1}{c}{} \\ \cline{1-1} a_3 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

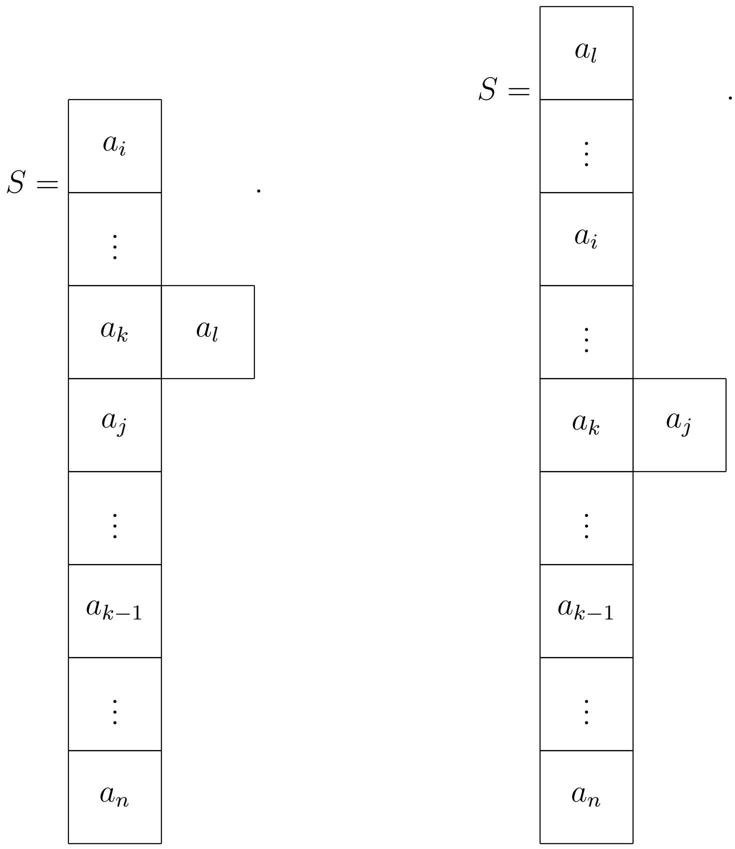
Else, if some $a_i > a_k$ for $1 \leq i \leq k-2$ then we will have (after some possible column "bumps"):

$$S_i = \begin{array}{|c|c} \cline{1-1} a_i & \\ \cline{1-1} \vdots & \\ \hline a_k & \multicolumn{1}{c|}{a_j} \\ \hline \vdots & \\ \cline{1-1} a_{k-1} & \\ \cline{1-1} \vdots & \\ \cline{1-1} a_n & \\ \cline{1-1} \end{array}.$$

Note, that we cannot have $a_i < a_j$ for $1 \leq i \leq j$, since this would extend the row which a_j occupies, resulting in an undesired T shape. Further, we can also not have $a_l < a_i$ and $a_l > a_k$ for $1 \leq l \leq i$ since that would force a_l in a row with a_i, as shown below, which violates the desired shape of T again.

$$
S_l = \begin{array}{|c|c|}
\hline
a_i & a_l \\
\hline
\vdots & \multicolumn{1}{|c}{} \\
\hline
a_k & a_j \\
\hline
\vdots & \multicolumn{1}{|c}{} \\
\cline{1-1}
a_{k-1} & \multicolumn{1}{|c}{} \\
\cline{1-1}
\vdots & \multicolumn{1}{|c}{} \\
\cline{1-1}
a_n & \multicolumn{1}{|c}{} \\
\cline{1-1}
\end{array}.
$$

Thus, we have the possible S shapes below:



(a) $a_j < a_l < a_k$ (b) $a_l > a_i$

Figure 3.6: Possible S shapes

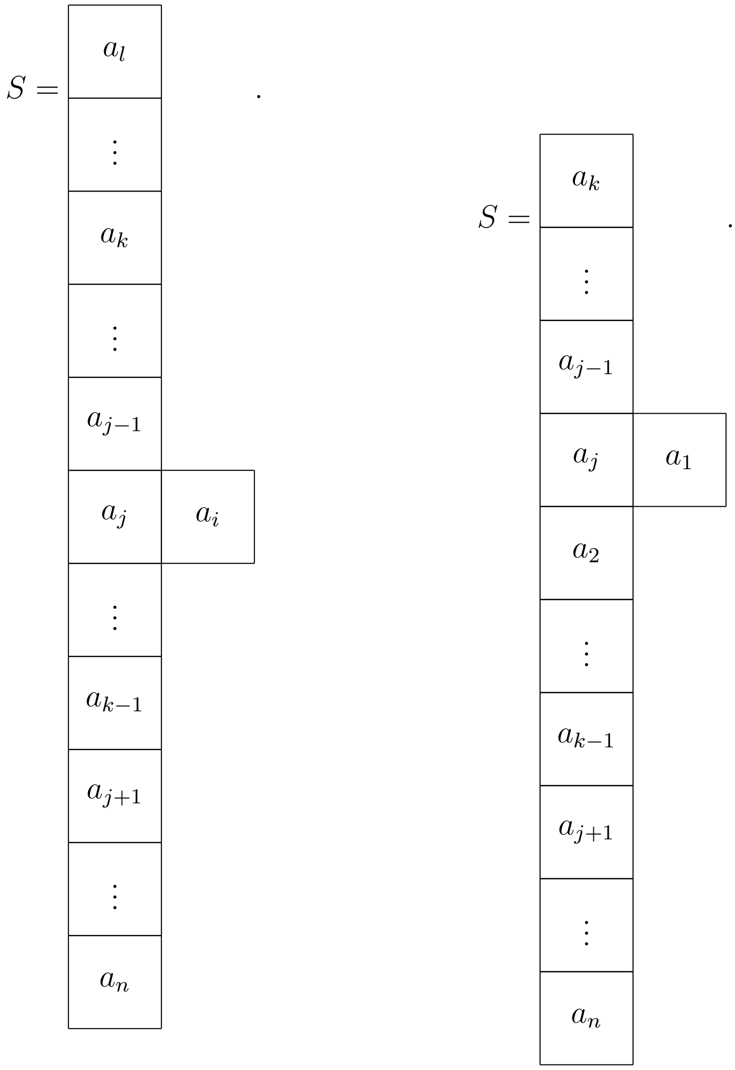
Case 2: $a_{k-1} < a_j$ for some $k + 1 \leq j \leq n$

Note, we can no longer have the same shape as the previous case, since we would violate the triple rule with the interaction between a_k, a_j, a_{k-1}. Choose the lowest a_j possible to also avoid breaking the triple rule.

We must have,

$$S_{k+1} = \begin{array}{|c|c} \hline a_k & \\ \hline \vdots & \\ \hline a_{j-1} & \\ \hline a_j & \multicolumn{1}{|c|}{a_{k-1}} \\ \hline a_{j+1} & \\ \cline{1-1} \vdots & \\ \cline{1-1} a_n & \\ \cline{1-1} \end{array} .$$

Similar to Case 1, we can have an a_i such that $a_j > a_i > a_{k-1}$ for $1 \leq i \leq k-2$, forcing a_i to take the place of a_{k-1} and a_{k-1} move to the first column below a_k. Similar to Case 1, the S shapes that result are:



(a) Not all elements "bump"

(b) All elements "bump"

Figure 3.7: Possible S shapes

From Case 1: We have our permutation $a_1 a_2 \cdots a_l \cdots a_i \cdots a_j \cdots a_{k-1} a_k \cdots a_n$ with pattern 231 formed by $a_k > a_{k-1} > a_n$. We have pattern 213 from $a_k > a_1 > a_{k-1}$ or from $a_k > a_j > a_{k-1}$. We also have the pattern 132 formed from $a_i > a_k > a_l$, when we have S as in Figure 3.6(a). The only other pattern that occurs is 321, formed from anything in a descending column in S.

From Case 2: We obtain all the same patterns as the previous case in any of the S shapes due to the similar logic used to obtain the S shapes. ■

Lemma 3.5. Let $\pi \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (1^{n-2}, 2)$ then π can only contain the length three patterns $213, 312,$ and 321.

Proof. Assume $\pi = a_1a_2\cdots a_n \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (1^{n-2}, 2)$. We must have

$$T = \begin{array}{|c|c|} \hline n & n-1 \\ \hline n-2 & \multicolumn{1}{c}{} \\ \cline{1-1} n-3 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

since n must go in position $(n-1,1)$ to have the column strictly increase from bottom to top and cannot go in position $(n-1,2)$ since otherwise an element in position $(n-1,1)$ would be smaller and force the $n-1$th row to increase. The element $n-1$ must go in position $(n-1,2)$ to avoid breaking the triple rule, which would be broken with any other element placed there. Lastly, the column must strictly increase from bottom to top forcing the position of all other elements.

From this filling of T we know that $a_n > a_{n-1}$ in the permutation $\pi = a_1\cdots a_n$.

So at step 2 of insertion we must have

$$S_2 = \begin{array}{|c|c|} \hline a_n & a_{n-1} \\ \hline \end{array}$$

.

Now in order to obtain a straight column in T we must have a straight column in S, so we have the following cases:

Case 1: ($a_j > a_n$ for $1 \leq j \leq n-2$ and $a_j > a_{j+1}$ for this range)

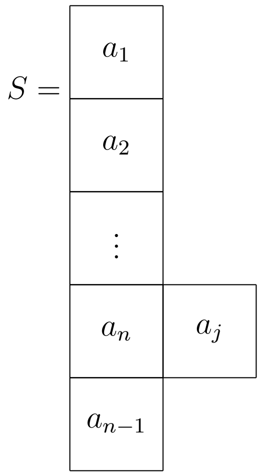
Then

$$S = \begin{array}{|c|c|}\hline a_1 & \\ \hline a_2 & \\ \hline \vdots & \\ \hline a_{n-2} & \\ \hline a_n & a_{n-1} \\ \hline \end{array}$$

Case 2:

$$S_3 = \begin{array}{|c|c|}\hline a_{j+1} & \\ \hline \vdots & \\ \hline a_n & a_j \\ \hline a_{n-1} & \\ \hline \end{array}$$

so we have $a_n > a_j > a_{n-1}$ where $1 \leq j \leq n-2$ and $a_i > a_n$ for $j+1 \leq i \leq n-2$. For the rest of the a_j's we must have them larger than a_n's and successively increasing as in case 1 to obtain a straight column. Thus,



Note: We cannot have the a_j's larger than a_{n-2} but smaller than a_n since they would break the triple rule in this case with a_j. Additionally, this gives $n-2$ options of j which may be helpful in the future when counting coefficients.

Letting $\pi = a_1 \cdots a_n$, from Case 1 we have:

$a_1 > a_n > a_{n-1}$ giving the pattern 312 and any other interaction giving the pattern 321 by the shape of S having a straight column.

From Case 2 we have:

$a_n > a_j > a_{n-1}$ giving the pattern 213 and $a_1 > a_n > a_j$ giving the pattern 312 and any other interaction giving 321. ■

Lemma 3.6.

1. Let $\pi \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (n)$ then π can only contain the length three pattern 123.

2. Let $\pi \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (1^n)$ then π can only contain the length three pattern 321.

Proof. Proof of 1: Assume $\pi = a_1 a_2 \cdots a_n \xrightarrow[cRSK]{} (S,T)$ and $Sh(T) = (n)$. Then, by the row of T decreasing from left to right, we must have the filling of in Figure 3.7(a).

Since we insert the entries of π in reverse order, this results in the S tableau in figure 3.7(b).

n	$n-1$	$n-2$	$\cdots$	1

(a) Filling of T

a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1

.

(b) Resulting S tableau

Figure 3.8: Resulting Tableaux

Thus, we have the relationship $a_1 < a_2 < \cdots < a_n$, so only the pattern 123 in π.

Proof of 2: Assume $\pi = a_1 a_2 \cdots a_n \xrightarrow[cRSK]{} (S, T)$ and $Sh(T) = (1^n)$. Then, by the column of T strictly increasing from bottom to top, we must have the filling in Figure 3.8(a). Since we insert the entries of π in reverse order and we want the resulting column shape of T, we must have a column shape in S, which results in the S tableau in figure 3.8(b).

n
$n-1$
$n-2$
$\vdots$
1

(a) Filling of T

a_1
a_2
a_3
$\cdots$
a_n

.

(b) Resulting S tableau

Figure 3.9: Resulting Tableaux

Thus, we have the relationship $a_1 > a_2 > \cdots > a_n$, so only the pattern 321 in π. ■

Chapter 4: Permutations resulting in quasisymmetric Schur positivity

We will now utilize the lemmas from the previous chapter to prove the main results from Figure 1.3. We will invoke a few results from Hamacker, et al in order to also prove the results, so we list those here:

Theorem 4.1. [3] If $\Pi = \{123, 132, 312\}$ or $\Pi = \{123, 213, 231\}$ or $\Pi = \{123, 231, 312\}$ then for $n \geq 3$, $Q_n(\Pi) = s_{1^n} + s_{2,1^{n-2}}$

Theorem 4.2. [3] If $\Pi = \{132, 213, 321\}$ or $\Pi = \{132, 312, 321\}$ or $\Pi = \{213, 231, 321\}$ then for $n \geq 3$, $Q_n(\Pi) = s_n + s_{n-1,1}$

Now we list the three results from Figure 1.3:

Theorem 4.3. If $\Pi = \{213, 231, 321, 312\}$ or $\Pi = \{132, 321, 312, 213\}$ then for $n \geq 3$ $Q_n(\Pi) = S_n + S_{n-1,1}$

Proof. Recall that $s_{n-1,1} = S_{n-1,1} + S_{1,n-1}$ and $s_n = S_n$ so $s_n + s_{n-1,1} = S_n + S_{n-1,1} + S_{1,n-1}$. Now, we notice that for $\Pi_1 = \{213, 231, 321\}$ and $\Pi_2 = \{132, 312, 321\}$ we have $Q_n(\Pi_1) = Q_n(\Pi_2) = s_n + s_{n-1,1} = S_n + S_{n-1,1} + S_{1,n-1}$ by Theorem 4.2.

Let $\Pi = \{213, 231, 321, 312\} = \Pi_1 \cup \{312\}$. We know which Quasisymmetric Shur factors appear with just Π_1 so we check which are eliminated with the inclusion of 312. By Lemma 3.2, since we are now avoiding 312 and were already avoiding 213 in Π_1, we cannot have the shape $(1, n-1)$ for our SRCT, thus we cannot have $S_{1,n-1}$ in $Q_n(\Pi)$. Further, Lemma 3.1 and Lemma 3.6 show that avoiding 312 does *not* rule out shapes $(n-1, 1)$ or (n) for the SRCT, thus $S_{n-1,1}$ and S_n are retained in $Q_n(\Pi)$.

Thus, $Q_n(\Pi) = S_n + S_{n-1,1}$.

Let $\Pi = \{132, 321, 312, 213\} = \Pi_2 \cup \{213\}$. We know which Quasisymmetric Shur factors appear with just Π_2 so we check which are eliminated with the inclusion of 213. By Lemma 2, since we are now avoiding 213 and were already avoiding 312 in Π_2, we cannot have the shape $(1, n-1)$ for our SRCT, thus we cannot have $S_{1,n-1}$ in $Q_n(\Pi)$. Further, Lemma 1 and Lemma 6 show that avoiding 213 does *not* rule out shapes $(n-1, 1)$ or (n) for the SRCT, thus $S_{n-1,1}$ and S_n are retained in $Q_n(\Pi)$. Thus, $Q_n(\Pi) = S_n + S_{n-1,1}$. ■

Theorem 4.4. If $\Pi = \{123, 231, 312, 213\}$ or $\Pi = \{123, 132, 312, 213\}$ then for $n \geq 3$ $Q_n(\Pi) = S_{1^n} + S_{2,1^{n-2}}$

Proof. Recall that $s_{2,1^{n-2}} = S_{2,1^{n-2}} + S_{1,2,1^{n-3}} + \cdots + S_{1^{n-3},2,1} + S_{1^{n-2},2}$ and $s_{1^n} = S_{1^n}$ so $s_{1^n} + s_{2,1^{n-2}} = S_{1^n} + S_{2,1^{n-2}} + S_{1,2,1^{n-3}} + \cdots + S_{1^{n-3},2,1} + S_{1^{n-2},2}$.

Now, we notice that for $\Pi_1 = \{123, 231, 312\}$ and $\Pi_2 = \{123, 132, 312\}$ we have,

$$Q_n(\Pi_1) = Q_n(\Pi_2) = s_{1^n} + s_{2,1^{n-2}} = S_{1^n} + S_{2,1^{n-2}} + S_{1,2,1^{n-3}} + \cdots + S_{1^{n-3},2,1} + S_{1^{n-2},2}$$

by Theorem 4.1.

Let $\Pi = \{123, 231, 312, 213\} = \Pi_1 \cup \{213\}$. We know which Quasisymmetric Shur factors appear with just Π_1 so we check which are eliminated with the inclusion of 213. By Lemma 3.5, since we are now avoiding 213 and were already avoiding 321 and 312 in Π_1, we cannot have the shape $(1^{n-2}, 2)$ for our SRCT, thus we cannot have $S_{1^{n-2},2}$ in $Q_n(\Pi)$. By Lemma 3.4, since we are now avoiding 213, which by the proof of Lemma 3.4 excludes all possible S shapes, we cannot have the shape $(1^{n-2-k}, 2, 1^k)$ where $1 \leq k \leq n-1$. Thus, we cannot have $S_{1^{n-2-k},2,1^k}$ for $1 \leq k \leq n-1$ in $Q_n(\Pi)$.

Further, Lemma 3.3 and Lemma 3.6 show that avoiding 213 does *not* rule out shapes $(2, 1^{n-2})$ or (1^n) for the SRCT, thus $S_{2,1^{n-2}}$ and S_{1^n} are retained in $Q_n(\Pi)$. Thus, $Q_n(\Pi) = S_{1^n} + S_{2,1^{n-2}}$.

The proof is similar for $\Pi = \{123, 132, 312, 213\}$. ■

Theorem 4.5. If $\Pi = \{123, 231, 312, 132\}$ or $\Pi = \{123, 213, 231, 132\}$ then for $n \geq 3$ $Q_n(\Pi) = S_{1^n} + S_{1^{n-2},2}$

Proof. Recall that $s_{2,1^{n-2}} = S_{2,1^{n-2}} + S_{1,2,1^{n-3}} + \cdots + S_{1^{n-3},2,1} + S_{1^{n-2},2}$ and $s_{1^n} = S_{1^n}$ so $s_{1^n} + s_{2,1^{n-2}} = S_{1^n} + S_{2,1^{n-2}} + S_{1,2,1^{n-3}} + \cdots + S_{1^{n-3},2,1} + S_{1^{n-2},2}$.

Now, we notice that for $\Pi_1 = \{123, 231, 312\}$ and $\Pi_2 = \{123, 213, 231\}$ we have,

$$Q_n(\Pi_1) = Q_n(\Pi_2) = s_{1^n} + s_{2,1^{n-2}} = S_{1^n} + S_{2,1^{n-2}} + S_{1,2,1^{n-3}} + \cdots + S_{1^{n-3},2,1} + S_{1^{n-2},2}$$

by Theorem 4.1.

Let $\Pi = \{123, 231, 312, 132\} = \Pi_1 \cup \{132\}$. We know which Quasisymmetric Shur factors appear with just Π_1 so we check which are eliminated with the inclusion of 132. By Lemma 3.3, since we are now avoiding 132 and were already avoiding 321 and 231 in Π_1, we cannot have the shape $(2, 1^{n-2})$ for our SRCT, thus we cannot have $S_{2,1^{n-2}}$ in $Q_n(\Pi)$. By Lemma 4, since we are now avoiding 132, which by the proof of Lemma 3.4 excludes all possible S shapes, we cannot have the shape $(1^{n-2-k}, 2, 1^k)$ where $1 \leq k \leq n-1$. Thus, we cannot have $S_{1^{n-2-k},2,1^k}$ for $1 \leq k \leq n-1$ in $Q_n(\Pi)$.

Further, Lemma 3.5 and Lemma 3.6 show that avoiding 132 does *not* rule out shapes $(1^{n-2}, 2)$ or (1^n) for the SRCT, thus $S_{1^{n-2},2}$ and S_{1^n} are retained in $Q_n(\Pi)$. Thus, $Q_n(\Pi) = S_{1^n} + S_{1^{n-2},2}$.

The proof is similar for $\Pi = \{123, 213, 231, 132\}$. ■

Chapter 5: Possible Future Work

The conjectured part of the table still remains to be proven. We have verified the conjectures up to $n = 9$ via Sage calculations. In particular, we feel that a new approach needs to be developed to handle the generating functions that have coefficients larger than 1 in them, as the current method only rules out patterns but doesn't count how many permutations with a given S tableau could show up. Additionally, the current method strongly relied on the results of Hamacker, et al, which for the conjectured entries there are no obvious reductions of the pattern quasisymmetric function with those results.

It is also interesting that only hook shapes are appearing in the tableaux that are generated by the pattern quasisymmetric function. It would be interesting to prove more characteristics about possible shapes with patterns larger than length three.

Additionally, via Sage, it was found that although other subsets of permutations might not be quasisymmetric Schur nonnegative, there could be patterns in the expansion that still occur and may be of interest.

There is another class of quasisymmetric functions that are in the literature, Young quasisymmetric Schur functions. Sage calculations showed positivity in different sets for these functions, which are generated by a different, but closely related, class of tableaux. It is possible to use similar methods to this thesis to prove positivity results for these functions and finding out why positivity on certain sets differs would be excellent.

In Hamacker, et al [3], a discussion of the representation theory of subsets of the symmetric group is discussed. It would be interesting to know what the Quasi-Schur positivity of the sets in the earlier table means in this regard.

In 2018, Bloom and Sagan [1] revisited the work in the original paper and questions about superstandard hooks were answered, as well as a previous conjecture about partial shuffles was proven. It remains to be seen if these ideas could be applied to the quasisymmetric case and Standard Reverse Composition Tableaux.

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Appendix A: Sage Code for generating quasisymmetric Schur functions from permutations

```
NCSF = NonCommutativeSymmetricFunctions(QQ['q','t'].fraction_field())
(q,t) = NCSF.base_ring().gens()
QSym=NCSF.dual()
R = PolynomialRing(NCSF.base_ring(), 10, 'x')
x = R.gens()
QS=QSym.QS()
YQS=QSym.YQS()
F=QSym.F()
M = QSym.M()
def avoiders(L,n):
    # produces the permutations of n that avoid the elements of list L
    LL=deepcopy(L)
    global M
    M=Permutations(n,avoiding=LL)

    return list(M)
def polly(Q):
    # produces the polynomial we are looking for. f is in fundamentals, g
       is in QS
    QQ=deepcopy(Q)
    S=[]
    f = 0
    global g
    global h
    for i in range(0,len(QQ)):
        S.append(Permutation(QQ[i]).descents())
        f=f+F(Permutation(QQ[i]).descents_composition())
        g=QS(f)
        h=YQS(f)
    return g
```

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