TAYLOR RESOLUTIONS OVER SKEW POLYNOMIAL RINGS

BY

DESIREE MARTIN

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Approved By:

Frank Moore, Ph.D., Advisor

Sarah Raynor, Ph.D., Chair

Jeremy Rouse, Ph.D.

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Abstract

Let I be a monomial ideal in the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ over a field $\Bbbk$. In her Ph.D. thesis [Tay66], Taylor introduced a complex which provides a multigraded free resolution for R/I as an R-module. Later, Gemeda [Gem76] provided a differential graded algebra structure on Taylor's complex. We generalize each of these results to monomial ideals in a skew polynomial ring.

Chapter 1: Introduction

Algebraic properties of polynomials have been studied since the early days of mathematics. Homological algebra (an area inspired by similar work in topology) over a polynomial ring was studied by Hilbert in his well-known Hilbert's Syzygy Theorem. Since then, mathematicians have been interested in studying finite free resolutions over polynomial rings. A free resolution is essentially a sequence of matrices which measure how far a given module is from having a basis. These resolutions are typically difficult to compute, and can be fairly complicated to describe. In [Tay66], Taylor used a combinatorial model to construct a free resolution for a monomial ideal over a commutative polynomial ring.

One generalization of the polynomial ring is that of a skew polynomial ring. In this context, the polynomial variables do not commute with one another, instead a scalar factor is created when variables interchange. That is, $xy = qyx$ for some constant q. Such rings arise naturally in the study of noncommutative algebraic geometry. Our first result generalizes Taylor's complex to monomial ideals in a skew polynomial ring.

One way the complexity of the matrices in a resolution can be controlled is via a product structure such that the matrices in the resolution satisfy an analogue of the product rule. Such a structure is called a differential graded algebra structure or DG-algebra for short. In [Gem76], Gemeda proved there is a DG-algebra structure defined on Taylor's complex over commutative rings. We also generalize this result to monomial ideals in a skew polynomial ring.

We briefly outline the contents of this thesis. Chapter 2 will cover the background needed for the results of the thesis. In this chapter we will mention definitions of commutative algebraic structures such as modules and monomials. We will also introduce what a complex and a resolution are, as well as other homological ideas.

This chapter will also provide some background and structures needed for the skew polynomial ring case.

In Chapter 3 we will define the Taylor complex and provide a proof of its acyclicity in the commutative case. We will then define our extension of the Taylor complex to the noncommutative setting, as well as prove it is a resolution of the quotient by a monomial ideal.

Chapter 4 will provide background information on DG-algebra structures. We will then present and prove Gemeda's theorem that the Taylor complex has a DG-algebra structure in the commutative case. We finally generalize the theorem to the skew polynomial ring case and provide our proof that the color Taylor complex admits a DG-algebra structure defined by a new product.

Chapter 2: Background

We begin by providing background information on some concepts from Taylor's thesis, which we will later generalize to a multiplicatively skew polynomial ring. In this thesis we will define $\Bbbk$ to be a field and all $\Bbbk$-algebras to be associative that possesses multiplicative identity. We will also let all tensor products be defined over $\Bbbk$ unless otherwise noted.

2.1 Commutative Polynomial Rings

Let $\Bbbk$ be a field. Then $\Bbbk[x_1, \ldots, x_n]$ will denote the *polynomial ring of n variables* with coefficients in $\Bbbk$.

For the remainder of the thesis we will let $R = \Bbbk[x_1, \ldots, x_n]$.

Definition 2.1.1. We define a *monomial* in R by $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{N}$ for all i.

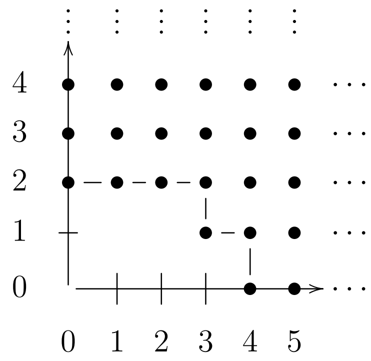
Definition 2.1.2. Let $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $\mathbf{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ be monomials in R where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. We define

$$\begin{aligned} \frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\mathbf{x}^{\boldsymbol{\beta}}} &= \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \text{ , provided } \boldsymbol{\alpha}-\boldsymbol{\beta} \in \mathbb{N}^n, \text{and} \\ \mathbf{x}^{\boldsymbol{\alpha}} * \mathbf{x}^{\boldsymbol{\beta}} &= \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \end{aligned}$$

Definition 2.1.3. Let $\mathbf{x}^{\mathbf{a_1}}, \cdots, \mathbf{x}^{\mathbf{a_m}}$ be monomials in R, where $\mathbf{a_i} = (a_{1,i} \ldots, a_{n,i})$. We define the *least common multiple* of $\mathbf{x}^{\mathbf{a_1}}, \cdots, \mathbf{x}^{\mathbf{a_m}}$, or $\operatorname{lcm}(\mathbf{x}^{\mathbf{a_1}}, \cdots, \mathbf{x}^{\mathbf{a_m}})$, to be equal to $\mathbf{x}^{\max \mathbf{a_i}}$ where $\max\ \mathbf{a_i} = (\max_i\ a_{1,i}\ , \ldots,\ \max_i\ a_{n,i})$. We will denote $\operatorname{lcm}(\mathbf{x}^{\mathbf{a_1}}, \cdots, \mathbf{x}^{\mathbf{a_m}})$ by m_F, where F is the set of monomials $\{\mathbf{x}^{\mathbf{a_1}}, \cdots, \mathbf{x}^{\mathbf{a_m}}\}$.

Definition 2.1.4. A *monomial ideal* is an ideal of R of the form $\langle \mathbf{x}^{\boldsymbol{\alpha}_1}, \ldots, \mathbf{x}^{\boldsymbol{\alpha}_m} \rangle$. Equivalently, it is the $\Bbbk$-span of a set of monomials closed under multiplication from the set of all monomials.

Example 2.1.5. Staircase Diagram of a monomial ideal.



The diagram above represents the monomial ideal $\langle x^4, x^3y, y^2\rangle \subseteq \Bbbk[x, y]$, where the y-axis represents powers of the monomial y and the x-axis represents powers of the monomial x. All monomials above and to the right of each of the generators of our monomial ideal will be included in $\langle x^4, x^3y, y^2\rangle$.

Definition 2.1.6. An R*-module* is a set M with

- a binary operation "+" on M, where $(M, +)$ is an abelian group, and
- an associative and distributive action of R on M denoted by rm for all $r \in R$ and $m \in M$ that satisfies:
  - $r(sm) = (rs)m$ for all $r, s \in R$ and $m \in M$,
  - $r(m + n) = rm + rn$ for all $r \in R$ and $m, n \in M$, and
  - $(r + s)m = (rm + sm)$ for all $r, s \in R$ and $m \in M$.

  Since R has an identity, then we also have the extra condition $1m = m$ for all $m \in M$.

Definition 2.1.7. An R-module M is said to be *free* if it has a basis. That is, there exists $m_1, \ldots, m_n \in M$ such that for any $m \in M$ there exists unique $r_1, \ldots, r_n \in R$ such that

$$m = r_1m_1 + \cdots + r_nm_n.$$

2.2 Graded Rings and Modules

Definition 2.2.1. Let G be an abelian group with an additive operation. We say a $\Bbbk$-algebra R is *G-graded* if $R = \bigoplus_i R_i$, for $i \in G$, where each R_i is a $\Bbbk$-module and $R_i R_j \subseteq R_{i+j}$. We say a graded $\Bbbk$-algebra is *bigraded* if each R_i has a further vector space decomposition $R_i = \bigoplus_n R_{i,n}$ such that $R_{i,n} R_{j,m} \subseteq R_{i+j,n+m}$. Thus we may write $R = \bigoplus_{i,n} R_{i,n}$. Later in the thesis we define G-gradings where G has a multiplicative operation.

Example 2.2.2. Let $R = \Bbbk[x_1, \ldots, x_n]$. Then R is $\mathbb{Z}$ graded, which corresponds to the degree of the polynomials in R. In this instance, R_i vector space of the polynomials of degree i.

We define the $\mathbb{Z}$-degree of a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ to be $\alpha_1 d_1 + \cdots + \alpha_n d_n$ where $\deg(x_i) = d_i \in \mathbb{Z}^+$.

Definition 2.2.3. A *graded A-module* is an A-module M such that $M = \bigoplus_{n=1}^{\infty} M_n$ as abelian groups, and $A_m M_n \subseteq M_{m+n}$, where each M_n is an A-module.

Rings in this paper will often have a bigrading which comes from an internal grading and a homological grading. A third grading, the color grading, will appear later in the thesis. It is important to note the constants relating to the homological grading in most of the proofs in the noncommutative case can be traced back to the commutative case of each of the results.

2.3 Homological Algebra

Definition 2.3.1. Let R be a ring. A sequence of homomorphisms of R-modules

$$\cdots \to M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \to \cdots$$

is called a *complex* if for each n we have $\partial_n \partial_{n+1} = 0$. We call a complex *exact* if at each node where there is a map entering and leaving we have $\mathrm{im}\partial_{i+1} = \ker\partial_i$. If

$M_n = 0$ for $n < k$ for some k, we write

$$\cdots \to M_{k+2} \to M_{k+1} \to M_k \to 0$$

Definition 2.3.2. We say a nonzero element $x \in M_i$, from above, has *homological degree* i, denoted by $|x|$.

Definition 2.3.3. Let R be a ring and M_n an R-module for each n. Given the complex

$$\mathbb{M} : \cdots \to M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \to \cdots$$

We define $H_n(\mathbb{M}) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

Note that $H_n(\mathbb{M})$ measures a complex's deviation from exactness. In fact a complex is exact iff $H_i(\mathbb{M}) = 0$ for all i.

Definition 2.3.4. We say that $\mathbb{M}$ is *acyclic* if $H_n(\mathbb{M}) = 0$ for all $n > 0$.

Definition 2.3.5. Suppose M is an R-module. A *resolution of* M is an exact complex of the form

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

where F_i is a free R-module for each i.

This data can be equivalently formulated as a complex

$$\mathbb{F} : \cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0$$

where each F_i is free, and $H_0(\mathbb{F}) = M$. In other words, the above complex is acyclic with 0^{th} homology isomorphic to M.

Now that we have defined a complex and a resolution, here is an example.

Example 2.3.6. Let R be a polynomial ring and $\langle m \rangle$ be a monomial ideal. Then we generate the resolution:

$$0 \leftarrow R/\langle m \rangle \xleftarrow{\pi} R^1 \xleftarrow{(m)} R^1 \leftarrow 0$$

Where (m) is the one by one matrix with entry m representing a column and π is the quotient map. It is easy to see $\ker(\pi) = \mathrm{im}(m)$, therefore we have the sequence above is indeed a complex. One should note that the Rs in the sequence are free with basis $\{1\}$.

Example 2.3.7. Let R be a polynomial ring and $I = \langle x^2z, xyz, y^2\rangle$. We will build a resolution of R/I. We first start with the sequence;

$$0 \leftarrow R/I \leftarrow R^1 \xleftarrow{\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}} R^3.$$

Let $\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix} = \partial_1$. To extend the sequence, we need to find a map ∂_2, such that the im $\partial_2 = \ker \partial_1$. To do this we need to find all relations among the generators of I. One way we might relate the generators is through their least common multiples.

For example, if we wish to relate the monomials x^2z and xyz, one may consider the $\mathrm{lcm}(x^2z, xyz) = x^2yz$. Using the least common multiple, we can find an R-linear combination of the generators that lands us in the kernel of $\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}$. In this case we have

$$x^2z(-y) + xyz(x) + 0 \;=\; 0.$$

Using the same process, we also have the relations

$$\begin{aligned} x^2z(-y^2) + 0 + y^2(x^2z) &= 0 \\ 0 + xyz(-y) + y^2(xz) &= 0, \end{aligned}$$

which gives us the map

$$\partial_2 = \begin{pmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2z & xz \end{pmatrix}.$$

The equations above imply that $\partial_1\partial_2 = 0$, which gives im $\partial_2 \subseteq \ker \partial_1$. We will assume the other containment as fact for now, and will prove it holds later in the thesis.

Our sequence now becomes;

$$0 \leftarrow R/I \leftarrow R^1 \xleftarrow{\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2z & xz \end{pmatrix}} R^3.$$

We again will extend the sequence by finding a map ∂_3 such that im $\partial_3 = \ker \partial_2$. To do this we need to define relations among the relations we have already made on the generators.

Our previous relations we made were through the least common multiples of the monomials. Each column of ∂_2 corresponds to a least common multiple above. We will describe the relations among those using least common multiples again. For instance, the first column of ∂_2 defines the relation of the $\text{lcm}(x^2z, xyz) = x^2yz$, and the second column defines the relation of the $\text{lcm}(x^2z, y^2) = x^2y^2z$. We relate those relations using $\text{lcm}(x^2yz, x^2y^2z) = x^2y^2z$, which gives the R-linear combination

$$x^2yz(y) + x^2y^2z(-1) + 0 \;=\; 0.$$

Using the same process, we also have the relations

$$\begin{aligned} x^2yz(y) + 0 + xy^2z(x) &= 0 \\ 0 + x^2y^2z(-1) + xy^2z(x) &= 0. \end{aligned}$$

It turns out the columns of ∂_2 correspond to the relations of the lcms, as seen below:

$$\begin{pmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2z & xz \end{pmatrix} \begin{pmatrix} y \\ -1 \\ x \end{pmatrix} = y \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + -1 \begin{pmatrix} -y^2 \\ 0 \\ x^2z \end{pmatrix} + x \begin{pmatrix} 0 \\ -y \\ xz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives us the map

$$\partial_3 = \begin{pmatrix} y \\ -1 \\ x \end{pmatrix},$$

that implies $\partial_2\partial_3 = 0$. This gives im $\partial_3 \subseteq$ ker ∂_2, and we will assume the other containment holds.

Our sequence now becomes;

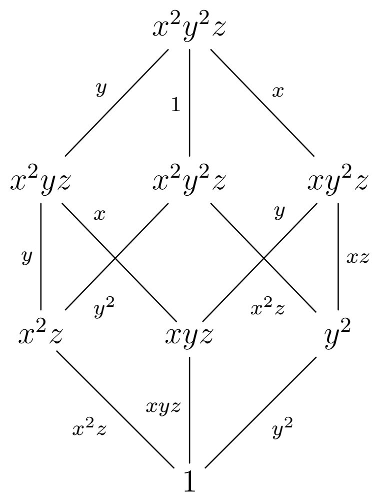
$$0 \leftarrow R/I \leftarrow R^1 \xleftarrow{\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2z & xz \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} y \\ -1 \\ x \end{pmatrix}} R^1.$$

Since R is a domain and ∂_3 is a map that scales by a variable, we know ∂_3 is injective. Thus, $0 =$ ker $\partial_3 =$ im ∂_4, which finishes our complex;

$$0 \leftarrow R/I \leftarrow R^1 \xleftarrow{\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2z & xz \end{pmatrix}} R^3 \xleftarrow{\begin{pmatrix} y \\ -1 \\ x \end{pmatrix}} R^1 \leftarrow 0.$$

This construction is an example of the Taylor resolution given in Definition 3.1.2.

The following is the lcm lattice of the list of monomials in the example, with the edges indicating divisibility labeled with the quotient. Note that the labels of the edges in the diagram are the same as those appearing in the resolution above.



2.4 Color Commutative Polynomial Rings

Definition 2.4.1. Let $\mathbf{q} = (q_{i,j})$ be a $n \times n$ matrix with entries in $\Bbbk^*$ such that $q_{i,i} = 1$ for all $i = 1,..,n$ and $q_{i,j} = q_{j,i}^{-1}$ for all $i,j = 1,...,n$. Let $\Bbbk\langle x_1, \ldots x_n\rangle$ be the non-commutative polynomial ring on n variables. Let I be the two-sided ideal of $\Bbbk\langle x_1, \ldots x_n\rangle$ generated by $\{x_i x_j - q_{i,j} x_j x_i \mid i,j = 1, \ldots, n\}$. Then the *skew polynomial ring* associated to the matrix $\mathbf{q}$ is

$$\Bbbk_{\mathbf{q}}[x_1, ..., x_n] = \frac{\Bbbk\langle x_1, \ldots x_n\rangle}{I}.$$

Note that in $\Bbbk_{\mathbf{q}}[x_1, ..., x_n]$, the relations $x_i x_j = q_{i,j} x_j x_i$ hold for all $i,j \in \{1, \ldots, n\}$.

For the remainder of the thesis, we will let $S = \Bbbk_{\mathbf{q}}[x_1, ..., x_n]$.

Definition 2.4.2. Let R be a ring. We say $f \in R$ is *normal* if there exists $\sigma \in \operatorname{Aut}(R)$ such that $fg = \sigma(g)f$ for all $g \in R$. We call σ the *normalizing automorphism* of f.

Example 2.4.3. Let $S = \Bbbk_{\mathbf{q}}[x_1, ..., x_n]$ and $G = \langle \sigma_1, \ldots, \sigma_n\rangle \leqslant \operatorname{Aut}(S)$. Let $\sigma_i \in G$ be the normalizing automorphism of x_i, then by definition $\sigma_i(x_j) = q_{ij} x_j$.

Remark 2.4.4. Let $G = \langle \sigma_1, \ldots, \sigma_n\rangle \leqslant \operatorname{Aut}(S)$, where σ_i is the normalizing automorphism of x_i for each $i = 1, \ldots, n$. Note that G is abelian since,

$$\sigma_a\sigma_b(x_i) = \sigma_a(q_{bi}x_i) = q_{bi}\sigma_a(x_i) = q_{bi}q_{ai}x_i = q_{ai}\sigma_b(x_i) = \sigma_b(q_{ai}x_i) = \sigma_b\sigma_a(x_i)$$

for all $\sigma_a, \sigma_b \in G$.

We define the G-degree of $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ to be $\sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n} \in G$. It is useful to notice that the G-degree of a monomial is exactly the normalizing automorphism of that monomial, i.e. $x_1^{\alpha_1} \cdots x_n^{\alpha_n} f = \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}(f) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for all $f \in S$.

We define a $\mathbb{Z} \times G$ grading on S. Let $S = \bigoplus_{i \geq 0} S_i$ be the usual $\mathbb{Z}$-grading and $S = \bigoplus_{\sigma \in G} S_\sigma$ be the color grading over the ring S, where S_i is the span of the monomials of degree i and S_σ is the span of monomials with normalizing automorphism σ. Then each $S_i = \bigoplus_{\sigma \in G} S_{i,\sigma}$ such that for $\sigma, \tau \in G$ $S_{i,\sigma} S_{j,\tau} \subseteq S_{i+j,\sigma\tau}$.

Definition 2.4.5. Let G be an abelian group. A *bicharacter* of G with values in a field $\Bbbk$ is a map $\chi : G \times G \to \Bbbk^*$ such that $\forall a, b, c \in G$

$$\chi(ab, c) = \chi(a, c)\chi(b, c)$$

$$\chi(a, bc) = \chi(a, b)\chi(a, c).$$

We say χ is *alternating* if for all $a \in G$ one has:

$$\chi(a, a) = 1.$$

Note that alternating implies *skew*, which says that

$$\chi(a, b) = \chi(b, a)^{-1}.$$

Remark 2.4.6. Let $G = \langle \sigma_1, \ldots, \sigma_n \rangle \subseteq \mathrm{Aut}(S)$. Define

$$\chi : G \times G \to \Bbbk^*$$

$$(\sigma_i, \sigma_j) \mapsto q_{i,j}$$

and extend it to all of G so that χ is a bicharacter. If $m, n \in R$ are G-homogeneous and the G-degree of m and n is σ and τ respectively we abuse notation and write $\chi(m, n)$ for $\chi(\sigma, \tau)$. Notice that if $m, n \in S$ are G-homogeneous then $mn = \chi(m, n)nm$. These are examples of color commutative rings, see [FM].

All modules in this paper are right color modules where a color module is a module with a compatible G-grading. Note that a color right module can be turned into a color bimodule by letting $rm = \chi(r, m)mr$. We define the quotients and $*$ products of monomials of S as in definition 2.1.2.

We now give another pairing which is related to χ.

Definition 2.4.7. Let $\mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathbf{x}^{\boldsymbol{\beta}}$ be monomials in S. We define $C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}})$ to be the constant such that $\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}} = C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}})\mathbf{x}^{\boldsymbol{\alpha}} * \mathbf{x}^{\boldsymbol{\beta}}$.

Example 2.4.8. Let S be a skew polynomial ring and $m_1 = x^2z, m_2 = xyz$ be monomials in S. Then

$$m_1 * m_2 = x^3yz^2$$

and

$$\begin{aligned} m_1m_2 &= x^2zxyz \\ &= \chi(z,x)x^3zyz \\ &= \chi(z,x)\chi(z,y)x^3yz^2 \\ &= C(m_1,m_2)m_1 * m_2. \end{aligned}$$

From the example we see that $C(m_1, m_2) = \chi(z, x)\chi(z, y)$. One should wonder if the pairing $C(-,-)$ will always be defined as a product of the bicharacter χ, or what other properties $C(-,-)$ has. The pairing $C(-,-)$ has the following properties:

Lemma 2.4.9. *Let X be the set of monomials of S. Then $(X, *)$ is a monoid. Let $\mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathbf{x}^{\boldsymbol{\beta}}$ be monomials in S.*

(a) $C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}) = \prod_{i>j} \chi(x_i, x_j)^{\alpha_i\beta_j}$.

(b) $C : X \times X \to \Bbbk^*$ *is a bicharacter, where X is the monoid of monomials of R.*

(c) $\frac{C\left(\mathbf{x}^{\boldsymbol{\alpha}},\mathbf{x}^{\boldsymbol{\beta}}\right)}{C\left(\mathbf{x}^{\boldsymbol{\beta}},\mathbf{x}^{\boldsymbol{\alpha}}\right)} = \chi\left(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\right)$.

Proof.

(a) Let $\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}} = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} x_1^{\beta_1} \cdots x_{n-1}^{\beta_{n-1}} x_n^{\beta_n}$, we move $x_n^{\alpha_n}$ past all $x_j^{\beta_j}$ where $j < n$. In doing so we pick up a factor of $\chi(x_n, x_j)^{\alpha_n\beta_j}$ for all $j < n$. Thus

$$\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}} = \prod_{n>j} \chi(x_n, x_j)^{\alpha_n,\beta_j} x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_1^{\beta_1} \cdots x_{n-1}^{\beta_{n-1}} x_n^{\alpha_n+\beta_n}.$$

In general when we move $x_i^{\alpha_i}$ past all $x_j^{\beta_j}$ where $i > j$ we pick up a factor of $\chi(x_i, x_j)^{\alpha_i\beta_j}$. In total we pick up $\prod_{i>j} \chi(x_i, x_j)^{\alpha_i\beta_j}$, thus

$$\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}} = \prod_{i>j} \chi(x_i, x_j)^{\alpha_i\beta_j} x_1^{\alpha_1+\beta_1} \cdots x_n^{\alpha_n+\beta_n}.$$

Therefore, by definition, $C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}) = \prod_{i>j} \chi(x_i, x_j)^{\alpha_i\beta_j}$.

(b) Let $\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\boldsymbol{\gamma}}$ be monomials in S. Then

$$\begin{aligned} C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}})C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\gamma}}) &= \prod_{i>j} \chi(x_i, x_j)^{\alpha_i\beta_j} \prod_{i>j} \chi(x_i, x_j)^{\alpha_i\gamma_j} \\ &= \prod_{i>j} \chi(x_i, x_j)^{\alpha_i(\beta_j+\gamma_j)} \\ &= C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}}), \end{aligned}$$

and

$$\begin{aligned} C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}})C(\mathbf{x}^{\boldsymbol{\gamma}}, \mathbf{x}^{\boldsymbol{\beta}}) &= \prod_{i>j} \chi(x_i, x_j)^{\alpha_i\beta_j} \prod_{i>j} \chi(x_i, x_j)^{\gamma_i\beta_j} \\ &= \prod_{i>j} \chi(x_i, x_j)^{(\alpha_i+\gamma_i)\beta_j} \\ &= C(\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}}, \mathbf{x}^{\boldsymbol{\beta}}). \end{aligned}$$

Thus, C is a bicharacter of the monoid X.

(c) By definition $\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}} = C\left(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\right) \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. However,

$$\begin{aligned} \mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}} &= \chi\left(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\right) \mathbf{x}^{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\alpha}} \\ &= \chi\left(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\right) C\left(\mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\boldsymbol{\alpha}}\right) \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \end{aligned}$$

Thus the result follows.

□

Remark 2.4.10. The bicharacter C is often not alternating.

Remark 2.4.11. We extend C to the group of monomials in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ by setting

$$C(\mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}, \mathbf{x}^{\gamma-\boldsymbol{\delta}}) = C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\gamma})C(\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\delta}})^{-1}C(\mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\gamma})^{-1}C(\mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\boldsymbol{\delta}}).$$

Chapter 3: The Taylor Complex

In this chapter we will be first providing the construction and proof of the Taylor complex, first originally proven in Taylor's thesis, [Tay66]. Then we will use this to prove we may construct a Taylor complex over the skew polynomial ring. One should note the usual Taylor complex is defined using exterior powers; however, we will be using the following definition so we may remain consistent once we define the color Taylor complex.

3.1 The Commutative Taylor Complex

Definition 3.1.1. Let F be a subset of $[s] = \{1, \ldots, s\}$. We define $F_i = F \backslash \{i\}$ and $F_{i,s} = F \backslash \{i\} \cup \{s\}$.

Definition 3.1.2. The *Taylor complex* $\mathbb{T}$ of $m_1, \ldots, m_s$ is a graded R-module with $T_i = R^{\binom{s}{i}}$. A basis of T_i is $\mathbf{e}_F$ where $|F| = i$ and $F \subset [s]$.

We define the differential of $\mathbb{T}$ by setting

$$\partial(\mathbf{e}_F) = \sum_{i \in F} \mathbf{e}_{F_i} (-1)^{\sigma(i,F)} \frac{m_F}{m_{F_i}},$$

where $\sigma(i, F) = |\{j \in F : j < i\}|$ and m_F denotes $\text{lcm}(m_i)$, where $i \in F$.

Example 3.1.3. Let R be a polynomial ring and $I = \langle x^2z, xyz, y^2 \rangle$, as in Example 2.3.7. Then the Taylor complex of R/I is,

$$0 \longleftarrow R \xleftarrow{\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}} R^{\binom{3}{1}} \xleftarrow{\begin{pmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2z & xz \end{pmatrix}} R^{\binom{3}{2}} \xleftarrow{\begin{pmatrix} y \\ -1 \\ x \end{pmatrix}} R^{\binom{3}{3}} \longleftarrow 0$$

Basis:	$e_\varnothing$	e_1, e_2, e_3	e_{12}, e_{13}, e_{23}	e_{123}

We computed this resolution in Example 2.3.7 without using basis elements, so it would be helpful to demonstrate the use of the differential from the definition of the Taylor complex.

Let $F = \{1, 2\}$ then,

$$\begin{aligned}\partial(e_F) &= e_1(-1)^{\sigma(2,F)}\frac{m_F}{m_1} + e_2(-1)^{\sigma(1,F)}\frac{m_F}{m_2} \\ &= e_1(-1)\frac{x^2yz}{x^2z} + e_2\frac{x^2yz}{xyz} \\ &= e_1(-y) + e_2x,\end{aligned}$$

which corresponds to the first column of ∂_2.

Theorem 3.1.4. *The Taylor complex is in fact a complex.*

Proof. In order to show the Taylor complex is a complex, we need to show $\partial^2 = 0$.

When we apply the differential twice we have,

$$\partial^2(\mathbf{e}_F) = \sum_{i\in F}\sum_{j\in F_i} \mathbf{e}_{F\setminus\{i,j\}}(-1)^{\sigma(i,F)+\sigma(j,F_i)}\frac{m_F}{m_{F\setminus\{i.j\}}}.$$

One may see in the sum we first pull out an i for every j in F, but we also pull out j then i from the set F. Whether we choose i or j first does not change the monomial on the end. However, it does change our homological degree. Thus, it suffices to show,

$$(-1)^{\sigma(i,F)+\sigma(j,F_i)} = -(-1)^{\sigma(j,F)+\sigma(i,F_j)}.$$

Without loss of generality let $i < j$. Since $\sigma(i, F) = |\{w \in F : w < i\}|$ and $i < j$, then we know $\sigma(i, F) = \sigma(i, F_j)$. Since $i < j$, then $\sigma(j, F) = \sigma(j, F_i) - 1$. Therefore we conclude,

$$\begin{aligned}(-1)^{\sigma(i,F)+\sigma(j,F_i)} &= (-1)^{\sigma(j,F)-1+\sigma(i,F_j)} \\ &= -(-1)^{\sigma(j,F)+\sigma(i,F_j)}.\end{aligned}$$

Therefore $\partial^2 = 0$. □

Theorem 3.1.5. *The Taylor complex is an acyclic complex, meaning $H_0(\mathbb{T}) = R/I$ where $I = \langle m_1, \ldots, m_s \rangle$ and $H_i(\mathbb{T}) = 0$ if $i > 0$.*

Proof. We prove this by inducting on s, with $s = 1$ being clear because R is a domain. Now assume $s > 1$ and the statement holds true for the sequence $m_1, \ldots, m_{s-1}$. Let $\mathbb{T}'$ be equal to the Taylor complex of the sequence of monomials $m_1 \ldots, m_{s-1}$, then $\mathbb{T}'$ is a subcomplex of $\mathbb{T}$ with basis elements e_F where $F \subseteq [s-1]$. Consider $\mathbb{G} = \mathbb{T}/\mathbb{T}'$, then $G_0 = 0$ and $G_i = T_i/T_i'$ for $i > 0$ is a free module generated by the basis $e_{F\cup\{s\}}$ where $|F| = i - 1$ and $F \subseteq [s-1]$. The differential on $\mathbb{G}$ is given by

$$\partial(e_{F\cup\{s\}}) = \sum_{i\in F} e_{F_{i,s}}(-1)^{\sigma(i,F)} \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}.$$

We prove $\mathbb{G}$ is isomorphic to the Taylor complex shifted by 1 on the sequence of monomials $v_1, \ldots, v_{s-1}$ where $v_i = \operatorname{lcm}(m_i, m_s)/m_s$ through the isomorphism

$$\varphi(e_{F\cup\{s\}}) = e_F(-1)^{|F|}.$$

Indeed,

$$\begin{aligned} \varphi\partial^{\mathbb{G}}(e_{F\cup\{s\}}) &= \varphi\left(\sum_{i\in F} e_{F_{i,s}}(-1)^{\sigma(i,F)} \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right) \\ &= \sum_{i\in F} e_{F_i}(-1)^{\sigma(i,F)+|F_i|} \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}} \end{aligned}$$

and, denoting by $\mathbb{V}$ the shift of the Taylor complex on $v_1, \ldots, v_{s-1}$,

$$\begin{aligned} \partial^{\mathbb{V}}\varphi(e_{F\cup\{s\}}) &= \partial^{\mathbb{V}}\left(e_F(-1)^{|F|}\right) \\ &= \sum_{i\in F} e_{F_i}(-1)^{\sigma(i,F)+|F|+1} \frac{v_F}{v_{F_i}} \end{aligned}$$

Thus, in order to show the isomorphism commutes with the differential we need to show the monomials and constants match. We start by proving the monomials are the same. Let $m_i = \mathbf{x}^{\boldsymbol{\alpha}_i}$ then

$$m_{F\cup\{s\}} = \mathbf{x}^{\max_{i\in F\cup\{s\}}\boldsymbol{\alpha}_i}.$$

Thus,

$$\frac{m_{F\cup\{s\}}}{m_s} = \mathbf{x}^{\max_{i\in F\cup\{s\}}(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_s)}.$$

Similarly by definition

$$\begin{aligned} v_i = \frac{m_{\{i,s\}}}{m_s} &= \mathbf{x}^{\max(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_s)-\boldsymbol{\alpha_s}} \\ &= \mathbf{x}^{\max(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_s,0)}. \end{aligned}$$

Therefore,

$$\begin{aligned} v_F &= \mathbf{x}^{\max_{i\in F}(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_s,0)} \\ &= \mathbf{x}^{\max_{i\in F\cup\{s\}}(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_s)}. \end{aligned}$$

Thus, $v_F = \frac{m_{F\cup\{s\}}}{m_s}$ which implies the monomials are the same.

Since $\mathbb{G}$ is isomorphic to the shift of the Taylor complex by 1 on the sequence of monomials $v_1, \ldots, v_{s-1}$, by induction $H_i(\mathbb{G}) = 0$ when $i \neq 1$ and $H_1(\mathbb{G}) = R/(v_1, \ldots, v_{s-1})$. The exact sequence induced in homology by the exact sequence

$$0 \to \mathbb{T}' \to \mathbb{T} \to \mathbb{G} \to 0 \tag{3.1.1}$$

yields $H_i(\mathbb{T}) = 0$ when $i > 1$, and there is an exact sequence

$$0 \to H_1(\mathbb{T}) \to H_1(\mathbb{G}) \to H_0(\mathbb{T}') \to H_0(\mathbb{T}) \to 0. \tag{3.1.2}$$

To conclude $H_1(\mathbb{T}) = 0$ we prove $\eth : H_1(\mathbb{G}) \to H_0(\mathbb{T}')$ is injective. To this end, let $\eth$ be the connecting map of (3.1.1). A computation shows $\eth(\bar{r}) = \overline{m_s r}$, therefore it suffices to prove that if $m_s r \in \langle m_1, \ldots, m_{s-1}\rangle$ then $r \in \langle v_1, \ldots, v_{s-1}\rangle$. Without loss of generality we may assume r is a monomial. Then $m_s r = a m_i$, for some monomial $a \in R$ and $m_i \in \langle m_1, \ldots, m_{s-1}\rangle$. Since m_s is not contained in $\langle m_1, \ldots, m_{s-1}\rangle$, we may then write $r = \frac{a m_i}{m_s}$. Since $a m_i$ is a multiple of m_i and m_s, therefore r is a multiple of v_i. Thus, $\eth$ is injective. Hence, $\ker \eth = 0$ implies $\operatorname{im}(H_1(\mathbb{T}) \to H_1(\mathbb{G})) = 0$ by exactness, proving $H_1(\mathbb{T}) = 0$. To finish the proof, we must show $H_0(\mathbb{T}) \cong R/\langle m_1, \ldots, m_s\rangle$. This follows from the fact that $H_0(\mathbb{T}) \cong H_0(\mathbb{T}')/(m) \cong R\langle m_1, \ldots, m_{s-1}\rangle/(m) \cong R\langle m_1, \ldots, m_s\rangle$, finishing the proof. $\square$

3.2 The Color Taylor Complex

We now present the modifications necessary to make the previous construction work in the skew commutative setting.

Construction 3.2.1. The *color Taylor complex* $\mathbb{T}$ of $m_1, \ldots, m_s$ is a graded right S-module with $T_i = S^{\binom{s}{i}}$. A basis of T_i is $\mathbf{e}_F$ where $|F| = i$ and $F \subset [s]$.

We define the differential $\mathbb{T}$ by setting

$$\partial(\mathbf{e}_F) = \sum_{i \in F} \mathbf{e}_{F_i}(-1)^{\sigma(i,F)} C\left(m_{F_i}, \frac{m_F}{m_{F_i}}\right)^{-1} \frac{m_F}{m_{F_i}},$$

where $\sigma(i, F) = |\{j \in F : j < i\}|$ and m_F denotes $\operatorname{lcm}^*(m_i)$, where $i \in F$. In order for the differential to be bihomogeneous we set the G-degree and the internal degree of e_F equal to the G-degree and the internal degree of m_F respectively.

Example 3.2.2. Let S be the Skew polynomial ring $S = \mathbb{k}_{\mathbf{q}}[x, y, z]$ defined by the matrix $\mathbf{q} = \begin{pmatrix} 1 & q_{1,2} & q_{1,3} \\ q_{2,1} & 1 & q_{2,3} \\ q_{3,1} & q_{3,2} & 1 \end{pmatrix}$ and $I = \langle x^2z, xyz, y^2\rangle$.

Recall the Taylor resolution of I in $R = \mathbb{k}[x, y, z]$ is:

$$0 \leftarrow R/I \leftarrow R \xleftarrow{\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}} R^{\binom{3}{1}} \xleftarrow{\begin{pmatrix} -y & -y^2 & 0 \\ x & 0 & -y \\ 0 & x^2z & xz \end{pmatrix}} R^{\binom{3}{2}} \xleftarrow{\begin{pmatrix} y \\ -1 \\ x \end{pmatrix}} R^{\binom{3}{3}} \leftarrow 0.$$

The color Taylor resolution of I in S is:

$$0 \leftarrow S/I \leftarrow S \xleftarrow{\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix}} S^{\binom{3}{1}} \xleftarrow{\begin{pmatrix} -q_{2,3}y & -q_{2,3}^2y^2 & 0 \\ q_{1,2}q_{1,3}x & 0 & -q_{2,3}y \\ 0 & q_{1,2}^4x^2z & q_{1,2}^2xz \end{pmatrix}} S^{\binom{3}{2}} \xleftarrow{\begin{pmatrix} q_{2,3}y \\ -1 \\ q_{1,2}^2q_{1,3}x \end{pmatrix}} S^{\binom{3}{3}} \leftarrow 0.$$

We can see the only difference between the two complexes are the constants in the color Taylor resolution that comes from the bicharacter $C(-,-)$.

Let's calculate the first entry of

$$\begin{pmatrix} x^2z & xyz & y^2 \end{pmatrix} \begin{pmatrix} -q_{2,3}y & -q_{2,3}^2y^2 & 0 \\ q_{1,2}q_{1,3}x & 0 & -q_{2,3}y \\ 0 & q_{1,2}^4x^2z & q_{1,2}^2xz \end{pmatrix}$$

to see why each $q_{i,j}$ is needed.

Our matrix $\mathbf{q}$ tells us the following equalities:

$$\begin{aligned} xy &= q_{1,2}yx \\ yz &= q_{2,3}zy \\ xz &= q_{1,3}zx. \end{aligned}$$

Thus,

$$\begin{aligned} -q_{2,3}x^2zy + q_{1,2}q_{1,3}xyzx + 0 &= -q_{2,3}q_{3,2}x^2yz + q_{1,2}q_{1,3}q_{3,1}xyxz \\ &= -x^2yz + q_{1,2}q_{2,1}x^2yz \\ &= -x^2yz + x^2yz = 0. \end{aligned}$$

Theorem 3.2.3. *The color Taylor complex is a complex.*

Proof. In order to prove $\mathbb{T}$ is a complex, we show $\partial \circ \partial = 0$. It follows from the definition of the differential that

$$\partial^2(\mathbf{e}_F) = \sum_{i \in F} \sum_{j \in F \setminus \{i\}} e_{F \setminus \{i,j\}} (-1)^{\sigma(i,F)+\sigma(j,F_i)} \widetilde{C}_{i,j} \frac{m_F}{m_{F \setminus \{i,j\}}},$$

where

$$\widetilde{C}_{i,j} = C\left(m_{F\setminus\{i,j\}}, \frac{m_{F_i}}{m_{F\setminus\{i,j\}}}\right)^{-1} C\left(m_{F_i}, \frac{m_F}{m_{F_i}}\right)^{-1} C\left(\frac{m_{F_i}}{m_{F\setminus\{i,j\}}}, \frac{m_F}{m_{F_i}}\right).$$

By the commutative case it suffices to show $\widetilde{C}_{i,j} = \widetilde{C}_{j,i}$. To do that we prove the following lemma.

Lemma 3.2.4. *If* $\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\boldsymbol{\gamma}}, \mathbf{x}^{\boldsymbol{\delta}}$ *are monomials in* R *such that* $\mathbf{x}^{\boldsymbol{\delta}}|\mathbf{x}^{\boldsymbol{\beta}}$, $\mathbf{x}^{\boldsymbol{\delta}}|\mathbf{x}^{\boldsymbol{\gamma}}$, $\mathbf{x}^{\boldsymbol{\beta}}|\mathbf{x}^{\boldsymbol{\alpha}}$, $\mathbf{x}^{\boldsymbol{\gamma}}|\mathbf{x}^{\boldsymbol{\alpha}}$. *Then*

$$\frac{C(\mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}})}{C(\mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}})} \frac{C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\delta}})}{C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\delta}})} \frac{C(\mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}})}{C(\mathbf{x}^{\boldsymbol{\gamma}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}})} = 1.$$

The proof follows from the lemma by allowing $m_{F\setminus\{i,j\}} = \mathbf{x}^{\boldsymbol{\delta}}, m_{F_j} = \mathbf{x}^{\boldsymbol{\beta}}, m_F = \mathbf{x}^{\boldsymbol{\alpha}}$, and $m_{F_i} = \mathbf{x}^{\boldsymbol{\gamma}}$.

Proof. Indeed,

$$\begin{aligned}\frac{C(\mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}})}{C(\mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}})} \frac{C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\delta}})}{C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\delta}})} \frac{C(\mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}})}{C(\mathbf{x}^{\boldsymbol{\gamma}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}})} &= C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}})C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\gamma}})C(\mathbf{x}^{-\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}}) \\ &= C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}})C(\mathbf{x}^{\boldsymbol{\delta}}, \mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}})^{-1} \\ &= 1.\end{aligned}$$

□

We have shown $\widetilde{C}_{i,j} = \widetilde{C}_{j,i}$, therefore $\mathbb{T}$ is a complex. □

Theorem 3.2.5. *Let* $\mathbb{T}$ *be the color Taylor complex in* $m_1, \ldots, m_s$. *Then* $H_0(\mathbb{T}) = R/I$ *where* $I = \langle m_1, \ldots, m_s \rangle$ *and* $H_i(\mathbb{T}) = 0$ *if* $i > 0$.

Proof. We prove this by inducting on s, with $s = 1$ being clear. Now assume $s > 1$ and the statement holds true for the sequence $m_1, \ldots, m_{s-1}$. Let $\mathbb{T}'$ be equal to the Taylor complex of the sequence of monomials $m_1 \ldots, m_{s-1}$, then $\mathbb{T}'$ is a subcomplex of $\mathbb{T}$ with basis elements e_{j_l} with $j_l = 1, \ldots, s-1$. Consider $\mathbb{G} = \mathbb{T}/\mathbb{T}'$, then $G_0 = 0$ and $G_i = T_i/T_i'$ for $i > 0$ is a free module generated by the basis $e_{F\cup\{s\}}$ where $|F| = i - 1$ and $F \subseteq [s-1]$. The differential on $\mathbb{G}$ is given by

$$\partial(e_{F\cup\{s\}}) = \sum_{i\in F} e_{F_{i,s}}(-1)^{\sigma(i,F)} C\left(m_{F_{i,s}}, \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)^{-1} \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}.$$

We prove $\mathbb{G}$ is isomorphic to the Taylor complex shifted by 1 on the sequence of monomials $v_1, \ldots, v_{s-1}$ where $v_i = \operatorname{lcm}(m_i, m_s)/m_s$ through the isomorphism

$$\varphi(e_{F\cup\{s\}}) = e_F(-1)^{|F|} C\left(m_s, v_F\right)^{-1}.$$

Indeed,

$$\varphi\partial^{\mathbb{G}}(e_{F\cup\{s\}}) = \varphi\left(\sum_{i\in F} e_{F_{i,s}}(-1)^{\sigma(i,F)}C\left(m_{F_{i,s}}, \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)^{-1}\frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)$$

$$= \sum_{i\in F} e_{F_i}(-1)^{\sigma(i,F)+|F_i|}C\left(m_s, v_{F_i}\right)^{-1}C\left(m_{F_{i,s}}, \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)^{-1}\frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}$$

and, denoting by $\mathbb{V}$ the shift of the Taylor complex on $v_1, \ldots, v_{s-1}$,

$$\partial^{\mathbb{V}}\varphi(e_{F\cup\{s\}}) = \partial^{\mathbb{V}}\left(e_F(-1)^{|F|}C\left(m_s, v_F\right)^{-1}\right)$$

$$= \sum_{i\in F} e_{F_i}(-1)^{\sigma(i,F)+|F|+1}C\left(v_{F_i}, \frac{v_F}{v_{F_i}}\right)^{-1}C\left(m_s, v_F\right)^{-1}\frac{v_F}{v_{F_i}}$$

Thus, in order to show the isomorphism commutes with the differential we need to show the monomials and constants match. The monomials and (-1) signs are the same in the commutative case, since we have shown that case works, it suffices to show the constants are the same. Indeed,

$$\frac{C\left(m_s, \frac{m_{F\cup\{s\}}}{m_s}\right)}{C\left(m_s, \frac{m_{F_{i,s}}}{m_s}\right)}\frac{C\left(\frac{m_{F_{i,s}}}{m_s}, \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)}{C\left(m_{F_{i,s}}, \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)} = C\left(m_s, \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)C\left(m_s, \frac{m_{F\cup\{s\}}}{m_{F_{i,s}}}\right)^{-1} = 1.$$

Therefore we have that $\mathbb{G}$ is isomorphic to the shift of the Taylor complex by 1 on the sequence of monomials $v_1, \ldots, v_{s-1}$. The rest of the proof now follows from the commutative case. $\square$

Chapter 4: Differential Graded Algebra Structure

Once we have shown we can construct a color Taylor resolution on our skew polynomial ring, a natural question to ask is: does there exist a differential graded algebra structure on the color Taylor complex as well? In the commutative case one has a differential graded algebra structure, originally proven in Gemeda's thesis [Gem76]. In this chapter we will provide the background and proof of this theorem, and then provide a differential graded algebra structure of the color Taylor complex defined in the previous section.

4.1 DG Structure of Commutative Taylor Complex

Definition 4.1.1. Let A be a graded algebra over a ring R. We say A is a differential graded (DG) R-algebra if there is a degree -1 map

$$\partial : A \to A$$

such that for each $\partial_i : A_i \to A_{-i}$ we have $\partial_i \partial_{i+1} = 0$, and ∂ satisfies the Leibniz Rule:

$$\partial(ab) = \partial(a)b + (-1)^{|a|} a\partial(b).$$

To provide the DG algebra structure, we set up some notation and prove a combinatorial lemma.

Definition 4.1.2. Let V and W be subsets of $[s]$. We define $\sum(V, W) = \{(v, w) \in V \times W \ : \ w < v\}$ and $\sigma(V, W) = |\sum(V, W)|$.

Lemma 4.1.3. *Let* V,V_1, V_2, W, W_1, *and* W_2 *be subsets of* $[s]$ *with* $V_1 \cap V_2 = \varnothing$ *and*

$W_1 \cap W_2 = \varnothing$. Let $\sum$ and σ be defined as above. Then one has equalities:

$$\begin{aligned}
\sigma(W,i) &= \begin{cases} |W| - \sigma(i,W) & i \notin W \\ |W| - 1 - \sigma(i,W) & i \in W. \end{cases} \\
\sigma(V_1 \cup V_2, W) &= \sigma(V_1, W) + \sigma(V_2, W) \\
\sigma(V, W_1 \cup W_2) &= \sigma(V, W_1) + \sigma(V, W_2).
\end{aligned}$$

The proof of this lemma follows directly from the definitions.

Theorem 4.1.4. *The Taylor complex has a differential graded algebra structure given by the product*

$$e_V e_W = \begin{cases} e_{V\cup W}(-1)^{\sigma(V,W)} \dfrac{m_V * m_W}{m_{V\cup W}} & V \cap W = \varnothing \\ 0 & V \cap W \neq \varnothing. \end{cases}$$

Proof. We need to check that ∂ satisfies the Leibniz rule with respect to the product above, and our product is associative and commutative with respect to the homological grading.

First we will prove the Leibniz rule holds. Let V and W be subsets of $[s]$. If we expand the left hand side we get,

$$\begin{aligned}
\partial\left(e_V e_W\right) &= \partial\left(e_{V\cup W}\left(-1\right)^{\sigma(V,W)} \frac{m_V * m_W}{m_{V\cup W}}\right) \\
&= \left(-1\right)^{\sigma(V,W)} \frac{m_V * m_W}{m_{V\cup W}} \sum_{i\in V\cup W} e_{(V\cup W)_i}\left(-1\right)^{\sigma(i,V\cup W)} \frac{m_{V\cup W}}{m_{(V\cup W)_i}} \\
&= \sum_{i\in V\cup W} e_{(V\cup W)_i}\left(-1\right)^{\sigma(i,V\cup W)+\sigma(V,W)} \frac{m_V * m_W}{m_{(V\cup W)_i}}.
\end{aligned}$$

Now we expand the right hand side,

$$\begin{aligned}
&\partial(e_V)e_W + (-1)^{|V|} e_V \partial(e_W) \\
&= \sum_{i\in V} e_{V_i} e_W (-1)^{\sigma(i,V)} \frac{m_V}{m_{V_i}} + \sum_{i\in W} e_V e_{W_i} (-1)^{|V|+\sigma(i,W)} \frac{m_W}{m_{W_i}} \\
&= \sum_{i\in V} e_{V_i\cup W} (-1)^{\sigma(i,V)+\sigma(V_i,W)} \frac{m_V * m_W}{m_{V_i\cup W}} + \sum_{i\in W} e_{V\cup W_i} (-1)^{|V|+\sigma(i,W)+\sigma(V,W_i)} \frac{m_V * m_W}{m_{V\cup W_i}},
\end{aligned}$$

where $V_i \cap W = \varnothing$ and $V \cap W_i = \varnothing$ respectively in each sum. To finish this proof we need to understand how the elements in V overlap with the elements in W. We check that the coefficients of each basis element agree in each case.

Case 1: Suppose V and W share more than one element in common. Then by the definition of the product we have $\partial(e_V e_W) = \partial(0) = 0$ on the left hand side. Similarly on the right hand side we have $e_{V_i} e_W = e_V e_{W_i} = 0$. Therefore, $\partial(e_V)e_W + (-1)^{|V|} e_V \partial(e_W) = 0$, and the Leibniz rule holds.

Case 2: Now suppose $V \cap W = \{i\}$. By the definition of the product we know that the left hand side will be zero like the previous case. However, it is not obvious the right hand side will also be zero.

If $V \cap W = \{i\}$, then it is easy to see $V_i \cap W = V \cap W_i = \varnothing$. Therefore our basis elements $e_{V_i \cap W}$ and $e_{V \cap W_i}$, and our monomials $\frac{m_V * m_W}{m_{V_i \cup W}}$ and $\frac{m_V * m_W}{m_{V \cup W_i}}$ are the same. Thus, it suffices to show $\sigma(i, V) + \sigma(V_i, W) \equiv 1 + |V| + \sigma(i, W) + \sigma(V, W_i) \mod 2$.

By the lemma above we know,

$$\begin{aligned}
\sigma(i, V) + \sigma(V_i, W) &= |V| - \sigma(V, i) - 1 + \sigma(V, W) - \sigma(i, W) \\
&= |V| - (\sigma(V, W) - \sigma(V, W_i)) - 1 - \sigma(V, W) - \sigma(i, W) \\
&= |V| + \sigma(V, W_i) - \sigma(i, W) - 1 \\
&\equiv |V| + \sigma(V, W_i) + \sigma(i, W) + 1 \mod 2.
\end{aligned}$$

Therefore, we have shown the right hand side will also be zero.

Case 3: Suppose $V \cap W = \varnothing$ and $i \in V \backslash W$. Since V and W share no common elements, then the left hand side of our equation gives,

$$\sum_{i \in V \cup W} e_{(V \cup W)_i} (-1)^{\sigma(i, V \cup W) + \sigma(V, W)} \frac{m_V * m_W}{m_{(V \cup W)_i}}.$$

One may notice that $(V \cup W)_i = V_i \cup W$, which means the basis element $e_{(V \cup W)_i} = e_{V_i \cup W}$ and the monomial $\frac{m_V * m_W}{m_{(V \cup W)_i}} = \frac{m_V * m_W}{m_{V_i \cup W}}$. It is important to note that $V_i \cup W \neq V \cup W_j$ for any $j \in W$, since V and W are disjoint. Therefore when we compare the basis elements on the right hand side with the left hand side, we will not include the basis elements of the form $e_{V\cup W_i}$ for $i \in W$.

Thus, it suffices to show $\sigma(i, V \cup W) + \sigma(V, W) \equiv \sigma(i, V) + \sigma(V_i, W) \mod 2$. Notice,

$$\begin{aligned}
\sigma(i, V \cup W) + \sigma(V, W) &= \sigma(i, V) + \sigma(i, W) + \sigma(V, W) \\
&= \sigma(i, V) + \sigma(i, W) + \sigma(i, W) + \sigma(V_i, W) \\
&= \sigma(i, V) + 2\sigma(i, W) + \sigma(V_i, W) \\
&\equiv \sigma(i, V) + \sigma(V_i, W) \mod 2.
\end{aligned}$$

Case 4: Suppose $V \cap W = \varnothing$ and $i \in W \backslash V$. The proof of this case follows similarly above except $(V \cup W)_i = V \cup W_i$. We note the basis element $e_{(V\cup W)_i} = e_{V\cup W_i}$ and the monomial $\frac{m_V * m_W}{m_{(V\cup W)_i}} = \frac{m_V * m_W}{m_{V\cup W_i}}$. By the same reasoning it suffices to show $\sigma(i, V \cup W) + \sigma(V, W) \equiv |V| + \sigma(i, W) + \sigma(V, W_i) \mod 2$. Notice,

$$\begin{aligned}
\sigma(i, V \cup W) + \sigma(V, W) &= \sigma(i, V) + \sigma(i, W) + \sigma(V, W) \\
&= \sigma(i, V) + \sigma(i, W) + \sigma(V, W_i) + \sigma(V, i) \\
&= |V| + \sigma(i, W) + \sigma(V, W_i).
\end{aligned}$$

Therefore, we have shown ∂ satisfies the Leibniz rule.

To finish the proof, we must show our product is associative and commutative with respect to the homological grading, which gives us the algebra structure on the Taylor resolution. First we will show the product is associative.

Let V, W, and X be subsets of $[s]$. Then we want the show $e_V\left(e_W e_X\right) = \left(e_V e_W\right) e_X$. One should note we may assume $V \cap W \cap X = \varnothing$, otherwise the proof is trivial. Now, if we expand the left hand side we get,

$$\begin{aligned}
e_V\left(e_W e_X\right) &= e_V e_{W\cup X}(-1)^{\sigma(W,X)} \frac{m_W * m_X}{m_{W\cup X}} \\
&= e_{V(W\cup X)}(-1)^{\sigma(V,W\cup X)+\sigma(W,X)} \frac{m_V * m_{W\cup X}}{m_{V\cup(W\cup X)}} \frac{m_W * m_X}{m_{W\cup X}} \\
&= e_{V(W\cup X)}(-1)^{\sigma(V,W\cup X)+\sigma(W,X)} \frac{m_V * m_W * m_X}{m_{V\cup(W\cup X)}}.
\end{aligned}$$

Similarly on the right hand side,

$$\begin{aligned}(e_V e_W)\, e_X &= e_{V\cup W}(-1)^{\sigma(V,W)}\frac{m_V * m_W}{m_{V\cup W}}e_X \\ &= e_{V\cup W}e_X(-1)^{\sigma(V,W)}\frac{m_V * m_W}{m_{V\cup W}} \\ &= e_{(V\cup W)\cup X}(-1)^{\sigma(V\cup W,X)+\sigma(V,W)}\frac{m_{V\cup W} * m_X}{m_{(V\cup W)\cup X}}\frac{m_V * m_W}{m_{V\cup W}} \\ &= e_{(V\cup W)\cup X}(-1)^{\sigma(V\cup W,X)+\sigma(V,W)}\frac{m_V * m_W * m_X}{m_{(V\cup W)\cup X}}.\end{aligned}$$

It is easy top see our basis elements and monomials are the same. The only thing left to check are the signs of the equation, i.e.

$$\sigma(V, W\cup X) + \sigma(W, X) = \sigma(V\cup W, X) + \sigma(V, W).$$

Notice,

$$\begin{aligned}\sigma(V, W\cup X) + \sigma(W, X) &= \sigma(V, W) + \sigma(V, X) + \sigma(W, X) \\ &= \sigma(V, W) + \sigma(V\cup W, X).\end{aligned}$$

Hence, our product is associative.

Finally, we will show our product is commutative with respect to the homological grading. Let V and W be sets. We want to show $e_V e_W = (-1)^{|V||W|}e_W e_V$, where we may assume $V\cap W = \varnothing$, since otherwise the proof is trivial.

If we expand out either side, as above, then one may see it suffices to show $\sigma(V, W) \equiv |V||W| + \sigma(W, V) \mod 2$. Indeed, if we consider the set $\sum(V, W) = \{(v, w)|w < v\}$, the compliment of $\sum(V, W)$ in $V\times W$ would be the set $\{(v, w)|v \leq w\}$. Since $V\cap W = \varnothing$, then $\{(v, w)|v \leq w\} = \{(v, w)|v < w\} = \sum(W, V)$. Therefore, we have

$$\begin{aligned}\sigma(V, W) &= |V||W| - \sigma(W, V) \\ &\equiv |V||W| + \sigma(W, V) \mod 2.\end{aligned}$$

Hence our product is commutative with respect to the homological grading, proving the Taylor complex has a differential graded algebra structure with the product above. □

4.2 DG Structure of Color Taylor Complex

Theorem 4.2.1. *The color Taylor complex has a color commutative differential graded algebra structure given by the product*

$$e_V e_W = \begin{cases} e_{V\cup W}(-1)^{\sigma(V,W)}\dfrac{m_V * m_W}{m_{V\cup W}} C\left(m_V, m_W\right) C\left(m_{V\cup W}, \dfrac{m_V * m_W}{m_{V\cup W}}\right)^{-1} & V\cap W = \varnothing \\ 0 & V \cap W \neq \varnothing. \end{cases}$$

Proof. We start by showing the product satisfies the Leibniz rule. We will break the proof into similar cases as in the commutative proof.

Case 1: If $V \cap W$ have more than one element in common is trivial.

Case 2: Let $V \cap W = \{i\}$. Similarly to the commutative proof, we know the left hand side of the equation will be zero. Thus, we need to prove the right hand side will also be zero. If we expand the right hand side we see,

$$\begin{aligned}
&\partial(e_V)e_W + (-1)^{|V|}e_V\partial(e_W) \\
&= \sum_{i\in V} e_{V_i}(-1)^{\sigma(i,V)} C\left(m_{V_i}, \frac{m_V}{m_{V_i}}\right)^{-1}\frac{m_V}{m_{V_i}} e_W \\
&\quad + e_V \sum_{i\in W} e_{W_i}(-1)^{|W|+\sigma(i,W)} C\left(m_{W_i}, \frac{m_W}{m_{W_i}}\right)^{-1}\frac{m_W}{m_{W_i}} \\
&= \sum_{i\in V} e_{V_i}e_W(-1)^{\sigma(i,V)}\frac{\chi(\frac{m_V}{m_{V_i}}, m_W)}{C\left(m_{V_i}, \frac{m_V}{m_{V_i}}\right)}\frac{m_V}{m_{V_i}} \\
&\quad + \sum_{i\in W} e_V e_{W_i}(-1)^{|W|+\sigma(i,W)} C\left(m_{W_i}, \frac{m_W}{m_{W_i}}\right)^{-1}\frac{m_W}{m_{W_i}} \\
&= \sum_{i\in V} e_{V_i\cup W}(-1)^{\sigma(i,V)}\frac{m_{V_i} * m_W}{m_{V_i\cup W}}\frac{m_V}{m_{V_i}}\frac{C\left(m_{V_i}, m_W\right)}{C\left(m_{V_i\cup W}, \frac{m_{V_i} * m_W}{m_{V_i\cup W}}\right)}\frac{\chi(\frac{m_V}{m_{V_i}}, m_W)}{C\left(m_{V_i}, \frac{m_V}{m_{V_i}}\right)} \\
&\quad + \sum_{i\in W} e_{V\cup W_i}(-1)^{|W|+\sigma(i,W)}\frac{m_{W_i} * m_V}{m_{V\cup W_i}}\frac{m_W}{m_{W_i}}\frac{C\left(m_V, m_{W_i}\right)}{C\left(m_{W_i}, \frac{m_W}{m_{W_i}}\right)} C\left(m_{V\cup W_i}, \frac{m_V * m_{W_i}}{m_{V\cup W_i}}\right)^{-1}
\end{aligned}$$

$$= \sum_{i \in V} e_{V_i \cup W} (-1)^{\sigma(i,V)} \frac{m_V * m_W}{m_{V_i \cup W}} \frac{C\left(\frac{m_{V_i} * m_W}{m_{V_i \cup W}}, \frac{m_V}{m_{V_i}}\right)}{C\left(m_{V_i \cup W}, \frac{m_{V_i} * m_W}{m_{V_i \cup W}}\right)} \frac{C\left(\frac{m_V}{m_{V_i}}, m_W\right)}{C\left(m_W, \frac{m_V}{m_{V_i}}\right)} \frac{C\left(m_{V_i}, m_W\right)}{C\left(m_{V_i}, \frac{m_V}{m_{V_i}}\right)}$$

$$+ \sum_{i \in W} e_{V \cup W_i} (-1)^{|W| + \sigma(i,W)} \frac{m_V * m_W}{m_{V \cup W_i}} \frac{C\left(\frac{m_{W_i} * m_V}{m_{V \cup W_i}}, \frac{m_W}{m_{W_i}}\right)}{C\left(m_{V \cup W_i}, \frac{m_V * m_{W_i}}{m_{V \cup W_i}}\right)} \frac{C\left(m_V, m_{W_i}\right)}{C\left(m_{W_i}, \frac{m_W}{m_{W_i}}\right)}.$$

Therefore, by the commutative case it suffices to prove

$$\frac{C\left(\frac{m_{V_i} * m_W}{m_{V_i \cup W}}, \frac{m_V}{m_{V_i}}\right)}{C\left(m_{V_i \cup W}, \frac{m_{V_i} * m_W}{m_{V_i \cup W}}\right)} \frac{C\left(\frac{m_V}{m_{V_i}}, m_W\right)}{C\left(m_W, \frac{m_V}{m_{V_i}}\right)} \frac{C\left(m_{V_i}, m_W\right)}{C\left(m_{V_i}, \frac{m_V}{m_{V_i}}\right)} = \frac{C\left(\frac{m_{W_i} * m_V}{m_{V \cup W_i}}, \frac{m_W}{m_{W_i}}\right)}{C\left(m_{V \cup W_i}, \frac{m_V * m_{W_i}}{m_{V \cup W_i}}\right)} \frac{C\left(m_V, m_{W_i}\right)}{C\left(m_{W_i}, \frac{m_W}{m_{W_i}}\right)}.$$

Indeed,

$$\frac{C\left(\frac{m_{V_i} * m_W}{m_{V_i \cup W}}, \frac{m_V}{m_{V_i}}\right)}{C\left(m_{V_i \cup W}, \frac{m_{V_i} * m_W}{m_{V_i \cup W}}\right)} \frac{C\left(\frac{m_V}{m_{V_i}}, m_W\right)}{C\left(m_W, \frac{m_V}{m_{V_i}}\right)} \frac{C\left(m_{V_i}, m_W\right)}{C\left(m_{V_i}, \frac{m_V}{m_{V_i}}\right)}$$

$$= \frac{C\left(\frac{m_{V_i} * m_W}{m_{V_i \cup W}}, \frac{m_V}{m_{V_i}}\right)}{C\left(m_{V_i \cup W}, \frac{m_{V_i} * m_W}{m_{V_i \cup W}}\right)} \frac{C\left(m_V, m_W\right)}{C\left(m_W * m_{V_i}, \frac{m_V}{m_{V_i}}\right)}$$

$$= \frac{C\left(m_{V_i \cup W}, \frac{m_V}{m_{V_i}}\right)^{-1}}{C\left(m_{V_i \cup W}, \frac{m_{V_i} * m_W}{m_{V_i \cup W}}\right)} C\left(m_V, m_W\right)$$

$$= C\left(m_{V_i \cup W}, \frac{m_V * m_W}{m_{V_i \cup W}}\right)^{-1} C\left(m_V, m_W\right).$$

Similarly,

$$\frac{C\left(\frac{m_V * m_{W_i}}{m_{V\cup W_i}}, \frac{m_W}{m_{W_i}}\right)}{C\left(m_{W_i}, \frac{m_W}{m_{W_i}}\right)} \frac{C\left(m_V, m_{W_i}\right)}{C\left(m_{V\cup W_i}, \frac{m_V * m_{W_i}}{m_{V\cup W_i}}\right)}$$

$$= C\left(\frac{m_{V\cup W_i}}{m_V}, \frac{m_W}{m_{W_i}}\right)^{-1} \frac{C\left(m_V, m_{W_i}\right)}{C\left(m_{V\cup W_i}, \frac{m_V * m_{W_i}}{m_{V\cup W_i}}\right)}$$

$$= C\left(m_V, m_W\right) C\left(m_V, m_{W_i}\right)^{-1} C\left(m_{V\cup W_i}, \frac{m_W}{m_{W_i}}\right)^{-1} \frac{C\left(m_V, m_{W_i}\right)}{C\left(m_{V\cup W_i}, \frac{m_V * m_{W_i}}{m_{V\cup W_i}}\right)}$$

$$= C\left(m_V, m_W\right) C\left(m_{V\cup W_i}, \frac{m_V * m_W}{m_{V\cup W_i}}\right)^{-1}.$$

Since $V \cap W = \{i\}$, it is easy to see $V_i \cup W = V \cup W_i$. Hence, $C\left(m_{V_i\cup W}, \frac{m_V * m_W}{m_{V_i\cup W}}\right)^{-1} C\left(m_V, m_W\right) = C\left(m_V, m_W\right) C\left(m_{V\cup W_i}, \frac{m_V * m_W}{m_{V\cup W_i}}\right)^{-1}$, finishing the proof.

Case 3: Suppose $V \cap W = \varnothing$ and $i \in V \backslash W$. If we expand the left hand side of our equation, then

$$\partial(e_V e_W) = \partial\left(e_{V\cup W}(-1)^{\sigma(V,W)} \frac{m_V * m_W}{m_{V\cup W}} C\left(m_V, m_W\right) C\left(m_{V\cup W}, \frac{m_V * m_W}{m_{V\cup W}}\right)^{-1}\right)$$

$$= \sum_{i\in V\cup W} e_{(V\cup W)_i}(-1)^{\sigma(V,W)+\sigma(i,V\cup W)} \frac{m_{V\cup W}}{m_{(V\cup W)_i}} \frac{m_V * m_W}{m_{V\cup W}} \widetilde{C}_{V,W,i}$$

$$= \sum_{i\in V\cup W} e_{(V\cup W)_i}(-1)^{\sigma(V,W)+\sigma(i,V\cup W)} \frac{m_V * m_W}{m_{(V\cup W)_i}} C\left(\frac{m_{V\cup W}}{m_{(V\cup W)_i}}, \frac{m_V * m_W}{m_{V\cup W}}\right) \widetilde{C}_{V,W,i},$$

where $\widetilde{C}_{V,W,i} = C\left(m_{(V\cup W)_i}, \frac{m_{V\cup W}}{m_{(V\cup W)_i}}\right)^{-1} C\left(m_V, m_W\right) C\left(m_{V\cup W}, \frac{m_V * m_W}{m_{V\cup W}}\right)^{-1}$. By the commutative case it suffices to show

$$C\left(\frac{m_{V\cup W}}{m_{(V\cup W)_i}}, \frac{m_V * m_W}{m_{V\cup W}}\right) \widetilde{C}_{V,W,i} = \frac{C\left(\frac{m_{V_i} * m_W}{m_{V_i\cup W}}, \frac{m_V}{m_{V_i}}\right)}{C\left(m_{V_i\cup W}, \frac{m_{V_i} * m_W}{m_{V_i\cup W}}\right)} \frac{C\left(\frac{m_V}{m_{V_i}}, m_W\right)}{C\left(m_W, \frac{m_V}{m_{V_i}}\right)} \frac{C\left(m_{V_i}, m_W\right)}{C\left(m_{V_i}, \frac{m_V}{m_{V_i}}\right)}.$$

Notice,

$$C\left(\frac{m_{V\cup W}}{m_{(V\cup W)_i}}, \frac{m_V * m_W}{m_{V\cup W}}\right)\widetilde{C}_{V,W,i} = \frac{C\left(\frac{m_{V\cup W}}{m_{(V\cup W)_i}}, \frac{m_V * m_W}{m_{V\cup W}}\right)}{C\left(m_{V\cup W}, \frac{m_V * m_W}{m_{V\cup W}}\right)} \frac{C\left(m_V, m_W\right)}{C\left(m_{V\cup W_i}, \frac{m_{V\cup W}}{m_{V\cup W_i}}\right)}$$

$$= C\left(\frac{1}{m_{(V\cup W)_i}}, \frac{m_V * m_W}{m_{V\cup W}}\right) C\left(\frac{1}{m_{(V\cup W)_i}}, \frac{m_{V\cup W}}{m_{(V\cup W)_i}}\right) C\left(m_V, m_W\right)$$

$$= C\left(m_{(V\cup W)_i}, \frac{m_V * m_W}{m_{(V\cup W)_i}}\right)^{-1} C\left(m_V, m_W\right).$$

Since $V\cap W = \varnothing$ and $i \in V\backslash W$, it is easy to see $(V\cup W)_i = V_i \cup W$. Therefore, the proof follows from the simplification made in case 2.

Case 4: Suppose $V\cup W = \varnothing$ and $i \in W\backslash V$. Then it is easy to see $(V\cup W)_i = V\cup W_i$. Hence, the proof follows similarly to case 3. Thus, the Leibniz rule holds for the product.

The proof for associativiy is similar to the format of the proof of the Leibniz rule. By the commutative case it suffices to show the constants of the products $(e_V e_W)\, e_X$ and $e_V\, (e_W e_X)$ are the same. Expanding out $(e_V e_W)\, e_X$ we have the constants,

$$C\left(m_{V\cup W}, m_X\right)\chi\left(\frac{m_V * m_W}{m_{V\cup W}}, m_X\right) \frac{C\left(m_V, m_W\right)}{C\left(m_{V\cup W\cup X}, \frac{m_{V\cup W} * m_X}{m_{V\cup W\cup X}}\right)} \frac{C\left(\frac{m_{V\cup W} * m_X}{m_{V\cup W\cup X}}, \frac{m_V * m_W}{m_{V\cup W}}\right)}{C\left(m_{V\cup W}, \frac{m_V * m_W}{m_{V\cup W}}\right)}$$

$$= C\left(m_{V\cup W}, m_X\right) \frac{C\left(\frac{m_V * m_W}{m_{V\cup W}}, m_X\right)}{C\left(m_X, \frac{m_V * m_W}{m_{V\cup W}}\right)} \frac{C\left(m_V, m_W\right)}{C\left(m_{V\cup W\cup X}, \frac{m_{V\cup W} * m_X}{m_{V\cup W\cup X}}\right)} C\left(\frac{m_X}{m_{V\cup W\cup X}}, \frac{m_V * m_W}{m_{V\cup W}}\right)$$

$$= C\left(m_V * m_W, m_X\right) C\left(\frac{1}{m_X}, \frac{m_V * m_W}{m_{V\cup W}}\right) \frac{C\left(m_V, m_W\right)}{C\left(m_{V\cup W\cup X}, \frac{m_{V\cup W} * m_X}{m_{V\cup W\cup X}}\right)} C\left(\frac{m_X}{m_{V\cup W\cup X}}, \frac{m_V * m_W}{m_{V\cup W}}\right)$$

$$= C\left(m_V * m_W, m_X\right) C\left(m_V, m_W\right) C\left(m_{V\cup W\cup X}, \frac{m_{V\cup W} * m_X}{m_{V\cup W\cup X}}\right)^{-1} C\left(m_{V\cup W\cup X}, \frac{m_V * m_W}{m_{V\cup W}}\right)^{-1}$$

$$= C\left(m_V * m_W, m_X\right) C\left(m_V, m_W\right) C\left(m_{V\cup W\cup X}, \frac{m_V * m_W * m_X}{m_{V\cup W\cup X}}\right)^{-1}.$$

Expanding the product $e_V\left(e_We_X\right)$ we have constants,

$$C\left(m_W,m_X\right)\frac{C\left(m_V,m_{W\cup X}\right)}{C\left(m_{V\cup W\cup X},\frac{m_V*m_{W\cup X}}{m_{V\cup W\cup X}}\right)}\frac{C\left(\frac{m_V*m_{W\cup X}}{m_{V\cup W\cup X}},\frac{m_W*m_X}{m_{W\cup X}}\right)}{C\left(m_{W\cup X},\frac{m_W*m_X}{m_{W\cup X}}\right)}$$

$$=C\left(m_W,m_X\right)\frac{C\left(m_V,m_{W\cup X}\right)}{C\left(m_{V\cup W\cup X},\frac{m_V*m_{W\cup X}}{m_{V\cup W\cup X}}\right)}C\left(\frac{m_V}{m_{V\cup W\cup X}},\frac{m_W*m_X}{m_{W\cup X}}\right)$$

$$=C\left(m_W,m_X\right)\frac{C\left(m_V,m_{W\cup X}\right)}{C\left(m_{V\cup W\cup X},\frac{m_V*m_{W\cup X}}{m_{V\cup W\cup X}}\right)}\frac{C\left(m_V,\frac{m_W*m_X}{m_{W\cup X}}\right)}{C\left(m_{V\cup W\cup X},\frac{m_W*m_X}{m_{W\cup X}}\right)}$$

$$=C\left(m_W,m_X\right)C\left(m_V,m_W*m_X\right)C\left(m_{V\cup W\cup X},\frac{m_V*m_W*m_X}{m_{V\cup W\cup X}}\right)^{-1}$$

$$=C\left(m_W,m_X\right)C\left(m_V,m_X\right)C\left(m_V,m_W\right)C\left(m_{V\cup W\cup X},\frac{m_V*m_W*m_X}{m_{V\cup W\cup X}}\right)^{-1}$$

$$=C\left(m_W*m_V,m_X\right)C\left(m_V,m_W\right)C\left(m_{V\cup W\cup X},\frac{m_V*m_W*m_X}{m_{V\cup W\cup X}}\right)^{-1}.$$

Thus, the constants are the same, so associativity holds.

We have proven the color Taylor Complex has a differential graded algebra structure given by the product above, therefore we only need to show color commutativity, i.e. $e_Ve_W=(-1)^{|V|\cdot|W|}\chi\left(e_V,e_W\right)e_We_V$. Similar to the proofs for the Leibniz rule and associativity, by the commutative case it suffices to show the constants are the same. By expanding the right side we obtain the constants

$$\chi\left(e_V,e_W\right)C\left(m_W,m_V\right)C\left(m_{V\cup W},\frac{m_W*m_V}{m_{V\cup W}}\right)^{-1}$$

$$=\chi\left(m_V,m_W\right)C\left(m_W,m_V\right)C\left(m_{V\cup W},\frac{m_W*m_V}{m_{V\cup W}}\right)^{-1}$$

$$=\frac{C\left(m_V,m_W\right)}{C\left(m_W,m_V\right)}C\left(m_W,m_V\right)C\left(m_{V\cup W},\frac{m_W*m_V}{m_{V\cup W}}\right)^{-1}$$

$$=C\left(m_V,m_W\right)C\left(m_{V\cup W},\frac{m_W*m_V}{m_{V\cup W}}\right)^{-1},$$

the constants from the left hand side. Therefore, the Taylor Complex has a color commutative differential graded algebra structure from the product above. □

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DESIREE MARTIN

163 South Central St, Broadway VA, 22815
(540)421-9809 ⋄ martdl18@wfu.edu

EDUCATION

Wake Forest University, Winston-Salem *August 2018 - Present*
Master's of Arts in Mathematics GPA: 3.84

James Madison University, Harrisonburg *August 2014 - May 2018*
Bachelor's of Science in Mathematics with a minor in Geophysics Overall GPA: 3.27

RESEARCH

Master's Thesis, WFU *March 2019 - Present*

For my master's thesis I worked under Dr. Luigi Ferraro and Dr. Frank Moore to generalize the construction of the Taylor complex, found in Taylor's thesis, to monomial ideals over a skew polynomial ring. We also provide an associative product that satisfies a generalized version of the Leibniz rule, making the Taylor complex a color differential graded (DG) algebra, generalizing a result of Gemeda. We have also defined a divided power structure on the complex, and we are currently investigating the color DG Lie algebra of derivations of the Taylor complex.

PROFESSIONAL EXPERIENCE AND SKILLS

Teaching Assistant and Tutor

- **Wake Forest University Teaching Assistant** *August 2019 - Present*
  - Calculus I, III
  - Differential Equations
  - Abstract Algebra
  - Discrete Mathematics
- **Wake Forest University Tutor** *August 2018 - Present*
  - Calculus I, II, III
  - Linear Algebra and Differential Equations
  - Discrete Mathematics
  - Abstract Algebra

FORMAL RESEARCH PRESENTATIONS

AWM Brown Bag Talk *November, 14th 2018*
"Taylor Resolutions and Skew Polynomial Rings"

AMS Gainesville Sectional Meeting (Invited) *November, 2nd 2018*
"Taylor Resolutions Over Skew Polynomial Rings"

AWM Summer Brown Bag Talk *August, 14th 2018*
"Commutative Constants"

EXTRACURRICULAR ACTIVITIES & VOLUNTEERING

WFU Association for Women in Mathematics President *May 2018 - Present*

WFU Pi Mu Epsilon Member *April 2018 - Present*

AWM Piedmont-Triad Conference Planning Committee *October 2018 - March 2019*

The Piedmont-Triad Conference primarily focused on presentations given by the undergraduate and graduate students of the surrounding area. We also prepared career and academia panels for students who were interested in applying to jobs or graduate schools in mathematics.

Volunteer for MathCounts at WFU *February 2018*

MathCounts is a regional middle school level mathematics competition we host at Wake Forest University each year. As volunteers we distribute, proctor, and collect the examinations for each section to be graded. We then host the 'Count Down Round' where students face off during speed problem solving rounds.

Volunteer Instructor for Math 167 at JMU *January 2018 - May 2018*

Math 167 is a collaborative class constructed to inform students of the different mathematics classes that might be offered at James Madison University. Each instructor picked multiple classes to give a hands on overview of different topics.

GRADUATE COURSES AND BOOKS AT WFU

Algebra and Number Theory

- **Algebra I (MST 721)**
  Dummit and Foote, *Abstract Algebra*, ed. 3
- **Algebra II (MST 722)**
  Dummit and Foote, *Abstract Algebra*, ed. 3
- **Algebraic Number Theory (MST 646)**
  Stewart and Tall, *Algebraic Number Theory and Fermats Last Theorem*, ed. 4
- **Linear Algebra II (MST 624)**
  Axler, *Linear Algebra Done Right*, ed. 3
- **Commutative Algebra (MST 728)**
  Atiyah and MacDonald, *Introduction to Commutative Algebra*

Analysis

- **Real Analysis I (MST 711)**
  Raynor and Robinson, *Analysis on Metric Spaces: A First Graduate Analysis*
- **Measure Theory (MST 712)**
  Royden and Fitzpatrick, *Real Analysis*, ed. 4

Topology

- **Topology (MST 731)**
  Lee, *An Introduction to Topological Manifolds*, ed. 2
- **Algebraic Topology (MST 732)**
  Lee, *An Introduction to Topological Manifolds*, ed. 2

Other

- **Generating Functions (MST 649)**
  Wilf, *generatingfunctionology*