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Abstract

Reverse Mathematics is a subfield of Computability Theory and mathematical logic concerned with one main question: “What are the necessary axioms for mathematics?” Reverse Mathematics allows us to characterize the logical strength of theorems, relating theorems across many mathematical disciplines. In this paper, we synthesize the historical work on the Reverse Math of Real Analysis by Simpson and Brown in a way that is accessible to non-logicians while a useful reference to those within the field. In particular, we describe the nuances of different definitions of closed sets, whether as complements of open sets or as sets that contain all their limit points. These distinct definitions connect to a relatively large gap of logical strength. In one case, we correct an error in the proof that “All closed sets are separably closed” implies $ACA_0$ over $RCA_0$ from Brown’s 1990 paper “Notions of Closed Subsets of a Complete Separable Metric Space in Weak Subsystems of Second Order Arithmetic.” We also consider how the dual definitions of closed sets affect the strength of the Baire Category Theorem. As a whole, this paper outlines a path of understanding Reverse Mathematics for those outside the field by exploring fundamental ideas from Real Analysis.
Chapter 1

Introduction

1.1 What is Reverse Mathematics?

Reverse Mathematics is a subfield of Computability theory and mathematical logic concerned with one main question: "What are the necessary axioms for mathematics?" Reverse Mathematics allows us to characterize the logical strength of theorems and relate theorems across different areas of mathematics. Though Reverse Mathematics as a discipline was not formalized until the 1990's, the field has a rich history.

1.2 Historical Motivations

1.2.1 Euclid's Parallel Postulate

In Ancient Greece, Euclid presented the axioms and theorems of geometry in his groundbreaking *Elements*. Euclidean geometry relies on five postulates. The first four of these consider the definitions and basic properties of straight lines, circles, and right angles; we present them here in translation from Cannon [5].

Euclid's Four Postulates

**Theorem 1.2.1.** "Each pair of points can be joined by one and only one straight line segment."

**Theorem 1.2.2.** "Any straight line segment can be indefinitely extended in either direction."

**Theorem 1.2.3.** "There is exactly one circle of any given radius with any given center."
Theorem 1.2.4. "All right angles are congruent to one another."

Mathematicians were comfortable with these first four postulates, as they are straightforward characterizations of lines, circles and right angles. However, the fifth postulate, the parallel postulate, is quite a bit different than the others.

Theorem 1.2.5 (Euclid’s Parallel Postulate -Modern Translation). "Given a line and a point not on it, there is exactly one line going through the given point that is parallel to the given line" [5]

The parallel postulate is much more complicated than the previous postulates. So the question arose: why is the parallel postulate so different? For thousands of years after Euclid, mathematicians tried to resolve this supposed loose-end of geometry, figuring that there must be a way to prove the parallel postulate directly from the other four. These mathematicians attempted various forms of proof by contradiction. In fact, German mathematician Georg Simon Klügel published 20 attempts of such proofs as his doctoral thesis in 1763. The hope of these explorations were both to understand the parallel postulate more deeply but also to show that assuming the parallel postulate is false results in a contradiction [5].

In 1868, Italian mathematician Eugenio Beltrami showed that several different approaches to non-Euclidean geometry (where a different form of the parallel postulate is assumed) were equivalent to the study of geodesics of surfaces with negative curvature. This effectively showed that the parallel postulate in Euclidean geometry was equiconsistent with the fifth postulate in hyperbolic geometry, showing that you cannot prove the Euclidean parallel postulate with only the first four postulates [11]. Hence, the Euclidean parallel postulate is a required assumption for Euclidean geometry, despite its more complicated structure.
1.2.2 The Foundational Crisis

These sort of foundational questions persisted outside of geometry as well. Hilbert, along with many other mathematicians, wondered if there was some set of essential axioms of arithmetic that are consistent, and from which all theorems of mathematics could be proved. He included it as the second of his 23 problems. Here is the full statement of the problem, as translated by Newson [8].

"Upon closer consideration the question arises: Whether, in any way, certain statements of single axioms depend upon one another, and whether the axioms may not therefore contain certain parts in common, which must be isolated if one wishes to arrive at a system of axioms that shall be altogether independent of one another. But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results."

However, Gödel’s 2nd completeness theorem showed that the axioms of arithmetic cannot be proven to be consistent within its own axioms. This was a setback for formalists like Hilbert searching for a consistent system of arithmetic, though not all mathematicians agree that this conclusion follows from Gödel’s results [6].

1.3 Development of Reverse Mathematics

So if the axioms of arithmetic cannot be shown to be consistent, how close can we get to determining what an “ideal” set of axioms would be? From initial work by Friedman, Simpson formalized an approach called Reverse Mathematics in Subsystems of Second Order Arithmetic, published in 1999 [15]. The techniques of Reverse Mathematics follow from Set Theory, Recursion Theory, and Computability Theory and focus on set
existence axioms. Ever since then, mathematicians all over the world have been working in the field using the structures and notation created by Simpson. In 2018, Stillwell published \textit{Reverse Mathematics: Proofs from the Inside Out}, which was one of the first books on the subject aimed towards a more general audience than Simpson, who often relies on background knowledge in logic and recursion theory [16]. For an extensive study on the history and development of Reverse Mathematics, see "The Prehistory of the Subsystems of Second-Order Arithmetic" by Dean and Walsh [7].

\section{1.4 Methodology}

This paper is meant to provide an introduction to Reverse Mathematics through the lens of Real Analysis. We provide details, and in one case a correction, to Brown's 1990 paper "Notions of Closed Subsets of a Complete Separable Metric Space in Weak Subsystems of Second Order Arithmetic" [3]. This paper also intends to serve as a companion to Simpson's and Stillwell's books where they regard these topics. This is not an exhaustive introduction to Reverse Mathematics as a whole; we direct readers to Simpson and Stillwell for such resources. However, we strive that this paper is a solid resource for learning Reverse Mathematics through its treatment of Real Analysis, collecting and elucidating a majority of the concepts required to understand the topic. In particular we hope to have clarified Brown's paper on closed sets.

In Chapter 2, we discuss some theorems about completeness from Real Analysis and consider their relative strength. This first look is outside the framework of Reverse Mathematics to introduce the general perspective. However, this "naive" approach may result in problems and we will describe those fallbacks. In Chapter 3, we introduce the formalisms of Reverse Mathematics, including a strict logical hierarchy of mathematical formulas and the collection of axiomatic subsystems used frequently in Reverse Mathematics. In Chapter 4, we explore these subsystems and the character of proofs within Reverse Mathematics while developing some helpful lemmas. In Chapter 5,
we detail the construction of a complete separable metric space within Reverse Mathematics and explore the nuances of completeness within weak subsystems of axioms. In Chapter 6, we introduce closed and separably closed sets, prove which subsystems imply equivalence of these definitions, and also explore the logical strengths of those equivalences. In Chapter 7, we explore how the nuances of the definitions of closed versus separably closed sets affects analysis proofs involving closed sets by examining the Baire Category Theorem within Reverse Mathematics.
Chapter 2

Notions of Completeness

In a standard Real Analysis course, completeness is a large focus, and for good reason. The core of analysis as a discipline is finding solutions to complicated problems by first approximating them by simpler solutions and then taking the limit of these approximations to find the solution to the original problem. Students are often first introduced to this approach in a first course in Calculus, where the slope of the tangent line at a point is approximated by a sequence of slopes of secant lines which are much easier to calculate. However, this approximation process relies on the completeness of the real line. Without completeness, the tool of the limit is unreliable. Hence understanding completeness is a large part of the understanding of how analysis and its tools work.

In fact, even theorems of Real Analysis that seem unconcerned with completeness on the surface are actually intrinsically linked to the concept of completeness. As Reverse Mathematics is interested in the logical strengths of theorems, we will now explore some familiar theorems from Real Analysis and compare their logical strength. We follow an approach from Propp’s paper "Real Analysis in Reverse," which considers these questions without any of the structures of Reverse Mathematics [12]. However, there are some flaws with this approach. We will discuss the possible downfalls later in this chapter and return to the issues in further chapters. In the theorems below, we will let our spaces be arbitrary ordered fields $X$, as we intend to explore completeness in general, not just in the case of the real line which has some extra properties.
2.1 Least Upper Bound is Equivalent to the Cut Property

Most students are introduced to the axiom of completeness as a statement of the least upper bound property.

Theorem 2.1.1 (Least Upper Bound Property). If \( S \) is a nonempty subset of \( X \) that is bounded above, then there exists a least upper bound \( c \) of \( S \) [12].

The least upper bound property specifically describes Dedekind-completeness. However, there is another property equivalent to the Least Upper Bound Property that provides some very useful structures that will make later proofs much easier.

Theorem 2.1.2 (Cut Property). Let \( A \) and \( B \) be a nonempty pair of disjoint sets such that \( A \cup B = X \) and \( \forall x \in A \) and \( \forall y \in B \), \( x < y \). There exists a cutpoint \( c \in X \) such that all \( x < c \) are elements of \( A \) and all \( y > c \) are elements of \( B \) [12].

In other words, if you divide our space \( X \) into two sections that are not intertwined in any way but are separate, then there is a cutpoint. This cutpoint serves as the least upper bound of one set and the greatest lower bound of the other (as we will prove in the next theorem) and is contained in one of the two sets. These cuts are sometimes referred to as Dedekind cuts.

The following diagrams illustrate a cut comprised of sets \( A \) and \( B \). The first diagram shows the case where the cutpoint \( c \) is contained in \( A \), and the second shows the case where \( c \) is contained in \( B \).

---

\[ A \quad \text{cutpoint} \quad B \]

\[ c \]

**Figure 2.1:** Cutpoint \( c \) is contained in \( A \).
Chapter 2. Notions of Completeness

\[ A \xrightarrow{c} B \]

Figure 2.2: Cutpoint \( c \) is contained in \( B \)

So we will first show that the Cut Property is equivalent to the Least Upper Bound Property, and we are justified in using it as our standard of Dedekind completeness.

**Theorem 2.1.3.** \( X \) has the Least Upper Bound Property \( \iff \) \( X \) has the Cut Property [12]

**Proof.** \((\Rightarrow)\) Suppose \( X \) satisfies the least upper bound property. Let \( A \) and \( B \) be as hypothesized in the cut property. Since \( A \) is a nonempty subset of \( X \) that is bounded above, the Least Upper Bound Property applies, and there exists some least upper bound \( c \) of \( A \). We claim this least upper bound is the desired cutpoint. If \( x > c \), then \( x \notin A \) so \( x \in B \) as otherwise \( c \) would not be the least upper bound. If \( x < c \), then \( x \) is not an upper bound of \( A \). Therefore there exists some \( a \in A \) such that \( x < a \), so \( x \notin B \). Therefore the Cut Property holds.

\((\Leftarrow)\) Suppose \( X \) satisfies the Cut Property. Consider some nonempty subset \( S \) of \( X \) that is bounded above. Then let \( B \) be the set of all upper bounds of \( S \), and let \( A = X \setminus B \).

By construction \( A \cup B = X \) and since all \( y \in B \) are upper bounds of \( S \), if \( x \in A \), then \( x \) is not an upper bound of \( S \), and there exists some \( s \in S \), such that \( x < s < y \). So \( A \) and \( B \) satisfy the hypotheses of the Cut Property. Therefore there is a cutpoint \( c \) such that if \( x < c \) then \( x \in A \) and if \( x > c \) then \( x \in B \).

We claim that \( c \) is the least upper bound of \( S \). First we will show that \( c \) is an upper bound of \( S \) by contradiction. Suppose there exists some \( s \in S \) such that \( s > c \). Consider some point in between \( s \) and \( c \), namely \( \frac{s + c}{2} \). Since \( \frac{s + c}{2} > c \), then \( \frac{s + c}{2} \) is an element of \( B \) as \( c \) is the cutpoint. Therefore \( \frac{s + c}{2} \) is an upper bound of \( S \). However, \( s > \frac{s + c}{2} \), which is a contradiction.

Now we show \( c \) is the least upper bound of \( S \), again by contradiction. Suppose there exists some \( d < c \) such that \( d \) is an upper bound of \( S \). However, since \( d < c \), then \( d \notin B \), so \( d \) cannot be an upper bound.

Therefore all nonempty subsets \( S \) of \( X \) that are bounded above have a least upper
bound, and the least upper bound is given by the cut point of the cut defined as above from $S$. ■

2.2 Cauchy Criterion is not Equivalent to the Cut Property

There are also theorems of Real Analysis that are related to Dedekind completeness, but are not equivalent. The **Cauchy Criterion** is another incredibly helpful notion of completeness, recall its definition as follows.

**Theorem 2.2.1** (Cauchy Criterion). *Every Cauchy Sequence in $X$ is convergent.*

However, the Cauchy criterion is distinct from Dedekind completeness, so we refer to it as **Cauchy completeness**. The set of real numbers is both Dedekind and Cauchy complete. However, there are some ordered spaces where all Cauchy sequences converge, but the cut property fails. One such space is the field of formal Laurent series.

2.2.1 The Field of Formal Laurent Series

**Definition 2.2.1.** *The Field of Formal Laurent Series* is denoted as $L = \mathbb{R}((x))$ where

$$
p \in L \iff p = \sum_{n=N}^{\infty} a_n x^n \text{ where } N \in \mathbb{Z} \text{ and } a_n \in \mathbb{R}
$$

[12]

$L$ is the set of "polynomials" in terms of $x$ with real coefficients where finitely many terms may have negative exponents. We will refer to the elements of $L$ as polynomials, though they are technically series.

**Definition 2.2.2.** For polynomials in $L$, we consider the **leading term** to be the non-zero term with the smallest power of $x$. The coefficient of the leading term is referred to as the **leading coefficient**. [12]

**Definition 2.2.3.** For polynomials $p$ in $L$, we consider the **degree** to be the smallest power of $x$, and denote it with $\deg(p)$ [12].
Chapter 2. Notions of Completeness

$L$ admits the following metric and ordering.

**Definition 2.2.4 (Metric on $L$).** For $p, q \in L$, let $d(p, q) = 2^{-\deg(p-q)}$ [12].

Note that $p - q$ will have degree equal to the smallest power where $p$ and $q$ disagree. Essentially, as we are working with formal polynomials, our powers of $x$ are simply placeholders, like places of a decimal or the index of sequence. A polynomial describes some sequence of coefficients. We will be working with sequences of various sorts throughout this paper, and the idea of a metric pointing out exactly where sequences differ will be a recurring concept throughout this paper.

**Definition 2.2.5 (Ordering on $L$).** Let $p > q$ if and only if $p - q$ has a positive leading coefficient. Note that $p = q$ if and only if $p - q$ is exactly the zero polynomial. [12]

We can classify polynomials $p \in L$ as one of three types considering the leading term and its coefficient.

**Definition 2.2.6.** $p \in L$ is **finite**, if there are no terms of $p$ with negative exponents. $p = \sum_{n=N}^{\infty} a_n x^n$ where $N \geq 0$. [12]

Note that a finite $p$ may be an infinite sum of powers of $x$. For example both $p_0 = 2 - 3x + 2x^2$ and $p_1 = 1 + x + x^2 + x^3 + \ldots$ are finite.

**Definition 2.2.7.** $p \in L$ is **positively infinite** if the degree is negative and the leading coefficient is positive. [12]

For example $p_2 = 3x^{-4} + 2x^2$ is positively infinite.

**Definition 2.2.8.** $p \in L$ is **negatively infinite** if the degree is negative and the leading coefficient is negative. [12]

For example $p_3 = -2x^{-3} + x^{-2} + x$ is negatively infinite. Note that these three different categories of polynomials in $L$ are distinct and partition all of $L$.

**Theorem 2.2.2.** $X$ satisfies the Cauchy Criterion $\iff X$ has the Cut Property [12]
Chapter 2. Notions of Completeness

Proof. We will prove this with a counterexample, the field of formal Laurent Series $L = \mathbb{R}((x))$, which satisfies the Cauchy Criterion but does not have the Cut Property. We claim that $L$ satisfies the Cauchy Criterion. Let $(p_n)$ be a Cauchy sequence of polynomials in $L$. We will show that $(p_n)$ converges, by constructing the limit of $(p_n)$. Since $(p_n)$ is Cauchy, for all $\varepsilon > 0$, there exists some $M \in \mathbb{N}$ such that $d(p_n, p_m) < \varepsilon$ for all $n, m > M$. Let $M_k \in \mathbb{N}$ be such that $d(p_n, p_m) < 2^{-k}$.

Then from the note above, $p_n$ and $p_m$ agree through the $x^k$ term. Eventually the coefficients of the powers of $x$ stabilize after some step in the sequence, and we would expect $(p_n)$ to converge. We can construct its limit $q$. Let $q = \sum_{n=N}^{\infty} a_n x^n$ where $a_n$ is the coefficient of $x^n$ in $p_{N_k}$; $(p_n)$ converges to $q$ since $d(p_n, q) \leq 2^{-k}$ for all $n > N_k$. Therefore $L$ satisfies the Cauchy Criterion.

However, we will show that $L$ does not satisfy the Cut Property. Let $A = \{ p \in L \mid p \text{ is finite or negatively infinite} \}$ and let $B = \{ p \in L \mid p \text{ is positively infinite} \}$. We will first show that $A$ and $B$ satisfy the hypotheses of the Cut Property, then we will show it has no cutpoint.

$L = A \cup B$ and $A \cap B = \emptyset$ since all polynomials in $L$ must be either finite, negatively infinite, or positively infinite, and can only fall into one category. Consider $p \in A$ and $q \in B$, $p - q$ must have a negative leading coefficient. If $p$ is negatively infinite, then the leading coefficient of $p - q$ is either the same as the leading coefficient of $p$, or it is a non positive number minus the positive leading coefficient of $q$. In both situations, $p - q$ has a negative leading coefficient, so $p < q$. If $p$ is finite, then the leading coefficient of $p - q$ is 0 minus some positive number, so $p - q$ has a negative leading coefficient, so $p < q$.

Therefore $A$ and $B$ is a cut of $L$. However, there is no cutpoint $c \in L$. We will show this by contradiction. Suppose that there exists some cutpoint $c = \sum_{n=N}^{\infty} c_n x^n$ of $A$ and $B$.

Consider two cases of the degree of $c$.

**Case 1:** $\deg(c) < 0$. Then, since $p < c$ for all $p \in A$, then the leading coefficient of $c$ must be greater than or equal to 0. Also, since $q > c$ for all $q \in B$, the leading coefficient of $c$ must be less than or equal to 0. So the leading coefficient of $c$ must be 0, which is a contradiction.
Case 2: \( \deg(c) \geq 0 \). Then \( c \) is finite and as such an element of \( A \). However, \( c < \sum_{n=N}^{\infty} (c_n + 1)x^n \in A \). So \( c \) is not a cut point of \( A \) and \( B \).

Therefore \( A \) and \( B \) do not have a cutpoint, and \( L \) does not satisfy the Cut Property. ■

So the common instinct of a Real Analysis student, that Cauchy completeness and Dedekind completeness are equivalent, does not hold in general. While the field of formal Laurent series might be unfamiliar to students, it is not an absurd, fringe counterexample. We will further explore the relationship between Cauchy completeness and Dedekind completeness later in Chapter 5, when we formally construct a complete metric space within the framework of Reverse Mathematics.

### 2.3 Negation of the Cut Property

For these proofs of completeness, the contrapositive is often a successful approach. So let’s now consider what the negation of the Cut property entails.

**Negation of the Cut Property:** If \( X \) does not have the Cut Property, there exists some "bad cut" of the space \( X \): a nonempty pair of disjoint \( A \) and \( B \) such that \( A \cup B = X \) and for all \( x \in A \) and \( y \in B \), where \( x < y \), such that there does not exist a cutpoint \( c \in X \).

If the cut property does not hold, the bad cut is a cut constructed of open sets, and we can express \( X \) as the union of two open sets.

**Lemma 2.3.1.** If \( A \) and \( B \) are a bad cut, then \( A \) and \( B \) are both open sets. [17]

**Proof:** We will prove this by contradiction for \( B \). If \( B \) is not open, then there exists some \( x \in B \) where \( \forall \epsilon > 0 \), the metric ball with center \( x \) and radius \( \epsilon \) is not a subset of \( B \).

Consider \( \epsilon = \frac{1}{n} \). Then there exists some \( a_n \in A \) such that \( a_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \). However, since \( x \in B \), it must be true that \( a_n < x \) for all \( n \). Thus \( x - \frac{1}{n} < a_n < x \). Therefore \( x \leq b \) for all \( b \in B \) and \( a < x \) for all \( a \in A \), so \( x \) would be a cut point for \( A \) and \( B \), which is a contradiction. ■

We can see visually how the negation of the cut property contradicts our sense of the completeness of the real line, as it highlights a gap in the real numbers.
2.4 Propp’s Claim: Intermediate Value Property is Equivalent to the Cut Property

Though the Intermediate Value Property is a standard principle from both Calculus and Real Analysis, its connection to completeness is not often explored. However, the Intermediate Value Property has an interesting relationship to completeness. We will present a proof from Propp that the Intermediate Value Property is equivalent to the Cut Property. However, this proof is only correct given certain axioms, from a result from Reverse Mathematics. Propp’s proof illustrates the general philosophy of Reverse Mathematics, but this proof makes some extra assumptions that we will explain in Chapter 5.

**Theorem 2.4.1** (Intermediate Value Property). If $f$ is a continuous function from $[a, b] \subseteq X$ to $X$, then for all $d \in [f(a), f(b)]$ there exists $c$ in $(a, b)$ with $f(c) = d$.

**Theorem 2.4.2** ([Propp]). $X$ has the Cut Property $\iff X$ has the Intermediate Value Property [12],[17]

*Proof*. $(\Rightarrow)$ First we note that, since the cut property follows (and is equivalent) to Dedekind completeness, then the forward direction is a standard proof from Real Analysis. ([13], pg.26).

$(\Leftarrow)$ We will now prove the reverse direction by contrapositive. Suppose the cut property does not apply. We will show that the Intermediate Value Theorem also fails by
building a counterexample.
Since the cut property does not apply, there exists some bad cut of $X$: a pair of nonempty disjoint subsets $A$ and $B$ of $X$ such that $A \cup B = X$ and $\forall x \in A, \forall y \in B, x < y$ such that there is no cut point of $A$ and $B$. Now we construct a function that seems discontinuous from our natural intuition, but will in fact be continuous by nature of the "openness" of the bad cut (and thus will satisfy the hypothesis of the Intermediate Value Theorem).
Let $f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$
We can have 0 and 1 be any arbitrary distinct elements of $X$, but we will denote them as 0 and 1 here for convenience.
Note that $f : X \to X$.

First we claim that $f$ is continuous.

**Lemma 2.4.3.** $f$ is continuous.

We will use the topological definition of continuity. $f$ is continuous if any only if, for all open sets $U$ in $X$, its preimage $f^{-1}(U)$ is open. So consider some arbitrary open $U \subseteq X$.

**Case 1:** $0 \notin U, 1 \notin U$. Then $f^{-1}(U) = \varnothing$ which is open. (An example of such a $U$ is displayed in red in the graph below)

**Case 2:** $0 \notin U, 1 \in U$. Then $f^{-1}(U) = B$, which is open by definition of bad cut. (An
example of such a \( U \) is displayed in blue in the graph below)

**Case 3:** \( 0 \in U, 1 \in U \). Then \( f^{-1}(U) = X \), which is open. (An example of such a \( U \) is displayed in gray in the graph below)

**Case 4:** \( 0 \in U, 1 \notin U \). Then \( f^{-1}(U) = A \) which is open by definition of bad cut. (An example of such a \( U \) is displayed in orange in the graph below)

![Graph of \( f(x) \) with possible open sets in the codomain](image)

**Figure 2.5:** Graph of \( f(x) \) with possible open sets in the codomain

Therefore the preimage of any open set is open, so \( f \) is continuous.

Now, choose some \( a \in A \) and \( b \in B \) where \( a < b \), which is possible by definition of bad cut. Consider \( f \) on the closed interval \([a, b]\). We have that \( f(a) = 0 \) and \( f(b) = 1 \). However, \( 0.5 \in (0, 1) \) but there does not exist any point in \([a, b]\) that maps to 0.5. In fact there are no points in \( X \) that map to 0.5. So \( f \) violates the Intermediate Value Property and the reverse implication follows by contrapositive.

From our work on this proof, it's unclear exactly where the error is. But from what we know from Reverse Mathematics (that the Intermediate Value Property is provable in \( RCA_0 \)), Propp must be making an assumption that provides more logical strength than the Intermediate Value Property "innately" offers. The fact that this error is so hard to identify underscores the importance of the formalizations of Reverse Mathematics, which allow us to precisely identify our assumptions.
The proofs in this section have introduced the approach of Reverse Mathematics without its formal structures, but this approach has some flaws, as we just demonstrated through Propp's claim about the Intermediate Value Theorem, that we will explore in Chapter 5. We will now introduce the language of Reverse Mathematics which will allow us to rigorously prove equivalences and strengths of theorems of Real Analysis.
Chapter 3

Language of Reverse Mathematics

In the previous section, we discussed theorems of completeness with a Reverse Mathematical perspective. We were able to use fairly familiar language to those outside of logic. However, the work of Reverse Mathematics necessitates working more closely with sets and axioms. We will work carefully through the language of Reverse Mathematics that will be used to build our familiar concepts from Real Analysis.

3.1 What is Second Order Arithmetic?

In second order arithmetic we only work with natural numbers and subsets of the natural numbers. All other mathematical objects, like real numbers, functions, and the like must be constructed from those essential concepts. Second order denotes the fact that we are working with two kinds of objects, natural numbers and sets of natural numbers. The full set of axioms for second order arithmetic is often referred to as Z_2. Third order arithmetic works with numbers, sets of natural numbers, and sets of subsets of the natural numbers, also known as classes.

3.2 The Arithmetic Hierarchy

The arithmetic hierarchy is a way to compare the logical strength of different mathematical formulas. This strength is determined by the number, types, and order of
the quantifiers within the statement. First we will formally define what we mean by formula.

**Definition 3.2.1.** A atomic formula is a mathematical statement in one of the following forms: \( a = b, a < b, \) and \( a \in X \) where \( a, b \) are any numerical terms, and \( X \) is any set term. [15]

**Definition 3.2.2.** A formula is a mathematical statement built from a combination of atomic formulas, numerical variables, set variables, connectives \( (\land, \lor, \neg, \rightarrow, \leftrightarrow) \), and quantifiers. [15]

**Definition 3.2.3.** A sentence is a formula with no free variables. [15]

Note that we use \( \varphi(x) \) as shorthand to represent "the formula \( \varphi \) is true for \( x \)." We will continue to use this shorthand for the remainder of the paper.

### 3.2.1 Types of Quantifiers

We distinguish quantifiers by the kinds of objects over which they quantify.

**Definition 3.2.4.** A Number quantifier is an existential modifier or universal modifier over a number (an element of \( \mathbb{N} \)). Examples are \( \exists n \) and \( \forall n \) [15]

**Definition 3.2.5.** A Bounded quantifier is a number quantifier that only concerns numbers below some bound. Examples are (for arbitrary fixed \( t \in \mathbb{N} \)) \( \forall n < t, \exists n < t, \) and \( \exists n < t \). [15]

**Definition 3.2.6.** A Set quantifier is an existential modifier or universal modifier over a set (an element of \( P(\mathbb{N}) \)). Examples are \( \exists X \) and \( \forall X \). [15]

If we were to rank the strength of these three kinds of quantifiers, bounded quantifiers are the weakest quantifiers, followed by number quantifiers, and set quantifiers being the strongest. In this sense, we are considering a stronger quantifier to be a quantifier that requires more "searching". A bounded quantifier requires only a finite number of natural numbers to be checked (either for existence or universal statements). A number quantifier may require all of the natural numbers to be checked. A set quantifier may require all possible subsets of the natural numbers, which is an uncountable collection.
3.2.2 Strength Levels of Formulas

The distinctions between the strengths of formulas depend on the alternation of $\exists$ and $\forall$ quantifiers.

First, quantifiers can be moved to the beginning of a statement, to present it in prenex normal form as explained in Stillwell [16]. Consider a statement of the form

$$\forall x(\varphi_1(x) \lor \exists y(\varphi_1(y) \land \varphi_2(x)))$$

From logic, we know if a formula $\varphi_1$ doesn't depend on variable $x$, then $\exists x(\varphi_1 \land \varphi_2(x))$ is equivalent to $\varphi_1 \land \exists x\varphi_2(x)$. So our original statement can be expressed in prenex form as follows:

$$\forall x \exists y(\varphi_1(x) \lor (\varphi_1(y) \land \varphi_2(x)))$$

Also note that any repeated quantifiers of the same type can be simplified into a single quantifier with a standard procedure taken from Stillwell [16]. First consider the expression $\exists x\exists y(x + y \geq 2y)$. We could combine the two repeated existential quantifiers into an existential quantifier over an ordered pair variable $\exists(x, y)(x + y \geq 2y)$. Note that we can consider $\exists(x, y)$ as a number quantifier even though $(x, y)$ is an ordered pair. Since the set of ordered pairs of $\mathbb{N}$ is also a countable set, and we could represent any ordered pair with a single natural number and change the quantifier to concern the natural numbers representing the pairs. A standard procedure of representing ordered pairs as natural numbers is by using a pairing map, see Simpson ([15], pg.66). Similarly for universal quantifiers, $\forall x\forall y(x + y \geq 2)$ can be converted to $\forall(x, y)(x + y \geq 2)$. So repeated quantifiers of the same type can be considered as a single quantifier when determining the strength of a statement.
3.2.3 Arithmetic Formulas

Our weakest type of formulas are arithmetic formulas, and they will be the focus for the majority of the paper.

**Definition 3.2.7.** A formula is arithmetic if its only quantifiers are number quantifiers. [15]

All of the following formulas are arithmetic. First, we have our weakest formulas, bounded formulas.

**Definition 3.2.8.** A bounded formula is a formula where all quantifiers are bounded number quantifiers. These formulas are sometimes denoted as $\Sigma^0_0$ formulas. [15]

**Example:** $\exists n < 3(n = 2)$ "There exists a natural number less than 3 that is equal to 2." Note that any formula without any quantifiers at all is a bounded formula.

**Definition 3.2.9.** A $\Pi^0_k$ formula has $k$ number quantifiers which alternate $(\forall$ then $\exists$) where the outer most quantifier is a universal modifier and the most internal statement is a bounded formula. [15]

**Example:** "$(a_n)$ is a Cauchy sequence." $\forall \epsilon \in \mathbb{Q}^+(\exists M(\forall n > M \forall i(d(a_n, a_{n+i} < \epsilon))))$. This formula is $\Pi^0_3$. Note that, even though $\forall n > M$ may appear at first glance to be a bounded quantifier, it is a number quantifier, because it requires checking all natural numbers above a bound, a countable collection.

**Definition 3.2.10.** A $\Sigma^0_k$ formula has $k$ number quantifiers which alternate $(\exists$ then $\forall$) where the outer most quantifier is an existential modifier and the most internal statement is a bounded formula. [15]

**Example:** $\exists n(m = 2n)$ "There exists a natural number $n$ such that $m = 2n$, or in other words, $m$ is even." This formula is $\Sigma^0_1$.

**Definition 3.2.11.** A $\Delta^0_k$ formula is a formula that can be described as both a $\Sigma^0_k$ and $\Pi^0_k$ formula. [15]

**Example:** "$n$ is even." We can express this statement as both the $\Sigma^0_1$ formula $\exists m(n = 2m)$ and the $\Pi^0_1$ formula $\forall m(n \neq 2m + 1)$.
3.2.4 Formulas with Set Quantifiers

Once we allow for quantifiers over sets, the formulas get stronger.

**Definition 3.2.12.** A $\Pi^1_k$ formula has $k$ set quantifiers which alternate ($\forall$ then $\exists$), where the outer most quantifier is a universal $\forall$ quantifier, and the most internal statement is arithmetic. [15]

**Definition 3.2.13.** A $\Sigma^1_k$ formula has $k$ set quantifiers which alternate ($\forall$ then $\exists$), where the outer most quantifier is an existential $\exists$ quantifier, and the most internal statement is arithmetic. [15]

**Definition 3.2.14.** A $\Delta^1_k$ formula is a formula that can be described as both a $\Sigma^1_k$ and $\Pi^1_k$ formula. [15]

Note that the exponent in the notation above denotes whether the formula uses only number quantifiers (with 0) or includes set quantifiers (with 1).

As a standard convention, a set described by a formula of a given strength, suppose some $\Sigma^1_k$ formula, then that set is considered a $\Sigma^1_k$ set. The same convention follows for sets defined by $\Pi^1_k$ formulas.

3.2.5 Lemmas Concerning Relationships between Kinds of Formulas

Here are a few useful lemmas about how these different kinds of formulas are related.

**Lemma 3.2.1.** Any $\Pi^0_k$ statement can be expressed as the negation of a $\Sigma^0_k$ formula (and vice versa). [15]

*Proof.* For example, $\forall x \exists y \varphi$ is equivalent to the negation of $\exists x \forall y \neg \varphi$. The quantifiers switch types, so the outer most quantifier is the opposite type, however the number of quantifiers is the same. ■

**Corollary 3.2.1.** Any $\Pi^1_k$ statement can be expressed as the negation of a $\Sigma^1_k$ formula (and vice versa). [15]
Below we will use $\Pi_k^0$ to represent the set of all $\Pi_k^0$ formulas, and similarly for $\Sigma_k^0$.

**Lemma 3.2.2.** $\Pi_k^0 \subseteq \Pi_{k+1}^0$ for all $k \geq 0$. [15]

*Proof.* If a statement is $\Pi_k^0$, it can be expressed in the form $\forall x_1 \exists x_2 \ldots (\phi)$ with $k$ alternating quantifiers and $\phi$ a bounded formula. Directly to the right of $\phi$ we can add either a $\forall$ or $\exists$ quantifier over a dummy variable. The kind of quantifier will depend on whether or not $k$ is even or odd.

$$\forall x_1 \exists x_2 \ldots \forall y(\phi) \lor \forall x_1 \exists x_2 \ldots \exists y(\phi)$$

So $\theta$ can be expressed with $k+1$ alternating quantifiers with a universal outermost quantifier, making $\theta$ a $\Pi_{k+1}^0$ formula. $\blacksquare$

**Lemma 3.2.3.** $\Sigma_k^0 \subseteq \Sigma_{k+1}^0$ for all $k \geq 0$. [15]

*Proof.* The proof follows similarly to above. Directly to the right of $\phi$ we can add either a $\forall$ or $\exists$ quantifier over a dummy variable. The kind of quantifier will depend on whether or not $k$ is even or odd.

$$\exists x_1 \forall x_2 \ldots \forall y(\phi) \lor \exists x_1 \forall x_2 \ldots \exists y(\phi)$$

So we can express $\theta$ as a statement with $k+1$ alternating quantifiers with an $\exists$ as the outermost quantifier. Therefore $\theta$ is $\Sigma_{k+1}^0$. $\blacksquare$

**Lemma 3.2.4.** $\Pi_k^0 \cup \Sigma_k^0 \subseteq \Delta_{k+1}^0$ [15]

*Proof.* Consider a formula $\theta$ that is in the union $\Pi_k^0 \cup \Sigma_k^0$. If this statement is $\Pi_k^0$, it can be expressed in the form $\forall x_1 \exists x_2 \ldots (\phi)$ with $k$ alternating quantifiers and $\phi$ a bounded formula. For $\theta$ to be $\Delta_{k+1}^0$, we must be able to express it as both a $\Sigma_{k+1}^0$ formula and a $\Pi_{k+1}^0$ formula. From Lemma 2, we know that $\theta$ is $\Pi_{k+1}^0$. On the left side of the formula, we can add an $\exists$ modifier over a dummy variable to express the formula with $k+1$
alternating quantifiers with $\exists$ as the outer most quantifier.

$$\exists y \forall x_1 \exists x_2 \ldots (\phi)$$

Therefore $\theta$ can be expressed with $k+1$ alternating quantifiers, with an outermost existential quantifier, so $\theta$ is also $\Sigma^0_{k+1}$. Therefore $\theta$ is $\Lambda^0_{k+1}$.

If the statement $\theta$ is $\Sigma^0_k$, then it can be expressed in the form $\exists x_1 \forall x_2 \ldots (\phi)$ with $k$ alternating quantifiers. For $\theta$ to be $\Lambda^0_{k+1}$, we must be able to express it as both a $\Sigma^0_{k+1}$ formula and a $\Pi^0_{k+1}$ formula. From Lemma 3.2.3, we know that $\theta$ is $\Sigma^0_{k+1}$. On the left side of the formula, we can add an $\forall$ modifier over a dummy variable to express the formula with $k+1$ alternating quantifiers with $\exists$ as the outer most quantifier.

$$\forall y \forall x_1 \exists x_2 \ldots (\phi)$$

Therefore $\theta$ can be expressed with $k+1$ alternating quantifiers, with an outermost universal quantifier, so $\theta$ is also $\Pi^0_{k+1}$. Therefore $\theta$ is $\Lambda^0_{k+1}$. ■

**Corollary 3.2.2.** $\Pi^1_k \cup \Sigma^1_k \in \Delta^1_{k+1}; [15]$  

**Proof.** This proof follows identically to the proof above, the only difference using set quantifiers instead of number quantifiers. ([15], pg. 16) ■

### 3.3 Definitions of Subsystems of $Z_2$

For all subsystems, we assume an array of basic axioms about natural numbers (Simpson 4). The subsystems of $Z_2$ are sets of axioms consisting of these basic axioms in addition to some induction and comprehension scheme which may be significantly limited in scope. The axioms of these subsystems effectively restrict the complexity, computability, and cardinality of the sets that can be described. These subsystems are ordered in increasing strength, though intermediate subsystems exist.
First, we will explain the concepts of comprehension, induction, and universal closure within the context of Reverse Mathematics.

3.3.1 What is Comprehension?

Comprehension is the act of defining a set by its properties. We can comprehend the set of even natural numbers by writing $A = \{ x \in \mathbb{N} \mid \exists y \in \mathbb{N}(x = 2y) \}$, or if we're working with a finite set, we can list out the set's elements.

In standard mathematics, the existence of any given set is assumed. Mathematicians usually don't worry about the existence of sets, except perhaps in discussion of Russell's paradox in an introduction to proofs course. (Russell's paradox explains how rigorous set theory must avoid the existence of the set of all sets that do not contain themselves.)

This nuance connects to the diversity in infinity. The different subsystems that we work with are essentially denoting what sizes of infinity we can work with. Suppose we only want to work with countable infinity, and any uncountable infinities are incomprehensible, or we want to avoid using them in our problems as much as possible. Then working with a set like $\mathbb{R}$ would be complicated. Perhaps we could describe it as some sort of other structure on a countable set or a collection of sets, but we'd have to ensure it follows in some way from the countable infinities on which we would like to base our work.

For many hundreds of years, infinity was not comfortable for mathematicians. Now, set theorists working in large cardinals work with incredibly large sets but mathematicians like Archimedes skirted around limits. Determining how much infinity is needed for a certain theorem or certain subfield of mathematics is one important meta-analysis possible with Reverse Mathematics.
Restricted set comprehension limits which sets we can use in our mathematics. We must be able to describe sets with a simple enough mathematical formula. As we are interested in weak axioms, mathematics that is computable by simple machines, we only guarantee the existence of a set if it can be described by a formula of a certain strength or simplicity.

3.3.2 Why Restrict Induction?

Induction as a valid mathematical argument is a standard concept for math students who have taken an introduction to proofs course. Standard and Strong Mathematical induction are helpful tools for working with statements about natural numbers. However, induction is not limited to these two types. For example, those who have taken a course in set theory may be familiar with transfinite induction. Transfinite induction involves induction on ordinals and cardinal numbers. It could be argued that you don’t need such a powerful tool if you don’t work with very large sets like ordinals and cardinals.

Hence, as we explained for comprehension, if we’re interested in minimal axioms, we may want to restrict induction to simpler mathematical statements, like in standard mathematics when we only utilize standard or strong induction, and rarely need transfinite induction for non set-theoretic applications.

3.3.3 What is Universal Closure?

Universal closure is the process of adding universal quantifiers to some mathematical statement for all free variables, which implies that the statement is true for all possible free variables.

Consider the statement: If an arbitrary, fixed set $X$ is nonempty, it contains some element. $(X \neq \varnothing \rightarrow \exists x \in X.)$ Here, $X$ is a free set variable, and $x$ is a number quantifier.
You may consider our previous statement to be a definition of a nonempty set, so we would claim that we have universal closure of this statement, that this statement is true for any nonempty set. For all sets \( X \), if \( X \) is nonempty, it contains some element \( x \).

\[
(\forall X(X \neq \emptyset \rightarrow \exists x \in X))
\]

In general, for statement \( \varphi(x_1, x_2, \ldots, x_n) \), the universal closure of \( \varphi \) is the statement \( \forall x_1 \ldots \forall x_j(\varphi(x_1, x_2, \ldots, x_n)) \), where \( x_i, \ldots, x_j \) are all the free variables of \( \varphi \). Now we will explain the subsystems of second order arithmetic commonly used in Reverse Mathematics.

### 3.3.4 \( \text{RCA}_0 \)

\( \text{RCA}_0 \) contains all basic axioms and the following induction and comprehension schemes.

The basic axioms include properties of natural numbers such as \( n + 1 \neq 0 \), \( m + 0 = m \), and similar statements. For a full list, see Simpson ([15], pg. 4).

**Theorem 3.3.1 (\( \Sigma^0_1 \) Induction Axiom).** The universal closure of the following: (or in other words, the following statement is always true).

\[
(\varphi(0) \land \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n(\varphi(n))
\]

if \( \varphi(n) \) is a \( \Sigma^0_1 \) formula. [15]

**In other words:** Let's break down this expression. First consider the left side:

\[(\varphi(0) \land \forall n(\varphi(n) \rightarrow \varphi(n+1)))\].

The portion before the \( \land \) means the statement \( \varphi \) is true for 0. This is the base case of induction. The next part states that for arbitrary \( n \), if the statement is true for \( n \), then the statement is true for \( n + 1 \). This is the induction step. The right side states that the statement is true for all natural numbers \( n \). So this induction axiom claims, if \( \varphi \) is a \( \Sigma^0_1 \) statement, then induction is a valid argument to prove \( \varphi \) is true for all \( n \).
Theorem 3.3.2 ($\Delta^0_1$ comprehension). The universal closure of the following:

$$\forall n (\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

if $\varphi(n)$ is a $\Sigma^0_1$ formula and $\psi(n)$ is a $\Pi^0_1$ formula. [15]

Note that $\forall n (\varphi(n) \rightarrow \psi(n))$ implies that $\varphi(n)$ is a $\Delta^0_1$ formula, hence the name of this comprehension scheme.

Using set builder notation, this axiom can also be expressed as follows: If $\varphi$ is a $\Delta^1_0$ formula, then $\exists X (X = \{n \in \mathbb{N} \mid \varphi(n)\})$.

In other words: Let’s break down this expression. The left side, $\forall n (\varphi(n) \leftrightarrow \psi(n))$, means that some formula $\varphi(n)$ can be described as a both a $\Sigma^0_1$ and a $\Pi^0_1$ formula, or in other words, $\varphi(n)$ can be described as a $\Delta^0_1$ formula.

The right side $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$, means that there exists a set that contains all the natural numbers for which $\varphi(n)$ is true. This is the comprehension part of the axiom. If $\varphi(n)$ is $\Delta^0_1$, then we can comprehend or collect together the numbers that satisfy that condition.

These definitions follow from standards in set theory. However, from within Computability Theory, we consider sets in $\text{RCA}_0$ to be computable.

Definition 3.3.1. A computable set is a $\Delta^0_1$ set. [15],[16]

These sets are referred to as computable within Computability Theory since the set has an algorithm that answers the question: Is $k$ an element of this set? These sets are sometimes referred to as recursive sets in Recursion Theory or Turing decidable languages within Computer Science. Turing decidable languages (or sets) are so named because we can directly decide if any element is contained the set with an algorithm.

3.3.5 $\text{ACA}_0$

$\text{ACA}_0$ consists of all basic axioms, and the following induction and comprehension schemes.
Theorem 3.3.3 (Arithmetic Induction). The universal closure of the induction scheme for arithmetic formulas.

\[(\varphi(0) \land \forall n (\varphi(n) \rightarrow \varphi(n + 1))) \rightarrow \forall n (\varphi(n))\]

if \(\varphi(n)\) is an arithmetic formula. [15]

Theorem 3.3.4 (Arithmetic Comprehension). The universal closure of the following:

\[\exists X \forall n (n \in X \leftrightarrow \varphi(n))\]

if \(\varphi(n)\) is an arithmetic formula. [15]

Similarly to the connection for \(RCA_0\), within Computability Theory, sets in \(ACA_0\) are the **computably enumerable** sets.

Definition 3.3.2. A **computably enumerable set** is a \(\Sigma^0_1\) set. [15], [16]

In other terms, we have an algorithm that will answer the question: What is the list of members of this set? This algorithm may not halt, but if we can describe such a set by a \(\Sigma^0_1\) formula, then the set is computably enumerable. In Chapter 4, we will show that a \(\Sigma^0_1\) set can be defined as the range of some \(\Sigma^0_1\) function within \(RCA_0\). This function can be thought of as the enumeration of the set (Lemma 4.2.1).

These sets are sometimes referred to as **recursively enumerable sets** in Recursion Theory, and **Turing recognizable languages** in Computer Science. Turing recognizable languages are so named because we can eventually recognize whether an element is in the set by checking the list of elements of the set created by the algorithm. However this process may not halt, and so we won’t necessarily be able to conclude if an element is **not** in the set.
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It may seem strange that computably enumerable sets are defined to be \( \Sigma^0_1 \) sets, not arithmetic sets. However, we will see that arithmetic comprehension is equivalent to \( \Sigma^0_1 \) in Chapter 4 (Lemma 4.3.1).

### 3.3.6 \( \Pi^1_1 - CA_0 \)

\( \Pi^1_1 - CA_0 \) consists of all basic axioms, induction on all second order formulas, and the \( \Pi^1_1 \) comprehension axiom, where comprehension is restricted to \( \Pi^1_1 \) formulas.

**Theorem 3.3.5 (Second Order Induction).** The universal closure of the induction scheme for second order formulas.

\[
(\varphi(0) \land \forall n (\varphi(n) \rightarrow \varphi(n + 1))) \rightarrow \forall n (\varphi(n))
\]

if \( \varphi(n) \) is a second order formula [15].

Recall that a \( \Pi^1_1 \) statement is one of the form \( \forall X \theta \) where \( X \) is a set and \( \theta \) is an arithmetic formula (with only number quantifiers).

**Theorem 3.3.6 (\( \Pi^1_1 \) Comprehension).** The universal closure of the following:

\[
\exists X (\forall n (n \in X \rightarrow \varphi(n))
\]

if \( \varphi(n) \) is an \( \Pi^1_1 \) formula [15].

### 3.4 Big Five Subsystems

As mentioned before, these subsystems are strictly ranked by strength, with \( RCA_0 \) being the weakest and \( \Pi^1_1 - CA_0 \) being the strongest.

\[
\Pi^1_1 - CA_0 \implies ACA_0 \implies RCA_0
\]
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The **Big Five** denotes these three systems and the two most common intermediate subsystems that make up a strict hierarchy.

\[ \Pi^1_1 - CA_0 \implies ATR_0 \implies ACA_0 \implies WKL_0 \implies RCA_0 \]

While \( WKL_0 \) and \( ATR \) are important intermediate subsystems, they are beyond the scope of this paper. See Simpson for more on these intermediate subsystems [15].

These five subsystems are considered the Big Five as the majority of theorems are logically equivalent to one of these five subsystems. However, more current research is moving away from the focus on these five subsystems and has shifted more towards theorems with strengths outside and between these levels, many of which originated within the field of Combinatorics.

Now that we are more familiar with the language and structures of Reverse Mathematics, we can construct a complete metric space.
Chapter 4

Exploring Subsystems $\text{RCA}_0, \text{ACA}_0,$ and $\Pi^1_1 - \text{CA}_0$

Now that we have become more familiar with the language of Reverse Mathematics, we will get familiar with how we use these concepts within the various subsystems of second order arithmetic. We will be using the base system $\text{RCA}_0$, as well as the stronger subsystems $\text{ACA}_0$ and $\Pi^1_1 - \text{CA}_0$ to construct complete separable metric spaces and examine theorems of analysis.

When proving the equivalence of different statements to one of the Big Five subsystems, it is often helpful to build an equivalence chain using a different theorem whose structure connects more easily to those we can build with the theorem under consideration. We will present a few of these theorems that we will use to bridge in our future proofs involving closed sets and prove their equivalence with the relevant subsystems.

4.1 A Computably Enumerable Set that is not Computable.

All computable sets are computably enumerable, (since $\Delta^0_1 \subseteq \Sigma^0_1$) but there exist computably enumerable sets that are noncomputable. Such a set is the center of this counterexample, which will be vital for a proof in Chapter 5 (Lemma 5.4.4). This counterexample is best presented from a slightly different perspective, which also provides an
opportunity to show how certain concepts from Reverse Mathematics are interpreted within Computability Theory and Theoretical Computer Science.

**Lemma 4.1.1.** There exists a computably enumerable but noncomputable set. [16]

**Proof.** There are countably many algorithms, so we can enumerate these algorithms with a unique natural number \( k \). So let \( \Phi_k \) be the \( k \)th algorithm in some ordering of the set of algorithms. Then \( \Phi_k(n) \) represents the output of algorithm \( \Phi_k \) on input \( n \), if any. Asking the question \( \Phi_k(n) \) is a computable process, as an algorithm would only need proceed finitely many steps on the countable set of algorithms, and then compute the output with input \( n \).

We claim \( D = \{ k \mid \Phi_k(k) = 0 \} \) is a computably enumerable but noncomputable set. For the sake of contradiction, suppose \( D \) is computable.

If \( D \) is computable, its characteristic function is computable (considered here as a computable formula recognizing membership).

\[
d(m) = \begin{cases} 
1 & m \in D \\
0 & m \notin D 
\end{cases}
\]

Since this function is computable, \( d \) is on the list of algorithms and can be represented as some \( \Phi_k \). Suppose \( k \notin D \), then \( d(k) = 0 \). However, \( \Phi_k(k) = 0 \). So by definition, \( k \in D \). This is a contradiction, so \( D \) is noncomputable. \( \Box \)

### 4.2 Useful Theorems that can be Proved in \( \text{RCA}_0 \)

Even with the weakness of the base system \( \text{RCA}_0 \), we still have some useful structures we can utilize. From the previous section, we now know that computable and computably enumerable sets are distinct. However, the structures of computably enumerable \( \Sigma^0_1 \) sets are still of interest within \( \text{RCA}_0 \), though we know such sets will not comprehensible. So now we will present a lemma that describes how we can work with computably enumerable \( \Sigma^0_1 \) sets within \( \text{RCA}_0 \).
4.2.1 $\Sigma_1^0$ Sets are Computably Enumerable

**Lemma 4.2.1 (RCA$_0$).** If $S$ is a nonempty $\Sigma_1^0$ set then either $S$ is finite or $S$ is the range of a one to one function $f : \mathbb{N} \to \mathbb{N}$. Or in other words, $\Sigma_1^0$ sets are computably enumerable. ([15],[16], pg. 106)

Proof. First, note that if $S$ is finite, it can be trivially enumerated, as a finite list of elements could also be considered as a sequence. So suppose $S$ is nonempty, infinite and described by some $\Sigma_1^0$ formula $\phi$. Then, by definition of $\Sigma_1^0$, we have that $n \in S \iff \exists m \phi(m,n)$ where $\phi$ is a $\Sigma_1^0$ formula. Note that the set $Y = \{(m,n) \mid \phi(m,n) \wedge \exists k < m(\phi(k,n))\}$ is described by a bounded formula, and is therefore comprehensible in RCA$_0$. $Y$ is the set of "minimal" pairs $(m,n)$, where we select the least $m$ such that $\phi(m,n)$, so each $n$ has one matching $m$ within $Y$. By this description, $Y$ is infinite as $S$ is. $\mathbb{N} \times \mathbb{N}$ has a standard ordering, so there exists a bijective function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ that orders $\mathbb{N} \times \mathbb{N}$. Let $h(m)$ be the least $n \geq m$ such that $g(n) \in Y$. Note that $h$ counts through the elements of $Y$ according to the ordering $g$. This function $h$ exists since the fact that $Y$ is infinite, allows us to check $n + 1, n + 2, \ldots$ until we find a natural number that corresponds to a pair in $Y$. However, $h$ is not one to one. For example, if $g(1) \in Y$ but $g(2) \in Y$, we would have that $h(1) = g(2)$ and $h(2) = g(2)$. To capture a one to one correspondence, we define $\pi(0) = h(0)$, $\pi(1) = h(\pi(0) + 1)$, and in general $\pi(k + 1) = h(\pi(k) + 1)$. This is a recursive process that is possible within RCA$_0$. There is unique $m$ such that $g(m) = (i,j) \in Y$, so there is a unique $k \in \mathbb{N}$ such that $\pi(k) = m$. Thus $\pi$ is a bijection from $\mathbb{N}$ to $\{m \mid g(m) \in Y\}$ We are looking to construct our range set within $\mathbb{N}$ not $\mathbb{N} \times \mathbb{N}$, so we need to recover the 2nd coordinate of the pair which are actually the elements of $S$. Let $p_2$ be the projection function from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $p_2(x,y) = y$. So $p_2$ recovers the 2nd coordinate of the pair. Therefore our desired function is $f : \mathbb{N} \to S$ where $f(m) = p_2(g(\pi(m)))$. This function $f$ enumerates $S$ as desired.

**Claim 1:** range$f = S$. Suppose $n \in S$, then $\phi(n)$, so there is a unique $m$ such that $(m,n) \in Y$. Since $g$ and $\pi$ are both bijections, there is a unique $k$ such that $g(\pi(k)) = (m,n)$. Therefore range$f = S$. 
Claim 2: \textbf{f is one to one}. We will prove this by contradiction. Let \( f(m) = f(n) \) for some \( m \neq n \). So \( p_2(g(\pi(m))) = p_2(g(\pi(n))) \), therefore the second coordinates of \( g(\pi(m)) \) and \( g(\pi(n)) \) are the same. However, since \( g \) and \( \pi \) are both one to one, so \( m = n \). \hfill \blacksquare

### 4.3 Theorems Equivalent to ACA\(_0\)

The following theorem will introduce a very useful statement that we will use to more easily prove that a statement is equivalent to ACA\(_0\), this statement about the range of injective, well-defined functions will be more easily connected to our statements about closed sets. We will also prove the equivalence between ACA\(_0\) and \( \Sigma^0_1 \) comprehension which will also prove useful.

#### 4.3.1 ACA\(_0\) ⇔ The Range of Injective, Well-defined Functions Exist.

**Lemma 4.3.1.** (RCA\(_0\)) The following are equivalent.

1. ACA\(_0\)

2. \( \Sigma^0_1 \) comprehension, or in other words \( \exists X(n \in X \leftrightarrow \varphi(n)) \) where \( \varphi \) is \( \Sigma^0_1 \) and \( X \) is not a set variable.

3. If \( f : \mathbb{N} \to \mathbb{N} \) is total (or in other words, defined for all inputs) and one-to-one, then \( \exists X[n \in X \leftrightarrow \exists m(f(m) = n)] \) So, the range of injective, well-defined functions exist.

([15], pg. 105)

**Proof.** Our first implication is a direct consequence from the definition of arithmetic formulas. (1 ⇒ 3) Since the statement \( \exists m(f(m) = n) \) is a \( \Sigma^0_1 \) condition, \( X \) exists by arithmetic comprehension.

(3 ⇒ 2) Let some \( \Sigma^0_1 \) set \( X \) be given. By the lemma in the previous section, there exists a one to one function \( f : \mathbb{N} \to \mathbb{N} \) such that the range of \( f \) is \( X \). Therefore, by (3) since the range of any function exists, then \( X \) exists.

(2 ⇒ 1) Any arithmetic statement can be expressed as a \( \Sigma^0_1 \) statement for some \( k \), since if
some arithmetic $\varphi$ was a $\Pi^0_1$ statement, then it is also a $\Sigma^0_{j+1}$ statement by the addition of an existential modifier over a dummy variable. So we will show that $\Sigma^0_i$ comprehension implies $\Sigma^0_k$ comprehension for arbitrary $k$ by induction. As a base case, if $k \leq 1$, the implication follows trivially. So let $k > 1$. Assume $\Sigma^0_k$ comprehension, and suppose $\varphi(n)$ is a $\Sigma^0_{k+1}$ formula. $\varphi(n)$ can be represented as $\exists m \varphi(n, m)$ where $\varphi(n, m)$ is $\Pi^0_k$. So by $\Sigma^0_k$ comprehension, we can construct a set $Y$ such that $(p, q) \in Y \iff \neg \varphi(p, q)$ since $\neg \varphi$ is a $\Sigma^0_k$ formula. Then we define $X$ where $n \in X \iff \exists m (n, m) \not\in Y$ by $\Sigma^0_i$ comprehension. Therefore $X$ is the set containing all $n$ that satisfies $\varphi(n)$. Therefore, by induction, $\Sigma^0_k$ comprehension, and thus arithmetic comprehension holds.

### 4.4 Theorems Equivalent to $\Pi^1_1 - CA_0$

#### 4.4.1 Definitions

$\Pi^1_1 - CA_0$ is a much stronger subsystem than $RCA_0$ and $ACA_0$, so perhaps it is not too surprising that the natural mathematical structures we use within $\Pi^1_1 - CA_0$ are different than those we have considered before. Now we will now consider trees.

**Definition 4.4.1.** $2^{<\mathbb{N}}$ is the set of finite and nonempty sequences of $\{0, 1\}$, so $\sigma \in 2^{<\mathbb{N}}$ where $\sigma : \{1, \ldots, n\} \to \{0, 1\}$. This is called the full binary tree. [3]

**Definition 4.4.2.** $2^\mathbb{N}$ is the Cantor space. We represent $\sigma \in 2^\mathbb{N}$ as a function $\sigma : \mathbb{N} \to \{0, 1\}$. So $2^\mathbb{N}$ is the set of all countable sequences of 0’s and 1’s. [3]

In the next chapter, we will construct the Cantor Space formally and show it is completion of $2^{<\mathbb{N}}$

**Definition 4.4.3.** $\mathbb{N}^{<\mathbb{N}}$ is the set of finite sequences of natural numbers. [3],[15]

**Definition 4.4.4.** The Baire Space $\mathbb{N}^\mathbb{N}$ is the completion of $\mathbb{N}^{<\mathbb{N}}$ (via the method that will be described in Chapter 5) [3],[15]

**Definition 4.4.5.** $\ell(\sigma)$ denotes the length of sequence $\sigma$. [3]
Definition 4.4.6. \( \sigma[i] \) is the "truncation" of the sequence \( \sigma \). \( \sigma[i] = (\sigma(0), \sigma(1), \ldots, \sigma(i)) \), the sequence \( \sigma \) cut off after index \( i \). [3]

Definition 4.4.7. \( \tau \) extends \( \sigma \) \( \tau \circ \sigma \): \( \tau \circ \sigma \) if \( \tau(i) = \sigma(i) \) for all \( i < lh(\sigma) \). For \( \sigma, \tau \in 2^{\mathbb{N}} \). In other words, this means that all the elements of the two sequences match through the length of \( \sigma \). [3]

Note that "extends" is a bounded statement because the only quantifier necessary is \( \forall i < \ell(\sigma) \), a bounded quantifier.

Definition 4.4.8. The concatenation of two sequences is denoted as \( \sigma \cdot \tau = (\sigma(0), \ldots, \sigma(\ell(\sigma) - 1), \tau(0), \ldots \tau(\ell(\tau) - 1)) \).[3],[15]

Definition 4.4.9. A tree is a set \( T \subseteq \mathbb{N}^{<\mathbb{N}} \) such that any initial segment of a sequence in \( T \) belongs to \( T \). [15]

\[ (\sigma \in T \to \sigma[i] \in T, \forall i < lh(\sigma)) \]

Definition 4.4.10. A path is function \( f : \mathbb{N} \to \mathbb{N} \) such that \( \forall k \in \mathbb{N} \)

\[ f[k] = (f(0), f(1), \ldots, f(k - 1)) \in T \]

[3],[15]

The function \( f \) describes the step by step direction of which nodes (sequences) to follow in the tree. For example, if our tree contains the sequence \( \langle a, b, c, \ldots \rangle \), we could have a path \( f \) where \( f(1) = \langle a \rangle \in T, f(2) = \langle a, b \rangle \in T, f(3) = \langle a, b, c \rangle \in T \) and so on. In our context, we always consider a path to be infinite. Also note that a tree may not have a path, for example, all finite trees do not contain a path.

Definition 4.4.11. A finitely branching tree is a tree where for all \( \sigma \in T \) there are only finitely many \( n \) such that \( \sigma^\langle n \rangle \in T \). [15]

One of the most important lemmas concerning trees is König's lemma, and it is of particular interest in Reverse Mathematics.
Lemma 4.4.1 (König's Lemma (ACA₀)). Every infinite, finitely branching tree has an infinite path. [15]

Proof. See Simpson ([15], pg.122).

In fact, in an even stronger result, König's Lemma is equivalent to ACA₀ over RCA₀ ([15], pg.122). Next we will present a lemma that will allow us to more easily prove equivalence to Π₁¹ comprehension.

4.4.2 Π₁¹ Comprehension ↔ Any Sequence of Trees has a Set Containing all Indices of Trees with Paths.

Lemma 4.4.2 (RCA₀). Π₁¹ comprehension ↔ For any sequence of trees (Tₖ : k ∈ ℕ), Tₖ ⊆ ℕ<ℕ, there exists a set X such that ∀k(k ∈ X ↔ Tₖ has a path). ([15], pg.217)

Proof. Note that, as all Π₁¹ statements are negations of Σ₁¹ statements (see Corollary 3.2.1), then Π₁¹ comprehension is equivalent to Σ₁¹ comprehension.

(⇒) First assume Π₁¹ comprehension. The description of X is the following Σ₁¹ formula, and so X exists.

\[ k ∈ X ↔ ∃(xₙ)∀n((x₁, x₂, \ldots, xₙ) ∈ Tₖ) \]

(⇐) We will first prove arithmetic comprehension as both a proof of concept and to allow us to use arithmetic comprehension later in the proof. We will prove arithmetic comprehension using the bridging theorem proved in the last section. Assume the set X exists for all sequences of trees. Let g : ℕ → ℕ be an arbitrary function. Let (Tₖ) be the sequence of trees such that

\[ τ ∈ Tₖ ↔ (∀m < lh(τ))(g(m) ≠ k) \]

We put all finite sequences in Tₖ where the length is at longest the natural number that maps to k via g, if one exists. This sequence exists by Δ₀¹ comprehension, since \((∀m < lh(τ))(g(m) ≠ k)\) is a bounded formula and is therefore Δ₀¹.

By hypothesis, there exists an X such that k ∈ X ↔ Tₖ has a path. We claim that X is
the range of $g$. Using the definition of $(T_k)$, it must be that $T_k$ has a path if and only if
$\neg \exists m (g(m) = k)$. Otherwise, there would be a bound on any possible sequence, and the
infinite sequence required to construct a path would not exist. Therefore the range of $g$
exists and by the Lemma earlier in this chapter, arithmetic comprehension holds.

Now we will show $\Pi^1_1$ comprehension by showing $\Sigma^1_1$ comprehension holds. Let $\varphi(k)$
be a $\Sigma^1_1$ formula. By the Kleene normal form theorem ([15], Lemma V.1.4), we can
express $\varphi$ with a arithmetical formula $\theta(k, \tau)$ where

$$\forall k (\varphi(k) \leftrightarrow \exists f \forall m \theta(k, f[m]))$$

Let $(T_k)$ be the sequence of trees such that

$$\tau \in T_k \iff \forall m < lh(\tau) (\theta(k, \tau[m]))$$

This sequence exists by arithmetic comprehension. By hypothesis, we have a set $X$
such that $k \in X$ if and only if $T_k$ has a path. And we have that $T_k$ has a path if and only
if $\varphi(k)$. This is because if $T_k$ has a path, there is a path $f$, and that can only exist if $\varphi(k)$.
Therefore the set $X$ is also the set where

$$k \in X \iff \varphi(k)$$

Therefore we have $\Sigma^1_1$ comprehension which implies $\Pi^1_1$ comprehension. ■
Chapter 5

Construction of a Complete Separable Metric Space

We would like to work with complete spaces such as $\mathbb{R}$ so we can study theorems of Real Analysis. However, we must construct complete spaces from a countable set, so we can prove statements about complete spaces using the countable tools we have. These complete spaces will be separable as they will be built from a countable dense subset. This construction may differ from what the reader has seen in other contexts, so we will notate and explain any expected differences. We will first construct the real numbers and then construct general complete metric spaces including the Cantor space as an example. This construction holds in $RCA_0$.

5.1 Construction of the Real Numbers

We will construct $\mathbb{R}$ as the completion of $\mathbb{Q}$, the rational numbers. $\mathbb{Q}$ is our countable dense set that we have the tools to work with. As $\mathbb{Q}$ is countable, we can bijectively map $\mathbb{Q}$ into $\mathbb{N}$. So with the proper coding, $\mathbb{Q}$ can be considered as a subset of $\mathbb{N}$, though the rational numbers themselves are not technically members of $\mathbb{N}$. As we are using $\mathbb{Q}$ as our countable framework for $\mathbb{R}$, we will denote $\mathbb{R}$ as created by our construction by $\hat{\mathbb{Q}}$.

Also, we will construct our metric on $\mathbb{R}$ by adapting the standard metric on $\mathbb{Q}$.

Definition 5.1.1. Let $d$ be the standard metric on $\mathbb{Q}$ where $d(x, y) = |x - y|$ [15]
Now we can define points in $\hat{Q}$ as follows.

**Definition 5.1.2.** A point in $\hat{Q}$, in other words, a real number, is defined by a function $f : \mathbb{N} \rightarrow \mathbb{Q}$ where
\[
\forall n \forall i [d(f(n), f(n + i)) \leq 2^{-n}]
\]

[3],[15]

Note that the preceding definition is $\Pi^0_1$.

In other words, a point in $\hat{Q}$ is defined by a sequence of points $(a_n : n \in \mathbb{N})$ in $\mathbb{Q}$ where the distances between points are bounded by the Cauchy sequence $(2^{-n})$. These sequences are called rapidly converging Cauchy sequences since the sum $\sum_{n \in \mathbb{N}} |a_{n+1} - a_n|$ converges.

### 5.1.1 Why do we Use Rapidly Converging Cauchy Sequences?

For those who have constructed the completion of an incomplete set in the context of a course in Real Analysis, the question arises: Why do we use rapidly converging Cauchy sequences and not consider Cauchy sequences in general? We make this particular choice for our sequences because it allows us to define the completion with weaker axioms.

Recall that a sequence $(a_n)$ is Cauchy if and only if:
\[
\forall \varepsilon > 0, \exists M (\forall n > M \forall i (d(a_n, a_{n+i}) < \varepsilon))
\]

This is a very strong statement. First of all $\varepsilon$ is not necessarily a natural number, and so the quantifier $\forall \varepsilon$ is not well defined in second order arithmetic without the further construction we are currently defining. So we would have to first define how to describe a real valued epsilon to use it in the definition of how to describe real valued numbers in general.

However we could swap the $\forall \varepsilon > 0$ for $\forall \varepsilon \in \mathbb{Q}^+$, to have epsilon range over a countable set instead of an uncountable one. This definition is equivalent to the one stated
previously.
\[ \forall \epsilon \in \mathbb{Q}^+ \left( \exists M \left( \forall n > M \forall i (d(a_n, a_{n+i} < \epsilon)) \right) \right) \]

We still have some concerns with this definition. Note that the above statement is $\Pi^0_3$, which is much stronger than the $\Pi^0_1$ definition of rapidly Cauchy.

There are many other ways to represent real numbers in Reverse Mathematics (and natural mathematics), including open cuts, Dedekind cuts, and decimal expansions. For a single real number, one kind of representation can be converted into another over $\text{RCA}_0$. However, for sequences of real numbers, the conversions require stronger axioms. Therefore we use rapidly Cauchy sequences as a convenient option, both for the weakness of the $\Pi^0_1$ definition as described above and for its easy connection to other representations. We could also develop the reals using another representation, but we use rapidly Cauchy sequences for their many benefits. For further discussion on the different ways to represent real numbers, see Hirst's paper "Representations of Reals in Reverse Mathematics." [9]

### 5.1.2 Equivalence of Points in $\hat{\mathbb{Q}}$

Now we will define when points are equivalent.

**Definition 5.1.3.** Two points $x = \langle a_n : n \in \mathbb{N} \rangle$ and $y = \langle b_n : n \in \mathbb{N} \rangle$ in $\hat{\mathbb{Q}}$ are equal ($x \sim y$) if and only if $\lim d(a_n, b_n) = 0$. [15]

Note that the sequence representing $x$ and the sequence representing $y$ are not necessarily pairwise equal. For clarity, we will represent this equivalence as $x \sim y$. We will now show that the property of equality as stated above is indeed an equivalence relation.

**Lemma 5.1.1.** $\sim$ is an equivalence relation.

**Proof.** Reflexive: Since $d$ is a metric, $d(a_k, a_k) = 0$ for all $k$. Therefore $\lim d(a_k, a_k) = 0$, so $\langle a_n \rangle \sim \langle a_n \rangle$. 
Symmetric: Suppose \((a_k) \sim (b_k)\). Then \(\lim d(a_k, b_k) = 0\), so by the symmetry of the metric \(d\), \(\lim d(b_k, a_k) = 0\). Therefore \((b_k) \sim (a_k)\).

Transitive: Suppose \((a_k) \sim (b_k)\) and \((b_k) \sim (c_k)\). Then \(\lim d(a_k, b_k) = 0 = \lim d(b_k, c_k)\). So, by the triangle inequality of the metric \(d\):

\[
d(a_k, c_k) \leq d(a_k, b_k) + d(b_k, c_k)
\]

By the non-negativity of \(d\):

\[
0 \leq d(a_k, c_k) \leq [d(a_k, b_k) + d(b_k, c_k)]
\]

Since the limit of both the left and right sides of the equation goes to 0, then \(\lim d(a_k, c_k) = 0\) and \((a_k) \sim (c_k)\). Therefore \(\sim\) has all three properties of an equivalence relation. \(\blacksquare\)

Note that we can embed the dense set \(\mathbb{Q}\) into \(\hat{\mathbb{Q}}\) by letting \(a \in \mathbb{Q}\) be represented by the sequence \((a : n \in \mathbb{N})\).

### 5.1.3 A Note about The Set of Real Numbers as an Object

The standard approach would be to define \(\hat{\mathbb{Q}}\) as the set of equivalence classes of these sequences and real numbers as representatives of equivalence classes. However, we can not actually utilize that structure within second order arithmetic. A class, a set of sets, is a third order object, and using representatives requires a strong version of the axiom of choice. Second order arithmetic is strictly contained within third order arithmetic, so allowing ourselves to use equivalence classes would be equivalent to adding all the axioms of second order arithmetic. However, we would like to conduct our analysis within weak subsystems of 2nd order arithmetic. ([15], pg. 75)

So while we can work with real numbers, we can not work with the set of real numbers. For example, this nuance prevents us from defining the irrational numbers as \(\mathbb{R} \setminus \mathbb{Q}\) because we cannot comprehend \(\mathbb{R}\). If we wanted to use the set of irrational
numbers we would have to construct them separately. Even then, notice that if we could work with the set of irrational numbers, that would imply that we could comprehend the real numbers as the union of the rational and irrational numbers, which is a contradiction.

We propose an alternative definition of equality, using the particular bound in the definition of a point in $\mathbb{Q}$.

**Lemma 5.1.2.** Given real numbers $x = (a_n : n \in \mathbb{N})$ and $y = (b_n : n \in \mathbb{N})$, $x \sim y$ if and only if $\forall n[d(a_n, b_n) \leq 2^{-n+1}]$ [3], [15]

In other words, two points are equal if the pairwise distance between their sequences is bounded by $2^{-n+1}$.

**Why do we Bound the Distance by $2^{-n+1}$?**

Consider $x$ and $y$ on the real number line. Essentially, we're supposing that the limits of the two sequences is the same, some $L \in \mathbb{R}$. For any step in the sequence, the distance of the corresponding $a_k$ to the limit is $2^{-k}$ and the distance of the corresponding $b_k$ to the limit is $2^{-k}$. However the $a_k$ could be to the left of $L$ and the $b_k$ could be to the right of $L$.

![Figure 5.1: Demonstration of $2^{-n+1}$ bound](image)

Then we would expect that it would be possible for $d(a_k, b_k) = 2^{-k+1}$, even if the two sequences are converging to the same point as expected.
Proof of Lemma 5.1.2

Proof. First consider that \( x \sim y \). Then \( \lim d(a_n, b_n) = 0 \). For arbitrary \( i \), by the triangle inequality:

\[
d(a_n, b_n) \leq d(a_n, a_{n+i}) + d(a_{n+i}, b_{n+i}) + d(b_{n+i}, b_n)
\]

Since \( x, y \in \hat{Q} \), \( d(a_n, a_{n+i}) < 2^{-n} \) and \( d(b_n, b_{n+i}) < 2^{-n} \). Substituting into the inequality:

\[
\leq 2^{-n} + d(a_{n+i}, b_{n+i}) + 2^{-n} = d(a_{n+i}, b_{n+i}) + 2^{-n+1}
\]

The \( i \) value was chosen arbitrarily, so take the limit as \( i \) goes to infinity. Since only the \( d(a_{n+i}, b_{n+i}) \) term depends on \( i \):

\[
d(a_n, b_n) \leq 0 + 2^{-n+1} = 2^{-n+1}
\]

Now suppose \( \forall n [d(a_n, b_n) \leq 2^{-n+1}] \). Since that inequality holds for all \( n \), take the limit of both sides:

\[
\lim d(a_n, b_n) \leq \lim (2^{-n+1})
\]

\[
\lim d(a_n, b_n) \leq 0
\]

So since \( d \) is a metric, \( \lim d(a_n, b_n) \) is non-negative, so \( \lim d(a_n, b_n) = 0 \) and \( x \sim y \). □

5.1.4 Constructing a Metric on \( \hat{Q} \)

Definition 5.1.4. Let the metric \( \hat{d} \) on \( \hat{Q} \) be defined where \( \hat{d}((x_n), (y_n)) = \lim d(x_n, y_n) \). [15]

Note that this metric could result in the irrational limit of a rational sequence \( d(x_n, y_n) \).

Example, let \( x \) be a decimal approximation of \( \pi \) and \( y \) be a decimal approximations of \( 2\pi \). The distance between \( \pi \) and \( 2\pi \) is \( \pi \), which is irrational.

First we will show this metric is well-defined and has all the required qualities.

Lemma 5.1.3. \( \forall x, y \in \hat{Q}, \hat{d}(x, y) = z \) is a real number
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Proof. We must show that there is a sequence of rational numbers \( \langle z_n \rangle \) where \( d(z_n, z_{n+1}) \leq 2^{-n} \) and \( \lim z_n = \hat{d}(x, y) \). Let \( \langle z_n \rangle = \langle d(x_{n+1}, y_{n+1}) \rangle \). Note that \( \lim d(x_{n+1}, y_{n+1}) = \lim d(x_n, y_n) = z \). We will show that \( \langle z_n \rangle \) is a sequence that satisfies the rapidly Cauchy definition of a point in \( \hat{Q} \). Let \( n \) and \( i \) be arbitrary. Consider \( z_n \):

\[
z_n = \hat{d}(x_{n+1}, y_{n+1}) \leq \hat{d}(x_{n+1}, x_{n+i+1}) + \hat{d}(x_{n+i+1}, y_{n+i+1}) + \hat{d}(y_{n+i+1}, y_{n+1})
\]

\[
\leq 2^{-n+1} + \hat{d}(x_{n+i+1}, y_{n+i+1}) + 2^{-n+1}
\]

\[
= 2^{-n} + z_{n+i}
\]

\[z_n - z_{n+i} \leq 2^{-n}\]

We need to show that \( d(z_n, z_{n+1}) \leq 2^{-n} \), so we have constructed half of the absolute value. Now consider \( z_{n+i} \):

\[
z_{n+i} = \hat{d}(x_{n+i+1}, y_{n+i+1}) \leq \hat{d}(x_{n+i+1}, x_{n+1}) + \hat{d}(x_{n+1}, y_{n+i+1}) + \hat{d}(y_{n+i+1}, y_{n+1})
\]

\[
\leq 2^{-n+1} + \hat{d}(x_{n+1}, y_{n+1}) + 2^{-n+1}
\]

\[
= 2^{-n} + z_n
\]

\[z_{n+i} - z_n \leq 2^{-n}\]

So \( d(z_n, z_{n+1}) = |z_n - z_{n+i}| \leq 2^{-n} \). Therefore \( z = \langle z_n \rangle \) is a real number as desired. \( \Box \)

Lemma 5.14. If \( x \sim x' \) and \( y \sim y' \) then \( \hat{d}(x, y) = \hat{d}(x', y') \).

Proof. Let \( x = \langle x_n \rangle, x' = \langle x'_n \rangle, y = \langle y_n \rangle, y' = \langle y'_n \rangle \). Consider \( \hat{d}(x, y) \):

\[
\hat{d}(x, y) = \lim d(x_n, y_n)
\]

By the triangle inequality of \( d \):

\[
= \lim (d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n))
\]
By properties of limits:

\[
= \lim \ d(x_n, x'_n) + \lim \ d(x'_n, y'_n) + \lim \ d(y'_n, y_n)
\]

Since \( x \sim x' \), we have that \( \lim \ d(x_n, x'_n) = 0 \). Since \( y \sim y' \), we have that \( (y_n, y'_n) = 0 \). So therefore we have that

\[
= \lim \ d(x'_n, y'_n) = \hat{d}(x', y')
\]

Therefore the mapping \( \hat{d} \) is well-defined.

**Lemma 5.1.5.** \( \hat{d} : \hat{Q} \times \hat{Q} \rightarrow \hat{Q} \) is a metric.

**Proof.** We will show \( \hat{d} \) has all the required properties of a metric.

**Non-negativity:** Since \( d \) is non-negative, \( d(x_n, y_n) \geq 0 \) for all \((x_n), (y_n) \in \hat{Q}\). Therefore \( \lim d(x_n, y_n) \geq 0 \) for all \((x_n), (y_n) \in \hat{Q}\).

**Definite:** Suppose that \( \hat{d}(x, y) = 0 \), then \( \lim d(x_n, y_n) = 0 \). Then by definition of the equality relation on points of \( \hat{Q} \), it must be true that \( x = y \). Now suppose \( x = y \), then \( \lim d(x_n, y_n) = 0 \), so \( \hat{d}(x, y) = 0 \).

**Symmetry:** Since \( d \) is symmetric, \( d(x_n, y_n) = d(y_n, x_n) \), therefore \( \hat{d}(x, y) = \lim d(x_n, y_n) \) = \( \lim d(y_n, x_n) = \hat{d}(y, x) \).

**Triangle inequality:** Consider \( \hat{d}(x, z) \).

\[
\hat{d}(x, z) = \lim d(x_n, z_n)
\]

By the triangle inequality of \( d \):

\[
\leq \lim (d(x_n, y_n) + d(y_n, z_n)) = \lim d(x_n, y_n) + \lim d(y_n, z_n) = \hat{d}(x, y) + \hat{d}(y, z)
\]

So the triangle inequality holds and \( \hat{d} \) is a metric on \( \hat{Q} \).
In Brown’s paper "Notions of Closed Sets...", the metric on the completion \( \hat{A} \) is defined differently [3].

\[
\hat{d}(x, y) = \lim d(x_{n+3}, y_{n+3})
\]

This definition is equivalent, but we use the definition given by Simpson as we find it to be more straightforward.

### 5.1.5 Standard Operations and Field Properties of \( \mathbb{R} \)

We will assume that our construction of \( \mathbb{R} \) as \( \hat{\mathbb{Q}} \) has well-defined operations such as addition, multiplication, inequalities, as well as all the properties of an ordered field. Also, we are able to prove the Archimedean property of \( \mathbb{R} \) within \( RCA_0 \), as \( \mathbb{R} \) is a Archimedean ordered field. We will take these properties as given. For full constructions of the operations on \( \mathbb{R} \) and proofs of the Archimedean field properties see ([15], pg. 76).

### 5.2 Construction of a General Complete Separable Metric Space

Before we prove that our construction is in fact complete, we will show how the construction of \( \mathbb{R} \) from \( \mathbb{Q} \) can be generalized to any subset \( A \subseteq \mathbb{N} \).

**Definition 5.2.1.** A complete separable metric space \( \hat{A} \) is encoded by an \( A \subseteq \mathbb{N} \) and a metric \( d \). \((A, d)\) is considered the code for \( \hat{A} \), where \( A \) is the dense set that is completed to build \( \hat{A} \) and \( d \) is the metric function. [3],[15]

**Definition 5.2.2.** A point in the completion \( \hat{A} \) is defined by a function \( f: \mathbb{N} \to A \) where

\[
\forall n \forall i [d(f(n), f(n+i)) \leq 2^{-n}]
\]

[3],[15]

All the previous lemmas hold true for this general construction. Note that the metric \( \hat{d} \) on \( \hat{A} \) will be defined as \( \hat{d}: \hat{A} \times \hat{A} \to \mathbb{R} \). Then the proofs above for arbitrary \( A \),
which relies on the use of the standard metric on \( \mathbb{Q} \) and \( \mathbb{R} \), still allows for an arbitrary metric on \( \mathbb{A} \).

### 5.3 Construction of the Cantor Space \((2^{\mathbb{N}})\)

Other than the real numbers, a common example of a complete metric space in both analysis and set theory is the Cantor Space. We will show how we can use the above method to construct the Cantor space, the set of all countable sequences of 0's and 1's, as the completion of the set of all finite sequences of 0's and 1's. This will allow us to explore the process of the construction of a complete space as well as introduce us to the Cantor Space which is relevant to our later discussions.

**Definition 5.3.1.** Let \( 2^{\mathbb{N}} \) denote the set of all finite sequences of 0's and 1's. We denote an arbitrary element \( a \) of \( 2^{\mathbb{N}} \) as \( \langle a_i : i \leq n \rangle \) where \( n \in \mathbb{N} \). [3]

**Definition 5.3.2.** For \( a \in 2^{\mathbb{N}} \), the length \( \ell(a) = \begin{cases} 0 & \text{if } a_i = 0, (\forall i \leq n) \\ \max\{i \leq n | a_i = 1\} & \text{else} \end{cases} \)

[3]

Note that \( \ell(a) \) may not be the \( n \) bound denoted in the encoding for \( a \), but will be some number less than or equal to \( n \). For simplicity, we will use \( a' \) to represent the "extension" of \( a \) by zeros to create an infinite sequence.

**Definition 5.3.3.** Given \( a \in 2^{\mathbb{N}} \), let \( a' \) be \( \langle a'_i : n \in \mathbb{N} \rangle \) where \( a'_i = \begin{cases} a_i & i \leq \ell(a) \\ 0 & i > \ell(a) \end{cases} \). [15]

This will allow us to work with our elements of \( 2^{\mathbb{N}} \) more easily within our set of countable sequences \((2^{\mathbb{N}})\) we are about to construct. Now we will define a suitable metric \( d \) on \( 2^{\mathbb{N}} \). First, let \( \delta \) be the discrete metric on \( \{0, 1\} \) that we will use to construct \( d \).

**Definition 5.3.4.** Let the **discrete metric** on \( \{0, 1\} \) be \( \delta(a, b) = \begin{cases} 0 & a = b \\ 1 & a \neq b \end{cases} \) [15]
Chapter 5. Construction of a Complete Separable Metric Space

Definition 5.3.5. Let \( d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{Q} \subseteq \mathbb{R} \) be the metric on \( 2^{\mathbb{N}} \), where \( d(a, b) = \sum_{i=1}^{N} \frac{1}{2^i} \delta(a'_i, b'_i) \) where \( N = \max\{\ell(a), \ell(b)\} \). [15]

First we will show this function has the required properties of a metric.

Lemma 5.3.1. \( d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{Q} \) is a metric.

Proof. We will show \( d \) has all the required properties of a metric.

Non-negativity: Given the formula, \( d \) is always a sum of non-negative terms and such is non-negative.

Definite: Suppose that \( d(\langle a_i \rangle, \langle b_i \rangle) = 0 \), then \( \sum_{i=1}^{\infty} \frac{1}{2^i} \delta(a_i, b_i) = 0 \). Therefore \( \frac{1}{2^i} \delta(a_i, b_i) = 0 \) for all \( i \), so \( \delta(a_i, b_i) = 0 \) for all \( i \). Thus \( \langle a_i \rangle \) and \( \langle b_i \rangle \) have identical terms for all \( i \). Note that \( \langle a_i \rangle \) and \( \langle b_i \rangle \) must be the same length or else be identical to the sequence with additional trailing zeros. This same argument follows in reverse to show that if \( \langle a_i \rangle = \langle b_i \rangle \) then \( d(x, y) = 0 \).

Symmetry: Since \( \delta \) is symmetric (as equality is symmetric), \( d \) is also symmetric.

Triangle inequality: Consider \( d(a, c) \).

\[
d(a, c) = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta(a_i, c_i)
\]

By the triangle inequality of \( \delta \):

\[
\leq \sum_{i=1}^{\infty} \frac{1}{2^i} (\delta(a_i, b_i) + \delta(b_i, c_i))
\]

\[
= \sum_{i=1}^{\infty} \frac{1}{2^i} \delta(a_i, b_i) + \sum_{i=1}^{\infty} \frac{1}{2^i} \delta(b_i, c_i)
\]

\[
= d(a, b) + d(b, c)
\]

So the triangle inequality holds, and \( d \) is a metric on \( A \). ■

Note that if \( d(a, b) = 0 \), then \( a \) as a sequence may not be identical to \( b \) as a sequence. For example, if \( a = (0, 1, 0) \) and \( b = (0, 1, 0, 0) \), \( a \) would technically not be identical to \( b \) as a sequence, but their extensions would be equal \( (a' = b' = (0, 1, 0, 0, 0, \ldots)) \).
Lemma 5.3.2. \( d(a, b) < 2^{-k-1} \iff a'_i = b'_i \) for all \( i < k + 1 \).

Proof. \((\Rightarrow)\) If \( d(a, b) < 2^{-k-1} \), then we have that \( \sum_{i=1}^{\infty} \frac{1}{2^i} \delta(a_i, b_i) < 2^{-k-1} \). All terms in this sum are in the form \( 2^{-i} \) and decrease as \( i \) increases. Each unique \( 2^{-i} \) can only appear at most once. Since the largest term must be less than \( 2^{-k-1} \), it must be that \( \delta(a_i, b_i) = 0 \) for all \( i < k + 1 \). So \( a'_i = b'_i \) for all \( i < k + 1 \) and the sequence corresponding to \( a \) must agree with the sequence corresponding to \( b \) up to the \( k + 1 \)st position.

\((\Leftarrow)\) Suppose \( a'_i = b'_i \) for all \( i < k + 1 \). Then \( \delta(a'_i, b'_i) = 0 \) for all \( i < k + 1 \).

\[
d(a, b) = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta(a'_i, b'_i)
\]

\[
= \sum_{i=k+1}^{\infty} \frac{1}{2^i} \delta(a'_i, b'_i)
\]

\[
\leq \sum_{i=k+1}^{\infty} \frac{1}{2^{i+1}}
\]

\[
< \sum_{i=k+1}^{\infty} \frac{1}{2^{i+1}} = 2^{-k-1}
\]

We now define the Cantor Space \( 2^\mathbb{N} \) as the completion of \( 2^{\mathbb{N}} \) with metric \( d \) as detailed in Section 5.2. Similarly to \( \mathbb{Q} \), since \( 2^{\mathbb{N}} \) is countable, it can represented as a subset of \( \mathbb{N} \) with some code, and so the construction follows. It is helpful to show what the definition of a point in \( 2^\mathbb{N} \) implies given the properties of \( \tilde{d} \). The result of this lemma connects nicely to thinking of the elements of \( A \) and \( 2^\mathbb{N} \) as numbers in binary digits. With this lemma, we have a sequence \( \langle a_n \rangle \) is a point of \( 2^\mathbb{N} \) if and only if the elements of the sequence agree with each other up to the index of the earliest element of the sequence.

5.3.1 Equality in \( 2^\mathbb{N} \)

Recall that \( \tilde{d} \) is a pseudo-metric. So \( \tilde{d}(a, b) = 0 \), does not imply that \( a = b \), but simply that they can be described with the same sequence of 0’s and 1’s. As we did for \( \mathbb{R} \), we
will reframe the \( \hat{d} \) metric to more concretely connect with our definition of \( d \) and the structures of the elements of \( A \).

**Lemma 5.3.3.** \( \hat{d}(a, b) = 0 \) if and only if \( d(a_k, b_k) \leq 2^{-k+1} \) for all \( k \in \mathbb{N} \).

**Proof.** Suppose \( \hat{d}(a, b) = 0 \). First consider \( d(a_k, b_k) \). We can apply the triangle inequality as follows, for arbitrary \( j \in \mathbb{N} \).

\[
d(a_k, b_k) \leq d(a_k, a_{k+j}) + d(a_{k+j}, b_{k+j}) + d(b_{k+j}, b_k)
\]

By the definition of \( a \) and \( b \) as points in the Cantor space we can substitute.

\[
\leq 2^{-k} + d(a_{k+j}, b_{k+j}) + 2^{-k} = d(a_{k+j}, b_{k+j}) + 2^{-k+1}
\]

Since \( j \) is arbitrary, we take the limit as \( j \) approaches infinity, and by assumption, \( \lim d(a_{k+j}, b_{k+j}) = 0 \). Then we have \( d(a_k, b_k) \leq 2^{-k+1} \) as desired.

The backwards direction follows, as the sequence \( d(a_k, b_k) \) is dominated by the sequence \( 2^{-k+1} \), that is converging to 0 (and \( \hat{d} \) is non-negative by properties of metrics).

\[\blacksquare\]

### 5.3.2 The Cantor Space does not Always Behave like Binary Representations of Real Numbers.

For those familiar with binary representations of numbers may recall that 0.011... = 0.1 (which follows for the exact same reason that 0.9999... = 1 as baffles many early math students.) However this reduction does not apply to \( 2^\mathbb{N} \). Consider the example proposed above.

**Example:** Let \( a = \langle 0.1, 0.1, 0.1, \ldots \rangle \) and \( b = \langle 0.01, 0.011, 0.0111, \ldots \rangle \). We expect that \( b \) converges to 0.0111..., however \( a \neq b \) considering our metric \( \hat{d} \) on \( 2^\mathbb{N} \).

\[
\hat{d}(a, b) = \lim \hat{d}(a_k, b_k) = \lim_{k \to \infty} \sum_{i=1}^{k} \frac{1}{2^i} \delta(a'_i, b'_i) = \lim \left( \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \ldots \right) = \frac{1}{2^5} \neq 0
\]
5.4 Showing $\hat{A}$ is Complete in $\text{RCA}_0$

Now we have constructed both $\hat{Q}$ and $\hat{A}$ for arbitrary $A \subseteq \mathbb{N}$ which we claim to be the completion of $Q$ and $A$ respectively. However, showing that these spaces are indeed complete within the context of Reverse Mathematics reveals crucial differences between the subsystems $\text{RCA}_0$ and $\text{ACA}_0$.

Though we consider $\mathbb{R}$ to be the archetypal complete metric space, the weakness of $\text{RCA}_0$ creates an interesting nuance. Recall in Chapter 2, where we explored the differences between Cauchy completeness and Dedekind completeness. Within $\text{RCA}_0$, we can not prove that $\mathbb{R}$ is sequentially complete (or in other words Dedekind complete). However, we can prove that $\hat{A}$ is Cauchy complete within $\text{RCA}_0$.

5.4.1 Proof that $\hat{A}$ Satisfies the Cauchy Criterion

In Chapter 2, we discussed how the Cauchy Criterion is not sufficient for having the least upper bound property. Here is a statement of the Cauchy Criterion in the Reverse Mathematics context.

**Theorem 5.4.1.** Given a sequence $(x_n)$ of points in $\hat{A}$ such that $\exists(r_n)$ where $\lim r_n \to 0$, such that $\forall m \forall n[\hat{d}(x_n, x_m) < r_m]$, then $\exists x \in \hat{A}$ such that $x = \lim_n x_n$. [15]

First, we will prove a lemma that will allow us to bridge from the generic $r_n$ bound and the $2^{-n}$ bound that frequently appears in our constructions of $\hat{A}$. Next we will prove a lemma that if a subsequence of a Cauchy sequence converges to a point, then the original sequence converges to that point as well.

**Definition 5.4.1.** Given a sequence $(x_n)$ of points in $\hat{A}$, a subsequence is a function $f : \mathbb{N} \to \hat{A}$ such that $f(n) = x_m$ for some $m \geq n$, and can be represented by a sequence $(y_n)$.

**Lemma 5.4.2.** Given a sequence as in the previous theorem, $(x_n)$ of points in $\hat{A}$ where $\exists(r_n)$ where $\lim r_n \to 0$, such that $\forall m \forall n[\hat{d}(x_n, x_m) < r_m]$, there exists a subsequence $(y_n)$ of $(x_n)$ such that $(2^{-n})$ serves as a bound in the same matter as $(r_n)$.
Proof. Since \( \lim r_n \to 0, \forall n, \exists M \) such that \( r_M < 2^{-n} \). Otherwise \( \langle r_n \rangle \neq 0 \). Let \( M_n \) be the least of such bounds \( M \). This designation is possible by minimization which holds in \( \text{RCA}_0 \). (For more about minimization, see Chapter 7). Then let \( \langle y_n \rangle \) be the subsequence of \( \langle x_n \rangle \) defined by the function \( f : \mathbb{N} \to \hat{A} \) where \( f(n) = x_{M_n} \). We claim this subsequence \( \langle y_n \rangle \) is bound by \( 2^{-n} \) as desired.

We know:

\[
d(y_n, y_m) = d(x_{M_n}, x_{M_m})
\]

Since \( \langle x_n \rangle \) is Cauchy bounded by \( \langle r_n \rangle \):

\[
\leq r_{M_n}
\]

By choice of \( M_n \):

\[
\leq 2^{-n}
\]

\[\blacksquare\]

**Lemma 5.4.3.** If a sequence \( \langle x_n \rangle \) in \( \hat{A} \) is Cauchy, in other words if \( \exists (r_n) \) in \( A \) where \( \lim r_n \to 0 \) such that \( \forall m \forall n [d(x_n, x_m)] < r_m \), and there exists a subsequence \( \langle y_n \rangle \) of \( \langle x_n \rangle \) that converges to some \( x \), then \( \langle x_n \rangle \) also converges to \( x \).

Proof. Let \( \langle x_n \rangle \) and \( \langle y_n \rangle \) be as hypothesized. Consider \( d(x_n, x) \). By the triangle inequality:

\[
d(x_n, x) \leq d(x_n, y_n) + d(y_n, x)
\]

Since \( \langle y_n \rangle \) is a subsequence of \( \langle x_n \rangle \), for some \( m \geq n \)

\[
d(x_n, x) \leq d(x_n, x_m) + d(y_n, x)
\]

From the bounding sequence of \( \langle x_n \rangle \):

\[
d(x_n, x) \leq r_n + d(y_n, x)
\]
Taking the limit of both sides:

\[ \lim d(x_n, x) \leq \lim r_n + \lim d(y_n, x) \]

By the convergence of \( (y_n) \) to \( x \), and the convergence of \( (r_n) \) to 0 we have:

\[ \lim d(x_n, x) = 0 \]

So \( (x_n) \) also converges to 0. \hfill \square

Now we will prove that \( \hat{A} \) is Cauchy complete.

**Proof of Theorem 5.4.1**

*Proof.* By Lemma 5.4.2, there is some subsequence of \( (x_n) \) with the bounding sequence \( (2^{-n}) \). We rename that subsequence \( (x_n) \). Now if we show that \( \lim x_n = x \) for some \( x \in \hat{A} \) then we will have shown that the original sequence also converges to the same \( x \) by Lemma 5.4.3. We will prove this statement for a special case first.

**Case 1:** Suppose for all \( n \) that \( x_n = (x_n : n \in N) \) for some \( x_n \in A \). In other words, \( x_n \) is an element of \( A \) in its sequence form in \( \hat{A} \). Let \( x = (x_n) \). First, note that \( x \in \hat{A} \). This is because \( d(a_n, a_m) < 2^{-n} \) from the choice of the subsequence. So \( x \) can be represented by some \( (a_n) \in \hat{A} \).

Next we claim that \( \lim x_n = x \). For arbitrary \( n \), \( d(x_n, x) = \lim_{k \to \infty} d(x_n, x_k) = \lim_{k \to \infty} d(x_n, x_k) \).

For large enough \( k \), which will occur in the limit, \( k > n \). So by the definition of the \( x_n \) sequence, \( \lim_{k \to \infty} d(x_n, x_k) \leq 2^{-n} \).

Since the \( \lim 2^{-n} = 0 \), therefore \( \lim d(x_n, x_k) = 0 \) as distances are nonnegative. Therefore \( (x_n) \) converges to \( x \) in \( \hat{A} \).

**Case 2:** Consider some general \( (x_n) \). We know \( A \) is dense in \( \hat{A} \). So for any \( x_n \) in the sequence, we can choose an \( a_n \in A \) where \( d(x_n, a_n) \leq \frac{1}{3(2^n)} \). (This choice of bound will be clear in a later inequality.) We claim that \( x = (a_n) \) is the limit of the \( x_n \) sequence. First
we will show that $x \in \hat{A}$. Let $m > n$ without loss of generality. By the triangle inequality:

$$\hat{d}(a_n, a_m) \leq \hat{d}(a_n, x_n) + \hat{d}(x_n, x_m) + \hat{d}(x_m, a_m)$$

By the choice of $a_n$:

$$\leq \frac{1}{3(2^n)} + \frac{1}{3(2^n)} + \frac{1}{3(2^m)}$$

Since $m > n$, $\frac{1}{3(2^n)} < \frac{1}{3(2^n)}$, therefore

$$\leq \frac{3}{3(2^n)} = 2^{-n}$$

So $x \in \hat{A}$.

Next we will show that $(x_n) \to x$. Consider $\lim_{n \to \infty} \hat{d}(x_n, x)$. By the triangle inequality:

$$\lim_{n \to \infty} \hat{d}(x_n, x) \leq \lim_{n \to \infty} (\hat{d}(x_n, a_n) + \hat{d}(a_n, x))$$

By the choice of $a_n$:

$$\leq \lim_{n \to \infty} \left(\frac{1}{3(2^n)} + \hat{d}(a_n, x)\right)$$

In the limit, since $(a_n) \to x$, $\lim \hat{d}(a_n, x) \to 0$ So

$$\leq 0$$

So $\lim x_n = x$. 

5.4.2 $\mathbb{R}$ is not Sequentially Complete in $\mathbb{RCA}_0$

We will show that $\mathbb{R}$ is not sequentially complete in $\mathbb{RCA}_0$ with a counterexample from Stillwell and Simpson. Recall from Chapter 3 and Chapter 4 that the sets in $\mathbb{RCA}_0$ are computable and sets in $\mathbb{ACA}_0$ are computably enumerable. This dichotomy captures the distinction between $\mathbb{RCA}_0$ and $\mathbb{ACA}_0$. We will now use the example of a computably enumerable set that is not computable that we created in Lemma 4.1.1. This
set allows us to construct a well-behaved sequence of rationals that cannot converge to a point in \( \hat{Q} \).

**Theorem 5.4.4.** \( \hat{Q} \) is not sequentially complete. In other words, there exists some bounded, increasing sequence \( \langle a_k \rangle \) of rational numbers where \( \langle a_k \rangle \) does not converge to a \( x \in \hat{Q} \) [15],[16]

**Proof.** Let \( D \) be a non-recursive set that is recursively enumerable, like the set constructed in lemma 4.1.1. Then let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be a one to one recursive function where \( D \) is the range of \( f \). This function exists since \( D \) is recursively enumerable. Let \( \langle a_k \rangle \) be the sequence of numbers in \( Q \) where

\[
a_k = \sum_{m=0}^{k} \frac{1}{2f(m)}
\]

Note this sequence is bounded and increasing. This sequence is recursive and all elements are in \( Q \). If the limit \( x \) existed in \( \hat{Q} \), the binary expansion of \( x \) would exactly encode the elements of \( D \).

For any number \( n \in \mathbb{N} \), we could tell if \( n \) was a member of \( D \) by looking at the \( n \)th position of the binary expansion of \( x \). If there was a 1 in the \( n \)th position of the binary expansion of \( x \), then \( \frac{1}{2^n} \) is a term in the sum, and \( n \) must be an element of \( D \). If there is a 0 in the \( n \)th position of the binary expansion of \( x \), then \( \frac{1}{2^n} \) was not a term in the sum, and \( n \) cannot be in \( D \). This is a computable way to construct \( D \) but \( D \) is noncomputable. Therefore, \( x \) is not an element of \( \hat{Q} \), and there is no limit of \( \langle a_k \rangle \) existing in \( \hat{Q} \). \( \Box \)

### 5.5 Showing \( \hat{A} \) is Complete in ACA\(_0\).

With the extra strength afforded to us by ACA\(_0\), we are no longer restricted to only the Cauchy sense of completeness. ACA\(_0\) is strong enough to imply most of the standard theorems of completeness from Real Analysis.
5.5.1 R is Sequentially Complete in ACA₀

In the previous section, we considered a counterexample to sequential completeness. We will now show that this counterexample is no longer a counterexample within ACA₀.

Our computably enumerable sets that are not comprehensible under RCA₀ are described by Σᵡ¹₀ formulas, but ACA₀ includes arithmetic comprehension, so these sets now exist once we assume the axioms of ACA₀. Given a set is computably enumerable but noncomputable, as the set D described in lemma 5.4.7, then the sequence \( a_k = \sum_{m=0}^{2^k-1} \frac{1}{2^m} \) does have a limit in \( \hat{A} \) within ACA₀, since the contradiction does not apply with access to arithmetic comprehension. Therefore, our previous counterexample no longer holds. Now we provide the proof that, within ACA₀, we can show that \( \mathbb{R} \) is sequentially complete.

**Definition 5.5.1.** Let \( \langle a_n \rangle \) be a bounded sequence of real numbers. \( x = \lim \sup a_n \) if given any \( \epsilon \in \mathbb{Q} \) such that \( \epsilon > 0 \) the following two conditions hold [15],[16]

1. \( \exists m \in \mathbb{N} \) such that \( n > m \rightarrow x_n \leq x + \epsilon \)

2. \( \forall m \in \mathbb{N}, \exists n \in \mathbb{N} \) such that \( n > m \) and \( x - \epsilon < a_n \).

**Theorem 5.5.1 (ACA₀).** \( \mathbb{R} \) is sequentially complete, in other words, all bounded sequences \( \langle a_n \rangle \subset \mathbb{R} \) have a \( x \in \mathbb{R} \) such that \( \lim \sup a_n = x \). [15],[16]

**Proof.** Without loss of generality, let \( \langle a_n \rangle \) be a bounded sequence in \([0,1]\), as any bounded sequence in \( \mathbb{R} \) can be scaled to be a sequence within \([0,1]\). We will construct a nested collection of intervals containing infinitely many terms of the sequence. Let \( A_k = \{ i \mid (i < 2^k) \land (\forall m \in \mathbb{N} (\exists n > m (\frac{i}{2^m} \leq a_k \leq \frac{i+1}{2^m}))) \} \). \( A_k \) contains all possible kth partitions of \([0,1]\) that contain infinitely many elements of the sequence \( \langle a_k \rangle \). Since \( a_k \) is an infinite sequence, and the elements within \( A_k \), the \( i \) are bounded by \( 2^k \), \( A_k \) is a nonempty finite set for all \( k \). Therefore, we can computably find the minimum of these elements, let \( f(k) = \max \{ i \mid i \in A_k \} \). We claim \( x = \langle q_k \rangle = (\frac{f(k)}{2^k}) \) is our desired \( \lim \sup a_k \), which will show with three lemmas.
Lemma 5.5.2. $x = \langle q_k \rangle$ is a real number.

Let $k$ be fixed, and $j$ be arbitrary. There are infinitely many $n$ such that $q_k + \frac{1}{2^k} \leq a_n$ and infinitely many $n$ such that $q_k \leq a_n$. There are finitely many $i$ such that $q_k \leq \frac{1}{2^k} \leq q_k + \frac{1}{2^k}$. So therefore we have that $q_k \leq q_i \leq q_k + \frac{1}{2^k}$ which implies that $d(q_k, q_{k+1}) \leq \frac{1}{2^k}$. So $\langle q_k \rangle$ is a real number by definition.

Lemma 5.5.3. Given any $e \in \mathbb{Q}$ such that $e > 0$, $\exists m \in \mathbb{N}$ such that $a_n = x + e$ for all $n > m$.

Choose $k \in \mathbb{N}$ such that $\frac{1}{2^k} < e$. We know that $\{ n \mid a_n > q_k + \frac{1}{2^k} \}$ is finite, as this describes the indices of elements of $a_n$ outside of the chosen partition from $f$. Since this set is finite, let $m = \max \{ n \mid a_n > q_k + \frac{1}{2^k} \}$. So then, it must be true that $\forall n > m$, $x_n \leq q_k + \frac{1}{2^k} < q_k + e \leq x + e$.

Lemma 5.5.4. Let $e \in \mathbb{Q}$ such that $e > 0$ and $k \in \mathbb{N}$ such that $\frac{1}{2^k} < e$. Then, given any $m \in \mathbb{N}$, $\exists n > m$ such that $q_k \leq a_n \leq q_k + \frac{1}{2^k}$ and so $|x - a_k| \leq e$.

This follows since by construction, $\{ n \mid q_k \leq a_n \leq q_k + \frac{1}{2^k} \}$ is infinite for all $k$.

With these three lemmas, $x = \limsup a_n$.

5.5.2 Where did we use ACA₀ in this Proof?

From our earlier counterexample, we know that the assumption of the axioms of ACA₀ are essential for this proof. Since the definition of $f$ is arithmetic, we can only construct our limit point $x$ if we have arithmetic comprehension from ACA₀.

5.5.3 Theorems of Completeness Equivalent to ACA₀.

As ACA₀ is the weakest subsystem in the Big 5 where a full idea of completeness (including sequential completeness) appears, many theorems that are intrinsically statements about the completeness of the reals (like the theorems discussed in chapter 2) are equivalent to ACA₀ (over the base system RCA₀). These include Bolzano-Weierstrass Theorem, the Least Upper Bound Property, and the Cut Property. ([15],[16])
In Chapter 2 we discussed the Intermediate Value Theorem as a theorem of completeness. However, the intermediate value theorem is not equivalent to $ACA_0$ but is provable in $RCA_0$. Recall that we just claimed that the Cut property is equivalent to $ACA_0$, and Propp claimed in his paper that the Cut Property is equivalent to the Intermediate Value Theorem (as we explored in Chapter 2). However, was is unclear in that proof is the exact axioms we were assuming. In fact, our previous proof makes some extra assumptions beyond our usual subsystem $RCA_0$ that end up being equivalent to $ACA_0$. Note that, as $RCA_0$ is our base system, we cannot prove that the Intermediate Value Theorem is equivalent to $RCA_0$ unless we worked with a weaker base system. For the proof that the Intermediate Value Theorem is provable in $RCA_0$, see Simpson ([15], pg. 87).
Chapter 6

Closed and Separately Closed Sets

6.1 Topological Definitions

In standard analysis and topology, a closed set is either defined as a set that contains all of its limit points, or to be a set with an open complement, and the other definition is proved as a theorem. In a majority of mathematical settings, these two concepts are identical. However, we will see in the reverse mathematical context, there is a nuance between these two concepts. For these two concepts to be equivalent, you must assume the $\Pi^1_1 - CA_0$ level of axioms, the strongest set of axioms out of the Big 5.

We will define a "closed set" as the complement of an open set and a "separably closed set" as a set containing all its limit points. We can also defined open sets in two different ways, as an "open set" or a "separably open set", however these distinctions will not be utilized until we work with the Baire Category Theorem in Chapter 7.

We will now present some essential topological definitions along with the usual notations used in Reverse Mathematics. Recall that $\hat{A}$ denotes the completion of $A$ as described in the previous chapter.

Definition 6.1.1. A basic open set is an open metric ball with center $a$ and radius $r$ where $a \in A$ and $r \in Q^+$. In the Reverse Mathematics context, we will note such an open ball $(a, r) = \{x \in A \mid d(x, a) < r\}$. Note that this is not the open interval between endpoints $a$ and $r$. [3]
We will often denote $(a, r) \subset (b, s)$, as $(a, r) < (b, s)$. In other words, $(a, r) < (b, s) \iff d(a, b) < s$ and $\forall x \in (a, r), d(x, b) < s$.

In standard analysis, an open set is a set $O$ such that $\forall x \in O, \exists \varepsilon$ such that $(x, \varepsilon) \subseteq O$. In reverse math, we want to express an open set in a countable framework. So we define an open set with a countable code.

**Definition 6.1.2.** An open set $U$ has code $\langle (a_n, r_n) : n \in \mathbb{N} \rangle$, a countable sequence of basic open sets. An element $x \in U$ if $\exists (a, r)$ in the code for $U$ such that $x \in (a, r)$. [3]

As explained above, there are different definitions of closed sets, so we distinguish between them.

**Definition 6.1.3.** A closed set $C$ has code $\langle (a_n, r_n) : n \in \mathbb{N} \rangle$. An element $x \in C$ if $\forall n (x \notin (a_n, r_n))$. In other words, $C$ is the complement of an open set and is encoded by the code for that open set. [3]

This definition is known as the negative information representation of a closed set, since we define a closed set by the points that are not in the set [1].

**Definition 6.1.4.** A separably closed set $\overline{S}$ has code $S = \langle a_n : n \in \mathbb{N} \rangle$, a countable sequence of points. An element $x \in \overline{S}$ if $\forall r \in \mathbb{Q}^+ (\exists n [d(x, a_n) < r])$. In other words, a separably closed set is a countable set union its limit points. [3]

This definition is known as the positive information representation of a closed set since we define a separably closed set by the points that are contained in the set [1].

**Definition 6.1.5.** A separably open set $U$ is the complement of a separably closed set. Therefore $U$ has code $S = \langle a_n : n \in \mathbb{N} \rangle$ An element $x \in U$ if $x \notin \overline{S}$, or in other words, $\exists r \in \mathbb{Q}^+ (\forall n [d(x, a_n) \geq r])$. [3]

### 6.2 Unions and Intersections

As an exploration of our definitions and notation, we will first prove some statements about the unions and intersections of open and closed sets.
6.2.1 Countable Unions of Open Sets are Open

**Lemma 6.2.1 (RCA₀).** Let \( \langle U_n : n \in \mathbb{N} \rangle \) be a sequence of open sets in a complete separable metric space \( \hat{A} \). Then \( \bigcup_{n \in \mathbb{N}} U_n \) is an open set. [3]

**Proof.** We will show that the countable union is open, by defining \( \bigcup_{n \in \mathbb{N}} U_n \) with a \( \Sigma_1^0 \) formula. Let \( U = \{ (a, r) \mid \exists n[(a, r) \in U_n] \} \). Here we use \( (a, r) \in U_n \) to denote that \( (a, r) \) is an element in the code for \( U_n \). The formula defining \( U \) is a \( \Sigma_1^0 \) formula, as there is one \( \exists \) number quantifier in the formula on the right. Therefore, within \( RCA₀ \), there is a function \( f \) that enumerates \( U \), and the code for \( U \) is the enumeration \( \{ (a_n, r_n) \} \).

As \( U \) is the union of basic open sets, it is open. We will now show that \( U \) as defined above is exactly the desired union. Let \( x \in U \), then \( x \in (a_n, r_n) \) for some \( (a_n, r_n) \) in the code of \( U \). So \( x \in \bigcup_{n \in \mathbb{N}} U_n \). Now suppose \( x \in \bigcup_{n \in \mathbb{N}} U_n \). Then \( x \in (a, r) \) for some \( (a, r) \) in the code for some \( U_k \) for \( k \in \mathbb{N} \). Therefore \( x \in U \) by definition of \( x \). Thus, \( \bigcup_{n \in \mathbb{N}} U_n \) is open.

6.2.2 Finite Intersections of Open sets are Open

**Theorem 6.2.2 (RCA₀).** Let \( \langle U_i : 0 \leq i \leq n \rangle \) be a finite sequence of open sets (for some \( n \)) in a complete separable metric space \( \hat{A} \), then \( \bigcap_{i=0}^{n} U_i \) is open. [3]

**Proof.** We will prove this by invoking \( \Sigma_1^0 \) induction. Consider the \( n = 1 \) case. We will write a formula defining \( U = U_0 \cap U_1 \).

\[
U = \{ (a, r) \mid \exists (b_0, s_0) \in U_0 [\exists (b_1, s_1) \in U_1 [(r < s_0 - d(a, b_0) \land (r < s_1 - d(a, b_1)))] \}
\]

Or in other terms, \( U \) is the union of all the basic open sets contained in some open set in \( U_0 \) and some open set in \( U_1 \). Since the less than relation in \( \hat{A} \) is a \( \Sigma_1^0 \) formula, the description of \( U \) as above is a \( \Sigma_1^0 \) formula. So (from a similar argument to 2.3), \( U \) can be enumerated. We will show that \( U \) as defined above is exactly identical to \( U_0 \cap U_1 \).

Suppose \( x \in U \), then \( x \in (a, r) \) for some \( (a, r) \) in the code of \( U \). So by definition of \( U \), \( \exists (b_0, s_0) \in U_0 \) and \( \exists (b_1, s_1) \in U_1 \) that satisfy the statement. Then \( x \in (b_0, s_0) \) and
$x \in (b_1, s_1)$, therefore $x \in U_0$ and $x \in U_1$ and $x \in U_0 \cap U_1$ by definition of intersection.

Now suppose $x \in U_0 \cap U_1$. Then, $x \in U_0$ and $x \in U_1$, so there exists some $(b_0, s_0)$ in the code for $U_0$ and some $(b_1, s_1)$ in the code for $U_1$ such that $x \in (b_0, s_0)$ and $x \in (b_1, s_1)$. Therefore we can choose $r$ where

$$r = \frac{\min\{s_0 - d(a, b_0), s_1 - d(a, b_1)\}}{2}$$

Note that $r$ must be rational, since $s_i, b_i$ and $a$ are rational (for all $i$). Then, from the Archimedean principle, which is a theorem in $RCA_0$, there exists some rational $s$ such that $d(x, a) < r$. Therefore by the definition of $U$, we have that $(a, r) \subseteq U$. So since $x \in (a, r)$ it must be true that $x \in U$ as desired. Now, by applying $\Sigma^0_1$ induction, which is an axiom in $RCA_0$, the lemma follows.

You can similarly show that countable intersections of closed sets are closed and the finite union of closed sets are closed. ([3], pg. 43)

### 6.3 Separably Closed Sets are Closed

For the main work in this section, we will present the proofs about the relative strengths of the definitions of closed and separately closed sets in detail. For these proofs, we will be using tools we developed in Chapter 4.

#### 6.3.1 $ACA_0 \Rightarrow$ (Separably Closed $\rightarrow$ Closed)

**Theorem 6.3.1 ($ACA_0$).** All separably closed sets are closed. [3]

**Proof.** Let $S = (x_n : n \in \mathbb{N})$ be a code for a separably closed subset $\bar{S}$ of a complete separable metric space $\bar{A}$. First, we consider two simple cases. If $\bar{S} = \bar{A}$, then $\varnothing$ is the open set that is complement of $\bar{A}$. The empty set is vacuously open, since $\varnothing$ can be encoded by an empty sequence of basic open sets. Also, if $\bar{S} = \varnothing$, then $\bar{A}$ is the open set that is the complement of $\varnothing$. The entire set $\bar{A}$ is open, since it can be encoded by the sequence $((a, r) | r \in \mathbb{Q}^+)$ where $a$ is some arbitrary fixed element of $A$. 
Now we consider the case where $\overline{S}$ is a nonempty proper subset of $\hat{A}$. We will build an open set and show that it must be the complement of $\overline{S}$, proving that $\overline{S}$ is closed. Let the code for $\mathcal{U}$ be $\{(a_k, r_k)\}$, an enumeration of the following

$$\{(a, r) \in A \times \mathbb{Q}^+ \mid \forall i[d(a, x_i) \geq r]\}$$

Note that $\forall i[d(a, x_i) \geq r]$ is a $\Pi_1^0$ statement, as it has one $\forall$ number quantifier, and no set quantifiers. Therefore $\mathcal{U}$ can be enumerated by arithmetic comprehension which is an axiom of ACA$_0$. We claim that $\mathcal{U} = (\overline{S})^c$.

First suppose $x \in \mathcal{U}$, we will show that $x \notin \overline{S}$. Note that the negation of the definition of a separably closed set is as follows:

$$x \notin \overline{S} \iff \exists r \in \mathbb{Q}^+ \forall n[d(x, x_n) \geq r]$$

In other words $x$ is not a member of $\overline{S}$ if there exists some positive rational radius $r$ such that the ball $(x, r)$ contains no elements of the code. So we will find such a radius. Since $x \in \mathcal{U}$, by definition of open set, $\exists k \in \mathbb{N}$ such that $x \in (a_k, r_k)$. So $x$ is contained in at least one of the open balls in the code for $\mathcal{U}$. By the Archimedean property (which is a theorem in ACA$_0$), choose $s \in \mathbb{Q}^+$ such that $s \leq r_k - d(x, a_k)$. We know that for all $x_i$ in the code of $\overline{S}$ it must be true that $x_i \in S$.

We would like to make an argument about $d(x, x_i)$ and extend it to all $i \in \mathbb{N}$ as explained above. So $d(a_k, x_i) \leq d(a_k, x) + d(x, x_i)$ by the triangle inequality. Rearranging this inequality we find the following:

$$d(x, x_i) \geq d(a_k, x_i) - d(a_k, x)$$

Due to the construction of $\mathcal{U}$ and its code, $x_i \notin (a_k, r_k)$ for any $k$, therefore we can substitute as follows.

$$d(x, x_i) \geq r_k - d(a_k, x) \geq s$$

Note that $i$ was arbitrary. So we have just shown that there exists a radius $s \in \mathbb{Q}^+$ from
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$x$ within which all elements of the code $S$ are excluded. Therefore $x \notin \overline{S}$.

Next suppose $x \notin \overline{S}$. We will show that $x \in U$. We need to show that $x$ is contained in one of the balls in the code for $U$. Since $x \notin \overline{S}$, $\exists r \in \mathbb{Q}^+$ such that $\forall i(d(x, x_i) \geq r)$. As $x$ is in the completion of $S$, $x$ can be represented by a sequence. Let $x = (b_n : n \in \mathbb{N})$. By definition of completion, note that $d(b_n, b_{n+1}) \leq 2^{-n} \forall i$. By the Archimedean property, choose $n$ such that $2^{-n} \leq \frac{r}{2}$. So, $\forall i \in \mathbb{N}$

$$r \leq d(x, x_i) < d(x, a_n) + d(a_n, x_i)$$

$$\leq 2^{-n} + d(a_n, x_i)$$

$$\leq \frac{r}{2} + d(a_n, x_i)$$

Rearranging the equation:

$$d(a_n, x_i) \geq r - \frac{r}{2} = \frac{r}{2} \geq 2^{-n}$$

Therefore $2^{-n}$ is a radius that avoids all $x_i \in S$, so $(a_n, 2^{-n}) \subseteq U$ by definition. Thus $x \in U$.

So $U = \overline{S}$, and by definition, $\overline{S}$ is closed.

6.3.2 Where did we Need ACA₀ in this Proof?

Here we used ACA₀ when building the set $U$ from a $\Pi^0_1$ statement. However, once we justified the construction of $U$ and its code in the proof, the remainder of the proof follows with a fairly typical approach from Real Analysis, applying the triangle inequality in key points of the proof.

6.3.3 RCA₀ + (Separably Closed → Closed) ⇒ ACA₀

We will prove the reversal of the previous theorem by applying some lemmas from Stillwell, Simpson, and Brown, some of which where proved in Chapter 4, and the remaining will be briefly explained. First, we will employ a lemma we proved in Chapter 4, Lemma 4.3.1, that ACA₀ is equivalent to the fact that the ranges of all total one-to-one functions are comprehensible. Faced with the challenge of proving a subsystem of
second order arithmetic from a theorem, it will be easier to prove an equivalence with $\text{ACA}_0$ by proving a theorem we know is equivalent to $\text{ACA}_0$ instead that uses concepts that are more natural to the hypothesis of the theorem. Using Lemma 4.3.1 allows us to work with functions instead of set comprehension directly to prove $\text{ACA}_0$.

The second lemma we will need is the following.

**Lemma 4.2.1** (RCA$_0$) *We can find a function whose values satisfy a $\Sigma^0_1$ condition.*

See Chapter 4 for proof.

The last lemma we will use is from Brown’s doctoral thesis, which allows us to construct the function that we will use to prove the existence of the range of injective, well-defined functions.

**Lemma 6.3.2** (RCA$_0$). *Given a sequence of open sets $\{U_n, n \in \mathbb{N}\}$, there exists a sequence of continuous functions $\{f_n : \tilde{A} \to [0,1]\}$ such that $f_n(x) > 0$ if and only if $x \in U_n$ for some $n$. ([2], Lemma 1.29)

*Proof.* For proof, see [2], Lemma 1.29.

This lemma proves that we can carefully construct a sequence of continuous functions that achieve 0 only on points in a closed set, something similar to a characteristic function on that closed set that will be very useful.

**Theorem 6.3.3** (RCA$_0$). *All separably closed sets are closed $\Rightarrow$ ACA$_0$ [3]*

*Proof.* We will apply Lemma 4.3.1, and show that this statement about closed sets is equivalent to the existence of the range of one-to-one functions. Let $f : \mathbb{N} \to \mathbb{N}$ be total and one-to-one. Recall that while working in RCA$_0$, we only have access to $\Delta^1_0$ comprehension. So if we hope to build the range set of a function, we will need to describe elements by both a $\Sigma^1_0$ and $\Pi^1_0$ formula.

Let $\tilde{A} = [0,1]$. Let $S = \langle 2^{-f(m)} : m \in \mathbb{N} \rangle$ be the code for separately closed set $\overline{S}$. By hypothesis, $\overline{S}$ is closed. By a lemma from Brown’s thesis, since $\overline{S}$ is closed, there exists
some continuous \( g : [0,1] \to \mathbb{R} \) such that \( g(x) = 0 \iff x \in \overline{S} \). So we can define the range of \( f, X \) as follows: \( n \in X \iff (g(2^{-n}) = 0 \iff \exists m[f(m) = n]) \). Note that \( g(2^{-n}) = 0 \) is a \( \Pi^0_1 \) formula, since we can add a for all quantifier over a dummy variable \( (\forall y g(2^{-n}) = 0) \). So we have a \( \Pi^0_1 \) formula equivalent to a \( \Sigma^0_1 \) formula \( \exists m[f(m) = n] \). This is exactly what is required for \( \Delta^0_1 \) comprehension which is an axiom within \( \text{RCA}_0 \). Therefore \( X \) exists. So \( \text{ACA}_0 \) is true by extension via Lemma 2.8

\[ \text{Theorem 6.3.4 (RCA}_0 \text{). } \text{ACA}_0 \iff (\text{All separably closed sets are closed}) \ [3] \]

\[ \text{Proof: } (\Rightarrow) \text{ Theorem 6.3.1} \]

\[ (\Leftarrow) \text{ Theorem 6.3.3} \]

From all the previous proofs, we have that, within \( \text{ACA}_0 \), all separably closed sets are closed, and that conclusion is as logically strong as \( \text{ACA}_0 \). The next section will show that \( \Pi^1_1 - \text{CA}_0 \) is required to have these definitions be equivalent, and in weaker subsystems than \( \Pi^1_1 - \text{CA}_0 \) these two definitions must be distinct.

### 6.4 Closed Sets are Separately Closed

#### 6.4.1 \( \Pi^1_1 - \text{CA}_0 \Rightarrow (\text{Closed} \to \text{Separably Closed}) \)

\[ \text{Theorem 6.4.1 (} \Pi^1_1 - \text{CA}_0 \text{). Let } (a_n, r_n) : n \in \mathbb{N} \text{ be a code for a closed set } C. \text{ Then if } C \text{ is nonempty, there exists a separably closed set } \overline{S} \text{ such that } \overline{S} = C \ [3] \]

\[ \text{Proof. Let } (b_k, s_k) : k \in \mathbb{N} \text{ be the enumeration of the following set.} \]

\[ U = \{(b, s) : \exists x \in \hat{A} [\forall n[d(x, a_n) \geq r_n] \land d(x, b) < s]\} \]

Note that this set is \( \Sigma^1_1 \) since the \( \exists x \in \hat{A} \) is a set quantifier. As \( x \) is an element of the completion \( \hat{A} \), \( x \) is a sequence, which is not a natural number but a set of natural numbers (precisely rational numbers as a subset of the natural numbers).
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Therefore by $\Sigma^1_1$ comprehension, which is an axiom of $\Pi^1_1 - CA_0$, this set $U$ can be enumerated as claimed above. Since $C$ is nonempty, $\forall k \exists x \left( x \in C \land x \in (b_k, s_k) \right)$. By construction, every ball $(b_k, s_k)$ has some element of $C$ contained within it.

The countable axiom of choice for $\Sigma^1_1$ formulas holds in $\text{ATR}_0$, so it also holds in $\Pi^1_1 - CA_0$ (since $\text{ATR}_0 \subseteq \Pi^1_1 - CA_0$). Therefore we can construct a sequence by choosing one such $x$ from each ball. Let this sequence be $S = \langle x_i : i \in \mathbb{N} \rangle$, such that $x_i \in C$ and $x_i \in (b_i, s_i)$. We claim that $\overline{S} = C$, which we will prove by a double set inclusion argument.

(1) $\overline{S} \subseteq C$

First let $x \in \overline{S}$. We will prove $x \in C$ by contradiction. Suppose $x \notin C$. Since $x \in \overline{S}$, $\forall q \in \mathbb{Q}^+ \exists i [d(x, x_i) < q]$. Since $x \notin C$, $\exists n [d(a_n, x) < r_n]$. In other words, there exists some open ball in the code for $C$ that contains $x$. By the Archimedean property, choose $q \in \mathbb{Q}^+$ such that $q < r_n - d(a_n, x)$.

We will show that $x$ is contained in one of the open balls in the code for $C$ (which will be a contradiction.) Then by the triangle inequality

$$d(a_n, x_i) \leq d(a_n, x) + d(x, x_i)$$

By the choice of $q$, $d(a_n, x) > r_n - q$, and by $x \in \overline{S}, x$ must be within a $q$ radius of $a_n$.

$$\leq r_n - q + q = r_n$$

Since $d(a_n, x_i) \leq r_n, x_i \in (a_n, r_n)$ for all $i$. Therefore $x \notin C$. This is a contradiction.

(2) $C \subseteq \overline{S}$

Now we will prove that if $x \in C$ then $x \in \overline{S}$. Let $x \in C$, where $x = \langle c_k : k \in \mathbb{N} \rangle$. Let $q \in \mathbb{Q}^+$ be arbitrary. We want to show that some term in the $x_i$ sequence is within a radius of $q$ from $x$. Choose $n$ such that $2^{-n} < q$. Consider $(c_{n+1}, 2^{-(n+1)})$. By the definition of real numbers, $x \in (c_{n+1}, 2^{-(n+1)})$.

Since $x \in C$, by definition $\forall n [d(a_n, x) \geq r_n]$. (x is not contained in any of the open sets in the code $\langle a_k, n_k \rangle$.) Therefore $(c_{n+1}, 2^{-(n+1)}) \in U$, so $(c_{n+1}, 2^{-(n+1)})$ is one of the open
sets in the code for $U$, $(c_{n+1}, 2^{-(n+1)}) = (b_k, s_k)$ for some $k$. By construction of the code of $U$, $(b_k, s_k)$ contains some element of $\langle x_n \rangle_{n \in \mathbb{N}}$. In other words, $\exists i [d(c_{n+1}, x_i) < 2^{-(n+1)}]$. Fix this $i$. Therefore, by the triangle inequality:

$$d(x, x_i) \leq d(x, c_{n+1}) + d(c_{n+1}, x_i)$$

$$\leq 2^{-(n+1)} + 2^{-(n+1)}$$

The first substitution is due to the fact that $x \in (c_{n+1}, s2^{-(n+1)})$ and the the second substitution is because $i$ is the fixed value such that $d(c_{n+1}, x_i) < 2^{-(n+1)}$.

$$= 2^{-n} < q$$

Therefore for arbitrary $q$, there exists some $i$ such that $d(x, x_i) < q$. Hence $x \in \overline{S}$, as desired. Therefore $\overline{S} = C$, and the theorem is true.

6.4.2 Where did we Need $\Pi^1_1 - CA_0$ in this Proof?

Similarly for the proof that separately closed sets are closed within $ACA_0$, the key use of $\Pi^1_1 - CA_0$ was in the construction of the set $U$. We also take advantage of $\Sigma^1_1$ choice to choose the elements of our sequence that defines $\overline{S}$. Otherwise, the proof applies fairly standard techniques from analysis.

6.4.3 $RCA_0 + (\text{Closed } \Rightarrow \text{ Separably Closed}) \Rightarrow ACA_0$

We will use some of the theorems and definitions presented in the section of Chapter 4 about $\Pi^1_1 - CA_0$ to build our full proof of this statement. As for the previous proof, we will also rely on some familiarity with $2^{\mathbb{N}}$ from Chapter 5.

First we will show that closed sets being separably closed implies $ACA_0$. This bootstrapping approach will both serve as a proof of concept (since $ACA_0$ is a strictly weaker system of $\Pi^1_1 - CA_0$), but allow us to use an axiom of $ACA_0$ in the main proof.
We will be working very closely with the Cantor Space in this proof, which we introduced briefly in Chapter 4 and explored more deeply in Chapter 5. We will first propose a new relation $c_0$ on elements of the Cantor space that will help clarify certain arguments of the proof, and prove an important lemma.

**Definition 6.4.1 (Extends with 0).** For $\sigma \in 2^{<\mathbb{N}}$ and $x \in 2^{\mathbb{N}}$, $\sigma \prec_0 x$ if $\sigma \prec x$ and $\exists k > \ell(\sigma)$ such that $x_k = 0$. So $\sigma \prec_0 x$ if the two sequences agree up to the length of $\sigma$ and $x$ does not continue to be an infinite series of ones.

The next lemma explains how the $c_0$ relation corresponds to the metric, (similarly to $c$).

**Lemma 6.4.2.** If $\ell(\sigma) \leq \ell(x)$ and $d(\sigma, x) < \frac{1}{2^{|\ell(\sigma)|}} \Leftrightarrow \sigma \prec_0 x$

**Proof.** $(\Rightarrow)$ Suppose $d(\sigma, x) < \frac{1}{2^{|\ell(\sigma)|}}$. Then $\sigma[k] = x[k]$ for all $k \leq \ell(\sigma) + 1$ so surely $\sigma \prec x$. Since $\sigma[\ell(\sigma) + 1] = x[\ell(\sigma) + 1]$, then $\sigma$ and $x$ agree on the first place after the length of $\sigma$. Since $\sigma[\ell(\sigma) + 1] = 0$ by the extension required by the metric, it must be that $x[\ell(\sigma) + 1] = 0$, so $\sigma \prec_0 x$.

$(\Leftarrow)$ Suppose $\sigma \prec_0 x$, then $\sigma$ and $x$ agree up through the length of $\sigma$, therefore $d(\sigma, x) < \frac{1}{2^{|\ell(\sigma)|}}$ and $x$ must have some 0 digit after the length of $\sigma$.

$$d(x, \sigma) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} d(x_k, \sigma_k)$$

$$= \sum_{k=\ell(\sigma)+1}^{\infty} \frac{1}{2^{k+1}} d(x_k)$$

$$\leq \sum_{k=\ell(\sigma)+1}^{\infty} \frac{1}{2^{k+1}} < \frac{1}{2^{\ell(\sigma)+1}}$$

This $c_0$ operation allows us to more easily compare elements of $2^{<\mathbb{N}}$ to elements of the Cantor Space, and avoid problems that arise when we have an infinite string ending in repeated 1's. $c_0$ is not defined by Brown or Simpson, but we include it as we find it clarifies the proof.
Beyond the introduction of $c_0$, we have also made some minor alterations from the original proof from Brown in the construction of the relevant closed sets. The approach is very similar, but we define some of the sets slightly differently in order to avoid an error. We will explain this change in detail after we present our version.

**Theorem 6.4.3 (RCA$_0$).** Closed sets are separably closed $\rightarrow$ ACA$_0$ (adapted from [3])

**Proof.** As shown in previous sections, $2^{<\mathbb{N}}$ is a countable set, so we can enumerate the elements of $2^{<\mathbb{N}}$ and represent each element $\sigma$ with a unique natural number, so that $\langle \sigma_n \rangle$ enumerates $2^{<\mathbb{N}}$. We will prove this theorem via Lemma 4.3.1. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a one to one total function. We will show that the range of this function exists by $\Delta^0_1$ comprehension. In general, the range function is described by a $\Sigma^0_1$ formula $n$ is an element of the range if and only if $\exists m(f(m) = n)$. So all that is required is to show that there is a $\Pi^0_1$ description for the range set. However, it is convenient to work with the complement of the range instead, and show that this complement has a $\Sigma^0_1$ description.

We will show that $\exists X(n \in X^c \leftrightarrow \exists m(f(m) = n))$.

Let $O_j = \cup_{m=j}^{\infty}(\sigma_f(m), 2^{-f(\sigma_f(m))})$. Let $C_j$ denote the closed set encoded by $O_j$. Note that our $O_j$ sets correspond to our range set and our $C_j$ sets correspond to the complement of the range. For each $f(m)$, we are taking the corresponding $\sigma$ and excluding everything that extends it. So our closed $C_j$ avoids elements in the range of $f$, as well as points close to elements in the range, elements that extend range elements with 0.

However the collection $O_j$ is a subset of $2^\mathbb{N}$, not $\mathbb{N}$ as desired. So we will essentially recover the subscripts of the $\sigma_n$ in $O_j$ by taking advantage of the $c_0$ relation.

Since $C_j$ is closed, by hypothesis, $C_j$ is separably closed and there exists some $S_j = \langle x_{j,k} : k \in \mathbb{N} \rangle$ such that $S_j = C_j$.

Consider the set $X$ defined as follows:

$$n \in X \leftrightarrow \exists (j,k) \forall m < j[(n \neq f(m)) \land (\sigma_n \sim_0 x_{j,k})]$$

$X$ has a $\Sigma^0_1$ formula. Now, we argue $X$ is exactly the complement of the range of $f$.

**Lemma 6.4.4.** $n \in X \leftrightarrow \forall m(f(m) = n)$
Chapter 6. Closed and Separately Closed Sets

(\Leftarrow) First, suppose \( n \) is not in the range of \( f \) and \( \forall m \) we have that \( f(m) \neq n \).

Since \( f \) is one to one, there are only finitely many possible \( \sigma_{f(m)} \) such that \( \sigma_{f(m)} \subseteq \sigma_n \).

So there are only finitely many possible \( \sigma_{f(m)} \) such that \( d(\sigma_n, \sigma_{f(m)}) < \frac{1}{2^{\ell(\sigma_{f(m)})}} \). Therefore \( \sigma_n \in O_j \) for at most finitely many \( j \). So \( \sigma_n \in C_j \) for some large enough \( j \). Thus, for all \( r \in Q^+ \), \( \exists k \) such that \( d(\sigma_n, x_{j,k}) < r \). Therefore, choose \( r = \frac{1}{2^{\ell(\sigma_n)}} \). Then there exists a \( k \) such that \( d(\sigma_n, x_{j,k}) < \frac{1}{2^{\ell(\sigma_n)}} \) which by Lemma 6.4.2 implies that \( \sigma_n \subseteq x_{j,k} \) so \( n \in X \).

(\Rightarrow) Now, suppose \( n \in X \). We want to show that \( \sigma_n \in C_j \) for some \( j \) and there exists some \( x_{j,k} \) such that \( \sigma_n \subseteq x_{j,k} \). We will prove this by contradiction. Suppose \( \exists m \) such that \( f(m) = n \). By definition of \( X \), \( m \geq j \). However, since \( \sigma_n = \sigma_{f(m)} \), we have that \( \sigma_{f(m)} \subseteq x_{j,k} \).

So \( x_{j,k} \) agrees with \( \sigma_{f(m)} \) up to its length and then \( x_{j,k} \in (\sigma_{f(m)}, 2^{-\ell(\sigma_{f(m)})}) \subseteq O_j \). This is a contradiction to the fact that \( x_{j,k} \in C_j \). Therefore by Lemma 2.8, since the range set of \( f \) exists for all \( f \), \( ACA_0 \) holds.

6.4.4 What Changes did we Make to Brown’s Original Proof?

We defined the set \( X \) slightly differently. In Brown’s original proof, he defined \( X \) as follows [3].

\[
n \in X \iff (\sigma_n \in C \iff \exists k(\sigma_n \subseteq x_k))
\]

However, with this construction, it is not true that \( n \in X \iff \sigma_n \in C \).

If \( n \in X \), \( \sigma_n \) may be in finitely many \( (\sigma_{f(m)}, 2^{-\ell(\sigma_{f(m)})}) \), and therefore it is possible that \( \sigma_n \notin C \). There are finitely many \( \sigma_k \) such that \( lh(\sigma_k) < lh(\sigma_n) \). Since our function \( f \) is one to one, there are at most finitely many \( \sigma_{f(m)} \) such that \( lh(\sigma_{f(m)}) \geq lh(\sigma_n) \).

Then \( \sigma_n \in X \) but \( \sigma_n \notin C \).

The following counterexample demonstrates the preceding problem.

**Example:** Let \( \sigma_1, \ldots, \sigma_{16} \) enumerate all the elements of \( 2^{\mathbb{N}} \) with length 4 or less. We arbitrarily let \( \{\sigma_n \mid n \geq 17\} \) enumerate the remaining strings in \( 2^{\mathbb{N}} \). Let \( f(1) \in \{1, \ldots, 16\} \) such that \( \sigma_{f(1)} = 01 \). Let \( \phi \) be a computably enumerable and 1 to 1 function that is not computable. Let \( f(k) = \phi(k) + 16 \) for all \( k > 1 \). By construction, we have that \( \sigma_{f(1)} = 01 \) but \( \sigma_{f(m)} \) for all \( m > 1 \) must be a string of length greater than 4. Then for \( \sigma_j = 011 \),
$j \notin f(m)$ for any $m > 1$. But 011 extends 01, so $\sigma_j \in (\sigma_{f(1)}, 2^{-f(\sigma_j)})$. This is a contradiction, demonstrating the flaw that we corrected in our proof.

6.4.5 \(\text{RCA}_0 + (\text{Closed} \rightarrow \text{Separably Closed}) \Rightarrow \Pi^1_1 \land \text{CA}_0\)

Now we continue to the goal of this section: showing that all closed sets being separably closed implies $\Pi^1_1 \land \text{CA}_0$. Similarly to our proofs of $\text{ACA}_0$, we will prove $\Pi^1_1 \land \text{CA}_0$ via a bridge theorem, specifically Lemma 4.4.2.

**Theorem 6.4.5 (RCA$_0$).** Closed sets are separably closed $\Rightarrow \Pi^1_1 \land \text{CA}_0$ [3]

**Proof.** Let $\langle T_n \rangle$ be a sequence of trees. We will construct a set $X$ containing the indices identifying trees containing paths. Let $T$ be a tree where $\sigma \in T \iff \langle \sigma(1), \sigma(2), \ldots, \sigma(\text{lth}({\sigma} - 1) \in T_{\sigma(0)}$. Essentially, we construct a tree with the paths from the trees in $\langle T_n \rangle$, using the first node in each sequence to denote which tree contains the sequence and "starting" the terms of the sequence at index 1.

Let $\langle \sigma_n : n \in \mathbb{N} \rangle$ enumerate $\{\sigma \in \mathbb{N}^{<\mathbb{N}} \mid \sigma \notin T\}$. This is possible since $\mathbb{N}^{<\mathbb{N}}$ is countable.

Let $C$ be coded by $\langle (\sigma_n, 2^{-\text{lth}({\sigma}_n)}) : n \in \mathbb{N} \rangle$. So $x \in C \iff \forall n (\exists i < \text{lth}(\sigma_n)[\sigma_n(i) \neq x(i)]$.

We claim that $C$ is the exact set we are looking for, in other words, $x \in C \iff x$ is a path through $T$. We will prove this by contradiction. Suppose $\exists x \in C$ such that $x$ is not a path. Then, for some $n \in \mathbb{N}$, $x[n] \notin T$, which implies $x[n] = \sigma_m$ for some $m$. So $\forall i < n$, $x(i) = \sigma_m(i)$, so $x \notin C$, which is a contradiction.

For the other direction, let $x$ be a path in $T$. $\forall n, x[n] \in T$, therefore $\forall n, x[n] = \sigma_m$ by how we defined $\sigma_m$ as the "complement" of $T$. So for all $\sigma_m$, there exists some index $i < n$ such that $x(i) \neq \sigma_m(i)$, they have to be different at some index less than the index of the coarseness of $C$. Therefore $x \in C$.

Let $\langle x_n \rangle$ be the code for $S$. By assumption, $S = C$. Define the desired $X$ with the following formula.

$$n \in X \iff \forall i \exists k \forall j < i[lh(x_k) \geq i \land x_k(0) = n \land x_k[j] \in T]$$

Let us break down this statement. $\exists k \forall j < i[lh(x_k) \geq i \land x_k(0) = n \land x_k[j] \in T]$ says there exists a sequence $x_k$ that is longer than upper bound $i$, has $n$ as its first node, and is
contained in the tree $T$ for all strings shorter than the bound $i$. Such an $x_k$ would be an infinite path through tree $T_n$ (as its first node is $n$), if this was true for all upper bounds $i$, hence the quantifier $\forall i$. So $T_n$ has a path $x_k$ and $n$ should be an element of our set $X$. Since the description of $X$ is an arithmetic statement, containing only number quantifiers, $X$ exists in $ACA_0$. We will show that $n \in X$ if and only if there is a path through $T_n$, in other words, that $X$ is the desired set. 

First, suppose $n \in X$. Then for all $i$ there is a $k$ such that $lh(x_k) \geq i$ and $x_k[i-1] \in T$ (where $i-1$ is a particular $j$ that we’ve chosen that is less than $i$.) $(x_k)$ for the $k$ corresponding to each $i$ is an infinite, finitely branching tree. This is because each $x_k$ denotes some fragment of a single path. Then we can apply Konig’s lemma to find a path through $x_k$ which is a path through $T$. From Konig’s lemma (an axiom of $ACA_0$), we have a path $x$ through $T$, making $(x(1), x(2), \ldots)$ a path through $T_n$.

Next, suppose $x$ is a path through $T_n$. We will show $n \in X$. $(n, x(0), \ldots) = (n)^\forall x \in T$ is a path in $T$. So $(n)^\forall x \in C$ by definition of $C$. So $\forall i \exists k(d((n)^\forall x, x_k) < 2^{\neg i}]$. Therefore we have $\forall i \exists k \forall j < i[x_k(0) = n \wedge x_k(j) = (n)^\forall x(j)]$. So $\forall i \exists k \forall j < i[lh(x_k) \leq i \wedge x_k(0) = n \wedge x_k[j] \in T]$ so $n \in X$. Thus $n$ satisfies the definition of $X$ and $n \in X$. So by lemma 4.3.2, we have that $\Pi_1^1 - CA_0$ follows.

**Theorem 6.4.6 (RCA_0).** $\Pi_1^1 - CA_0 \iff \text{Closed sets are separably closed.}$ [3]

**Proof:** ($\Rightarrow$) Theorem 6.4.1

($\Leftarrow$) Theorem 6.4.5

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### 6.5 Conclusion

Now we have shown that the two definitions of closed set utilized in standard analysis can only be equivalent under $\Pi_1^1 - CA_0$ or a strictly stronger axiom system. While most analysis assumes all of second order arithmetic which brings along all the axioms of $\Pi_1^1 - CA_0$, this nuance of definitions unveils an interesting dynamic distinguishing how analysts use closed sets. Interestingly enough, if we assume that the our metric space $\hat{A}$ is compact, then the equivalence holds in $ACA_0$ [3].
Chapter 7

Baire Category Theorem

Now we will show how the duality of closed sets in weak systems of axioms carries over to other theorems in Real Analysis. The Baire Category Theorem is a standard theorem from Real Analysis that is a stepping stone for proving other theorems including the Banach-Steinhaus Theorem and the Open Mapping Theorem. The Baire Category Theorem asserts that an intersection of open dense subsets is also dense. However, in the context of Reverse Mathematics, the duality of closed sets adds some complexity and we will explore that complexity here.

7.1 Definitions

Here are a few definitions and a lemma from Analysis that we will utilize in this section.

Definition 7.1.1. The interior of a set $S$ is the largest open set $U$ such that $U \subseteq S$. [13]

Definition 7.1.2. The closure of a set $S$ (denoted as $\overline{S}$) is the smallest closed set $C$ such that $S \subseteq C$. [13]

Definition 7.1.3. A subset $X$ is dense in metric space $A$ if all nonempty open subsets of $A$ contain some point of $X$. [13]

Definition 7.1.4. A subset $X$ is nowhere dense if its interior is empty. [13]
7.1.1 A Subset is Dense if and Only if its Complement is Nowhere Dense.

**Lemma 7.1.1.** A subset $S \subseteq \hat{A}$ where $\hat{A}$ is a complete separable metric space is dense if and only if $S^c$ is nowhere dense. \([13]\)

*Proof.* We will prove this by contradiction. Suppose $S$ is dense and $S^c$ is not nowhere dense. Then there exists some nonempty open set $U$ such that $U \subseteq S^c$. However, then $U$ is an open set that does not intersect $S$, so then $S$ is not dense, resulting in a contradiction. \(\blacksquare\)

7.2 Traditional Proof of the Baire Category Theorem

First, let us consider the standard proof of the Baire Category Theorem from Real Analysis.

**Theorem 7.2.1 (BCT 1).** Let $\hat{A}$ be a complete separable metric space. If $(U_n : n \in \mathbb{N})$ is a sequence of open dense subsets of $\hat{A}$, then $\bigcap_{n=1}^\infty U_n$ is dense in $\hat{A}$. \((13), \text{pg. 211}\)

*Proof.* To prove $\bigcap_{n=1}^\infty U_n$ is dense in $\hat{A}$, consider some arbitrary nonempty open set $X_0$. We must show that $(\bigcap_{n=1}^\infty U_n) \cap X_0$ is nonempty.

Since $U_1$ is dense in $\hat{A}$, there exists some $x_1 \in X_0$ that is also contained in $U_1$. Choose a $r_1 \in \mathbb{Q}^+$ where $r_1 < 1$ and $(x_1, r_1) \subseteq X_0 \cap U_1$. We know such an $r_1$ exists, since $X_0 \cap U_1$ is an intersection of two open sets and is therefore open. Let $X_1 = (x_1, r_1)$. We can continue this process as follows, letting $X_n = (x_n, r_n)$ where $r_n \leq \frac{1}{n}$, $X_n \subseteq X_{n-1} \cap U_n$, $x_n \in X_{n-1}$ and $x_n \in U_n$. All of these steps are possible since $U_n$ is dense in $\hat{A}$ for all $n$, and the intersection of $X_{n-1}$ and $U_n$ is always open. Therefore, $(\overline{X}_n)$ is a nested sequence of closed sets where $\overline{X}_n \subseteq X_0$ for all $n$ and $\overline{X}_n \subseteq \bigcap_{i=1}^n U_i$. From the Cantor Intersection theorem (see [13]), as $\bigcap_{n=1}^\infty \overline{X}_n$ is an intersection of nested closed intervals in a complete metric space, $\bigcap_{n=1}^\infty \overline{X}_n$ is nonempty. Let $x \in \bigcap_{n=1}^\infty \overline{X}_n$. Then $x \in \bigcap_{n=1}^\infty U_n$, and $x \in X_0$. Therefore $x \in \bigcap_{n=1}^\infty U_n \cap X_0$ and the intersection is nonempty. Therefore $\bigcap_{n=1}^\infty U_n$ is dense in $\hat{A}$ \(\blacksquare\)
7.3 Baire Category Theorem 1

In reverse math, we have two distinct versions of the Baire Category theorem, one version BCT 1 for closed sets, and another version BCT 2, for separably closed sets. First we will consider the proof of BCT 1. We will need two concepts from RCA0 to complete the proof: primitive recursion and minimization.

7.3.1 Primitive Recursion

Primitive Recursion allows us, given some function $g$, to construct a new function $h$ that loops through values of $g$ recursively. Simpson refers to minimization as the fact that the universe of functions is "closed under primitive recursion." ([15], pg. 69). Here is the formal statement of primitive recursion from Simpson.

**Lemma 7.3.1 (RCA0).** Given $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^{k+2} \to \mathbb{N}$, there exists a unique $h : \mathbb{N}^{k+1} \to \mathbb{N}$ where

\[
h(0, n_1, \ldots, n_k) = f(n_1, \ldots, n_k)
\]

\[
h(m + 1, n_1, \ldots, n_k) = g(h(m, n_1, \ldots, n_k), m, n_1, \ldots, n_k)
\]

**Proof.** For proof, see [15], pg. 69-70.

7.3.2 Minimization

Minimization allows us to simplify a function by using the least number that satisfies a given quality. Simpson refers to minimization as the fact that "the universe of functions is closed under the least number operator." ([15], pg. 70). This is a weak operation, since we can imagine an algorithm that starts with testing $n = 1$ and continues through the natural numbers until it encounters a natural number that satisfies the function. If we know there is such a number, we can find the least of these numbers with a finite number of computing steps. Here is the formal statement of minimization from Simpson.
Lemma 7.3.2 (RCA₀). Given a function \( f : \mathbb{N}^{k+1} \rightarrow \mathbb{N} \) such that for all \( (n_1, \ldots, n_k) \in \mathbb{N}^k \) there exists an \( m \in \mathbb{N} \) such that \( f(m, n_1, \ldots, n_k) = 1 \), then there exists \( g : \mathbb{N}^k \rightarrow \mathbb{N} \) defined by 
\( g(n_1, \ldots, n_k) = \text{the least } m \text{ such that } f(m, n_1, \ldots, n_k) = 1. \) [15]

Proof. For proof, see [15], pg.70.

7.3.3 Proof of Baire Category Theorem 1 in RCA₀

This proof follows very closely to the approach in the traditional proof presented earlier. However, we must predictably justify a few steps in more detail.

Theorem 7.3.3 (BCT 1 - (RCA₀)). Let \( \hat{A} \) be a complete separable metric space and let \( \langle U_n : n \in \mathbb{N} \rangle \) be a sequence of open dense subsets of \( \hat{A} \). For all nonempty open \( U \subseteq \hat{A} \), \( \exists x \in \hat{A} \) such that \( x \in U \) and \( x \in U_n \) \( \forall n \in \mathbb{N} \). (In other words, \( \cap U_n \) is dense in \( \hat{A} \)). [4]

Proof. To show \( \cap U_n \) is dense in \( A \), given an open set \( U \), we must find some \( x \in \hat{A} \) such that \( x \in U \) and \( x \in \cap U_n \). It will suffice to show that for an arbitrary basic open set \( (c, j) \), there exists an \( x \in \hat{A} \) such that \( x \in (c, j) \) and \( x \in \cap U_n \).

We will define \( x \) recursively. Since \( U_o \) is dense, \( \exists (a_0, r_0) \subseteq U_0 \) where \( (a_0, r_0) \leq (c, j) \) and \( r_0 \leq \frac{1}{2} \).

We will continue building nested open balls in this manner. In general, we will let \( (a_k, r_k) \) denote the open ball that is a subset of \( (a_{k-1}, r_{k-1}) \cap U_k \) where \( r_k < 2^{-k-1} \). We will show this process is valid within RCA₀ and helps us construct our \( x \).

Let \( \varphi(k, a, r, b, s) \) express the following statement:
\( (a, r) \in A \times \mathbb{Q}^+ \) and \( (b, s) \in A \times \mathbb{Q}^+ \) such that \( (b, s) \subseteq U_k \) and \( s \leq 2^{-k-1} \).

We have a pair of open rational balls where the ball about \( b \) is contained in \( U_k \) and has a smaller and smaller radius as \( k \) increases.

More formally, \( \varphi(k, a, r, b, s) \) is a \( \Sigma^0_1 \) formula in the form:

\[
(\exists a \in A(\exists r \in \mathbb{Q}^+((\exists b \in A((\exists s \in \mathbb{Q}^+(b, s) \subseteq U_k \wedge s \leq 2^{-k-1}))))))
\]

Note that \( k \) is our free variable of this formula. This statement is \( \Sigma^0_1 \) since all of the existential quantifiers can be reduced to \( \exists (k, a, r, b, s) \in \mathbb{N} \times A \times \mathbb{Q}^+ \times A \times \mathbb{Q}^+ \). Since \( \varphi \) is
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a $\Sigma^1_0$ formula it can be expressed as $\exists n \theta(k,a,r,b,s,n)$ where $\theta$ is $\Sigma^0_3$ formula. This is by definition of a $\Sigma^1_0$ statement. Since all of the $U_k$ are dense, for all $(k,a,r)$, we can always find some $(b,s)$ that is contained in $U_k$ and has small radius as described above, a $(b,s)$ such that $\varphi(k,a,r,b,s)$.

By minimization we can "choose" the least $(n,b,s)$ such that $\theta(k,a,r,b,s,n)$ holds and build a function $f : \mathbb{N} \times A \times \mathbb{Q}^+ \to \mathbb{N} \times A \times \mathbb{Q}^+$. Minimization allows us to justify the choice of the triple so that $f$ is well defined.

Let $f(k,a_k, r_k) = (n_k, a_{k+1}, r_{k+1})$. This follows the construction of the nested balls described above.

By primitive recursion, we have a function $g : \mathbb{N} \to A \times \mathbb{Q}^+$ such that $g(0) = (a_0, r_0)$ and $\forall k \in \mathbb{N}, g(k+1) = (a_{k+1}, r_{k+1})$, where $(a_{k+1}, r_{k+1})$ is the open ball constructed within $(a_k, r_k) \cap U_{k+1}$ as described in the beginning of the proof. Primitive recursion allows us to remove the variable $n$ and step through the $U_k$ sets as desired.

Therefore $\varphi(k,a_k, a_{k+1}, r_{k+1})$ is true for all $n$.

The sequence of the $a_k$ now recursively define a real $x$, as $d(a_k, a_{k+i}) < 2^{-k}$ for all $k$ and $i$, by our definition of real numbers. So $x = \{a_n : n \in \mathbb{N}\} \in \hat{A}$. By construction $x \in \cap_{k=1}^{\infty} U_k$ and $d(x, c) < j$ since $\lim_{k \to \infty} 2^{-k} = 0 < j$, which implies that $x \in (c, j)$.

7.3.4 Comparing the Traditional and RCA$_0$ proofs of the Baire Category Theorem 1

Both proofs build a sequence of nested open intervals. We use minimization to justify our choice of the new open ball. We use primitive recursion to show we can repeat this process infinitely. Our definition of real number allows us to make the conclusion directly without an appeal to the Cantor Intersection Theorem necessitating closed sets.

7.3.5 Corollary: Baire Category Theorem for Closed Nowhere Dense Sets.

It will be convenient to translate this statement about open dense sets to a statement about closed nowhere dense sets. As such we prove the following corollary.
Corollary 7.3.1 (RCA₀). Let \( \{ E_n : n \in \mathbb{N} \} \) be a sequence of closed nowhere dense subsets of \( \hat{A} \).
Given an nonempty open set \( U \) of \( \hat{A} \), \( \exists x \in U \) such that \( x \notin \bigcup_n E_n \). In other words, the union of nowhere dense closed sets is not the entire set \( \hat{A} \). [4]

Proof. Let \( U \subseteq \hat{A} \) be given. The codes for each of the \( E_n \) represent some open set \( O_n = E_n^c \).
As these open \( O_n \) are the complement of nowhere dense \( E_n \), all of the \( O_n \) are dense.
Note that \( \bigcup_n E_n = \bigcap_n O_n \). Therefore, by the Baire Category Theorem 1, \( \bigcap_n O_n \) is dense in \( \hat{A} \). Therefore there exists some \( x \in U \) such that \( x \in \bigcap_n O_n \). Then \( x \in U \) but \( x \notin \bigcap_n O_n = \bigcup_n E_n \) as desired. ■

This different format of the Baire Category Theorem will be very relevant in the proof of BCT 2.

7.4 Baire Category Theorem 2

Now what if the open sets in the statement of the Baire Category Theorem were specifically separably open sets? This is the BCT 2, and according to Brown and Simpson's paper, having this true of separably open sets requires a proof over ACA₀ instead of RCA₀.

Theorem 7.4.1 (Baire Category Theorem 2 (ACA₀)). Let \( \hat{A} \) be a complete separable metric space and let \( \{ U_n : n \in \mathbb{N} \} \) be a sequence of separably open dense subsets of \( \hat{A} \). If this sequence codes a non-empty open set, \( \exists x \in \hat{A} \) such that \( x \in U \) and \( x \in U_n \ \forall n \in \mathbb{N} \). (In other words, \( \bigcap U_n \) is dense in \( U \)). [4]

Proof. Each separably open \( U_n \) has a code \( (x_{n,i} : i \in \mathbb{N}) \). In ACA₀, we can create a code and closed set \( E_n \) such that \( E_n = \langle n_i \rangle \). \( E_n \) is the closed set that is identical to the separably closed set that is the complement of \( O_n \). Therefore by the previous Corollary 7.3.1 (valid in RCA₀), for any open set \( U \), we have an \( x \notin \bigcup_n E_n \), so the proposition holds. ■
Chapter 7. Baire Category Theorem

7.5 Further Results

There has been some further work studying the Baire Category Theorem within Reverse Mathematics and Computability Theory that is beyond the scope of this paper, but we will describe those results briefly. In the previous section, we proved BCT2 in ACA₀. But, keeping in mind the perspective of Reverse Mathematics, we should ask: how do we know we can’t prove this theorem in a weaker subsystem? Does the proof rely on the fact that separably open sets are open or is some other weaker structure at the heart of the proof?

It turns out that ACA₀ is not the weakest subsystem in which BCT2 can be proved. Brown and Simpson, in their paper on the Baire Category Theorem, show that BCT2 is not provable in WKL₀ (and thus the weaker system RCA₀.) [4]. Therefore, considering the Big 5, we may suspect that BCT2 is equivalent to ACA₀, as our previous theorems have followed that similar pattern. However, the exact strength of BCT2 is a bit more complex. Later in the same paper, Brown and Simpson introduce a new subsystem called RCA₀⁺. This subsystem contains the axioms of RCA₀ plus a new principle concerning sequences of arithmetically defined dense subsets. RCA₀⁺ is sufficient to prove BCT2, as well as a stronger, Baire Category-like property. Brown and Simpson wondered whether RCA₀⁺ was the minimal system of axioms for proving BCT2 [4].

Mytilinaios and Slaman answered this question by restating BCT2 with a recursion theoretic perspective. Then they showed that there is a model of this restatement of BCT2 that is not a model of RCA₀⁺. This result means that BCT2 does not imply RCA₀⁺[0] [10].

In 2012, Simpson built on his work with Brown on the Baire Category Theorem and found that, if RCA₀⁺, a subsystem weaker than RCA₀ is used as a base system, then BCT1 is equivalent to RCA₀ [14]. So the BCT1 is a very essential property of complete
metric spaces with our construction. This led Simpson and many other reverse mathematicians to wonder if $\text{RCA}_0^*$ or an even weaker subsystem would be more appropriate as our base system.

Now, when considering theorems from Analysis that use the Baire Category Theorem, in their proofs, the question arises: Which version of the Baire Category is sufficient? If a theorem only uses $\text{BCT}_1$, then it could be possible for the new theorem to be provable in $\text{RCA}_0$. However, if the new theorem requires the strength of $\text{BCT}_2$, then $\text{RCA}_0$ will not be sufficient.

Further work has been done considering the Reverse Mathematics of certain modifications of the Baire Category Theorem including more work in the paper by Mytilinaios and Slaman we mentioned before [10]. Brattka, Hendtlass, and Kruezer also considered the Weihrauch reducibility, which is related to but distinct from logical equivalence, of different versions of the Baire Category Theorem (including $\text{BCT}_1$ and $\text{BCT}_2$) [1].
Bibliography


Brattka, Hendtlass, and Kreuzer consider the Weihrauch reducibility of different forms of the Baire Category Theorem. While Reverse Mathematics concerns whether mathematical statements are logically equivalent, Weihrauch Reducibility studies whether a solution to a mathematical statement can allow the construction a solution to another mathematical statement. For further explorations of similar questions to Reverse Mathematics with even more connections to Computer Science, this resource may be a good next step.


Brown's doctoral thesis, completed while working with Simpson at Penn State University, was one of the first treatments of Reverse Mathematics while it was under development. Brown discusses a variety of topics throughout Real and Functional Analysis.


This paper, published before Simpson's Subsystems of Second Order Arithmetic, is a central resource for our work in this paper. Brown presents the definitions of closed
and separably closed sets and the proofs involving their equivalence. He also explores these equivalences for compact metric spaces, which we didn't describe in this paper. The work of this paper is also included in Brown's doctoral thesis [2].


Building on the characterizations of closed sets from Brown's "Notions," Brown and Simpson explore the strength of the Baire Category Theorem within Reverse Mathematics.


These authors explain the history of the development of Non-Euclidean geometry, focusing on Hyperbolic geometry.


Davis presents a nice overview of Gödel's Incompleteness Theorems and how their publication affected the study of the foundations of mathematics.


Dean and Walsh provide an extremely detailed history of the development of Reverse Mathematics. They describe connections between the main subsystems and their origins from before Reverse Mathematics was formalized by Simpson.


In 1900, Hilbert presented 23 open problems to the International Congress of Mathematicians at Paris. This paper is a translated version of his talk to the International Congress, detailing the problems as well as his philosophy for why he deemed them
so critical for future research. Hilbert's problems made such a large impact on the field of mathematics, that any mathematician would benefit from some familiarity with Hilbert's problems and how they helped progress the field to where it is today.


Hirst explores different ways to construct the real numbers including open cuts, Dedekind cuts, and decimal approximations. For further exploration on other approaches we could have used to construct the reals, and why rapidly Cauchy sequences are a particularly appealing choice, this is a great resource.


Mytilinaios and Slaman explore whether or not $\mathsf{RCA_0}$ is the weakest subsystem that can prove $\mathsf{BCT2}$, as claimed in Brown and Simpson's paper. Their approach relies on recursion theory and model theory, and may be a good next stop for those interested in those areas.


O'Connor and Robertson provide a summary of the contributions of Beltrami, including his work that connected Non-Euclidean geometry to the study of geodesics on surfaces with negative curvature.


Propp is a very readable primer on the equivalences of many theorems of completeness from Real Analysis. However, Propp is not a Reverse Mathematician by training and its unclear exactly what basic axioms he employs. This leads to some results that
do not match results from Reverse Mathematics. With that caution in mind, Propp's paper is an excellent resource for non-logicians and less experienced students to start exploring these ideas. Propp uses the variety of possible axioms of completeness to consider how to make introductory courses in Real Analysis and Calculus more well-rounded for students. These questions are also very interesting and would be of particular interest to current or future teachers.


There are many textbooks for the study of Real Analysis. We used Royden's text as our reference.


In this paper, Simpson ties off a loose end from his paper with Brown on the Baire Category Theorem [4], proving that $BCT1$ is equivalent to $RCA_0$ over $RCA^*_0$, a strictly weaker subsystem than $RCA_0$.


Simpson's book is the definitive resource on Reverse Mathematics. In this book, Simpson presents the language of reverse mathematics and explores each subsystem in great detail. In the later half of the text, he discusses the model theoretic aspects of reverse mathematics. For those inexperienced with higher level work in logic, Simpson's book can be difficult to parse. However, an understanding of the concepts and approaches that Simpson presents will lead to a deep mastery of Reverse Mathematics.


Stillwell's book is a fantastic treatment of Reverse Mathematics for a more general audience than Simpson's *Subsystems*. While Stillwell may leave out some details, Stillwell provides great insights on the concepts of Reverse Mathematics, including
connections to Computability Theory. Stillwell is a great complement to Simpson for learning Reverse Mathematics without extensive background in mathematical logic.


Teismann's paper discusses many theorems of Real Analysis and their relationship to completeness. In particular, Teismann provides extra details on the use of the negation of the cut property to prove the negation of the Intermediate Value Property, within some set of axioms that is not specified. This paper has a similar drawbacks to Propp's paper [12], but also offers similar benefits, and so would be worthwhile to consult alongside Propp's treatment.
Appendix A

Curriculum Vitae

GRADUATE
University of Connecticut
Storrs, Connecticut
Ph.D. Candidate, Mathematics

Wake Forest University
Winston-Salem, North Carolina
M.A., Mathematics, May 2020

POSTBACCALAUREATE
Smith College
Northampton, Massachusetts
PostBaccalaureate Certificate, Mathematics, May 2018

UNDERGRADUATE
College of the Atlantic
Bar Harbor, Maine

Honors and Awards

• Teaching Assistantship (University of Connecticut, 2020-2021)
• Teaching Assistantship (Wake Forest University, 2018-2020)
• Alumni Travel Award (Wake Forest University, 2019)
• Center for Women in Mathematics Fellowship
  (Smith College/National Science Foundation, 2017-2018)
• Presidential Scholarship (College of the Atlantic, 2012-2016)

Conference Attendance

• Reverse Mathematics of Combinatorial Principles
  Casa Matemática Oaxaca (BIRS) (Oaxaca, Mexico - Sep. 15-20, 2019)
• Triangle Lectures in Combinatorics
  Wake Forest University (Mar. 30, 2019)
Appendix A. Curriculum Vitae

Conference Attendance

- AWM Piedmont-Triad Conference  
  Wake Forest University (Mar. 2, 2019)

- Hudson River Undergraduate Math Conference  
  St. Lawrence University (Cantor, New York - Apr. 7, 2018)

- Joint Mathematics Meetings 2018  
  AMS/MAA (San Diego, California - Jan. 10-13, 2018)

- Women in Mathematics in New England Conference  
  Center for Women in Mathematics at Smith College (Sep. 16, 2017)

- Creative Writing in Mathematics and Mathematical Science Workshop  
  Banff International Research Station (BIRS) (Banff, Canada - Jan. 10-15, 2016)

Presentations

- “What is a Closed Set Anyway? - The Reverse Mathematics of Real Analysis”  
  Brown Bag Research Talks - Wake Forest University AWM Chapter  
  (Feb. 10, 2020)

- “An Introduction to Reverse Mathematics”  
  Summer Brown Bag Research Talks - Wake Forest University AWM Chapter  
  (Aug. 7, 2019)

- “Linking Numbers in Dihedral Covers”  
  Hudson River Undergraduate Math Conference - St. Lawrence University  
  (Apr. 7, 2018)

- “Rigidity and the Pebble Game Algorithm in 3D”  
  Joint Mathematics Meetings - AMS/MAA  
  (Jan. 13, 2018)

- “The Suspension of Disbelief and Other Mathematical Principles”  
  Women in Mathematics in New England Conference - Smith College  
  (Sep. 16, 2017)

- “Problem Set: Exploring Mathematics through Playwriting”  
  Senior Project Presentations to the Board of Trustees - College of the Atlantic  
  (Jun. 3, 2016)

Society Membership

- American Mathematical Society (AMS)

- Association for Women in Mathematics (AWM) – Wake Forest University Chapter