ON THE NUMBER OF REPRESENTATIONS BY POSITIVE-DEFINITE INTEGER-VALUED QUATERNARY QUADRATIC FORMS

BY

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Contents

Abstract

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Let $\{Q_1, Q_2, \ldots, Q_s\}$ be a finite set of positive-definite integer-valued quaternary quadratic forms. We show that there exists a primitive positive-definite integer-valued quaternary quadratic form Q and a positive integer n such that Q represents n more times than Q_i for all $1 \leq i \leq s$.

Chapter 1: Introduction and Statement of Result

In 1747, Leonhard Euler proved Pierre de Fermat's theorem stating that an odd prime p can be written as a sum of two squares if and only if $p \equiv 1 \pmod 4$ (see Chapter 6 in [3]). In 1770, Joseph-Louis Lagrange proved that every positive integer can be written as a sum of four squares (see [14]). In 1798, Adrien-Marie Legendre gave an elementary proof to show that a positive integer n can be written as a sum of three squares if and only if $n \neq 4^a(8b+7)$ for all non-negative integers a and b (see Chapter 7 in [3]). In 1916, Srinivasa Ramanujan [12] claimed that there are precisely 55 4-tuples of positive integers (a, b, c, d) such that every positive integer is of the form $ax^2 + by^2 + cz^2 + dw^2$. Later in 1927, Leonard Eugene Dickson [4] proved Ramanujan's claim with the modification that every positive integer besides 15 is of the form $x^2 + 2y^2 + 5z^2 + 5w^2$.

Efforts to prove theorems concerning homogeneous polynomials of degree 2 gave rise to the modern theory about which integers can be represented by a quadratic form. In 1993, John Horton Conway and William Alan Schneeberger proved the following theorem about general quadratic forms (see [16]).

Theorem 1 ("The 15-Theorem")**.** *A positive-definite integer-matrix quadratic form represents all positive integers if and only if it represents the numbers*

$$1, 2, 3, 5, 6, 7, 10, 14, \textit{and } 15.$$

In 2005, Manjul Bhargava and Jonathan Hanke proved a similar theorem (see [7]).

Theorem 2 ("The 290-Theorem")**.** *If a positive-definite integer-valued quadratic form rep-*

resents the 29 integers

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29,$$
$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \textit{ and } 290,$$

then it represents all positive integers.

Instead of choosing integers as the values of variables for a quadratic form Q, we could also consider Q over the local ring of p-adic integers where p is a prime number. More specifically, we look for solutions such that Q represents an integer modulo every power of p. In 1929, Vladimir Abramovich Tartakovskii proved the following theorem concerning quadratic forms in more than 4 variables (see [21]).

Theorem 3. *If Q is a positive-definite quadratic form over $\mathbb{Z}$ in more than 4 variables, then every sufficiently large positive integer that is locally represented by Q is represented by Q.*

Beyond the questions about whether a quadratic form Q represents a set of integers, mathematicians are also interested in how many different ways Q can represent them. For example, in 1834, Carl Gustav Jacob Jacobi strengthened Lagrange's four-square theorem and gave a formula for $r_4(n)$, which is the number of ways a positive integer n can be written as a sum of four squares:

$$r_4(n) = \begin{cases} 8\sum\limits_{d|n} d & \text{if } n \text{ is odd} \\ 24\sum\limits_{\substack{d|n \\ d \text{ odd}}} d & \text{if } n \text{ is even} \end{cases}$$

(see Theorem 386 in [22]).

We are not satisfied with only knowing the number of representations by a quadratic form and are interested in comparing it with others. This leads us to a further question: which quadratic form represents a positive integer the maximal number of times? It motivates us to prove the following result concerning quadratic forms in 4 variables.

Theorem 4. *Let $\{Q_1, Q_2, \ldots, Q_s\}$ be a finite set of positive-definite integer-valued quaternary quadratic forms. There exists a primitive positive-definite integer-valued quaternary quadratic form Q and a positive integer n such that Q represents n more times than Q_i for all $1 \leq i \leq s$.*

For a quadratic form Q, there exists a theta series $\theta(z)$ attached to it where the number of representations of n by Q, $r_Q(n)$, is the coefficient of the n^{th} term of $\theta(z)$. The theory of modular forms allows us to decompose $\theta(z)$ as a sum of an Eisenstein series and a cusp form. This also gives us a decomposition of $r_Q(n)$ as the sum of $a_E(n)$ and $a_C(n)$ where $a_E(n)$ is the coefficient of the "Eisenstein piece" and $a_C(n)$ is the coefficient of the "cusp form piece". It turns out that the Eisenstein coefficients are large and predictable. On the other hand, the cusp form coefficients are small and mysterious. Since every quadratic form has weaknesses at some prime numbers, then we could choose a sequence of appropriate prime numbers to amplify the weaknesses associated to the given quadratic forms. Such a choice of prime numbers constructs a positive integer n so that we can efficiently bound the Eisenstein pieces associated to the given quadratic forms and the numbers of ways they represent n.

In Chapter 2, we define notations about quadratic forms and provide the background on local representations, modular forms, and the decomposition of the number of representations. In Chapter 3, we prove a series of requisite lemmas and conclude the main result.

Chapter 2: Background and Notation

2.1 Basics and Notations about Quadratic Forms

Definition 5. *A quadratic form in r variables is a function $Q : \mathbb{R}^r \to \mathbb{R}$ given by*

$$Q(\vec{x}) = \frac{1}{2}\vec{x}^{\mathsf{T}} A\vec{x}$$

where A is a symmetric $r \times r$ matrix. If $r = 4$, then Q is called a quaternary quadratic form.

If we expand a quadratic form Q, then Q is also a homogenous polynomial of degree 2. In the remainder of the thesis, we will assume that a given quadratic form Q is quaternary.

Definition 6. *A quadratic form Q is positive-definite if $Q(\vec{x}) \geq 0$ for all $\vec{x} \in \mathbb{R}^4$ and $Q(\vec{x}) = 0$ if and only if $\vec{x} = \vec{0}$.*

Definition 7. *A positive-definite quadratic form Q is an integer-matrix form if $Q(\vec{x}) = \frac{1}{2}\vec{x}^{\mathsf{T}} A\vec{x}$ where $\frac{1}{2}A$ is a matrix with integer entries. We say that Q is integer-valued if A has integer entries and even diagonal entries.*

For the rest of the thesis, a positive-definite integer-valued quaternary quadratic form will be abbreviated as PDIVQQF.

Definition 8. *A quadratic form Q is primitive if the content of Q is 1, that is, if its coefficients are coprime.*

Definition 9. *If $Q = \frac{1}{2}\vec{x}^{\mathsf{T}} A\vec{x}$ is a quadratic form, We call $\det(A)$ the discriminant of a quadratic from Q and denote it as $D(Q)$. The level of a quadratic form Q is the smallest positive integer $N(Q)$ such that $N(Q)A^{-1}$ has integer entries and even diagonal entries.*

For example, the primitive PDIVQQF

$$Q(\vec{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_4 + x_2x_4 + x_3x_4$$

can be written as $Q = \frac{1}{2}\vec{x}^{\mathsf{T}}A\vec{x}$ where

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

We can compute that $D(Q) = 4$ and $N(Q) = 2$.

Definition 10. *A lattice is an abelian group L that is isomorphic to $\mathbb{Z}^r$ for some positive integer r, together with an inner product $\langle \cdot, \cdot \rangle$ satisfying the following properties:*

(i) For all $\vec{x}, \vec{y}, \vec{z}$ in L, $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$.

(ii) For all $\vec{x}, \vec{y}$ in L, $\langle x, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

(iii) For all $\vec{x}$ in L, $\langle \vec{x}, \vec{x} \rangle \geq 0$.

If L' is a subgroup of L, then we call L' a sublattice of L.

Given a positive-definite quadratic form Q, we can construct a lattice L from Q by letting $L = \mathbb{Z}^4$ and defining

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{2}\big(Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y})\big).$$

Meanwhile, given a lattice L, the function $Q(\vec{x}) = \langle \vec{x}, \vec{x} \rangle$ is a quadratic form.

Definition 11. *A lattice L is integral if $\langle \vec{x}, \vec{x} \rangle \in \mathbb{Z}$ for all $\vec{x} \in L$. We say that an integral lattice L is maximal if there is no integral lattice $L' \neq L$ such that $L \subseteq L'$.*

Definition 12. *Suppose that R is a ring and 1 is its multiplicative identity. An R-module is an abelian group M together with a operation $\alpha : R \times M \to M$, for which we write $\alpha(r, m) = r \cdot m$, satisfying the following conditions:*

(i) For all $r, s \in R$ and $m \in M$, $(r + s) \cdot m = r \cdot m + s \cdot m$.

(ii) For all $r \in R$ and $m, n \in M$, $r \cdot (m + n) = r \cdot m + r \cdot n$.

(iii) For all $r, s \in R$ and $m \in M$, $r \cdot (s \cdot m) = (rs) \cdot m$.

(iv) For all $m \in M$, $1 \cdot m = m$.

Definition 13. *Let R be a ring. We define the characteristic of R to be the smallest positive integer c such that*

$$\sum_{i=1}^{c} 1 = 0.$$

If such a positive integer c does not exist, then we say that the characteristic of R is 0. A subset I of a commutative ring R is called an ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $rx \in I$ for every $r \in R$ and $x \in I$. If R is a nonzero commutative ring in which the product of any two nonzero elements is nonzero, then we say that R is an integral domain. If there exists $a \in I$ such that $I = aR$, then we say that I is principal. An integral domain R is called a principal ideal domain if every ideal of R is principal.

Let R be an integral domain. Given $(a_1, b_1), (a_2, b_2) \in R \times R$ where $b_1 \neq 0$ and $b_2 \neq 0$, we say that (a_1, b_1) is equivalent to (a_2, b_2) if $a_1 b_2 = a_2 b_1$. For $a, b \in R$ with $b \neq 0$, let $\frac{a}{b}$ denote the equivalence class of (a, b). Let Q be the set of all such equivalence classes. We define $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b)(c, d) = (ac, bd)$. Then we say that Q is the *quotient field* of R.

Definition 14. *Let R be a principal ideal domain of characteristic not 2 and F be the quotient field of R with $R \neq F$. If a lattice L is a R-module, then we say that L is an R-lattice.*

Definition 15. *We say that a function $\chi : \mathbb{Z} \to \mathbb{C}^*$ is a Dirichlet character modulo k if χ satisfies the following 3 conditions:*

(i) *There exists a positive integer k such that $\chi(n+k) = \chi(n)$ for all integers n.*

(ii) *$\chi(mn) = \chi(m)\chi(n)$ for all integers m and n.*

(iii) *$\chi(n) \neq 0$ if and only if $\gcd(n, k) = 1$.*

The *principal Dirichlet character* χ_0 modulo k is defined by $\chi_0(n) = 1$ if $\gcd(n, k) = 1$ and $\chi_0(n) = 0$ if $\gcd(n, k) > 1$.

Definition 16. *An integer D is called a fundamental discriminant if D satisfies one of the following conditions:*

(i) *$D \equiv 1 \pmod 4$ and D is square-free.*

(ii) *$D = 4m$ where $m \equiv 2$ or $3 \pmod 4$ and m is square-free.*

Definition 17. *Let p be an odd prime number. If x is an integer such that $x \equiv y^2 \pmod p$ for some integer y, then we call x a quadratic residue modulo p. If $x \not\equiv y^2 \pmod p$ for all integers y, then we call x a quadratic nonresidue modulo p. The Legendre symbol of an integer a and p is defined as*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod p \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p \\ 0 & a \equiv 0 \pmod p. \end{cases}$$

Besides simply determining whether an integer is a quadratic residue modulo an odd prime p, we can derive the number of such quadratic residues in the set of residue classes modulo p, that is, $\{0, 1, 2, , p-1\}$. If $a^2 \equiv b^2 \pmod p$ with $a \neq b$, then $(a-b)(a+b) \equiv 0 \pmod p$.

This implies that either $a-b \equiv 0 \pmod{p}$ or $a+b \equiv 0 \pmod{p}$. Since a and b are different elements in $\{0, 1, 2, , p-1\}$, then $a+b \equiv 0 \pmod{p}$. For each element $x \in \{0, 1, 2, , p-1\}$, $x \neq p-x$ and $x+(p-x) \equiv 0 \pmod{p}$. It follows that each quadratic residue modulo p is congruent to one of $0^2, 1^2, 2^2, \ldots, (\frac{p-1}{2})^2$. Thus, there are exactly $\frac{p+1}{2}$ quadratic residues modulo p and $\frac{p-1}{2}$ quadratic nonresidues modulo p in $\{0, 1, 2, , p-1\}$.

Furthermore, the Legendre symbol can be generalized to the Jacobi symbol defined below.

Definition 18. *Let a be any integer and n be a positive odd integer. Let the prime factorization of n be $n = \prod_{i=1}^{k} {p_i}^{\alpha_i}$. Then the Jacobi symbol of a and n is defined as*

$$\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{\alpha_i}$$

where $\left(\frac{a}{p_i}\right)$ is the Legendre symbol of a and p_i for each $1 \leq i \leq k$.

The law of quadratic reciprocity for Jacobi symbols shows that

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{(a-1)(b-1)}{4}}$$

if a and b are positive odd integers with $\gcd(a, b) = 1$ (see Theorem 9.11 in [2]). The Jacobi symbol can also be further extended to the Kronecker symbol.

Definition 19. *We define the Kronecker symbol $\left(\frac{a}{n}\right)$ for any $a, n \in \mathbb{Z}$ in the following way:*

(i)

$$\left(\frac{a}{0}\right) = \begin{cases} 1 & \text{if } a = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\frac{a}{-1}\right) = \begin{cases} -1 & \text{if } a < 0 \\ 1 & \text{if } a \geq 0, \end{cases}$$

and

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \text{ is even} \\ (-1)^{\frac{a^2-1}{8}} & \text{if } a \text{ is odd}. \end{cases}$$

(ii) If $n \neq 0$ *and* $n = u\prod_{i=1}^{k} p_i^{\alpha_i}$ *where* $u = \pm 1$ *indicates the sign of* n *and* $\prod_{i=1}^{k} p_i^{\alpha_i}$ *is the prime factorization of* $|n|$*, then*

$$\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right)\prod_{i=1}^{k}\left(\frac{a}{p_i}\right)^{\alpha_i}.$$

Here and throughout, $\chi_{D(Q)}$ denotes the Kronecker symbol defined by

$$\chi_{D(Q)}(n) = \left(\frac{D(Q)}{n}\right).$$

Section 9.3 in [13] shows that $\chi_{D(Q)}$ is a Dirichlet character if $D(Q)$ is a fundamental discriminant.

Lemma 20. *Let* Q *be a PDIVQQF. If* $D(Q)$ *is not a square, then* $\chi_{D(Q)}$ *is a non-principal Dirichlet character.*

Proof. Let Q be an arbitrary PDIVQQF such that $D(Q)$ is not a square. We will show that there exists a prime number p such that $p \nmid D(Q)$ and $D(Q)$ is a quadratic nonresidue modulo p. If $D(Q)$ is not square-free, then let $u > 1$ be the largest positive integer such that $u^2 \mid D(Q)$. Let $D'(Q) = \frac{D(Q)}{u^2}$. For any prime number $p \nmid D(Q)$,

$$\left(\frac{D(Q)}{p}\right) = \left(\frac{D'(Q)}{p}\right)\left(\frac{u^2}{p}\right) = \left(\frac{D'(Q)}{p}\right).$$

Therefore, without loss generality, we may assume that $D(Q)$ is square-free. Let $D(Q) = 2^{r_0}\prod_{j=1}^{n} {p_j}^{r_j}$ be the prime factorization of $D(Q)$ where $r_0 = 0$ or 1 and p_j is odd for all $1 \le j \le n$. Let t be a quadratic nonresidue modulo p_n. By the Chinese remain theorem (see Theorem 5.26 in [2]), the system of congruences

$$x \equiv 1 \pmod 8$$

$$x \equiv 1 \pmod{p_j} \text{ for all } 1 \le j \le n-1$$

$$x \equiv t \pmod{p_n}$$

has a solution $x \equiv s \pmod{8\prod_{i=1}^{n} p_i}$. Since $\gcd(s, 8\prod_{i=1}^{n} p_i) = 1$, then there exists some positive integer m such that $8m\prod_{i=1}^{n} p_i + s$ is a prime number by Theorem 10.9 in [10]. Let $p = 8m\prod_{i=1}^{n} p_i + s$. Thus,

$$\left(\frac{D(Q)}{p}\right) = \left(\frac{2}{p}\right)^{r_0} \prod_{j=1}^{n} \left(\frac{p_j}{p}\right)^{r_j}.$$

Since $a \equiv 1 \pmod 8$, then $\left(\frac{2}{p}\right) = 1$ by Theorem 9.5 in [2] and $\left(\frac{p_j}{p}\right) = \left(\frac{p}{p_j}\right) = \left(\frac{1}{p_j}\right) = 1$ for all $1 \le j \le n-1$ by the law of quadratic reciprocity. This implies that

$$\left(\frac{D(Q)}{p}\right) = \left(\frac{p_n}{p}\right) = \left(\frac{p}{p_n}\right) = \left(\frac{t}{p_n}\right) = -1.$$

Hence, $D(Q)$ is a quadratic nonresidue modulo p and $\chi_{D(Q)}$ is not a principal character. □

Definition 21. *Two quadratic forms Q_1 and Q_2 are equivalent over a ring R if there exists an invertible matrix P where all entries of P and P^{-1} are in R such that $Q_1(\vec{x}) = Q_2(P\vec{x})$.*

Let $Q_1(\vec{x}) = \frac{1}{2}\vec{x}^{\mathsf{T}} A_1 \vec{x}$ and $Q_2(\vec{x}) = \frac{1}{2}\vec{x}^{\mathsf{T}} A_2 \vec{x}$ be two quadratic forms. If $Q_1 \sim Q_2$ over R, then there exists an invertible matrix P where all entries of P and P^{-1} are in R such that

$A_1 = P^{\mathsf{T}} A_2 P$. We denote the equivalence of Q_1 and Q_2 as $Q_1 \sim Q_2$.

Definition 22. *The integral automorphism group, Aut(Q), of a quadratic form Q is*

$$\textit{Aut}(Q) = \{P \in M_4(\mathbb{Z}) : Q(\vec{x}) = Q(P\vec{x})\}.$$

where $M_4(\mathbb{Z})$ is the set of all 4×4 matrices with integer entries.

Definition 23. *Let n be a positive integer and Q be a PDIVQQF. If there exists a vector $\vec{x} \in \mathbb{Z}^4$ such that $Q(\vec{x}) = n$, then n is represented by Q. We let $r_Q(n) = \#\{\vec{x} \in \mathbb{Z}^4 : Q(\vec{x}) = n\}$ be the number of representations of n by Q.*

2.2 p-adic integers and Local Representation

Instead of working over $\mathbb{R}$, we can look at a quadratic form Q modulo powers of a prime number p to get local information.

Definition 24. *The p-adic valuation for $\mathbb{Z}$ is the function $\nu_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ given by*

$$\nu_p(n) = \begin{cases} \max\{v \in \mathbb{N} : p^v \mid n\} & \textit{if } n \neq 0 \\ \infty & \textit{if } n = 0. \end{cases}$$

The extension of the p-adic valuation to $\mathbb{Q}$ is the function $\nu_p : \mathbb{Q} \to \mathbb{Z}$ given by

$$\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b).$$

Definition 25. *The p-adic absolute value on $\mathbb{Q}$ is the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by*

$$|x|_p = \begin{cases} p^{-\nu_p(x)} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0. \end{cases}$$

The p-adic absolute value on $\mathbb{Q}$ gives a metric $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by

$$d(x, y) = |x - y|_p$$

and $\mathbb{Q}$ is not complete with respect to d. For each prime p, there exists a completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the associated p-adic metric d and $\mathbb{Q}_p$ is called the field of p-adic numbers. Every p-adic number x has a p-adic expansion:

$$x = a_{-m}p^{-m} + a_{-m+1}p^{-m+1} + \cdots + a_{-1}p^{-1} + a_0 + a_1p + a_2p^2 + \cdots$$

where $a_k \in \{0, 1, \ldots, p-1\}$ for all $k \geq -m$.

Definition 26. *x is called a p-adic integer if $a_k = 0$ for all $k < 0$ in the p-adic expansion of x, that is,*

$$x = a_0 + a_1 p + a_2 p^2 + \cdots$$

where $a_k \in \{0, 1, \ldots, p-1\}$ for all $k \geq 0$. The ring of all p-adic integers is denoted by $\mathbb{Z}_p$.

A non-negative integer n is said to be *locally represented by Q at a prime p* if there exists a vector $\vec{x}_p \in \mathbb{Z}_p^4$ such that $Q(\vec{x}_p) = n$. In other words, $Q(\vec{x}) = n$ has a solution modulo p^m for all $m \in \mathbb{N}$. If Q locally represents n at all prime numbers p, we say that n is *locally represented* by Q.

Definition 27. *A quadratic form Q is called anisotropic at the prime p if $Q(\vec{x}) = 0$ with $\vec{x} \in \mathbb{Z}_p^4$ implies that $\vec{x} = \vec{0}$.*

Now suppose that Q is a positive-definite integer-valued quadratic form and p is a prime number. The following definitions and notations about local representation are introduced by Jonathan Hanke in [6]. Let L be the $\mathbb{Z}_p$-lattice associated to Q with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{2}\big(Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y})\big).$$

Let m and k be positive integers. We define the sublattice

$$L^p = \{\vec{x} \in L : \langle \vec{x}, \vec{y} \rangle \equiv 0 \pmod{p} \text{ for all } \vec{y} \in L\}.$$

We also define

$$R_{p^k,Q}(m) = \{\vec{x} \in L/p^k L : Q(\vec{x}) \equiv m \pmod{p^k}\}$$

and

$$r_{p^k,Q}(m) = |R_{p^k,Q}(m)|.$$

If $L = {\mathbb{Z}_p}^4$, we say that a solution $\vec{x} \in R_{p^k,Q}(m)$ of $Q(\vec{x}) = m$ is of *good type* if $\vec{x} \in L$ and

$\vec{x} \notin L^p$; we say that $\vec{x}$ is of *bad type* if $\vec{x} \in L$, $\vec{x} \in L^p$, and $\vec{x} \notin pL$. We also define $r^{\text{Good}}_{p^k,Q}(m)$ to be the number of solutions of good type and $r^{\text{Bad}}_{p^k,Q}(m)$ to be the number of solutions of bad type.

Definition 28. *Let p be a prime number and Q be a positive-definite integer-valued quadratic form. We define a positive integer m to be p-stable for Q if m is locally represented by Q at p, and for all $k \gg 1$ the value of*

$$r^{\textit{Good}}_{p^k,Q}(p^{2v}m) + r^{\textit{Bad}}_{p^k,Q}(p^{2v}m)$$

is constant for all positive integers v.

2.3 Theta Series and Modular Forms

One of the main tools to obtain information about representations of a quadratic form Q is the theory of modular forms and the theta series attached to Q.

Definition 29. *Let Q be a quadratic form. The theta series of Q is the Fourier series generating function for the numbers of representations $r_Q(n)$ given by*

$$\theta(z) = \sum_{n=0}^{\infty} r_Q(n) q^n, \quad q = e^{2\pi i z}.$$

Definition 30. *Let N be a positive integer. We define*

$$\mathbb{H} = \{x + iy : y > 0 \textit{ and } x, y \in \mathbb{R}\},$$

$$SL_2(\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1\},$$

and

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \ (\mathrm{mod} N)\}.$$

Definition 31. *Let k be a nonnegative integer and N be a positive integer. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we define the weight-$k$ operator $[\gamma]_k$ on a function $g : \mathbb{H} \to \mathbb{C}$ by*

$$(g[\gamma]_k)(z) = (cz + d)^{-k} g\Big(\frac{az + b}{cz + d}\Big)$$

for all $z \in \mathbb{H}$. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is weakly modular of weight k, level N, and Dirichlet character χ if

$$(f[\gamma]_k)(z) = \chi(d) f(z) \textit{ for all } \gamma \in \Gamma_0(N) \textit{ and } z \in \mathbb{H}.$$

Note that any weakly modular function f of weight k, level N, and Dirichlet character

χ satisfies $f(z) = f(z+1)$ because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ and $f(z+1) = f(\frac{1 \cdot z+1}{0 \cdot z+1}) = \chi(1) \cdot 1^k \cdot f(z) = f(z)$. Therefore, $f(z)$ has the Fourier expansion

$$\sum_{n=-\infty}^{\infty} a_n q^n, \quad q = e^{2\pi i z}.$$

Definition 32. *A function $f : \mathbb{H} \to \mathbb{C}$ satisfying $f(z) = f(z+1)$ is called holomorphic at ∞ if the Fourier expansion of f is*

$$\sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z}.$$

Definition 33. *Let k be a nonnegative integer and N be a positive integer. A modular form of weight k, level N, and Dirichlet character χ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following 2 conditions:*

(i) f is weakly modular of weight k, level N, and Dirichlet character χ .

(ii) $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in SL_2(\mathbb{Z})$.

If Q is a quaternary quadratic form, then Theorem 10.9 in [8] implies that the theta series of Q is a modular form of weight 2, level $N(Q)$, and Dirichlet character $\chi_{D(Q)}$. Let k be an even integer greater than 2. For $z \in \mathbb{H}$,

$$G_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

is called an *Eisenstein series*. We call $f(z)$ a *cusp form*, if given $\gamma \in Sl_2(\mathbb{Z})$, the constant term of the Fourier expansion of $(f[\gamma])_k(z)$ is 0.

The theory of modular forms gives a decomposition of $\theta(z)$ as

$$\theta(z) = E(z) + C(z)$$

where $E(z) = \sum_{n=0}^{\infty} a_E(n)q^n$ is an Eisenstein series and $C(z) = \sum_{n=1}^{\infty} a_C(n)q^n$ is a cusp form (see Chapter 4 in [19]). Meanwhile, the n^{th} Fourier coefficient of the equation gives a decomposition of the number of representations of n by Q as

$$r_Q(n) = a_E(n) + a_C(n).$$

The Eisenstein piece in the decomposition contributes mostly to the size of $r_Q(n)$; the cusp form coefficient is very small compared to $a_E(n)$ and can be viewed as an error term.

Definition 34. *We define the genus of a quadratic form Q to be the set of all quadratic forms equivalent to Q over $\mathbb{Z}_p$ for all primes p and equivalent to Q over $\mathbb{R}$. The genus of Q is denoted by Gen(Q).*

The following theorem from the work of Von Carl Ludwig Siegel [20] shows that each Eisenstein coefficient corresponding to a quadratic form Q can be expressed as a weighted sum of theta series over the genus of Q.

Theorem 35. *Suppose that Q is a PDIVQQF and the decomposition of the theta series of Q is $\theta_Q(z) = E(z) + C(z)$. Then,*

$$E(z) = \frac{\sum_{R \in Gen(Q)} \frac{\theta_R(z)}{\#Aut(R)}}{\sum_{R \in Gen(Q)} \frac{1}{\#Aut(R)}} = \sum_{m=0}^{\infty} \prod_{p \leq \infty} \Big(\beta_p(Q; m)\Big) q^m,$$

where

$$\beta_p(Q; m) = \lim_{k \to \infty} \frac{\#\{\vec{x} \in (\mathbb{Z}/p^k\mathbb{Z})^4 : Q(\vec{x}) \equiv m \ (\pmod{p^k})\}}{p^{3k}}$$

is defined as the local density of Q at m over $\mathbb{Z}_p$.

More specifically, the Eisenstein coefficients $a_E(n)$ in the decomposition of the theta series of a PDIVQQF Q can be expressed as the infinite product of local densities, that is,

$$a_E(n) = \prod_{p \leq \infty} \beta_p(Q; m)$$

where $\beta_\infty(Q; m) = \frac{\pi^2 n}{\sqrt{D(Q)}}$ and p is a prime if $p \neq \infty$. This information is very important because bounding local densities is an efficient way to bound $a_E(n)$. In addition, given a primitive PDIVQQF Q such that $D(Q)$ is a fundamental discriminant, if p is a prime number, m is a positive odd integer, and n is a positive integer such that $m|n$, then Theorem 3.1 and Theorem 3.3 in [23] imply that

$$\beta_p(Q; m) \leq \beta_p(Q; n) \tag{1}$$

whenever p, m, and n satisfy one of the following conditions:

(i) $p \mid m$ and $\chi_{D(Q)}(p) = 1$.

(ii) $m \mid n$ and $\frac{n}{m}$ is a square modulo p.

Chapter 3: Proof

3.1 Preliminary Lemmas

In this section, we will prove a series of lemmas which will be used in the proof of Theorem 4 in the next section.

If F is a field, let F^* denote the multiplicative group of units in F; V denotes the union of the set of prime numbers and the symbol ∞, and we put $\mathbb{Q}_\infty = \mathbb{R}$; k denotes either $\mathbb{Q}_v$ for some $v \in V$ or $\mathbb{R}$; $\mathbb{F}_p$ denotes the field $\mathbb{Z}/p\mathbb{Z}$ where p is a prime number. If Q is a quadratic form, Theorem $1'$ in [17] shows that there exists a_1, a_2, a_3, and $a_4 \in k$ such that $Q \sim a_1{x_1}^2 + a_2{x_2}^2 + a_3{x_3}^2 + a_4{x_4}^2$.

Definition 36. *Let $Q \sim a_1{x_1}^2 + a_2{x_2}^2 + a_3{x_3}^2 + a_4{x_4}^2$ over a ring R be a quadratic form. We define the rank of Q to be the number of i such that $a_i \neq 0$.*

Definition 37. *Let $a, b \in k^*$ and $v \in V$. The Hilbert symbol is the function $(\cdot,\cdot) : k^* \times k^* \to \{-1, 1\}$ given by*

$$(a,b) = \begin{cases} 1 & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a nontrivial solution in } k^3 \\ -1 & \text{otherwise.} \end{cases}$$

There exists an embedding from $\mathbb{Q}$ to $\mathbb{Q}_v$ for each $v \in V$. If $a, b \in \mathbb{Q}^*$, then let a_v and b_v denote the images of a and b respectively under the embedding from $\mathbb{Q}$ to $\mathbb{Q}_v$. We define $(a,b)_v$ to be the value of (a_v, b_v) over $\mathbb{Q}_v$.

Definition 38. *Let $Q \sim a_1{x_1}^2 + a_2{x_2}^2 + a_3{x_3}^2 + a_4{x_4}^2$ over $\mathbb{Q}$ be a quadratic form of rank 4 and $v \in V$. We define*

$$\epsilon(Q) = \prod_{i<j} (a_i, a_j).$$

and

$$\epsilon_v(Q) = \prod_{i<j} (a_i, a_j)_v.$$

Lemma 39. *If Q is a positive-definite quadratic form of rank 4 and $D(Q)$ is a square, then R is anisotropic at some prime.*

Proof. In this proof, all the references are from [17]. Suppose that Q is an arbitrary positive-definite quadratic form of rank 4 and $D(Q)$ is a square. As shown in the middle of page 40, $Q \sim {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$ over $\mathbb{R}$. Since $z^2 - x^2 - y^2 = 0$ has nontrivial solutions over $\mathbb{R}$, then $(1,1)$=1. This implies that $\epsilon_\infty(Q) = 1$ because every coefficient of ${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$ is 1. Over $\mathbb{Q}_p$ where p is a prime, -1 is a unit. If p is odd, Theorem 1 on page 20 implies that

$$(-1,-1)_p = (-1)^{\nu(-1)\nu(-1)\epsilon(p)} \left(\frac{-1}{p}\right)^0 \left(\frac{-1}{p}\right)^0 = 1$$

where $\epsilon(p)$ denotes the class modulo 2 of -1. If $p = 2$,

$$(-1,-1)_2 = (-1)^{\epsilon(-1)\epsilon(-1)+\nu(-1)\omega(-1)+\nu(-1)\omega(-1)} = -1.$$

By Theorem 3,

$$\epsilon_\infty(Q)\epsilon_2(Q) \prod_{\substack{p \text{ prime} \\ p\neq 2}} \epsilon_p(Q) = 1.$$

Theorem 6 on page 37 implies that $Q(\vec{x}) = 0$ over $\mathbb{Q}_p$ for some nonzero $\vec{x}$ if $D(Q)$ is not a square or $D(Q)$ is a square and $\epsilon_p(Q) = (-1,-1)$. Thus, if $\epsilon_2(Q) = 1 = -(-1,-1)_2$, then Q has to be anisotropic over 2. If $\epsilon_2(Q) = -1$, there there exists some odd prime p such that $\epsilon_p(Q) = -1 = -(-1,-1)_p$. This implies that Q is anisotropic at p. □

An anisotropic prime associated to a given quadratic form Q where $D(Q)$ is a square or Q is a scalar multiple of a primitive quadratic form with square discriminant is useful to control the number of representations by Q.

Definition 40. *If $Q = \frac{1}{2}\vec{x}^T A\vec{x}$ is a quadratic form, we define the index of Q to be the number of positive eigenvalues of A and denote it as $s(\frac{1}{2}A)$. Also, we say that Q is reduced if Q satisfies the following condition: if there exists an integer-valued quadratic form R and an invertible matrix P with integer entries such that $Q(\vec{x}) = R(P\vec{x})$, then $|\det(P)| = 1$.*

Theorem 41. *Let $\sigma \leq 4$ be a positive even number, δ be a fundamental discriminant, and e_0, e_1 be square-free positive integers such that $e_0 \mid \delta$ and $(e_1, \delta) = 1$. Let r be the number of prime factors of e_0e_1. If $(-1)^{\frac{\sigma}{2}}\delta > 0$ and $\sigma - 4r \equiv 0 \pmod 8$, then there exists a reduced integer-valued quadratic form $Q = \frac{1}{2}\vec{x}^T A\vec{x}$ such that*

$$\det(A) = \delta e_1^2, s(\frac{1}{2}A) = \sigma, \text{and } N(Q) = \delta e_1.$$

Proof. See Theorem 6.4 in [18]. □

The following lemma implies the possible choice of a different quadratic form from the given ones. The existence of such a quadratic form allows us to construct an appropriate integer to be represented by it and the given ones.

Lemma 42. *If δ is a prime fundamental discriminant, then there exists a primitive PDIVQQF Q such that $D(Q) = \delta$.*

Proof. Suppose that δ is a prime fundamental discriminant. Let $\sigma = 4$, $e_0 = \delta$, $e_1 = 1$, and r be the number of prime factors of e_0e_1. Thus, $(-1)^{\frac{\sigma}{2}}\delta > 0$, $r = 1$ because δ is prime, and $\sigma - 4r \equiv 0 \pmod 8$. By Theorem 41, there exists an integer-valued quadratic form $Q = \frac{1}{2}\vec{x}^T A\vec{x}$ such that $|\det(A)| = |\delta|e_1^2 = \delta$, $s(\frac{1}{2}A) = \sigma = 4$, and $N(Q) = \delta e_1 = \delta$. It follows that A has 4 positive eigenvalues. This implies that A is positive-definite. Since $D(Q)$ is a prime, then Q is primitive. Otherwise, $Q = aQ'$ for some positive integer a and primitive PDIVQQF Q'. It follows that $a^4 \mid D(Q)$ but this is impossible. Thus, Q is a primitive PDIVQQF. □

Lemma 43. *If Q is a PDIVQQF and $D(Q)$ is a fundamental discriminant, then Q locally represents every positive integer.*

Proof. Let Q be an arbitrary PDIVQQF and $D(Q)$ is a fundamental discriminant. First, suppose that p is an odd prime. All the references in this case is from [17]. If $p \nmid D(Q)$, then Theorem 1 in [17] implies that there exists some nonzero $a_1, a_2, a_3, a_4 \in \mathbb{F}_p$ such that $Q \sim a_1{x_1}^2 + a_2{x_2}^2 + a_3{x_3}^2 + a_4{x_4}^2$ over $\mathbb{F}_p$. Thus, the rank of Q is 4 over $\mathbb{F}_p$. Proposition 4 shows that Q represents all elements of $\mathbb{F}_p$ because the rank of Q is greater than 3. If $p \mid D(Q)$, then $p^2 \nmid D(Q)$ since $D(Q)$ is a fundamental discriminant. It follows that there exist some nonzero $a_1, a_2, a_3 \in \mathbb{F}_p$ such that $Q \sim a_1{x_1}^2 + a_2{x_2}^2 + a_3{x_3}^2$ over $\mathbb{F}_p$. Thus, the rank of Q is 3 over $\mathbb{F}_p$. Proposition 4 again shows that Q represents all elements of $\mathbb{F}_p$. By the multidimensional Hensel's lemma on page 389 in [11], it follows that Q represents everything in $\mathbb{Z}_p$.

Next, suppose that $p = 2$. If $Q = \frac{1}{2}\vec{x}A\vec{x}$, then Lemma 1 in [9] shows the following 6 cases over $\mathbb{Z}_2$.

(i) A is equivalent to a diagonal form A' over $\mathbb{Z}_2$. This implies that $Q \sim Q'$ where $Q' = \frac{1}{2}\vec{x}A'\vec{x}$. Since $D(Q)$ is a fundamental discriminant in $\mathbb{R}$, then at least one of the diagonal entries of A' is a unit in $\mathbb{Z}_2$. Without loss of generality, we may assume that the diagonal entry in the first column of A' is a unit in $\mathbb{Z}_2$. It follows that Q represents $Q'(1,0,0,0)$ which is not in $\mathbb{Z}_2$. This contradicts the fact that Q is a PDIVQQF. Thus, this case is impossible.

(ii) $A \cong \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ where $A_1 = \begin{pmatrix} 2^{t_1}a & 0 \\ 0 & 2^{t_2}b \end{pmatrix}$ and $A_2 = 2^{t_3}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $2^{t_3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ for some $1 \le t_1 \le t_2 < t_3$ with $t_1, t_2, t_3 \in \mathbb{Z}$ and units $a, b \in \mathbb{Z}_2$. This implies that $4 \mid D(A_1)$ and $16 \mid D(A_2)$. Thus, $16 \mid D(Q) = D(A_1)D(A_2)$. This is impossible because $D(Q)$ is a fundamental discriminant.

(iii) $A \cong \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$ where $A_1 = \begin{pmatrix} 2^{t_1}a \end{pmatrix}$, $A_2 = 2^{t_2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $2^{t_2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and $A_3 = \begin{pmatrix} 2^{t_1}b \end{pmatrix}$ for some $1 \leq t_1 < t_2 < t_3$ with $t_1, t_2, t_3 \in \mathbb{Z}$ and units $a, b \in \mathbb{Z}_2$. It follows that $16 \mid D(Q)$. Thus, this case is impossible.

(iv) $A \cong \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = A'$ where $A_1 = 2^{t_1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $2^{t_1}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 2^{t_2}a & 0 \\ 0 & 2^{t_3}b \end{pmatrix}$ for some $0 \leq t_1 < t_2 \leq t_3$ with $t_1, t_2, t_3 \in \mathbb{Z}$ and units $a, b \in \mathbb{Z}_2$. It follows that $t_1 = 0$ and $t_2 = 1$. Thus,

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 2^{t_3}b \end{pmatrix} \text{ or } A' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 2^{t_3}b \end{pmatrix}.$$

This implies that $Q(x, y, z, w)$ is equivalent to a quadratic form $Q'(x, y, z, w) = xy + az^2 + 2^{t_3-1}bw^2$ or $x^2 + xy + y^2 + az^2 + 2^{t_3-1}bw^2$.

(v) $A \cong \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = A'$ where $A_1 = 2^{t_1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_2 = 2^{t_2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $2^{t_2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ for some $0 \leq t_1 \leq t_2$ with $t_1, t_2 \in \mathbb{Z}$. It follows that $t_1 = 0$. Thus,

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{t_2} \\ 0 & 0 & 2^{t_2} & 0 \end{pmatrix} \text{ or } A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2^{t_2+1} & 2^{t_2} \\ 0 & 0 & 2^{t_2} & 2^{t_2+1} \end{pmatrix}.$$

This implies that $Q(x, y, z, w)$ is equivalent to a quadratic form $Q'(x, y, z, w) = xy + 2^{t_2}zw$ or $xy + 2^{t_2}z^2 + 2^{t_2}zw + 2^{t_2}w^2$.

(vi) $A \cong \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = A'$ where $A_1 = 2^{t_1}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $A_2 = 2^{t_2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $2^{t_2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ for some $0 \leq t_1 < t_2$ with $t_1, t_2 \in \mathbb{Z}$. It follows that $t_1 = 0$ and $t_2 = 1$. Thus,

$$A' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \text{ or } A' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix}.$$

This implies that $Q(x,y,z,w)$ is equivalent to a quadratic form $Q'(x,y,z,w) = x^2 + xy + y^2 + 2zw$ or $x^2 + xy + y^2 + 2z^2 + 2zw + 2w^2$.

Therefore, over $\mathbb{Z}_2$, Q is equivalent to some quadratic form $Q'(x,y,z,w) = Q_1(x,y) + Q_2(z,w)$ where $Q_1(x,y) = xy$ or $x^2 + xy + y^2$ and $Q_2(z,w)$ is a quadratic form in two variables of z, w. Then we have the following 2 cases.

(i) Suppose that $Q_1(x,y) = xy$. Given $\alpha \geq 0$ and $n \in \mathbb{Z}$, there exist an odd integer x and a positive integer y such that $y \equiv x^{-1}n \pmod{2^\alpha}$. This implies that Q_1 represents n over $\mathbb{Z}_2$. Thus, Q' represents every positive integer over $\mathbb{Z}_2$ because $Q'(x,y,0,0) = Q_1(x,y)$.

(ii) Suppose that $Q_1(x,y) = x^2 + xy + y^2$. Let α be an arbitrary positive integer and n be an arbitrary odd integer. Given $x_1, x_2 \in \{0, 1, \ldots, 2^{\alpha-1} - 1\}$ such that $x_1 \neq x_2$,

$$\begin{aligned} x_1(x_1+1) - x_2(x_2+1) &= {x_1}^2 + x_1 - {x_2}^2 - x_2 \\ &= (x_1+x_2)(x_1-x_2) + (x_1-x_2) \\ &= (x_1+x_2+1)(x_1-x_2). \end{aligned}$$

If $x_1 - x_2$ is odd, then $x_1 + x_2 + 1$ is even. Since $0 \neq x_1 + x_2 + 1 \leq 2^{\alpha-1} - 1 + 2^{\alpha-1} - 2 + 1 < 2^\alpha$ and $x_1 \neq x_2$, then $2^\alpha \nmid (x_1+x_2+1)(x_1-x_2)$. If $x_1 - x_2$ is even, then $x_1 + x_2 + 1$ is odd. Since $x_1, x_2 \in \{0, 1, \ldots, 2^{\alpha-1} - 1\}$ and $x_1 \neq x_2$, then $2^\alpha \nmid (x_1 - x_2)$. It follows that $2^\alpha \nmid (x_1+x_2+1)(x_1-x_2)$. Therefore, $x_i(x_i+1) \not\equiv x_j(x_j+1) \pmod{2^\alpha}$ for all $x_i, x_j \in \{0, 1, \ldots, 2^{\alpha-1} - 1\}$ where $x_i \neq x_j$. Since there are exactly $2^{\alpha-1}$ even integers in $\{0, 1, \ldots, 2^\alpha - 1\}$, then there exists some x such that $x(x+1) \equiv n - 1 \pmod{2^\alpha}$. This implies $x^2 + xy + y^2$ represents every odd integer over $\mathbb{Z}_2$ where $y = 1$. It follows that Q' represents every positive odd integer over $\mathbb{Z}_2$ because $Q'(x,y,0,0) = Q_1(x,y) = x^2 + xy + y^2$.

As shown above in the cases (iv) and (vi), since $Q_1(x,y) = x^2 + xy + y^2$, then

$Q_2(z,w) = az^2 + 2^{t_3-1}bw^2$ for some integer $t_3 \geq 1$ and unit $a \in \mathbb{Z}_2$, $2zw$, or $2z^2 + 2zw + 2w^2$. If $Q_2(z,w) = az^2 + 2^{t_3-1}bw^2$, then $Q_2(z,0) = az^2$. Since $n \equiv 1^2 \pmod 2$, then Hensel's Lemma in [11] shows that there exists a corresponding integer z_α such that $z_\alpha^2 \equiv n \pmod{2^\alpha}$ for each positive integer α. This implies az^2 represents every odd integer over $\mathbb{Z}_2$ because a is a unit in $\mathbb{Z}_2$. Thus, Q' represents every positive even integer over $\mathbb{Z}_2$ because $Q'(x,y,z,0) = Q_1(x,y) + Q_2(z,0) = x^2 + xy + y^2 + az^2$. If $Q_2(z,w) = 2zw$ or $2z^2 + 2zw + 2w^2$, then Q_2 represents every positive even integer over $\mathbb{Z}_2$ because zw and $z^2 + zw + w^2$ represent every positive odd integer over $\mathbb{Z}_2$ as shown before. Since $Q'(0,0,z,w) = Q_2(z,w)$, then Q' also represents every positive even integer over $\mathbb{Z}_2$. Therefore, Q' represents every positive integer over $\mathbb{Z}_2$.

Hence, we can conclude that Q locally represents every positive integer. □

This lemma will be a sufficient condition for the lower bound of the Eisenstein coefficients of a quadratic form. Next, we need the following tools to construct a list of primes such that their product can be a possible choice represented by a quadratic form more times than the given ones.

Definition 44. *Let f be a real or complex valued function and g be a real valued function such that $g(x) > 0$ for all $x \geq x_0$. We define $f(x) = \mathcal{O}(g(x))$ to mean that there exists some positive real number M such that*

$$|f(x)| \leq Mg(x)$$

for all $x \geq x_0$.

Theorem 45 (Partial Summation)**.** *Let f and g be arithmetic functions. Let*

$$F(x) = \sum_{n \leq x} f(n).$$

Assume that a and b are nonnegative integers with $a < b$. Then

$$\sum_{n=a+1}^{b} f(n)g(n) = F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n)).$$

Let x and y be nonnegative real numbers with $x < y$, and let $h(t)$ be a continuous differentiable function on $[y, x]$. Then

$$\sum_{y<n\leq x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt.$$

Proof. See Theorem 6.8 in [10]. □

Lemma 46. *If $\gcd(a, q) = 1$ and $x \geq 2$, then*

$$\sum_{\substack{p\leq x \\ p\equiv a \pmod q \\ p \text{ prime}}} \frac{1}{p} = \frac{1}{\phi(q)} \log\log x + \mathcal{O}(1).$$

Proof. Suppose that $\gcd(a, q) = 1$ and $x \geq 2$. Let $g(m) = \frac{1}{\log m}$. For each positive integer m, let

$$f(m) = \begin{cases} \frac{\log m}{m} & \text{if } m \equiv a \pmod q \text{ and } m \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 45 shows that

$$\begin{aligned}
\sum_{\substack{p\le x\\ p\equiv a \pmod q\\ p \text{ prime}}} \frac{1}{p} &= \sum_{\substack{p\le x\\ p\equiv a \pmod q\\ p \text{ prime}}} \frac{\log p}{p}\frac{1}{\log p}\\
&= \sum_{2\le m\le x} f(m)g(m)\\
&= f(2)g(2) + \sum_{2<m\le x} f(m)g(m)\\
&= f(2)g(2) + g(x)\sum_{m\le x} f(m) - g(2)\sum_{m\le 2} f(m) - \int_2^x \Big(\sum_{m\le t} f(t)\Big)\Big(\frac{-1}{t(\log t)^2}\Big)dt\\
&= g(x) \sum_{\substack{p\le x\\ p\equiv a \pmod q\\ p \text{ prime}}} f(p) + \int_2^x \Big(\sum_{\substack{p\le t\\ p\equiv a \pmod q\\ p \text{ prime}}} f(t)\Big)\Big(\frac{1}{t(\log t)^2}\Big)dt + \mathcal{O}(1).
\end{aligned}$$

By Theorem 7.3 in [2], for each $t \in [2, x]$,

$$\sum_{\substack{p\le t\\ p\equiv a \pmod q\\ p \text{ prime}}} f(p) = \frac{1}{\phi(q)}\log t + \mathcal{O}(1).$$

This implies that

$$\begin{aligned}
\sum_{\substack{p\le x\\ p\equiv a \pmod q\\ p \text{ prime}}} \frac{1}{p} &= \Big(\frac{1}{\phi(q)}\log x + \mathcal{O}(1)\Big)\frac{1}{\log x} + \int_2^x \Big(\frac{1}{\phi(q)}\log t + \mathcal{O}(1)\Big)\Big(\frac{1}{t(\log t)^2}\Big)dt\\
&= \frac{1}{\phi(q)} + \mathcal{O}\Big(\frac{1}{\log x}\Big) + \frac{1}{\phi(q)}\int_2^x \frac{1}{t\log t}dt + \mathcal{O}\Big(\int_2^x \frac{1}{t(\log t)^2}dt\Big)\\
&= \frac{1}{\phi(q)} + \mathcal{O}\Big(\frac{1}{\log x}\Big) + \frac{1}{\phi(q)}(\log\log x - \log\log 2) + \mathcal{O}\Big(\frac{1}{\log 2} - \frac{1}{\log x}\Big)\\
&= \frac{1}{\phi(q)}\log\log x + \mathcal{O}(1).
\end{aligned}$$

□

Theorem 47. *A series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if, given* $\epsilon > 0$*, there exists a positive* *integer N such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ whenever $n > m \geq N$.*

Proof. See Theorem 2.7.2 in [1]. $\square$

Lemma 48. *If χ is a non-principal Dirichlet character, then $\sum_{p \text{ prime}} \frac{\chi(p)}{p^n}$ is convergent for each positive integer n.*

Proof. Suppose that n is an arbitrary positive integer. Let $g(m) = \frac{1}{\log m}$ and

$$f(m) = \begin{cases} \frac{\chi(m)\log m}{m^n} & \text{if } m \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Given $b > a \geq 2$, by Theorem 45,

$$\begin{aligned}
\sum_{\substack{a \leq p \leq b \\ p \text{ prime}}} \frac{\chi(p)}{p^n} &= \sum_{\substack{a \leq p \leq b \\ p \text{ prime}}} \frac{\chi(p)\log p}{p^n} \frac{1}{\log p} \\
&= \sum_{a \leq m \leq b} f(m)g(m) \\
&= f(a)g(a) + \sum_{a < m \leq b} f(m)g(m) \\
&= f(a)g(a) + g(b)\sum_{m \leq b} f(m) - g(a)\sum_{m \leq a} f(m) - \int_a^b \Big(\sum_{\substack{p \leq t \\ p \text{ prime}}} \frac{\chi(p)\log p}{p^n} \Big) \Big(\frac{-1}{t(\log t)^2} \Big) dt \\
&= g(b)\sum_{m \leq b} f(m) + \int_a^b \Big(\sum_{\substack{p \leq t \\ p \text{ prime}}} \frac{\chi(p)\log p}{p^n} \Big) \frac{1}{t(\log t)^2} dt - g(a) \sum_{\substack{p < a \\ p \text{ prime}}} \frac{\chi(p)\log p}{p^n} \\
&= \frac{1}{\log b} \sum_{\substack{p \leq b \\ p \text{ prime}}} \frac{\chi(p)\log p}{p^n} + \int_a^b \mathcal{O}\Big(\frac{1}{t(\log t)^2} \Big) dt + \mathcal{O}\Big(\frac{1}{\log a} \Big) \\
&= \mathcal{O}\Big(\frac{1}{\log b} \Big) + \mathcal{O}\Big(\frac{1}{\log a} - \frac{1}{\log b} \Big) + \mathcal{O}\Big(\frac{1}{\log a} \Big).
\end{aligned}$$

It follows that $\sum_{p \text{ prime}} \frac{\chi(p)}{p^n}$ is convergent by Theorem 47. $\square$

As shown by Theorem 10.3 in [10],

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^4}\right)^{-1} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p \text{ prime}} \left(1 + \frac{1}{p^2}\right)^{-1}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ are convergent, then $\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1}$ and $\prod_{p \text{ prime}} \left(1 + \frac{1}{p^2}\right)^{-1}$ are convergent. It follows that $\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)$ and $\prod_{p \text{ prime}} \left(1 + \frac{1}{p^2}\right)$ are also convergent.

Now let Q be a primitive PDIVQQF such that $D(Q)$ is a prime fundamental discriminant. Suppose that R is any PDIVQQF such that $D(R) \neq D(Q)$ and $D(R)$ is not a square. Let $\chi_{D(Q)}$ and $\chi_{D(R)}$ denote the Kronecker characters associated to Q and R respectively defined in Section 2.1. Theorem 6.15 in [2] shows that there are $\phi(D(R))$ Dirichlet characters modulo $D(R)$. Let

$$\chi_1, \chi_2, \dots, \chi_{\phi(D(R))}$$

be such $\phi(D(R))$ characters with χ_1 as the principal character. Define

$$C_{\chi_j} = \sum_{\substack{1 \leq a \leq D(R)-1 \\ \gcd(a, D(R))=1 \\ \chi_{D(Q)}(a)=1}} \frac{1}{\phi(D(R))} \chi_{D(R)}(a) \overline{\chi_j}(a)$$

for each $1 \leq j \leq \phi(D(R))$. Then we have the following 2 cases.

(1) If $\gcd(x, D(R)) = 1$ and $\chi_{D(Q)}(x) = 1$, then there exists some $1 \leq a_0 < D(R) - 1$ such that $x \equiv a_0 \pmod{D(R)}$. Theorem 6.16 in [2] implies that

$$\sum_{j=1}^{\phi(D(R))} \overline{\chi_j}(a_0) \chi_j(x) = \phi(D(R))$$

and

$$\sum_{j=1}^{\phi(D(R))} \overline{\chi_j}(a)\chi_j(x) = 0$$

for all $a \not\equiv x \pmod{D(R)}$. Thus,

$$\begin{aligned}
\sum_{j=1}^{\phi(D(R))} C_{\chi_j}\chi_j(x) &= \sum_{j=1}^{\phi(D(R))} \sum_{\substack{1\le a\le D(R)-1\\ \gcd(a,D(R))=1\\ \chi_{D(Q)}(a)=1}} \frac{1}{\phi(D(R))}\chi_{D(R)}(a)\overline{\chi_j}(a)\chi_j(x) \\
&= \sum_{\substack{1\le a\le D(R)-1\\ \gcd(a,D(R))=1\\ \chi_{D(Q)}(a)=1}} \sum_{j=1}^{\phi(D(R))} \frac{1}{\phi(D(R))}\chi_{D(R)}(a)\overline{\chi_j}(a)\chi_j(x) \\
&= \sum_{j=1}^{\phi(D(R))} \frac{1}{\phi(D(R))}\chi_{D(R)}(a_0)\overline{\chi_j}(a_0)\chi_j(x) \\
&= \chi_{D(R)}(a_0) \\
&= \chi_{D(R)}(x).
\end{aligned}$$

(2) If $\gcd(x, D(R)) \neq 1$ or $\chi_{D(Q)}(x) \neq 1$, then $x \not\equiv a \pmod{D(R)}$ for all $1 \le a \le D(R)-1$ with $\gcd(a, D(R)) = 1$ or $\chi_{D(Q)}(a) = 1$. It follows that

$$\sum_{j=1}^{\phi(D(R))} C_{\chi_j}\chi_j(x) = 0.$$

Let $\gamma : \mathbb{Z}/N\mathbb{Z} \to \{-1, 0, 1\}$ be defined by

$$\gamma(x) = \begin{cases} \chi_{D(R)}(x) & \text{if } \gcd(x, D(R)) = 1 \text{ and } \chi_{D(Q)}(x) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\sum_{j=1}^{\phi(D(R))} C_{\chi_j}\chi_j(x) = \gamma(x)$. Given a positive integer n,

$$\begin{aligned}\sum_{p \text{ prime}} \frac{\gamma(p)}{p^n} &= \sum_{p \text{ prime}} \sum_{j=1}^{\phi(D(R))} \frac{C_{\chi_j}\chi_j(p)}{p^n} \\ &= \sum_{j=1}^{\phi(D(R))} C_{\chi_j} \sum_{p \text{ prime}} \frac{\chi_j(p)}{p^n} \\ &= C_{\chi_1} \sum_{p \text{ prime}} \frac{1}{p^n} + \sum_{j=2}^{\phi(D(R))} C_{\chi_j} \sum_{p \text{ prime}} \frac{\chi_j(p)}{p^n}.\end{aligned}$$

Since $D(R)$ is not a square, then $\chi_{D(R)}$ is a non-principal Dirichlet character by Lemma 20. Thus,

$$\begin{aligned}C_{\chi_1} &= \sum_{\substack{1 \le a \le D(R)-1 \\ \gcd(a, D(R))=1 \\ \chi_{D(Q)}(a)=1}} \frac{1}{\phi(D(R))} \chi_{D(R)}(a)\overline{\chi_1}(a) \\ &= \frac{1}{\phi(D(R))} \sum_{\substack{1 \le a \le D(R)-1 \\ \gcd(a, D(R))=1 \\ \chi_{D(Q)}(a)=1}} \chi_{D(R)}(a) \\ &= 0.\end{aligned}$$

It follows that

$$\sum_{p \text{ prime}} \frac{\gamma(p)}{p^n} = \sum_{j=2}^{\phi(D(R))} C_{\chi_j} \sum_{p \text{ prime}} \frac{\chi_j(p)}{p^n}$$

which is convergent by Lemma 48. Therefore,

$$\lim_{x \to \infty} \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{\chi_{D(R)}(p)}{p^n} = \lim_{x \to \infty} \sum_{\substack{e^x \le p \le e^{x^2} \\ p \text{ prime}}} \frac{\gamma(p)}{p^n} = 0$$

because $\gamma(p) = \chi_{D(R)}(p)$ for all prime p greater than $D(R)$ with $\chi_{D(Q)}(p) = 1$.

Since

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

for $x \in (-1, 1)$, then

$$\begin{aligned}
&\log \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 + \frac{\chi_{D(R)}(p)}{p}\Big(1 - \frac{1}{p}\Big)\right) \\
&= \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \log\left(1 + \frac{\chi_{D(R)}(p)}{p}\Big(1 - \frac{1}{p}\Big)\right) \\
&= - \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=1}^{\infty} \frac{(-\chi_{D(R)}(p))^n}{np^n}\Big(1 - \frac{1}{p}\Big)^n \\
&= \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{\chi_{D(R)}(p)}{p}\Big(1 - \frac{1}{p}\Big) - \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{(-\chi_{D(R)}(p))^n}{np^n}\Big(1 - \frac{1}{p}\Big)^n \\
&\le \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{\chi_{D(R)}(p)}{p}\Big(1 - \frac{1}{p}\Big) + \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n}.
\end{aligned}$$

Lemma 49. $\displaystyle\sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n}$ *is convergent.*

Proof. Since

$$\begin{aligned}
0 &< \sum_{\substack{\chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n} \\
&< \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{1}{p^n} \\
&= \sum_{p \text{ prime}} \frac{1}{p(p-1)} \\
&< \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \\
&< \int_2^{\infty} \Big(\frac{1}{x-1} - \frac{1}{x}\Big) dx \\
&= \lim_{a\to\infty} \int_2^{a} \Big(\frac{1}{x-1} - \frac{1}{x}\Big) dx \\
&= \lim_{a\to\infty} (\log \Big(1 - \frac{1}{a}\Big) + \log 2) \\
&= \log 2,
\end{aligned}$$

then $\sum_{\substack{\chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n}$ converges. It follows that

$$\begin{aligned}
0 &< \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n} \\
&< \sum_{\substack{p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n} \\
&= \sum_{\substack{\chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n} - \sum_{\substack{p > e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n} \\
&< \sum_{\substack{\chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n}.
\end{aligned}$$

Hence, $\displaystyle\sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n}$ is convergent. □

Thus,

$$\begin{aligned}
&\log \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 + \frac{\chi_{D(R)}(p)}{p}\left(1 - \frac{1}{p}\right)\right) \\
&\le \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{\chi_{D(R)}(p)}{p}\left(1 - \frac{1}{p}\right) + \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n} \\
&= \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{\chi_{D(R)}(p)}{p} - \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{\chi_{D(R)}(p)}{p^2} + \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n}
\end{aligned}$$

which is convergent. Also,

$$\begin{aligned}
&\prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 + \frac{\chi_{D(R)}(p)}{p}\left(1 - \frac{1}{p}\right)\right) \\
&> \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 - \frac{1}{p}\left(1 - \frac{1}{p}\right)\right) \\
&> \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) \\
&> \prod_{\substack{p \le e^{x^2} \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).
\end{aligned}$$

Theorem 8.8 in [10] shows that there exists a constant γ such that

$$\prod_{\substack{p\le x\\ p \text{ prime}}} \left(1-\frac{1}{p}\right)^{-1} = e^{\gamma}\log x + \mathcal{O}(1).$$

This implies that

$$\prod_{\substack{p\le e^{x^2}\\ p \text{ prime}}} \left(1-\frac{1}{p}\right) = \left(e^{\gamma}x^2 + \mathcal{O}(1)\right)^{-1}.$$

It follows that

$$\lim_{x\to\infty} \log \prod_{\substack{e^x\le p\le e^{x^2}\\ \chi_{D(Q)}(p)=1\\ p \text{ prime}}} \left(1+\frac{\chi_{D(R)}(p)}{p}(1-\frac{1}{p})\right)$$

is finite. Therefore,

$$\lim_{x\to\infty} \prod_{\substack{e^x\le p\le e^{x^2}\\ \chi_{D(Q)}(p)=1\\ p \text{ prime}}} \left(1+\frac{\chi_{D(R)}(p)}{p}(1-\frac{1}{p})\right)$$

is also finite.

Lemma 50.

$$\lim_{x\to\infty} \prod_{\substack{e^x\le p\le e^{x^2}\\ \chi_{D(Q)}(p)=1\\ p \textit{ prime}}} \left(1+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right) = \infty.$$

Proof. Given a positive real number x,

$$1+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3} = \left(1+\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right).$$

Since

$$\prod_{\substack{e^x\le p\le e^{x^2}\\ \chi_{D(Q)}(p)=1\\ p \text{ prime}}} \left(1-\frac{1}{p^2}\right) > \prod_{p \text{ prime}} \left(1-\frac{1}{p^2}\right)$$

which is convergent, then it suffices to show that

$$\lim_{x\to\infty} \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \Big(1 + \frac{1}{p}\Big) = \infty.$$

The fact that $D(Q)$ is a prime fundamental discriminant implies that $D(Q) \equiv 1 \pmod 4$ and $\big(\frac{D(Q)}{a}\big) = \big(\frac{a}{D(Q)}\big)$ for all integers a by the law of quadratic reciprocity. Theorem 45 and Lemma 46 show that

$$\begin{aligned}
\sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{1}{p} &= \sum_{\left(\frac{a}{D(Q)}\right)=1} \sum_{\substack{e^x \le p \le e^{x^2} \\ p \equiv a \pmod{D(Q)} \\ p \text{ prime}}} \frac{1}{p} \\
&= \sum_{\left(\frac{a}{D(Q)}\right)=1} \Big(\sum_{\substack{p \le e^{x^2} \\ p \equiv a \pmod{D(Q)} \\ p \text{ prime}}} \frac{1}{p} - \sum_{\substack{p < e^{x} \\ p \equiv a \pmod{D(Q)} \\ p \text{ prime}}} \frac{1}{p} \Big) \\
&= \sum_{\left(\frac{a}{D(Q)}\right)=1} \Big(\frac{1}{\phi(D(Q))} (\log\log e^{x^2} - \log\log e^{x}) + \mathcal{O}(1) \Big) \\
&= \Big(\frac{D(Q)+1}{2} - 1 \Big) \Big(\frac{1}{D(Q)-1} \log x + \mathcal{O}(1) \Big) \\
&= \frac{1}{2} \log x + \mathcal{O}(1).
\end{aligned}$$

This implies that

$$
\begin{aligned}
0 &< \log \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1+\frac{1}{p}\right) \\
&= -\sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=1}^{\infty} \frac{(-1)^n}{np^n} \\
&= \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \frac{1}{p} - \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{(-1)^n}{np^n} \\
&\le \frac{1}{2}\log x + \mathcal{O}(1) + \sum_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \sum_{n=2}^{\infty} \frac{1}{np^n} \\
&= \frac{1}{2}\log x + \mathcal{O}(1).
\end{aligned}
$$

by Lemma 49. Thus,

$$
\lim_{x\to\infty} \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1+\frac{1}{p}\right) = \infty.
$$

□

In the next section, given a finite set of PDIVQQFs, we will show that

$$
\prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right)
$$

is a factor of the Eisenstein coefficient of a primitive PDIVQQF different from the given ones. Also, the quantity

$$\prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 + \frac{\chi_{D(Q_i)}(p)}{p}(1 - \frac{1}{p})\right)$$

will contribute mostly to the value of the Eisenstein coefficients for a subset of $\{Q_1, Q_2, \ldots, Q_s\}$. Then we can use the previous lemmas to choose a large enough x such that the Eisenstein coefficient of Q is much larger than the Eisenstein coefficients of the subset. The remaining part is to obtain a bound for the cusp form coefficients as shown in the following lemma

Lemma 51. *Let Q be a PDIVQQF and let $\theta_Q(z) = E(z) + C(z)$ be the decomposition of the theta series of Q where $E(z)$ is an Eisenstein series and $C(z)$ is a cusp form. Let $a_C(n)$ be the n^{th} cusp form coefficient and ϵ be an real number. Then there exists some constant C_ϵ only dependent on ϵ such that*

$$|a_C(n)| \le C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}$$

for all positive integers n coprime to $N(Q)$.

Proof. Let n be an arbitrary positive integer coprime to $N(Q)$. Section 5 and Theorem 3 in [15] together show that

$$\begin{aligned}
|a_C(n)| &\le C_\epsilon \Big(\max\{(N(Q)^2 D(Q))^{\frac{1}{4}+\epsilon}, N(Q)^{1+\epsilon}\}\Big)^{\frac{1}{2}} N(Q)^{1+\epsilon} d(n)\sqrt{n} \\
&= C_\epsilon \max\{N(Q)D(Q)^{\frac{1}{8}+\frac{\epsilon}{2}}, N(Q)^{\frac{1}{2}+\frac{\epsilon}{2}}\} N(Q)^{1+\epsilon} d(n)\sqrt{n} \\
&\le C_\epsilon \max\{D(Q)^{\frac{9}{8}+\frac{\epsilon}{2}}, D(Q)^{\frac{1}{2}+\frac{\epsilon}{2}}\} D(Q)^{1+\epsilon} d(n)\sqrt{n} \\
&\le C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}.
\end{aligned}$$

□

3.2 Proof of the Main Result

Proof. Let $Q_1, Q_2, \ldots, Q_s$ be arbitrary PDIVQQFs. For each $1 \le i \le s$, if Q_i is associated to a non-maximal lattice L_i, then there exists some PDIVQQF Q_i' which is associated to a maximal lattice containing L_i. This implies that the number of representations of each positive integer by Q_i' is greater than or equal to that of Q_i. Therefore, without loss of generality, we may assume that each of the given PDIVQQFs is associated to a maximal lattice and classify them into the following 4 groups with $1 \le s_1 \le s_2 \le s_3 \le s$:

(i) For $1 \le i \le s_1$, Q_i is primitive and $D(Q_i)$ is not a square.

(ii) For $s_1 < i \le s_2$, $Q_i = d_i Q_i'$ where d_i is an integer greater than 1 and Q_i' is primitive; $D(Q_i)$ is not a square.

(iii) For $s_2 < i \le s_3$. Q_i is primitive and $D(Q_i)$ is a square.

(iv) For $s_3 < i \le s$, $Q_i = d_i Q_i'$ where d_i is an integer greater than 1 and Q_i' is primitive; $D(Q_i')$ is a square.

For each $s_2 < i \le s_3$, there exists an anisotropic prime q_i corresponding to Q_i. For each $s_3 < i \le s$, there exists an associated anisotropic prime q_i' corresponding to Q_i'. Let d be the least common multiple of d_i for all $s_1 < i \le s_2$ or $s_3 < i \le s$. Remark $3.8.1$ in [6] shows that $q_i \mid N(Q_i)$ for each $s_2 < i \le s_3$ and $q_i' \mid N(Q_i')$ for each $s_3 < i \le s$. Let q be the least common multiple of q_i for all $s_2 < i \le s_3$ and q_j' for all $s_3 < j \le s$. Let q' be the least common multiple of 4, d, and q. Dirichlet's theorem on arithmetic progression shows that there are infinitely many prime numbers congruent to 1 modulo q' (see Theorem 10.9 in [10]). Let δ be a prime number congruent to 1 modulo q' such that δ does not divide $D(Q_i)$ for all $1 \le i \le s$. It follows that δ is a fundamental discriminant. Lemma 42 implies that there exists a primitive PDIVQQF Q with $D(Q) = \delta$.

Let

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n, \quad q = e^{2\pi i z}$$

be the theta series of Q and we may decompose $\theta_Q(z)$ as $\theta_Q(z) = E(z) + C(z)$ where $E(z) = \sum_{n=0}^{\infty} a_E(n) q^n$ is an Eisenstein form and $C(z) = \sum_{n=1}^{\infty} a_C(n) q^n$ is a cusp form. For each $1 \le i \le s$, let

$$\theta_{Q_i}(z) = \sum_{n=0}^{\infty} r_{Q_i}(n) q^n$$

to be the theta series of Q_i and decompose $\theta_{Q_i}(z)$ as $\theta_{Q_i}(z) = E_i(z) + C_i(z)$ where $E_i(z) = \sum_{n=0}^{\infty} a_{E_i}(n) q^n$ is an Eisenstein series and $C_i(z) = \sum_{n=1}^{\infty} a_{C_i}(n) q^n$ is a cusp form. Also, for each $s_1 < i \le s_2$ or $s_3 < i \le s$, let

$$\theta_{Q'_i}(z) = \sum_{n=0}^{\infty} r_{Q'_i}(n) q^n$$

to be the theta series of Q'_i and decompose $\theta_{Q'_i}(z)$ as $\theta_{Q'_i}(z) = E'_i(z) + C'_i(z)$ where $E'_i(z) = \sum_{n=0}^{\infty} a_{E'_i}(n) q^n$ is an Eisenstein series and $C'_i(z) = \sum_{n=1}^{\infty} a_{C'_i}(n) q^n$ is a cusp form.

Let ϵ be an arbitrary positive real number. Lemma 51 shows the following 3 cases:

(i) There exists a constant C_ϵ which only depends on ϵ such that

$$|a_C(n)| \le C_\epsilon D(Q)^{\frac{17}{8} + \frac{3\epsilon}{2}} d(n) \sqrt{n}. \tag{2}$$

(ii) For each $1 \le i \le s_1$ or $s_2 < i \le s_3$, there exists a corresponding constant $C_{i\epsilon}$ which only depends on ϵ such that

$$|a_{C_i}(n)| \le C_{i\epsilon} D(Q_i)^{\frac{17}{8} + \frac{3\epsilon}{2}} d(n) \sqrt{n}. \tag{3}$$

(iii) For each $s_1 < i \le s_2$ or $s_3 < i \le s$, there exists a corresponding constant $C'_{i\epsilon}$ which only depends on ϵ such that

$$|a_{C_i'}(n)| \leq C_{i\epsilon}' D(Q_i')^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}. \tag{4}$$

Let R be an arbitrary PDIVQQF. For each prime α, let $h(\alpha)$ be the largest nonnegative integer such that $\alpha^{h(\alpha)} \mid D(R)$. In this paragraph, all references are from [5]. Let p be an arbitrary odd prime such that $p \mid D(R)$. Inequality (8) shows that $\beta_p(R;n) \leq 2$ if $p \nmid n$; inequality (9) shows that $\beta_p(R;n) \leq p^{\frac{h(p)}{6}}\left(1+\frac{1}{p^2}\right)\left(1+\frac{1}{p}\right)$ if $p \mid n$. Let t be an arbitrary odd prime such that $t \nmid D(R)$. Inequality (11) shows that $\beta_t(R;n) \leq 1+\frac{1}{t^2}$ if $t \nmid n$; inequality (12) shows that $\beta_t(R;n) < 1+\frac{1}{t}$ if $t \mid n$. Meanwhile, inequality (10) shows that $\beta_2(R;n) \leq 4 \cdot 2^{\frac{h(2)-4}{6}}$.

There exist a positive constant m_1 such that, for all positive integers n,

$$m_1 \geq \beta_2(Q_i;n) \prod_{\substack{p \mid N(Q_i) \\ p \text{ odd prime}}} \beta_p(Q_i;n) \prod_{\substack{p \nmid N(Q_i) \\ p \mid dq \\ p \text{ odd prime}}} \beta_p(Q_i;n) \tag{5}$$

for all $1 \leq i \leq s_1$, and

$$m_1 \geq \beta_2(Q_i';n) \prod_{\substack{p \mid N(Q_i') \\ p \text{ odd prime}}} \beta_p(Q_i';n) \prod_{\substack{p \nmid N(Q_i') \\ p \mid \frac{dq}{d_i} \\ p \text{ odd prime}}} \beta_p(Q_i';n) \tag{6}$$

for all $s_1 < i \leq s_2$ because there are only finitely many local densities in the two finite products respectively and each of them is bounded above by a constant.

In addition, since $D(Q)$ is a prime number, then Theorem 6.4 in [18] implies that $N(Q) = D(Q)$. For all positive integers n which is not divisible by $D(Q)$,

$$\prod_{\substack{p \mid N(Q) \\ p \text{ prime}}} \beta_p(Q;n) = \beta_{D(Q)}(Q;n) \geq 1 - \frac{1}{D(Q)} \tag{7}$$

as shown by Lemma 2 in [15] because Q locally represents every positive integer by Lemma 43. Lemma 50 implies that there exists a sufficiently large positive real number x satisfying the following 3 conditions:

(i) For all $1 \leq i \leq s$,

$$e^x > \max\{D(Q), D(Q_i), q\}. \tag{8}$$

(ii)

$$\prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right) > \left(1+2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right)\left(\frac{\pi^2\left(1-\frac{1}{D(Q)}\right)}{\sqrt{D(Q)}}\prod_{p \text{ prime}}\left(1-\frac{1}{p^2}\right)\right)^{-1}. \tag{9}$$

(iii) For all $1 \leq i \leq s_1$,

$$\prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right) > \left(\frac{\pi^2 m_1}{\sqrt{D(Q_i)}}\prod_{p \text{ prime}}\left(1+\frac{1}{p^2}\right)\prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}}\left(1+\frac{\chi_{D(Q_i)}(p)}{p}\left(1-\frac{1}{p}\right)\right)\right.$$
$$\left.+\,2C_{i\epsilon}D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} + 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right)\left(\frac{\pi^2\left(1-\frac{1}{D(Q)}\right)}{\sqrt{D(Q)}}\prod_{p \text{ prime}}\left(1-\frac{1}{p^2}\right)\right)^{-1}. \tag{10}$$

(iv) For all $s_1 < i \leq s_2$,

$$\prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right) > \left(\frac{\pi^2 m_1}{\sqrt{D(Q_i')}}\prod_{p \text{ prime}}\left(1+\frac{1}{p^2}\right)\prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}}\left(1+\frac{\chi_{D(Q_i')}(p)}{p}\left(1-\frac{1}{p}\right)\right)\right.$$
$$\left.+\,2C_{i\epsilon}'D(Q_i')^{\frac{17}{8}+\frac{3\epsilon}{2}} + 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right)\left(\frac{\pi^2\left(1-\frac{1}{D(Q)}\right)}{\sqrt{D(Q)}}\prod_{p \text{ prime}}\left(1-\frac{1}{p^2}\right)\right)^{-1}. \tag{11}$$

Let m be the least common multiple of d, q, and

$$\prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} p.$$

Without loss of generality, we may have the following assumptions where $s_{2.5}$ and s_4 are integers such that $s_2 < s_{2.5} \le s_3 < s_4 \le s$.

(i) There exists some nonnegative integer v_1 such that Q_i locally represents mq^{2v_1} at q_i for each $s_2 < i \le s_{2.5}$

(ii) For all nonnegative integers u, Q_i does not locally represents mq^{2u} at q_i for each $s_{2.5} < i \le s_3$

(iii) There exists some nonnegative integer v_2 such that Q'_i locally represents $mq^{2v_2}d_i^{-1}$ at q'_i for each $s_3 < i \le s_4$.

(iv) For all nonnegative integers u, Q'_i does not locally represents $mq^{2u}d_i^{-1}$ at q'_i for each $s_4 < i \le s$.

For each prime α and positive integer β, let $\text{ord}_\alpha(\beta)$ denote the largest positive integer t such that $\alpha^t \mid \beta$. Let v_0 be a positive integer such that

$$v_0 \ge \max\left\{v_1, v_2, \frac{\text{ord}_{q_i}(N(Q_i)) - 1}{2}, \frac{\text{ord}_{q'_j}(N(Q'_j))}{2}\right\} \tag{12}$$

for all $s_2 < i \le s_{2.5}$ and for all $s_3 < j \le s_4$. Let v be a positive integer satisfying the following conditions.

(i) $v > v_0$.

(ii) For all $s_2 < i \leq s_{2.5}$,

$$v > \log_{q_i}\left[\Big(\frac{\pi^2}{\sqrt{D(Q_i)}}\prod_{p\text{ prime}}\beta_p(Q_i; nq_i^{-2(v-v_0)}) + 2C_{i\epsilon}D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}}\Big)\Big(\frac{\pi^2}{\sqrt{D(Q)}}\prod_{p\text{ prime}}\beta_p(Q; m) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\Big)^{-1}\right]. \tag{13}$$

(iii) For all $s_3 < i \leq s_4$,

$$v > \log_{q_i'}\left[\Big(\frac{\pi^2}{d_i\sqrt{D(Q_i')}}\prod_{p\text{ prime}}\beta_p(Q_i'; \frac{nq_i^{-2v}}{d_i}) + 2C_{i\epsilon}'D(Q_i')^{\frac{17}{8}+\frac{3\epsilon}{2}}\Big)\Big(\frac{\pi^2}{\sqrt{D(Q)}}\prod_{p\text{ prime}}\beta_p(Q; \frac{m}{d_i}) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\Big)^{-1}\right]. \tag{14}$$

Let $n = mq^{2v}$. This allows us to complete the proof by showing the following 6 cases.

(i) For each $1 \leq i \leq s_1$, Theorem 3.1 in [23] shows that

$$\beta_p(Q_i; n) \leq 1 + \frac{1}{p^2} \tag{15}$$

for each odd prime $p \nmid N(Q_i)n$. Also,

$$\beta_p(Q; n) \geq 1 - \frac{1}{p^2} \tag{16}$$

for each odd prime $p \nmid N(Q)n$, and

$$\beta_p(Q_i; n) = 1 + \frac{\chi_{D(Q_i)}(p)}{p}\Big(1 - \frac{1}{p}\Big) - \frac{1}{p^3} < 1 + \frac{\chi_{D(Q_i)}(p)}{p}\Big(1 - \frac{1}{p}\Big) \tag{17}$$

for each odd prime p such that $e^x \leq p \leq e^{x^2}$ and $\chi_{D(Q)}(p) = 1$. In addition,

$$\beta_p(Q; n) = 1 + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3} \tag{18}$$

for each odd prime p such that $e^x \leq p \leq e^{x^2}$ and $\chi_{D(Q)}(p) = 1$. Thus,

$$
\begin{aligned}
& r_{Q_i}(n) \\
&= a_{E_i}(n) + a_{C_i}(n) \\
&\leq \frac{\pi^2 n}{\sqrt{D(Q_i)}} \prod_{p \text{ prime}} \beta_p(Q_i;n) + C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n} && \text{(by (3))} \\
&= \frac{\pi^2 n}{\sqrt{D(Q_i)}} \Bigg(\beta_2(Q_i;n) \prod_{\substack{p \mid N(Q_i) \\ p \text{ odd prime}}} \beta_p(Q_i;n) \prod_{\substack{p \nmid N(Q_i) \\ p \mid dq \\ p \text{ odd prime}}} \beta_p(Q_i;n) \\
&\quad \prod_{\substack{p \nmid N(Q_i) \\ p \mid m \\ p \nmid dq \\ p \text{ odd prime}}} \beta_p(Q_i;n) \prod_{\substack{p \nmid N(Q_i) n \\ p \text{ odd prime}}} \beta_p(Q_i;n) \Bigg) + C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}
\end{aligned}
$$

$$< \frac{\pi^2 n m_1}{\sqrt{D(Q_i)}} \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \beta_p(Q_i; n) \prod_{\substack{p \nmid N(Q_i)n \\ p \text{ odd prime}}} \beta_p(Q_i; n) + 2C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} n \qquad \text{(by(5))}$$

$$< \frac{\pi^2 n m_1}{\sqrt{D(Q_i)}} \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 + \frac{\chi_{D(Q_i)}(p)}{p}\Big(1 - \frac{1}{p}\Big)\right) \prod_{p \text{ odd prime}} \Big(1 + \frac{1}{p^2}\Big)$$

$$+ 2C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} n \qquad \text{(by (15) and (17))}$$

$$= n\left(\frac{\pi^2 m_1}{\sqrt{D(Q_i)}} \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1 + \frac{\chi_{D(Q_i)}(p)}{p}\Big(1 - \frac{1}{p}\Big)\right)\right.$$

$$\left.\prod_{p \text{ prime}} \Big(1 + \frac{1}{p^2}\Big) + 2C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right)$$

$$< n\left(\frac{\pi^2\big(1 - \frac{1}{D(Q)}\big)}{\sqrt{D(Q)}} \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \Big(1 + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}\Big) \prod_{p \text{ prime}} \Big(1 - \frac{1}{p^2}\Big)\right.$$

$$\left. - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right) \qquad \text{(by (10))}$$

$$\le \frac{\pi^2 n}{\sqrt{D(Q)}} \left(\prod_{\substack{p \mid N(Q) \\ p \text{ prime}}} \beta_p(Q; n) \prod_{\substack{p \nmid N(Q) \\ p \mid n \\ p \text{ prime}}} \beta_p(Q; n)\right.$$

$$\left.\prod_{\substack{p \nmid N(Q)n \\ p \text{ prime}}} \beta_p(Q; n)\right) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} n \qquad \text{(by (7), (16) and (18))}$$

$$\le a_E(n) - C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}$$

$$\le a_E(n) + a_C(n) \qquad \text{(by (2))}$$

$$= r_Q(n).$$

(ii) For all $s_1 < i \le s_2$, Theorem 3.1 in [23] shows that

$$\beta_p\Big(Q_i'; \frac{n}{d_i}\Big) \le 1 + \frac{1}{p^2} \qquad (19)$$

for each odd prime $p \nmid N(Q_i')\frac{n}{d_i}$. Also,

$$\beta_p\Big(Q_i'; \frac{n}{d_i}\Big) = 1 + \frac{\chi_{D(Q_i')}(p)}{p}\Big(1 - \frac{1}{p}\Big) - \frac{1}{p^3} < 1 + \frac{\chi_{D(Q_i')}(p)}{p}\Big(1 - \frac{1}{p}\Big) \tag{20}$$

for each odd prime p such that $e^x \le p \le e^{x^2}$ and $\chi_{D(Q)}(p) = 1$. In addition,

$$\beta_p\Big(Q; \frac{n}{d_i}\Big) = 1 + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3} \tag{21}$$

for each odd prime p such that $e^x \le p \le e^{x^2}$ and $\chi_{D(Q)}(p) = 1$. Thus,

$$\begin{aligned}
& r_{Q_i}(n) \\
&= r_{Q_i'}\Big(\frac{n}{d_i}\Big) \\
&= a'_{E_i}\Big(\frac{n}{d_i}\Big) + a'_{C_i}\Big(\frac{n}{d_i}\Big) \\
&\le \frac{\pi^2 \frac{n}{d_i}}{\sqrt{D(Q_i')}} \prod_{p \text{ prime}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big) + C'_{i\epsilon} D(Q_i')^{\frac{17}{8}+\frac{3\epsilon}{2}} d\Big(\frac{n}{d_i}\Big)\sqrt{\frac{n}{d_i}} \qquad \text{(by (4))} \\
&= \frac{\pi^2 n}{\sqrt{D(Q_i')}d_i}\Bigg(\beta_2\Big(Q_i'; \frac{n}{d_i}\Big) \prod_{\substack{p \mid N(Q_i') \\ p \text{ odd prime}}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big) \prod_{\substack{p \nmid N(Q_i') \\ p \mid \frac{dq}{d_i} \\ p \text{ odd prime}}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big) \\
&\qquad \prod_{\substack{p \nmid N(Q_i') \\ p \mid m \\ p \nmid \frac{dq}{d_i} \\ p \text{ odd prime}}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big) \prod_{\substack{p \nmid N(Q_i')\frac{n}{d_i} \\ p \text{ odd prime}}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big)\Bigg) + C'_{i\epsilon} D(Q_i')^{\frac{17}{8}+\frac{3\epsilon}{2}} d\Big(\frac{n}{d_i}\Big)\sqrt{\frac{n}{d_i}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\pi^2 n m_1}{\sqrt{D(Q_i')} d_i} \prod_{\substack{p \nmid N(Q_i') \\ p \mid m \\ p \nmid \frac{dq}{d_i} \\ p \text{ odd prime}}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big) \prod_{\substack{p \nmid N(Q_i') \frac{n}{d_i} \\ p \text{ odd prime}}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big) \\
&\quad + C_{i\epsilon}' D(Q_i')^{\frac{17}{8} + \frac{3\epsilon}{2}} d\Big(\frac{n}{d_i}\Big) \sqrt{\frac{n}{d_i}} && \text{(by (6))} \\
&\leq \frac{\pi^2 n m_1}{\sqrt{D(Q_i')} d_i} \prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \beta_p\Big(Q_i'; \frac{n}{d_i}\Big) \prod_{p \text{ prime}} \Big(1 + \frac{1}{p^2}\Big) \\
&\quad + 2C_{i\epsilon}' D(Q_i')^{\frac{17}{8} + \frac{3\epsilon}{2}} \frac{n}{d_i} && \text{(by (19))} \\
&< \frac{n}{d_i} \Bigg(\frac{\pi^2 m_1}{\sqrt{D(Q_i')}} \prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \Big(1 + \frac{\chi_{D(Q_i')}}{p} \Big(1 - \frac{1}{p}\Big)\Big) \prod_{p \text{ prime}} \Big(1 + \frac{1}{p^2}\Big) \\
&\quad + 2C_{i\epsilon}' D(Q_i')^{\frac{17}{8} + \frac{3\epsilon}{2}} \Bigg) && \text{(by (20))} \\
&< n \Bigg(\frac{\pi^2 m_1}{\sqrt{D(Q_i')}} \prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \Big(1 + \frac{\chi_{D(Q_i')}}{p} \Big(1 - \frac{1}{p}\Big)\Big) \prod_{p \text{ prime}} \Big(1 + \frac{1}{p^2}\Big) \\
&\quad + 2C_{i\epsilon}' D(Q_i')^{\frac{17}{8} + \frac{3\epsilon}{2}} \Bigg) \\
&< n \Bigg(\frac{\pi^2 \big(1 - \frac{1}{D(Q)}\big)}{\sqrt{D(Q)}} \prod_{\substack{e^x \leq p \leq e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \Big(1 + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}\Big) \prod_{p \text{ prime}} \Big(1 - \frac{1}{p^2}\Big) \\
&\quad - 2C_\epsilon D(Q)^{\frac{17}{8} + \frac{3\epsilon}{2}} \Bigg) && \text{(by (11))} \\
&\leq \frac{\pi^2 n}{\sqrt{D(Q)}} \prod_{\substack{p \mid N(Q) \\ p \text{ prime}}} \beta_p\Big(Q; \frac{n}{d_i}\Big) \prod_{\substack{p \nmid N(Q) \\ p \mid \frac{n}{d_i} \\ p \text{ prime}}} \beta_p\Big(Q; \frac{n}{d_i}\Big) \prod_{\substack{p \nmid N(Q) \frac{n}{d_i} \\ p \text{ prime}}} \beta_p\Big(Q; \frac{n}{d_i}\Big) \\
&\quad - 2C_\epsilon D(Q)^{\frac{17}{8} + \frac{3\epsilon}{2}} n && \text{(by (7), (16), and (21))}
\end{aligned}
$$

$$\leq \frac{\pi^2 n}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p\Big(Q; \frac{n}{d_i}\Big) - C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}$$

$$\leq \frac{\pi^2 n}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p(Q; n) + a_C(n) \qquad \text{(by } (1))$$

$$\leq r_Q(n)$$

(iii) For all $s_2 < i \leq s_{2.5}$, since mq^{2v_1} is locally represented by Q_i at q_i, then $nq_i^{-2(v-v_0)}$ is also locally represented by Q_i at q_i because

$$nq_i^{-2(v-v_0)} = mq^{2v}q_i^{-2(v-v_0)} = m\Big(\frac{q}{q_i}\Big)^{2v} q_i^{2v} q_i^{-2(v-v_0)} = m\Big(\frac{q}{q_i}\Big)^{2v} q_i^{2v_0}$$

which is divisible by mq^{2v_1} and $\frac{nq_i^{-2(v-v_0)}}{mq^{2v_1}}$ is a square. Since $q_i^{\mathrm{ord}_{q_i}(N(Q_i))} \mid nq_i^{-2(v-v_0)}$ by (12), then Remark $3.6.1$ in [6] shows that $nq_i^{-2(v-v_0)}$ is q_i-stable for Q_i. By Corollary $3.8.2$ in [6],

$$r_{Q_i}(n) = r_{Q_i}(nq_i^{-2(v-v_0)}) \tag{22}$$

because Q_i is anisotropic at q_i. Thus,

$$\begin{aligned}
r_{Q_i}(n) &= r_{Q_i}(nq_i^{-2(v-v_0)}) && \text{(by (22))}\\
&= a_{E_i}(nq_i^{-2(v-v_0)}) + a_{C_i}(nq_i^{-2(v-v_0)})\\
&\leq \frac{\pi^2 nq_i^{-2(v-v_0)}}{\sqrt{D(Q_i)}} \prod_{p \text{ prime}} \beta_p(Q_i; nq_i^{-2(v-v_0)})\\
&\quad + C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(nq_i^{-2(v-v_0)}) \sqrt{nq_i^{-2(v-v_0)}} && \text{(by(3))}\\
&< \frac{\pi^2 nq_i^{-2(v-v_0)}}{\sqrt{D(Q_i)}} \prod_{p \text{ prime}} \beta_p(Q_i; nq_i^{-2(v-v_0)})\\
&\quad + q_i^{-(v-v_0)} C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}\\
&= n\left(\frac{\pi^2 q_i^{-2(v-v_0)}}{\sqrt{D(Q_i)}} \prod_{p \text{ prime}} \beta_p(Q_i; nq_i^{-2(v-v_0)})\right.\\
&\quad \left. + \frac{q_i^{-(v-v_0)} C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}}{n}\right)\\
&\leq n\left(\frac{\pi^2 q_i^{-v}}{\sqrt{D(Q_i)}} \prod_{p \text{ prime}} \beta_p(Q_i; nq_i^{-2(v-v_0)}) + 2q_i^{-v} C_{i\epsilon} D(Q_i)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right)\\
&< n\left(\frac{\pi}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p(Q; m) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right) && \text{(by (13))}\\
&= \frac{\pi n}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p(Q; m) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} n\\
&\leq \frac{\pi n}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p(Q; n) - C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n} && \text{(by (1))}\\
&\leq a_E(n) + a_C(n)\\
&= r_Q(n).
\end{aligned}$$

(iv) For each $s_{2.5} < i \leq s_3$, Q_i does not locally represent n at q_i. Thus Q_i also does not represent n. This implies that

$$
\begin{aligned}
r_{Q_i}(n) &= 0 \\
&< n \\
&< n\left(\frac{\pi^2\left(1-\frac{1}{D(Q)}\right)}{\sqrt{D(Q)}} \prod_{\substack{e^x \le p \le e^{x^2} \\ \chi_{D(Q)}(p)=1 \\ p \text{ prime}}} \left(1+\frac{1}{p}-\frac{1}{p^2}-\frac{1}{p^3}\right) \prod_{p \text{ prime}} \left(1-\frac{1}{p^2}\right)\right. \\
&\quad \left. - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\right) && \text{(by (9))} \\
&\le \frac{\pi^2 n}{\sqrt{D(Q)}}\left(\prod_{\substack{p \mid N(Q) \\ p \text{ prime}}} \beta_p(Q;n) \prod_{\substack{p \nmid N(Q) \\ p \mid n \\ p \text{ prime}}} \beta_p(Q;n)\right. \\
&\quad \left. \prod_{\substack{p \nmid N(Q)n \\ p \text{ prime}}} \beta_p(Q;n)\right) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} n && \text{(by (7), (16) and (18))} \\
&\le a_E(n) - C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n} \\
&\le a_E(n) + a_C(n) \\
&= r_Q(n).
\end{aligned}
$$

(v) For each $s_3 < i \le s_4$, since $mq^{2v_2}d_i^{-1}$ is locally represented by Q_i' at q_i', then $n{q'}_i^{-2(v-v_0)}d_i^{-1}$ is also locally represented by Q_i' at q_i' because

$$n{q'}_i^{-2(v-v_0)}d_i^{-1} = mq^{2v}{q'}_i^{-2(v-v_0)}d_i^{-1} = m\left(\frac{q}{q_i'}\right)^{2v} q_i'^{2v}{q'}_i^{-2(v-v_0)}d_i^{-1} = m\left(\frac{q}{q_i'}\right)^{2v}{q'}_i^{2v_0}d_i^{-1}$$

which is divisible by $m{q'}_i^{2v_2}d_i^{-1}$ and

$$\frac{n{q'}_i^{-2(v-v_0)}d_i^{-1}}{mq^{v_2}d_i^{-1}}$$

is a square. Also, ${q'}_i^{\mathrm{ord}'_{q^i}(N(Q_i'))} \mid n{q'}_i^{-2(v-v_0)}d_i^{-1}$ by (12). Remark $3.6.1$ in [6] shows that $n{q'_i}^{-2(v-v_0)}d_i^{-1}$ is q'_i-stable for Q'_i. By Corollary $3.8.2$ in [6],$r_{Q'_i}(\frac{n}{d_i}) = r_{Q_i}(n{q'_i}^{-2v}d_i^{-1})$ because Q'_i is anisotropic at q'_i. This implies that

$$
\begin{aligned}
r_{Q_i}(n) &= r_{Q'_i}\Big(\frac{n}{d_i}\Big) \\
&= r_{Q'_i}\Big(\frac{n{q'_i}^{-2(v-v_0)}}{d_i}\Big) \\
&= a_{E'_i}\Big(\frac{n{q'_i}^{-2(v-v_0)}}{d_i}\Big) + a_{C'_i}\Big(\frac{n{q'_i}^{-2(v-v_0)}}{d_i}\Big) \\
&\le \frac{\pi^2 n{q'_i}^{-2(v-v_0)}}{d_i\sqrt{D(Q'_i)}} \prod_{p \text{ prime}} \beta_p\Bigg(Q'_i; \frac{n{q'_i}^{-2(v-v_0)}}{d_i}\Bigg) \\
&\quad + C'_{i\epsilon} D(Q'_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d\Bigg(\frac{n{q'_i}^{-2(v-v_0)}}{d_i}\Bigg)\sqrt{\frac{n{q'_i}^{-2(v-v_0)}}{d_i}} && \text{(by (4))} \\
&= \frac{\pi^2 n{q'_i}^{-2(v-v_0)}}{d_i\sqrt{D(Q_i)}} \prod_{p \text{ prime}} \beta_p\Bigg(Q'_i; \frac{n{q'_i}^{-2(v-v_0)}}{d_i}\Bigg) \\
&\quad + {q'_i}^{-(v-v_0)} C'_{i\epsilon} D(Q'_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d\Big(\frac{n}{d_i}\Big)\sqrt{\frac{n}{d_i}} \\
&< \frac{\pi^2 n{q'_i}^{-2v}}{d_i\sqrt{D(Q_i)}} \prod_{p \text{ prime}} \beta_p\Bigg(Q'_i; \frac{n{q'_i}^{-2(v-v_0)}}{d_i}\Bigg) + {q'_i}^{-v} C'_{i\epsilon} D(Q'_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n} \\
&= n\Bigg(\frac{\pi^2 {q'_i}^{-2v}}{d_i\sqrt{D(Q'_i)}} \prod_{p \text{ prime}} \beta_p\Bigg(Q'_i; \frac{n{q'_i}^{-2v}}{d_i}\Bigg) + \frac{{q'_i}^{-v} C'_{i\epsilon} D(Q'_i)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n}}{n}\Bigg) \\
&< n\Bigg(\frac{\pi^2 {q'_i}^{-v}}{d_i\sqrt{D(Q'_i)}} \prod_{p \text{ prime}} \beta_p\Bigg(Q'_i; \frac{n{q'_i}^{-2v}}{d_i}\Bigg) + 2{q'_i}^{-v} C'_{i\epsilon} D(Q'_i)^{\frac{17}{8}+\frac{3\epsilon}{2}}\Bigg) \\
&\le n\Bigg(\frac{\pi^2}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p\Big(Q; \frac{m}{d_i}\Big) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}}\Bigg) && \text{(by (14))} \\
&= \frac{\pi n}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p\Big(Q; \frac{m}{d_i}\Big) - 2C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} n \\
&\le \frac{\pi n}{\sqrt{D(Q)}} \prod_{p \text{ prime}} \beta_p(Q; n) - C_\epsilon D(Q)^{\frac{17}{8}+\frac{3\epsilon}{2}} d(n)\sqrt{n} && \text{(by (1))} \\
&\le a_E(n) + a_C(n) \\
&= r_Q(n).
\end{aligned}
$$

(vi) For each $s_4 < i \leq s$, Q_i' does not locally represent $\frac{n}{d_i}$ at q_i. Thus, Q_i' also does not represent $\frac{n}{d_i}$. This implies that

$$r_{Q_i}(n) = r_{Q_i'}\left(\frac{n}{d_i}\right) = 0 < n < r_Q(n)$$

as shown in case (iv).

$\square$

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Curriculum Vitae

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Education

Wake Forest University, Winston-Salem, NC, May 2020

Master of Arts in Mathematics

Faculty Advisor: Dr. Jeremy Rouse

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Bachelor of Science in Mathematics

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Overall GPA: 3.782; Mathematics GPA: 3.942

Research Interest

Number Theory

Research Experience

Independent Researcher, Department of Mathematics, WFU, July 2019-Present

Research Advisor: Dr. Jeremy Rouse

- Investigate the relationship between the numbers of representations of a positive integer by positive-definite integer-valued quaternary quadratic forms and their local densities.

- Analyze the influence of anisotropic primes of such quadratic forms on the numbers of representations when the discriminants of the quadratic forms are squares.
- Give upper and lower bounds for the coefficients of the Eisenstein series and cusp forms in the decompositions of the theta series of the quadratic forms.
- Prove that there exists a primitive positive-definite integer-valued quaternary quadratic form Q represents a positive integer more times than each element in a given finite set of positive-definite integer-valued quaternary quadratic forms.

Independent Researcher, Department of Mathematics, WFU, August 2019-May 2019

Research Advisor: Dr. Jeremy Rouse

- Investigate modular properties of ordered Bell numbers.
- Show the heuristic probability that infinitely many of ordered Bell numbers are prime is 1.
- Construct a formula to show that there are infinitely many integers m such that each of m, $m+1$, $m+2$, $m+4$, $m+5$, and $m+8$ is a sum of two squares.

Conferences and Seminars

AMS/MAA Mathematics Meetings, Washington, DC, January 2021. (Invited Talk)

- Title: “On the number of representations by Primitive Positive-Definite Integer-Valued Quaternary Quadratic Forms.”

VaNTAGe, Online, January-March, 2020. (Attendee)

- A virtual seminar on open conjectures in number theory and arithmetic geometry.

SouthEast Regional Meeting on Numbers , Greensboro, NC, April 2020. (Attendee)

PAlmetto Number Theory Series XXXIII, Clemson, SC, December 2019. (Attendee)

Connecticut Number Theory Conference, Online, June 2020. (Attendee)

Teaching Experience

Teaching Assistant, Wake Forest University, August 2019-Present

- Calculus I, Calculus II, Calculus III, Linear Algebra, and Discrete Mathematics.

Tutor, Math Center, Wake Forest University, September 2018-Present

- Courses: calculus, discrete mathematics, linear algebra, and modern algebra.

Professional Experience

Study Guide Writer, Wake Forest University, May-August 2020

- Wrote the study guide for the Calculus II course.

Project Manager, Humblebee Global Youth Elite Academy, Beijing, Feb.-August 2014

- Collaborated with the CEO to set up an education academy.
- Recruited prospective cooperative partners.
- Designed the company website.
- Drafted contracts and translated files.

Volunteer and Leadership Activities

Honor Council Representative, WFU, September 2020-present

Graduate Student Association Representative, WFU, September 2019-present

Test Proctor, Winston-Salem Chapter Mathcounts Competition, NC, February 25, 2020

Lead Guitarist, Spring Awakening, Wake Forest University, February-April 2018

- Accompanied 7 shows of the Broadway musical.

Lead Tenor and Director, Opera Concert, Wake Forest University, January-April 2018

Computer Skills

Magma, Sage

Awards

- Outstanding Graduate Student Award in Mathematics, 2020.
- Full Tuition Scholarship, 2019-present.
- Dean's List at Wake Forest University, 2016-2019.
- The Joseph Pleasant and Marguerite Nutt Sloan Memorial Fund, 2017-2018.
- Boteler Prize for the Pursuit of Excellence in Music, 2018.