ON THE CLASSIFICATION OF GRADE THREE PERFECT IDEALS

BY

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Abstract

Logan Gray

In this thesis we will study a theorem by Avramov-Kustin-Miller found in *Poincaré series of modules over local rings of small embedding codepth or small linking number* that classifies the possible graded-commutative algebra structure of $A = H_\bullet(F_\bullet \otimes_Q k)$ where F is a minimal free resolution of a cyclic module of projective dimension 3. Our Macaulay2 package, MultFreeResThree, classifies which algebra structure A falls into, and generates a basis for this unique algebra structure.

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Chapter 1: Introduction

When one studies linear algebra they usually want to find a basis that behaves nicely with respect to some operations. As the reader likely knows, this is not typically an easy task. Constructing such a basis by hand can be difficult, time consuming, and almost impossible at times. It can become even more complicated when the algebra we are studying lacks a "nice" structure.

In this thesis we will study a theorem by Avramov-Kustin-Miller [AKM88] that classifies the possible graded-commutative algebra structure of $A = H_\bullet(F_\bullet \otimes_Q k)$ where F is a minimal free resolution of a cyclic module of projective dimension 3. Our Macaulay2 package, MultFreeResThree, will classify which algebra structure A falls into and generate a basis for this unique algebra structure.

In order to understand these structures we will first learn about complexes and differential graded algebras. This will then lead us into some interesting theorems and examples of the graded-commutative structure on A. We will then prove that our package does indeed generate a basis with the correct structure.

The motivation for this package came from a conjecture by Christensen, Veliche, and Weyman [CVW20] that states (in part) that if the minimal free resolution of an ideal is of Dynkin format, then it is in the linkage class of a complete intersection. This conjecture is discussed in Chapter 5 of this paper.

Chapter 2: Background

We will first start by setting up notation and definitions. For this paper let R denote a commutative noetherian ring and $\Bbbk$ be a field. $\Bbbk[x_1, x_2, \ldots, x_n]$ denotes the the polynomial ring of n variables with coefficients in $\Bbbk$.

2.1 Ring Theory

Definition 2.1.1. Let $\Bbbk$ be a field. A $\Bbbk$-algebra R is *graded* if

$$R = \bigoplus_{i \geq 0} R_i$$

where each R_i is a $\Bbbk$-vector space such that $R_i R_j \subseteq R_{i+j}$.

A *homogeneous element* of a graded ring R is an element that is in one of the R_i. Further, a *homogeneous ideal* of a ring R is an ideal generated by a set of homogeneous elements. Note here that the polynomial ring of n variables is a graded noetherian ring with unique homogeneous maximal ideal.

Definition 2.1.2. A *graded-commutative ring* R is a graded ring such that homogeneous elements $r, s \in R$ satisfy $rs = (-1)^{|r||s|} sr$ where $|r|$ and $|s|$ denote the degree of r and s.

In what follows, the ring R will be a graded $\Bbbk$-algebra with unique homogenous maximal ideal $\mathfrak{m} = R_{>0}$.

2.2 Complexes

Definition 2.2.1. Let R be a ring with an identity 1. An *R-module* F is an abelian group with a law of composition $+$ and scalar multiplication $R \times F \to F$ written

$(r, f) \rightsquigarrow rf$ such that:

1. $1f = f$ where $f \in F$;
2. $(rs)f = r(sf)$ where $r, s \in R$ and $f \in F$;
3. $(r + s)f = rf + sf$ where $r, s \in R$ and $f \in F$;
4. $r(f + f') = rf + rf'$ where $r \in R$ and f and $f' \in F$;

An R-module is said to be *free* if it has a basis.

Definition 2.2.2. Let R be a graded $\Bbbk$-algebra. F is a *graded R-module* if

$$F = \bigoplus_{i \in \mathbb{Z}} F_i$$

where each F_i is a vector space such that $R_i F_j \subseteq F_{i+j}$.

Definition 2.2.3. Let R be a ring. A *complex* of R-modules $F_\bullet$ is a sequence of modules F_i and R-linear maps $\partial_i : F_i \to F_{i-1}$ (which are called boundary maps) such that the composition of maps $\partial_{i-1}\partial_i : F_i \to F_{i-2}$ is zero for all $i \in \mathbb{Z}$.

For the code, that will be proven correct later in this thesis, we will want to preform a change of basis of the complex to make the matrices have a desired form. To do this we will use an analogue of the following definition of change of basis for linear maps.

Definition 2.2.4. Let $F_\bullet$ be complex of free R-modules with $F_i = R^{b_i}$. Let $\{P_i\}$ be a collection of invertible $b_i \times b_i$ matrices. The complex $F'_\bullet$ with $F'_i = F_i$ and $\partial'_i = P_{i-1}^{-1}\partial_i P_i$ is called the *change of basis* for $F_\bullet$ with respect to the $\{P_i\}$. In terms of a diagram, this has the form:

$$\begin{array}{lccccccccc}
F_\bullet : & 0 & \longleftarrow & R^{b_0} & \xleftarrow{\partial_1} & R^{b_1} & \xleftarrow{\partial_2} & R^{b_2} & \longleftarrow & \cdots . \\
 & & & \uparrow{\scriptstyle P_0} & & \uparrow{\scriptstyle P_1} & & \uparrow{\scriptstyle P_2} & & \\
F'_\bullet : & 0 & \longleftarrow & R^{b_0} & \xleftarrow{\partial'_1} & R^{b_1} & \xleftarrow{\partial'_2} & R^{b_2} & \longleftarrow & \cdots
\end{array}$$

Definition 2.2.5. The *homology* of the complex at F_i is the R-module $H_i(F) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$.

Note that a complex F is exact if $H_i(F) = 0$ for all i.

Definition 2.2.6. Suppose M is an R-module. A *free resolution of* M is an exact complex of the form:

$$0 \leftarrow M \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \leftarrow \cdots,$$

where F_i is a free R-module for each i.

If M produces a finite free resolution, then the minimal length of all finite free resolutions is called the *projective dimension* of M.

We now construct a complex that provides a useful example of a structure appearing in the next section.

Definition 2.2.7. The *tensor algebra* of an R-module M is the graded, non-commutative $\Bbbk$-algebra

$$T_R(M) := \bigoplus_{i \geq 0} M^{\bigotimes i},$$

where

$$M^{\bigotimes i} = \underbrace{M \otimes M \otimes \cdots \otimes M}_{i \text{ times}},$$

with product of $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ and $y_1 \otimes y_2 \otimes \cdots \otimes y_m$ given by $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_m$, for $x_i, y_j \in M$.

Definition 2.2.8. The *exterior algebra* of M is denoted $\bigwedge_R M$ and is obtained by taking the tensor algebra of M and modding out by the two sided ideal generated by square elements (i.e. elements with a repeated tensor factor) of $\bigwedge_R M = \bigoplus_{n \geq 0} \bigwedge_R^n M$

where

$$\bigwedge_R^n M = (M \otimes \cdots \otimes M)/\operatorname{span}\{(x_1 \otimes \cdots \otimes x_n) : x_i = x_j \text{ for some } i \neq j\}.$$

We let the symbol $x_1 \wedge \cdots \wedge x_n$ denote the coset corresponding to $x_1 \otimes \cdots \otimes x_n$ in $\bigwedge_R^n M$

Definition 2.2.9. On the exterior algebra we define the *Koszul complex* $K(x, R)$ on a sequence $x = (x_1, \ldots, x_r) \in R^r$. We let the underlying modules of the complex be the sets $\{\bigwedge_R^n R^r\}_{n \in \mathbb{Z}_{\geq 0}}$ and define the differential to be:

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \sum_{j=1}^{n} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_n}.$$

We will see in Definition 2.3.1 another way to think about this differential.

Example 2.2.10. Let $R = \Bbbk[x, y]$ and consider the Koszul complex on (x^2, xy). Note that $\partial(e_1 \wedge e_2) = x^2 e_2 - xy e_1$. The the Koszul complex yields the following:

$$K:\ 0 \leftarrow R \xleftarrow{\begin{array}{c|cc} & e_1 & e_2 \\ 1 & [x^2 & xy] \end{array}} R^2 \xleftarrow{\begin{array}{c|c} & e_1 \wedge e_2 \\ e_1 & \begin{bmatrix} -xy \\ x^2 \end{bmatrix} \\ e_2 & \end{array}} R \leftarrow 0$$

Note that $\begin{bmatrix} -y \\ x \end{bmatrix} \in \ker(\partial_1)$ but $\operatorname{im}(\partial_2)$ is generated by $\begin{bmatrix} -xy \\ x^2 \end{bmatrix}$. Thus $\begin{bmatrix} -y \\ x \end{bmatrix} \notin \operatorname{im}(\partial_2)$ which implies that $H_1(K) \neq 0$. Therefore, this complex is not exact.

2.3 Differential Graded Algebras

Definition 2.3.1. Let R be a graded $\Bbbk$-algebra. A *differential graded R-algebra A* is a complex (A, ∂) with an element $1 \in A_0$ (the unit), and an associative product that satisfies Leibniz rule. That is, for all homogeneous elements $a, b \in A$,

$$\partial(ab) = \partial(a)b + (-1)^{|a|} a \partial(b),$$

where $|a|$ is the degree of a. In addition, we assume A is graded commutative:

$$ab = (-1)^{|a||b|}ba \quad \textit{for} \quad a, b \in A \quad \textit{and} \quad a^2 = 0 \quad \textit{when} \quad |a| \text{ is odd}$$

and that $A_i = 0$ for $i < 0$.

For the duration of this paper we will refer to “differential graded R-algebras” as “DG algebras”.

Example 2.3.2. One example of a DG algebra that we have already seen in Example 2.2.10 is if we let A be the Koszul complex and let multiplication be defined by the wedge product. Then note:

$$\begin{aligned} \partial(e_1 e_2) &= \partial(e_1 \wedge e_2) \\ &= (-1)^{1-1} x^2 e_2 + (-1)^{2-1} xy e_1 \\ &= \partial(e_1)e_2 - \partial(e_2)e_1 \\ &= \partial(e_1)e_2 + (-1)^{|e_1|} e_1 \partial(b), \end{aligned}$$

which follows from the Leibniz rule.

Chapter 3: Multiplicative Structures

In this chapter we will first discuss the theorems that give us the multiplicative structures our code is building bases for, and then will explore some examples of these structures.

3.1 Theorems

Let Q denote a commutative noetherian graded $\Bbbk$-algebra with unique homogeneous maximal ideal $\mathfrak{m}$. For an ideal $I \subseteq Q$ with $\mathrm{pd}_Q(Q/I) = 3$, let $F_\bullet \to Q/I$ be a minimal free resolution over Q as follows:

$$F_\bullet : 0 \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} F_3 \leftarrow 0.$$

Let $m = \operatorname{rank}_Q F_1$ and $n = \operatorname{rank}_Q F_3$. Then, since the $\operatorname{rank}_Q F_0 = 1$, this forces $\operatorname{rank}_Q F_2 = m + n - 1$. Indeed, since $F_\bullet$ is acyclic with rank $H_0(F) = 0$, this implies that $F_\bullet$ has Euler characteristic 0, thus

$$0 = \operatorname{rank}_Q F_3 - \operatorname{rank}_Q F_2 + \operatorname{rank}_Q F_1 - \operatorname{rank}_Q F_0,$$

giving the desired equality of ranks.

It is known that some ideals have minimal free resolutions $F_\bullet \to Q/I$ that do not support a DG-algebra structure. However, this issue does not arise in projective dimension less than or equal to 3, as is seen in the following theorem:

Theorem 3.1.1. *Let $Q = \Bbbk[x_1, \ldots, x_n]$ and let I be a homogeneous ideal of Q such that Q/I has projective dimension 3. Then the minimal resolution of Q/I over Q has a graded-commutative DG algebra structure. [BE77]*

When proving Theorem 3.1.1 the issue is not defining the product structure, but whether or not said structure is associative. For projective dimension 4 and greater, associativity is not guaranteed, but the proof for Theorem 3.1.1 is as follows:

Proof. We verify the associative law, assuming the existence of a product that satisfies the Leibniz rule. How to find such a product is outlines in [Avr98]. First note that associativity is guaranteed for a set of elements a, b, c if, without loss of generality, a is of degree 2 or degree 3. (Note that b and c can be any degree.) Associativity is guaranteed because get we will get 0.

So, let a, b, c be degree 1 elements. Then:

$$\begin{aligned}\partial_3((ab)c)) &= \partial_2(ab)(c) + (ab)\partial_1(c)\\ &= (\partial_1(a)b - \partial_1(b)a)c + \partial_1(c)ab\\ &= \partial_1(a)bc - \partial_1(b)ac + \partial_1(c)ab\end{aligned}$$

and

$$\begin{aligned}\partial_3(a(bc)) &= \partial_1(a)(bc) + (a)\partial_2(bc)\\ &= \partial_1(a)bc + a(-\partial_1(b)c + \partial_1(c)b)\\ &= \partial_1(a)bc - \partial_1(b)ac + \partial_1(c)ab.\end{aligned}$$

Therefore,

$$\partial_3((ab)c) = \partial_3(a(bc))$$

which implies

$$\partial_3((ab)c - a(bc)) = 0$$

but since

$$F_\bullet : 0 \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} F_3 \leftarrow 0$$

is exact this implies $H_3 = \ker(\partial_3)/\operatorname{im}(0) = 0$. This forces ∂_3 to be injective, implying

$$(ab)c = a(bc).$$

□

Thus, $F_\bullet$ has a structure of a graded-commutative differential graded algebra. While the structure of $F_\bullet$ as a DG algebra is not unique, we do get the following theorem from Avramov-Kustin-Miller, and Weyman:

Theorem 3.1.2. *Let $A = H_\bullet(F_\bullet \otimes_Q \Bbbk)$. The graded-commutative algebra structure on A is unique up to isomorphism. Further, there exists bases $e_1, \ldots, e_m$ for A_1, $f_1, \ldots, f_{m+n-1}$ for A_2, and $g_1, \ldots, g_n$ for A_3 such that the multiplication is one of the following:*

$C(3)$	$e_1e_2 = f_3$ $e_2e_3 = f_1$ $e_3e_1 = f_2$	$e_if_i = g_1$ *for* $1 \leq i \leq 3$
T	$e_1e_2 = f_3$ $e_2e_3 = f_1$ $e_3e_1 = f_2$	
B	$e_1e_2 = f_3$	$e_if_i = g_1$ *for* $1 \leq i \leq 2$
$G(r)$	$[r \geq 2]$	$e_if_i = g_1$ *for* $1 \leq i \leq r$
$H(p, q)$	$e_{p+1}e_i = f_i$ *for* $1 \leq i \leq p$	$e_{p+1}f_{p+j} = g_j$ *for* $1 \leq j \leq q$.

If the product is not mentioned or given by the rules of graded-commutativity then the product is understood to be zero. We say that I is of class $H(p, q)$ if the multiplication structure on $A_\bullet$ is given by $H(p, q)$ in the previous table (respectively for $C(3), T, B$, and $G(r)$ [CVW20].

The following is an example that shows the DG Algebra structure of $F_\bullet$ is not unique, but $A = H_\bullet(F_\bullet \otimes_Q k)$, is unique:

Example 3.1.3. Let $\Bbbk$ be a field and $Q = \Bbbk[x, y, z]$ and $I = \langle x^2, y^2, z^2, xyz\rangle$ Then, our minimal free resolution $F_\bullet$ is as follows:

$$Q \leftarrow Q^4 \leftarrow Q^6 \leftarrow Q^3 \leftarrow 0$$

Using our Macaulay2 Package, we may produce the following different multiplication tables for the degree 1 elements when we look at the algebra structure on $F_\bullet$, showing that the algebra structure is not unique. Note that x, y, and z are variables in the product on $F_\bullet$:

	e_1	e_2	e_3	e_4
e_1	0	f_1	f_4	xf_2
e_2	$-f_1$	0	f_6	yf_3
e_3	$-f_4$	$-f_6$	0	zf_5
e_4	$-xf_2$	$-yf_3$	$-zf_5$	0

Table 3.1.1: Example 1 of $F_\bullet$ algebra structure not being unique

	e_1	e_2	e_3	e_4
e_1	0	$(z+1)f_1 - yf_2 + xf_3$	$zf_1 - yf_2 + xf_3 + f_4$	$zf_1 + (x-y)f_2 + xf_3$
e_2	$(-z-1)f_1 + yf_2 - xf_3$	0	$zf_1 - yf_2 + xf_3 + f_6$	$zf_1 - yf_2 + (x+y)f_3$
e_3	$-zf_1 + yf_2 - xf_3 - f_4$	$-zf_1 + yf_2 - xf_3 - f_6$	0	$zf_1 - yf_2 + xf_3 + zf_5$
e_4	$-zf_1 + (-x+y)f_2 - xf_3$	$-zf_1 + yf_2 + (-x-y)f_3$	$-zf_1 + yf_2 - xf_3 - zf_5$	0

Table 3.1.2: Example 2 of $F_\bullet$ algebra structure not being unique

But, Table 3.1.3 show the unique multiplication table of our degree 1 elements in $A_\bullet$. Note that when we take the tensor $A = H_\bullet(F_\bullet \otimes_Q k)$ we are really just zeroing out that variables in the products of $f_\bullet$. Thus to get from Tables 3.1.1 and 3.1.2 to Table 3.1.3 we just zero out x, y, and z

	e_1	e_2	e_3	e_4
e_1	0	f_1	f_4	0
e_2	$-f_1$	0	f_6	0
e_3	$-f_4$	$-f_6$	0	0
e_4	0	0	0	0

Table 3.1.3: Example of $A_\bullet$ algebra structure being unique

To further help us understand the structure of $A_\bullet$ we can consider the following invariants. Let

$$p_I = \operatorname{rank}_k A_1 \cdot A_1 \quad q_I = \operatorname{rank}_k A_1 \cdot A_2 \quad r_I = \operatorname{rank}_k \delta_2^A$$

where $\delta_2^A : A_2 \to \operatorname{Hom}_k(A_1, A_3)$ is defined by $\delta_2^A(f)(e) = fe$ for $e \in A_1$ and $f \in A_2$. Based on the class of our ideal I the values of the invariants are shown in Table 3.1.4.

Note that not all combinations of p and q are possible. We have the following theorems giving us conditions on these values.

Class of I	p_I	q_I	r_I
B	1	1	2
$C(3)$	3	1	3
$G(r)[r \geq 2]$	0	1	r
$H(p,q)$	p	q	q
T	3	0	0

Table 3.1.4: Invariants of $A_\bullet$

Theorem 3.1.4. *[CVW20] Let I be a grade 3 perfect ideal of class $H(p,q)$: One has:*

1. $p \leq m-1$ *and* $q \leq n$
2. $p \leq n+1$ *and* $q \leq m-2$

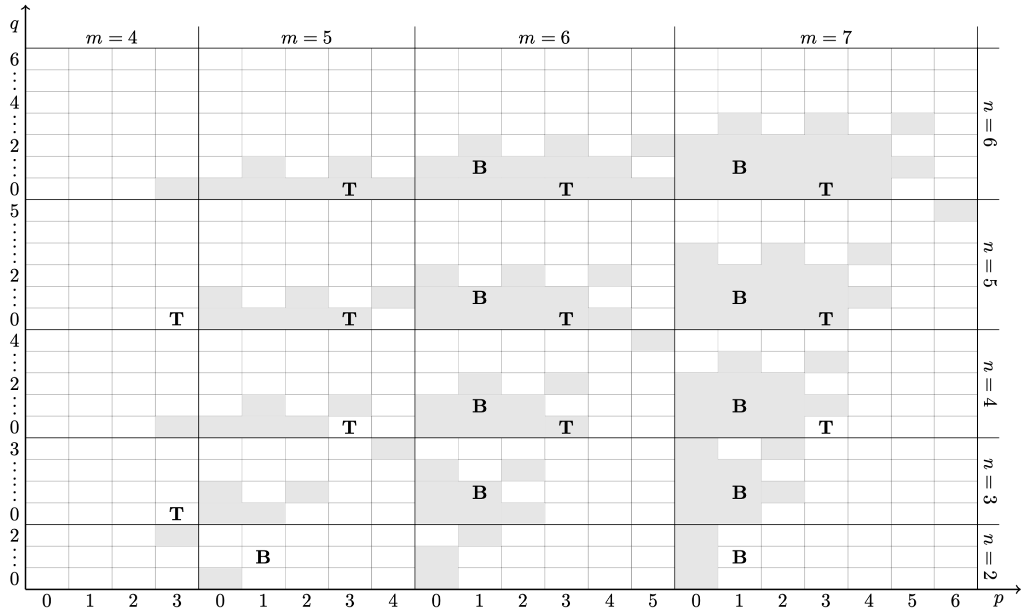
Moreover the following are equivalent

1. $p = n+1$
2. $q = m-2$
3. $p = m-1$ *and* $q = n$

Note that when this equivalence is satisfied $m - n = 2$.

Theorem 3.1.5. *[CVW19] Let I be a grade 3 perfect ideal of class $H(p,q)$. If $p \neq n+1$ or $q \neq m-1$, then $p \leq n-1$ and $q \leq m-4$. [CVW20]*

Further, the following table from gives possible values for the classes of $H(p,q)$, B, and T. The possible values for $H(p,q)$ are shaded gray, and the possible values for B and T are labeled respectively.



3.2 Examples

3.2.1 Example of T-Truncated Exterior Algebra

Let $\Bbbk = \mathbb{Z}/32003$ be our field, (note that 32003 is just a large prime to keep our example interesting, but not have extremely large coefficients). Then our ring will be the polynomial ring $Q = \Bbbk[x, y, z]$. Let $I = \langle x^2, y^3, z^4, xyz\rangle$, which is of class T, and $F_\bullet$ be the original minimal free resolution of this ideal. Note that our original free resolution $F_\bullet$ is as follows:

$$Q \xleftarrow{\partial_1} Q^4 \xleftarrow{\partial_2} Q^6 \xleftarrow{\partial_3} Q^3 \leftarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} x^2 & y^3 & xyz & z^4 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} -yz & y^3 & 0 & -z^4 & 0 & 0 \\ 0 & x^2 & -xz & 0 & 0 & -z^4 \\ x & 0 & y^2 & 0 & -z^3 & 0 \\ 0 & 0 & 0 & x^2 & xy & y^3 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} -y^2 & z^3 & 0 \\ z & 0 & 0 \\ x & 0 & -z^3 \\ 0 & -y & 0 \\ 0 & x & -y^2 \\ 0 & 0 & x \end{bmatrix}.$$

The problem is our multiplication table 3.2.1 does not match our multiplication definition of T.

0	e_1	e_2	e_3	e_4
e_1	0	f_2	0	$-f_4$
e_2	$-f_2$	0	f_6	0
e_3	0	0	0	0
e_4	$-f_4$	$-f_6$	0	0

Table 3.2.1: Class T Ex: Multiplication table of e's before change of coordinates

Macaulay2 makes the following change of basis:

$$\text{Basis for} A_1 : \{e_1, e_2 + 997e_3, e_4, e_3\},$$

$$\text{Basis for} A_2 : \{f_6, -f_4, f_2, f_1, f_3, f_5\},$$

$$\text{Basis for} A_3 : \{g_1, g_2, g_3\}.$$

This results in this new complex for the ideal with our new basis:

$$Q \overset{\partial_1'}{\leftarrow} Q^4 \overset{\partial_2'}{\leftarrow} Q^6 \overset{\partial_3'}{\leftarrow} Q^3 \leftarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} x^2 & y^3 + 997xyz & z^4 & xyz \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} 0 & z^4 & -y^3 & -yz & 0 & 0 \\ -z^4 & 0 & x^2 & 0 & -xz & 0 \\ y^3 & -x^2 & 0 & 0 & 0 & xy \\ 997z^4 & 0 & -997x^2 & x & y^2 + 997xz & -z^3 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \\ -y^2 & z^3 & 0 \\ x & 0 & -z^3 \\ 0 & x & -y^2 \end{bmatrix},$$

We also get the new multiplication table 3.2.2 with the desired values.

0	e_1	e_2	e_3	e_4
e_1	0	f_3	$-f_2$	0
e_2	$-f_3$	0	f_1	0
e_3	f_2	$-f_1$	0	0
e_4	0	0	0	0

Table 3.2.2: Class T Ex: Multiplication table of e's after change of coordinates

3.2.2 Example of B

Let $\Bbbk$ be a field and let $Q = \Bbbk[x, y, z]$ $I = \langle x^2, xy, z^2, yz \rangle$. Let F be the original minimal free resolution of this ideal. Further, our original free resolution F is as follows:

$$Q \xleftarrow{\partial_1} Q^4 \xleftarrow{\partial_2} Q^4 \xleftarrow{\partial_3} Q \leftarrow 0.$$

where

$$\partial_1 = \begin{bmatrix} x^2 & xy & yz & z^2 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} -y & 0 & 0 & -z^2 \\ x & -z & 0 & 0 \\ 0 & x & -z & 0 \\ 0 & 0 & y & x^2 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} -z^2 \\ -xz \\ -x^2 \\ y \end{bmatrix}.$$

Note that via the Leibniz rule:

$$\begin{aligned} \partial_2(e_1e_4) &= \partial_1(e_1)e_4 - \partial_1(e_4)e_1 \\ &= x^2e_4 - z^2e_1 \\ &= f_4. \end{aligned}$$

The reader should note that this is not the desired multiplication, as the only non-trivial multiplication of degree one elements should be $e_1e_2 = f_3$ One can see this problem, and the problem with multiplying degree 1 with degree 2 elements more clearly in the multiplication tables 3.2.3 and 3.2.4. These show that the first basis generated by Macaulay2 does not match the multiplication definition of B.

	e_1	e_2	e_3	e_4
e_1	0	0	0	f_4
e_2	0	0	0	0
e_3	0	0	0	0
e_4	$-f_4$	0	0	0

Table 3.2.3: Class B Ex: Multiplication table of e's before change of coordinates

Thus, Macaulay2 makes the following change of basis:

Basis forA_1 :$\{-e_1, -e_4, e_2, e_3\}$,

Basis forA_2 :$\{f_3, f_1, f_4, f_2\}$,

Basis forA_3 :$\{g_1\}$.

	f_1	f_2	f_3	f_4
e_1	0	0	$-g_1$	0
e_2	0	0	0	0
e_3	0	0	0	0
e_4	$-g_1$	0	0	0

Table 3.2.4: Class B Ex: Multiplication table of f's with e's before change of coordinates

This results in this new complex for the ideal with our new basis:

$$Q \xleftarrow{\partial_1'} Q^4 \xleftarrow{\partial_2'} Q^5 \xleftarrow{\partial_3'} Q^2 \leftarrow 0$$

$$\partial_1 = \begin{bmatrix} -x^2 & -z^2 & xy & yz \end{bmatrix}$$

$$\partial_2 = \begin{bmatrix} 0 & y & z^2 & 0 \\ -y & 0 & -x^2 & 0 \\ 0 & x & 0 & -z \\ -z & 0 & 0 & x \end{bmatrix}$$

$$\partial_3 = \begin{bmatrix} -x^2 \\ -z^2 \\ y \\ -xz \end{bmatrix}$$

Now note that via the Leibniz rule:

$$\begin{aligned} \partial_2(e_1e_2) &= \partial_1(e_1)e_2 - \partial_1(e_2)e_1 \\ &= -x^2e_2 + z^2e_1 \\ &= f_3. \end{aligned}$$

Thus we get the desired multiplication of degree 1 elements. Further, we also get the new multiplication table for degree 1 with degree 2. One can see the desired results were achieved in the Tables 3.2.5 and 3.2.6.

3.2.3 Example of $H(3, 2)$

Let $\mathbb{k}$ be a field and $Q = \mathbb{k}[x, y, z]$ and $I = \langle x^2, y^3, z^4, xy\rangle$. Let $F_\bullet$ be the original minimal free resolution of this ideal. Further, our original free resolution $F_\bullet$ is as

	e_1	e_2	e_3	e_4
e_1	0	f_3	0	0
e_2	$-f_3$	0	0	0
e_3	0	0	0	0
e_4	0	0	0	0

Table 3.2.5: Class B Ex: Multiplication table of e's after change of coordinates

	f_1	f_2	f_3	f_4
e_1	g_1	0	0	0
e_2	0	g_1	0	0
e_3	0	0	0	0
e_4	0	0	0	0

Table 3.2.6: Class B Ex: Multiplication table of f's with e's after change of coordinates

follows:

$$Q \xleftarrow{\partial_1} Q^4 \xleftarrow{\partial_2} Q^5 \xleftarrow{\partial_3} Q^2 \leftarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} x^2 & xy & y^3 & z^4 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} -y & 0 & -z^4 & 0 & 0 \\ x & -y^2 & 0 & -z^4 & 0 \\ 0 & x & 0 & 0 & -z^4 \\ 0 & 0 & x^2 & xy & y^3 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} z^4 & 0 \\ 0 & z^4 \\ -y & 0 \\ x & -y^2 \\ 0 & x \end{bmatrix}.$$

Note that via the Leibniz rule:

$$\begin{aligned} \partial_2(e_4 e_1) &= \partial_1(e_4)e_1 - \partial_1(e_1)e_4 \\ &= z^4 e_1 - x^2 e4 \\ &= -f_3 \end{aligned}$$

The reader should note that this is not the desired multiplication, as the multiplication of degree one elemnts should follow $e_4 e_i = f_i$ for $1 \leq i \leq 1$. The problem that our multiplication does not match our definition of $H(3,2)$ for degree 1 with degree 1 and degree 1 with degree 2 can be seen further in the multiplication tables 3.2.7 and 3.2.8.

	e_1	e_2	e_3	e_4
e_1	0	0	0	f_3
e_2	0	0	0	f_4
e_3	0	0	0	f_5
e_4	$-f_3$	$-f_4$	$-f_5$	0

Table 3.2.7: Class $H(3,2)$ Ex: Multiplication table of e's before change of coordinates

	f_1	f_2	f_3	f_4	f_5
e_1	0	0	0	0	0
e_2	0	0	0	0	0
e_3	0	0	0	0	0
e_4	g_1	g_2	0	0	0

Table 3.2.8: Class $H(3,2)$ Ex: Multiplication table of f's with e's before change of coordinates

Macaulay2 makes the following change of basis:

$$\text{Basis for} A_1 : \{-e_1, -e_2, -e_3, e_4\},$$

$$\text{Basis for} A_2 : \{f_3, f_4, f_5, f_1, f_2\},$$

$$\text{Basis for} A_3 : \{g_1, g_2\}.$$

This results in this new complex for the ideal with our new basis:

$$Q \xleftarrow{\partial_1'} Q^4 \xleftarrow{\partial_2'} Q^5 \xleftarrow{\partial_3'} Q^2 \leftarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} -x^2 & -xy & y^3 & z^4 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} z^4 & 0 & 0 & y & 0 \\ 0 & z^4 & 0 & -x & y^2 \\ 0 & 0 & z^4 & 0 & -x \\ x^2 & xy & y^3 & 0 & 0 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \\ z^4 & 0 \\ 0 & z^4 \end{bmatrix},$$

Now that via the Leibniz rule:

$$\begin{aligned} \partial_2(e_4 e_1) &= \partial_1(e_4)e_1 - \partial_1(e_1)e_4 \\ &= z^4 e_1 + x^2 e4 \\ &= f_1 \end{aligned}$$

as desired. We also get the new multiplication tables 3.2.9 and 3.2.10 with the desired values.

	e_1	e_2	e_3	e_4
e_1	0	0	0	$-f_1$
e_2	0	0	0	$-f_2$
e_3	0	0	0	$-f_3$
e_4	f_1	f_2	f_3	0

Table 3.2.9: Class $H(3,2)$ Ex :Multiplication table of e's after change of coordinates

	f_1	f_2	f_3	f_4	f_5
e_1	0	0	0	0	0
e_2	0	0	0	0	0
e_3	0	0	0	0	0
e_4	0	0	0	g_1	g_2

Table 3.2.10: Class $H(3,2)$ Ex: Multiplication table of f's with e's after change of coordinates

3.2.4 Example of $C(3)$

Let $\mathbb{k}$ be a field and $Q = \mathbb{k}[x, y, z]$. Then let our ideal be $I = \langle x^3, y^4, z^5 \rangle$. Note with ideals of class $C(3)$ there is no need for a change of basis because whichever basis Macaulay2 generates will have the desired multiplicative structure because the Tor algebra is an exterior algebra. Thus our minimal free resolution F is as follows

$$Q \xleftarrow{\partial_1} Q^3 \xleftarrow{\partial_2} Q^3 \xleftarrow{\partial_3} Q^1 \leftarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} x^3 & y^4 & z^5 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} -y^4 & -z^5 & 0 \\ x^3 & 0 & -z^5 \\ 0 & x^3 & y^4 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} z^5 \\ -y^4 \\ x^3 \end{bmatrix}.$$

Further, we have the multiplication tables 3.2.11 and 3.2.12 with the correct structure:

	e_1	e_2	e_3
e_1	0	f_1	f_2
e_2	$-f_1$	0	f_3
e_3	$-f_2$	$-f_3$	0

Table 3.2.11: Class $C(3)$ Ex: Multiplication table for e's

	f_1	f_2	f_3
e_1	0	0	g_1
e_2	0	$-g_1$	0
e_3	g_1	0	0

Table 3.2.12: Class $C(3)$ Ex: Multiplication table for f's with e's

3.2.5 Example of $G(2)$

Let $\Bbbk$ be a field and $Q = \Bbbk[x, y, z]$ be our ring. Then let $I = \langle xy^2, xyz, yz^2, x^4 - y^3z, xz^3 - y^4\rangle$. Let $F_\bullet$ be the original minimal free resolution of this ideal. Further, our original free resolution $F_\bullet$ is as follows:

$$Q \xleftarrow{\partial_1} Q^5 \xleftarrow{\partial_2} Q^6 \xleftarrow{\partial_3} Q^2 \leftarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} xy^2 & xyz & yz^2 & x^4 - y^3z & y^4 - xz^3 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} -z & 0 & -x^3 & 0 & y^3 & x^2y^2 \\ y & -z & z^3 & x^3 & 0 & 0 \\ 0 & x & 0 & -y^3 & -x^2z & -y^2z^2 \\ 0 & 0 & y^2 & -yz & 0 & -z^3 \\ 0 & 0 & yz & 0 & -xy & -x^3 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} 0 & x^4 - y^3z \\ -x^3z & -y^4 - xz^3 \\ 0 & -xz \\ -z^2 & -xy \\ -x^2 & -z^2 \\ y & 0 \end{bmatrix}.$$

The problem is this complex yields the multiplication table 3.2.13 does not match our multiplication definition of $G(2)$.

	f_1	f_2	f_3	f_4	f_5	f_6
e_1	0	0	0	0	0	0
e_2	0	0	0	0	0	0
e_3	0	0	0	0	0	0
e_4	g_2	0	0	0	0	0
e_5	0	$-g_2$	0	0	0	0

Table 3.2.13: Class $G(2)$ Ex: Multiplication table of f's with e's before change of coordinates

Thus, Macaulay2 makes the following change of basis:

$$\text{Basis for} A_1 : \{-e_5, e_4, e_1, e_2, e_3\},$$

$$\text{Basis for} A_2 : \{f_2, f_1, f_3, f_4, f_5, f_6\},$$

$$\text{Basis for} A_3 : \{g_2, g_1\}.$$

This results in this new complex for the ideal with our new basis:

$$Q \xleftarrow{\partial_1'} Q^5 \xleftarrow{\partial_2'} Q^6 \xleftarrow{\partial_3'} Q^2 \leftarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} -y^4 + xz^3 & x^4 - y^3 z & xy^2 & xyz & yz^2 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} 0 & 0 & -yz & 0 & xy & x^3 \\ 0 & 0 & y^2 & -yz & 0 & -z^3 \\ 0 & -z & -x^3 & 0 & y^3 & x^2y^2 \\ -z & y & z^3 & x^3 & 0 & 0 \\ x & 0 & 0 & -y^3 & -x^2 z & -y^2 z^2 \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} -y^4 - xz^3 & -x^3 z \\ x^4 - y^3 z & 0 \\ -xz & 0 \\ -xy & -z^2 \\ -z^2 & -x^2 \\ 0 & y \end{bmatrix}.$$

We also get the new multiplication table 3.2.14 with the desired values.

	f_1	f_2	f_3	f_4	f_5	f_6
e_1	g_1	0	0	0	0	0
e_2	0	g_1	0	0	0	0
e_3	0	0	0	0	0	0
e_4	0	0	0	0	0	0
e_5	0	0	0	0	0	0

Table 3.2.14: Class $G(2)$ Ex: Multiplication table of f's with e's after change of coordinates

Chapter 4: Proof of Correctness of the Code

Note that the code at the end of this paper classifies ideals into one of the classes of Theorem 3.1.2. Further, the code will then build a basis that matches the desired structure. In this chapter we will prove that the basis the code is building in each case does indeed match the correct multiplication structure in Theorem 3.1.2.

For this chapter we will let $e_1^*, \ldots, e_m^*$, $f_1^*, \ldots, f_{m+n-1}^*$, and $g_1^*, \ldots g_n^*$ respectively be the bases of A_1, A_2, and A_3 that our code is building. We will prove that the bases of each class, $H(p,q), T, G(r)$, and B, satisfies Theorem 3.1.2. Note, we need not show that the code for $C(3)$ satisfy Theorem 3.1.2 since the structure of $C(3)$ allows us to generically create a basis for A_1, A_2, and A_3.

4.1 $H(p,q)$

Our code uses the following algorithm to build a basis for $H(p,q)$:

1. Let $f_1^*, f_2^*, \ldots, f_p^*$ be a basis of the image of the multiplication map:

$$\mu_{1,1} : A_1 \otimes A_1 \to A_2.$$

2. Choose a generic element of A_1, which we call e_{p+1}^*. Such an element will be such that $\mu_1 = \mu_{1,1}(e_{p+1}^* \otimes -) : A_1 \to A_2$ has rank p.

3. Lift $f_1^*, f_2^*, \ldots, f_p^*$ under μ_1 to $\widehat{e_1}, \ldots, \widehat{e_p}$.

4. Let $1 \leq i \leq p$. Find constants x_i such that

$$e_i^* = \widehat{e_i} - x_i e_{p+1}^* \in \operatorname{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m).$$

5. Find a basis $g_1^*, \ldots, g_q^*$ of the image of $\mu_{1,2}(e_{p+1}^* \otimes -)$.

6. Arbitrarily lift g_i^* to f_{p+i}^* under $\mu_{1,2}(e_{p+1}^* \otimes -)$.

7. Find a basis the degree 1 elements of $\mathrm{Ann}(e_1^*, \ldots, e_{p+1}^*)$ to complete basis of A_1.

8. Take a basis of the elements of degree 2 in $\mathrm{Ann}(e_{p+1}^*)/\langle f_1^*, \ldots, f_p^* \rangle$ and lift to A_2 to complete A_2 to a basis.

9. Complete $g_1^*, \ldots, g_q^*$ to a basis of $\mathrm{span}(g_1, \ldots, g_n)$.

Proof of correctness of the algorithm: We know that if our ideal I is of class $H(p,q)$ then there must exist a basis of the Tor algebra A such that the multiplication structure matches that of Theorem 3.1.2. We will call these bases $e_1, \ldots, e_m$ for A_1, $f_1, \ldots, f_{m+n-1}$ for A_2, and $g_1, \ldots, g_n$ for A_3.

First let $f_1^*, f_2^*, \ldots, f_p^*$ be a basis of the image of the multiplication map

$$\mu_{1,1} : A_1 \otimes A_1 \to A_2.$$

Choose e_{p+1}^* such that $\mu_1 = \mu_{1,1}(e_{p+1}^* \otimes -) : A_1 \to A_2$ has rank p. Such a choice is generic, as is shown in the next lemma:

Proposition 4.1.1. *A generic element $e \in A_1$ satisfies* $\mathrm{rank}\, \mu_{1,1}(e \otimes -) = p$.

Proof. Let $e = c_1 e_1 + \cdots + c_m e_m$ where $c_{p+1} \neq 0$. Then, for $i \leq p$

$$\begin{aligned}
\mu_{1,1}(e \otimes e_i) &= e e_i \\
&= (c_1 e_1 + \cdots + c_m e_m) e_i \\
&= c_1 e_1 e_i + \cdots c_{p+1} e_{p+1} e_i + \cdots + c_m e_m e_i \\
&= c_{p+1} e_{p+1} e_i \\
&= c_{p+1} f_i
\end{aligned}$$

Further,

$$\begin{aligned}\mu_{1,1}(e \otimes e_{p+1}) &= ee_{p+1} \\ &= (c_1 e_1 + \cdots + c_m e_m) e_{p+1} \\ &= c_1 e_1 e_{p+1} + \cdots c_{p+1} e_{p+1} e_{p+1} + \cdots + c_m e_m e_{p+1} \\ &= -c_1 f_1 - \cdots - c_p f_p\end{aligned}$$

And, if $i \geq p+2$ then

$$\begin{aligned}\mu_{1,1}(e \otimes e_i) &= ee_i \\ &= (c_1 e_1 + \cdots + c_m e_m) e_i \\ &= c_1 e_1 e_i + \cdots c_{p+1} e_{p+1} e_i + \cdots + c_m e_m e_i \\ &= 0\end{aligned}$$

Thus the matrix of this map with respect to the bases $e_1, \ldots e_m$ and $f_1, \ldots, f_{m+n-1}$ is the following $(m+n-1) \times m$ matrix

$$\begin{bmatrix} c_{p+1} & 0 & \cdots & 0 & -c_1 & 0 & \cdots & 0 \\ 0 & c_{p+1} & \cdots & 0 & -c_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_{p+1} & -c_p & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

Where we have the $p \times p$ identity matrix multiplied by c_{p+1} which is nonzero by assumption, and the $p+1$ column has the negative of the first p coefficients of e. This matrix clearly has rank p, therefore rank $\mu_{1,1}(e \otimes -) = p$. □

Theorem 4.1.2. *Let $\widehat{e_1}, \ldots, \widehat{e_p}$ be lifts under μ_1 of $f_1^*, f_2^*, \ldots, f_p^*$. Then there exist constants $x_1, \ldots, x_p$ such that $e_i^* = \widehat{e_i} - x_i e_{p+1}^* \in \operatorname{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m)$. In particular, the elements $e_1^*, \ldots, e_p^*$ and $f_1^*, \ldots, f_p^*$ satisfy the conditions appearing in the definition of $H(p, q)$.*

One should note that we need $e_i^* e_j^* = 0$ for all $i \neq j$ where $i, j \in \{1, \ldots, p\}$. If we look at $i = 1$ and $j = 2$, then,

$$\begin{aligned} e_1^* e_2^* &= (\widehat{e_1} - x_1 e_{p+1}^*)(\widehat{e_2} - x_2 e_{p+1}^*) \\ &= \widehat{e_1}\widehat{e_2} - x_2 \widehat{e_1} e_{p+1}^* - x_1 e_{p+1}^* \widehat{e_2} + x_1 x_2 e_{p+1}^* e_{p+1}^* \\ &= \widehat{e_1}\widehat{e_2} + x_2 e_{p+1}^* \widehat{e_1} - x_1 e_{p+1}^* \widehat{e_2} \\ &= \widehat{e_1}\widehat{e_2} + x_2 f_1^* - x_1 f_2^*. \end{aligned}$$

Note we want this to be 0. If we did this for all $i \neq j$, as well as all $e_i^* f_j^*$, we end up with a system of linear equations with $\binom{p}{2}p$ equations with only p unknowns. Thus the set $\{x_1, \ldots, x_p\}$ is over determined. Therefore, in order prove this theorem holds and a solution exists we will first need a couple of lemmas.

Lemma 4.1.3. *Let $T : V \to W$ be a linear map and let $v_1, \ldots, v_p \in V$. Then $T(v_1), \ldots, T(v_p)$ linearly independent implies that $v_1, \ldots, v_p$ are linearly independent.*

Proof. Suppose $a_1 v_1 + \cdots + a_p v_p = 0$. Then $a_1 T(v_1) + \cdots + a_p T(v_p) = 0$. So $a_i = 0$. $\square$

Lemma 4.1.4. *Let $2 \leq c \leq p$ and suppose that $e_1^*, \ldots, e_c^*$ satisfy $e_i^* e_j^* = 0$ for all $i \neq j$ and $e_{p+1}^* e_i^*$ and $e_{p+1}^* e_j^*$ are not scalar multiples of one another. Then $e_i^* \in$ $\text{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m)$.*

Proof. We know that

$$\begin{aligned} e_i^* e_j^* &= (a_1 e_1 + \cdots + a_{p+1} e_{p+1} + \cdots + a_m e_m)(b_1 e_1 + \cdots + b_{p+1} e_{p+1} + \cdots + b_m e_m) \\ &= (a_{p+1} b_1 - a_1 b_{p+1}) f_1 + \cdots + (a_{p+1} b_p - a_p b_{p+1}) f_p \qquad (4.1.1) \\ &= 0 \end{aligned}$$

Thus since $e_{p+1}^* e_i^*$ and $e_{p+1}^* e_j^*$ are linearly independent, the elements e_i^* and e_j^* linearly independent so the following matrix is of rank 2:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_p & a_{p+1} & \cdots & a_m \\ b_1 & b_2 & \cdots & b_p & b_{p+1} & \cdots & b_m \end{bmatrix}$$

Since $e_{p+2}, \ldots, e_m$ do not participate in the multiplication, a lifts $a_{p+2} = \cdots = a_m = 0$ would also satisfy the hypothesis, so we know that the smaller matrix

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_p & a_{p+1} \\ b_1 & b_2 & \cdots & b_p & b_{p+1} \end{bmatrix}$$

has rank 2.

So $a_k b_l - a_l b_k \neq 0$ for some $l \neq k$. Hence since $a_k b_{p+1} = a_{p+1} b_k$ and $a_l b_{p+1} = a_{p+1} b_l$ by (4.1.1) we have that $a_{p+1} a_k b_l = a_k a_l b_{p+1} = a_{p+1} b_k a_l$.

Therefore $a_{p+1}(a_k b_l - a_l b_k) = 0$, which implies $a_{p+1} = 0$. Thus it follows that $e_i^* \in \operatorname{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m)$. The proof that $b_{p+1} = 0$ is similar. □

Proof of Theorem 4.1.2: By Lemma 4.1.3, since $f_1^*, f_2^*, \ldots f_p^*$ are linearly independent, then $\widehat{e_1}, \ldots, \widehat{e_p}$ is linearly independent. Recall that $e_{p+1}^* = c_1 e_1 + \cdots + c_m e_m$ where $c_{p+1} \neq 0$. Now consider $e_i^* = \widehat{e_i} - c_{p+1}^{-1} y_i e_{p+1}^*$ where y_i is the coefficient of e_{p+1} in $\widehat{e_i}$. One can check that the e_i^* satisfy the hypotheses of Lemma 4.1.4, hence $e_i^* \in \operatorname{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m)$. □

Remark 4.1.5. If we let x_i be a variable for each i and let $e_i^* = \widehat{e_i} - x_i e_{p+1}^*$. The conditions $e_i^* e_j^* = 0$ give linear equations in the x_i which have a solution by the previous theorem. Thus a solution exists, so we can ask Macaulay2 to find a solution to $x_1, \ldots x_p$ that satisfies the conditions for $H(p, q)$.

We will now continue to build our bases for A_2 and A_3.

Proposition 4.1.6. *A generic element $e \in A_1$ satisfies* $\operatorname{rank} \mu_{1,2}(e \otimes -) = q$.

Proof. Let $e = c_1e_1 + \cdots + c_me_m$ where $c_{p+1} \neq 0$. Then, for $i \leq q$

$$\begin{aligned}\mu_{1,2}(e \otimes f_{p+i}) &= ef_{p+i} \\ &= (c_1e_1 + \cdots + c_me_m)f_{p+i} \\ &= c_1e_1f_{p+i} + \cdots + c_{p+1}e_{p+1}f_{p+i} + \cdots + c_me_mf_{p+i} \\ &= c_{p+1}e_{p+1}f_{p+i} \\ &= c_{p+1}g_i\end{aligned}$$

And, if $i > q$ then

$$\begin{aligned}\mu_{1,2}(e \otimes f_{p+i}) &= ef_{p+i} \\ &= (c_1e_1 + \cdots + c_me_m)f_{p+1} \\ &= c_1e_1f_{p+i} + \cdots + c_{p+1}e_{p+1}f_{p+i} + \cdots + c_me_mf_{p+i} \\ &= 0\end{aligned}$$

Lastly, if $j \leq p$ then

$$\begin{aligned}\mu_{1,2}(e \otimes f_j) &= ef_j \\ &= (c_1e_1 + \cdots + c_me_m)f_j \\ &= c_1e_1f_j + \cdots + c_{p+1}e_{p+1}f_j + \cdots + c_me_mf_j \\ &= 0\end{aligned}$$

Thus the matrix of this map with respect to the bases $f_1, \ldots f_{m+n-1}$ and $g_1, \ldots, g_n$ is the following $n \times (m + n - 1)$ matrix

$$\begin{bmatrix} 0 & \cdots & 0 & c_{p+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & c_{p+1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & c_{p+1} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

Where the $q \times q$ identity matrix times $c_{p+1} \neq 0$ is starting in the 1^{st} row and the $p+1$ column. Thus our matrix has rank q as desired. $\square$

Find a basis $g_1^*, \ldots g_q^*$ of the image of $\mu_{1,2}(e_{p+1}^* \otimes -)$, where $\mu_{1,2} : A_1 \otimes A_2 \to A_3$ is the multiplication map. Next, we then arbitrarily lift g_i^* to f_{p+i}^* under $\mu_{1,2}(e_{p+1}^* \otimes -)$.

Claim 4.1.7. The set $\{f_1^*, \ldots, f_p^*, f_{p+1}^*, \ldots, f_{p+q}^*\}$ is linearly independent. Further, f_i^* for $i \leq p$ are in the kernel of multiplication by e_{p+1}^*. Also, for $i \neq p+1$, $e_i^* \in \operatorname{Ann}\langle f_1^*, \ldots, f_{p+q}^* \rangle$.

Proof. First, note that $f_i^* = e_i^* e_{p+1}^*$, so $e_{p+1}^* f_i^* = (e_{p+1}^*)^2 e_i^* = 0$. Next, Lemma 4.1.4 shows that for $i \neq p+1$, $e_i^* \in \operatorname{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m)$, which is annihilated by all elements of degree two.

Finally, suppose that b_i satisfy:

$$b_1 f_1^* + \cdots + b_p f_p^* + b_{p+1} f_{p+1}^* + \cdots + b_{p+q} f_{p+q}^* = 0.$$

Multiplying both sides by e_{p+1}^* on the right gives

$$b_{p+1} f_{p+1}^* e_{p+1}^* + \cdots + b_{p+q} f_{p+q}^* e_{p+1}^* = b_{p+1} g_1^* + \cdots + b_{p+q} g_q^* = 0$$

Since $\{g_1^*, \ldots, g_q^*\}$ is a basis, $b_{p+i} = 0$ for each i. Further f_i^* are independent for $i = 1, \ldots, p$ so $b_i = 0$ for each $i = 1, \ldots, p$ as well. $\square$

To finish the basis of A_1, we find a basis of the degree one elements of $\operatorname{Ann}(e_1^*, \ldots, e_{p+1}^*)$.

Proposition 4.1.8. *Let $p > 0$. Then the set of degree 1 elements in* $\operatorname{Ann}(e_1^*, \ldots, e_{p+1}^*)$ *is equal to* $\operatorname{span}(e_{p+2}, \ldots, e_m))$.

Proof. Let $\bar{e}$ be a degree 1 element of $\operatorname{Ann}(e_1^*, \ldots, e_{p+1}^*)$ Then, we know

$$\bar{e} = a_1 e_1 + \cdots + a_{p+1} e_{p+1} + a_{p+2} e_{p+2} + \cdots + a_m e_m$$

and by construction

$$e_{p+1}^* = c_1e_1 + \cdots + c_{p+1}e_{p+1} + c_{p+2}e_{p+2} + \cdots + c_me_m$$

where $c_{p+1} \neq 0$, so,

$$\begin{aligned} e_{p+1}^*\bar{e} &= (c_1e_1 + \cdots + c_me_m)(a_1e_1 + \cdots + a_me_m) \\ &= (a_1c_{p+1} - a_{p+1}c_1)f_1 + \cdots + (a_pc_{p+1} - a_{p+1}c_p)f_p \\ &= 0. \end{aligned}$$

Then since $f_1, \ldots, f_p$ is linearly independent, $a_kc_{p+1} - a_{p+1}c_k = 0$ which implies that $a_k = \frac{a_{p+1}c_k}{c_{p+1}}$ for all $1 \leq k \leq p$ (since $c_{p+1} \neq 0$). Also recall that $e_1^* \in \mathrm{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m)$ so

$$e_1^* = b_1e_1 + \cdots + b_pe_p + b_{p+2}e_{p+2} + \cdots + b_me_m.$$

Then

$$\begin{aligned} e_1^*\bar{e} &= b_1a_{p+1}f_1 + \cdots + b_pa_{p+1}f_p \\ &= a_{p+1}(b_1f_1 + \cdots + b_pf_p) \\ &= 0. \end{aligned}$$

But there must exist a $b_j \neq 0$ where $1 \leq j \leq p$ otherwise $e_1^*e_{p+i}^* = 0$. Thus $a_{p+1} = 0$ which implies that $a_k = \frac{a_{p+1}c_k}{c_{p+1}} = 0$ for all $1 \leq k \leq p$. Thus $\bar{e} \in \mathrm{span}(e_{p+2}, \ldots, e_m)$).

Conversely, let $\tilde{e} \in \mathrm{span}(e_{p+2}, \ldots, e_m)$. Then, since for each k with $1 \leq k \leq p$, $e_k^* \in \mathrm{span}(e_1, \ldots, e_p, e_{p+2}, \ldots, e_m)$ this implies

$$e_k^* = b_1e_1 + \cdots + b_pe_p + b_{p+2}e_{p+2} + \cdots + b_me_m.$$

Thus

$$\begin{aligned} \tilde{e}e_k^* &= (d_{p+2}e_{p+2} + \cdots + d_me_m)(b_1e_1 + \cdots + b_pe_p + b_{p+2}e_{p+2} + \cdots + b_me_m) \\ &= 0. \end{aligned}$$

Thus, $\tilde{e} \in \text{Ann}(e_1^*, \dots, e_{p+1}^*)$. Therefore, the set of degree 1 elements in $\text{Ann}(e_1^*, \dots, e_{p+1}^*)$ is equal to $\text{span}(e_{p+2}, \dots, e_m)$). □

Thus if we take a basis of $\text{Ann}(e_1^*, \dots, e_{p+1}^*)$ and label the elements $e_{p+2}^*, \dots, e_m^*$ we then obtain the set $\{e_1^*, \dots, e_m^*\}$ of A_1.

We will now prove that our choice of $\{e_1^*, \dots, e_m^*\}$ is linearly independent and thus a basis of A_1. Let

$$0 = a_1 e_1^* + \cdots + a_p e_p^* + a_{p+1} e_{p+1}^* + a_{p+2} e_{p+2}^* + \cdots + a_m e_m^*.$$

Then,

$$\begin{aligned} 0 =& e_{p+1}^*(a_1 e_1^* + \cdots + a_p e_p^* + a_{p+1} e_{p+1}^* + a_{p+2} e_{p+2}^* + \cdots + a_m e_m^*) \\ 0 =& a_1 f_1^* + \cdots + a_p f_p^*. \end{aligned}$$

By construction $f_1^*, \dots, f_p^*$ is linearly independent, thus $a_1 = \cdots = a_p = 0$. Thus,

$$0 = a_{p+1} e_{p+1}^* + a_{p+2} e_{p+2}^* + \cdots + a_m e_m^*.$$

Then

$$\begin{aligned} 0 =& e_1^*(a_{p+1} e_{p+1}^* + a_{p+2} e_{p+2}^* + \cdots + a_m e_m^*) \\ 0 =& - a_{p+1} f_1^*. \end{aligned}$$

Thus $a_{p+1} = 0$. Finally,

$$0 = a_{p+2} e_{p+2}^* + \cdots + a_m e_m^*.$$

Since $e_{p+2}^* \cdots, e_m^*$ is linearly independent, so $a_{p+2} = \cdots = a_m = 0$. Therefore $e_1^*, \dots e_m^*$ is linearly independent, and thus a basis of A_1.

We will now complete the basis of A_2. Take a basis of the elements of degree 2 in $\text{Ann}(e_{p+1}^*)/\langle f_1, \dots, f_p \rangle$. Note that we previously showed in Proposition 4.1.6 that $\mu_{1,2}(e_{p+1}^* \otimes -)$ has rank q. Let the set of degree 2 elements of $\text{Ann}(e_{p+1}^*)$ be denoted as B. Therefore by the rank plus nullity theorem we know $(m+n-1) = q+\dim(B)$. Thus $(m+n-1)-q = \dim(B)$ which implies $\dim(B/\langle f_1, \ldots, f_p\rangle) = m+n-1-q-p$. Thus take a basis of the degree 2 elements in $\mathrm{Ann}(e^*_{p+1})/\langle f_1, \ldots, f_p\rangle$ and label them $f^*_{p+q-1}, \ldots, f^*_{m+n-1}$, obtaining the set $\{f^*_1, \ldots, f^*_{m+n-1}\}$.

We will now prove that our choice of $\{f_1, \ldots, f_p, f^*_{p+1}, \ldots, f^*_{p+q}, f^*_{p+q+1}, \ldots, f^*_{m+n-1}\}$ is linearly independent, and therefore a basis. Let

$$0 = a_1 f^*_1 + \cdots + a_p f^*_p + a_{p+1} f^*_{p+1} + \cdots + a_{p+q} f^*_{p+q} + a_{p+q+1} f^*_{p+q+1} + \cdots + a_{m+n-1} f^*_{m+n-1}$$

be a dependence relation on $\{f^*_1, \ldots, f^*_{m+n-1}\}$. Then,

$$\begin{aligned} 0 =& e^*_{p+1}(a_1 f^*_1 + \cdots + a_p f^*_p + a_{p+1} f^*_{p+1} + \cdots + a_{p+q} f^*_{p+q} + \\ & a_{p+q+1} f^*_{p+q+1} + \cdots + a_{m+n-1} f^*_{m+n-1}) \\ 0 =& a_{p+1} g^*_1 + \cdots + a_{p+q} g^*_q. \end{aligned}$$

By construction $g^*_1, \ldots, g^*_q$ is linearly independent, thus $a_{p+1} = \cdots = a_{p+q} = 0$, Then,

$$0 = a_1 f^*_1 + \cdots + a_p f^*_p + a_{p+q+1} f^*_{p+q+1} + \cdots + a_{m+n-1} f^*_{m+n-1}.$$

Since $f^*_{p+q+1}, \ldots, f^*_{m+n-1}$ is a basis of the degree 2 elements of $\mathrm{Ann}(e^*_{p+1})/\langle f_1, \ldots, f_p\rangle$, then $f^*_1, \ldots, f^*_p, f^*_{p+q+1}, \ldots, f^*_{m+n-1}$ is linearly independent, which implies $a_1 = \cdots = a_p = a_{p+q+1} = \cdots = a_{m+n-1} = 0$. Thus $\{f^*_1, \ldots, f^*_{m+n-1}\}$ is linearly independent, which implies $\{f^*_1, \ldots, f^*_{m+n-1}\}$ is a basis of A_2.

Since g_i multiplies trivially in $H(p, q)$ for all $1 \leq i \leq n$, we can just complete $g^*_1, \ldots, g^*_q$ to a basis of $\mathrm{span}(g_1, \ldots, g_n)$.

Thus we now have a basis for $H(p, q)$ that satisfies the formulas in the definition.

4.2 T

Our code uses the following algorithm to build a basis for T :

1. Take generic choices of e_1^*, e_2^* and e_3^*.

2. Let $f_3^* = e_1^* e_2^*$, $f_1^* = e_2^* e_3^*$, and $f_2^* = e_3^* e_1^*$.

3. Arbitrarily complete $\{e_1^*, e_2^*, e_3^*\}$ to a basis of A_1.

4. Arbitrarily complete $\{f_1^*, f_2^*, f_3^*\}$ to a basis of A_2.

5. Create an arbitrary basis for A_3.

Proof of correctness of the algorithm: We know that if our ideal I is of class T then there must exist a basis such that the multiplication structure holds. We will call these bases $e_1, \ldots, e_m$ for A_1, $f_1, \ldots, f_{m+n-1}$ for A_2, and $g_1, \ldots, g_n$ for A_3.

Proposition 4.2.1. *For class T a generic choice of e_1^*, e_2^* and e_3^* will linearly independent.*

Proof. Take generic choices of e_1^*, e_2^* and e_3^*:

$$e_1^* = (a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + \cdots + a_{1m}e_m)$$
$$e_2^* = (a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + \cdots + a_{2m}e_m)$$
$$e_3^* = (a_{31}e_1 + a_{32}e_2 + a_{33}e_3 + \cdots + a_{3m}e_m).$$

Which implies that

$$e_1^* e_2^* = (a_{12}a_{23} - a_{13}a_{22})f_1 + (a_{13}a_{21} - a_{11}a_{23})f_2 + (a_{11}a_{22} - a_{12}a_{21})f_3$$
$$e_1^* e_3^* = (a_{12}a_{33} - a_{13}a_{32})f_1 + (a_{13}a_{31} - a_{11}a_{33})f_2 + (a_{11}a_{32} - a_{12}a_{31})f_3$$
$$e_2^* e_3^* = (a_{22}a_{33} - a_{23}a_{32})f_1 + (a_{23}a_{31} - a_{21}a_{33})f_2 + (a_{21}a_{32} - a_{22}a_{31})f_3.$$

The condition of linear independence of $\{e_1^*e_2^*, e_1^*e_3^*, e_2^*e_3^*\}$is determined by if the determinant of the matrix with the above coefficients is nonzero. To see this condition is not identically zero, it is enough to find a single nonzero value of a_{ij} that make this matrix nonzero. Take $a_{ij} = \delta_{ij}$. Then, 3×3 would be:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has a nonzero determinant. Thus our generic choice for e_1^*, e_2^* and e_3^* is linearly independent. □

4.3 $G(r)$

Our code uses the following algorithm to build a basis for $G(r)$:

1. Let $M : A_2 \to A_1^*$ where A_1^* is the dual space and M be the map given by $(M(e_i))(f_j) = e_i f_j$.
2. Choose f_1^* to be the preimage of a nonzero vector in A_1^*
3. Let e_1^* be an element such that $e_1^* f_1^* \neq 0$
4. Let $e_1^* f_1^* = g_1^*$.
5. Let f_2^* be a nonzero vector in $(e_1^*)^\perp \cap \text{preim}(M)$
6. Find $\tilde{e_2}$ such that $\tilde{e_2} f_2^* = m_2 \neq 0$
7. Let $e_2^* = m_2^{-1}\tilde{e_2} - m_2^{-1}c_{21}e_1^*$ where $e_2^* f_1^* = c_{21}g_1^*$.
8. Let f_3^* be a nonzero vector in $(e_1^*, e_2^*)^\perp \cap \text{preim}(M)$
9. Find $\tilde{e_3}$ such that $\tilde{e_3} f_3^* = m_3 \neq 0$
10. Let $e_3^* = m_3^{-1}\tilde{e_3} - m_3^{-1}c_{31}e_1^* - m_3^{-1}c_{32}e_2^*$ where $e_3^* f_1^* = c_{31}g_1^*$ and $e_3^* f_2^* = c_{32}g_1^*$.
11. Continue in this way until you have $\{e_1^*, \ldots, e_r^*\}$ and $\{f_1^*, \ldots, f_r^*\}$

12. Let $\{e^*_{r+1}, \dots, e^*_m\}$ be a basis of $\mathrm{Ann}(f^*_1, \dots, f^*_r)$ in degree one.

13. Let $\{f^*_{r+1}, \dots, f^*_{m+n-1}\}$ be a basis of $\mathrm{Ann}(e^*_1, \dots, e^*_r)$ in degree two.

14. Complete $\{g^*_1\}$ to a basis of A_3.

We first need a couple definitions before we begin the proof of $G(r)$:

Definition 4.3.1. Let V and W be vector spaces over a field $\Bbbk$. A linear map $\langle\ ,\ \rangle : V \otimes W \to k$ is called a *pairing*. A pairing is called *nondegenerate* if for every nonzero $v \in V$ there exists $w \in W$ such that $\langle v, w\rangle \neq 0$, and similarly for each $w \in W$, there exists $v \in V$ such that $\langle v, w\rangle \neq 0$.

Lemma 4.3.2. *Let $\langle\ ,\ \rangle : V \otimes W \to \Bbbk$ be a pairing with V and W finite dimensional vector spaces. Then there exists subspaces V' of V and W' of W such that:*

1) The restricted pairings $V' \otimes W \to \Bbbk$ and $V \otimes W' \to \Bbbk$ are zero.

2) The induced pairing $V/V' \otimes W/W' \to k$ is nondegenerate.

3) $\dim V/V' = \dim W/W'$,

4) There exists bases $\{\bar{v_1}, \dots, \bar{v_n}\}$ of V/V' and $\{\bar{w_1}, \dots, \bar{w_n}\}$ of W/W' such that $\langle v_i, w_j\rangle = \delta_{ij}$.

Proof. Let $V' = \{v \in V \mid \langle v, -\rangle = 0\}$, and let $W' = \{w \in W \mid \langle -, w\rangle = 0\}$. These subspaces satisfy (1) by construction. Further, the induced pairing $V/V' \otimes W/W' \to k$ given by $\langle \bar{v}, \bar{w}\rangle = \langle v, w\rangle$ is well-defined because the pairing vanishes when the left argument is in V' or the right argument is in W'.

Let $v \in V$. Suppose that for all $\bar{w} \in W/W'$, one has $\langle \bar{v}, \bar{w}\rangle = 0$. Then by definition, $\langle v, w\rangle = 0$ for all $w \in W$. By definition, $v \in V'$ so $\bar{v} = 0$. Similarly if $w \in W$ with $\langle \bar{v}, \bar{w}\rangle = 0$ for all $\bar{v} \in V/V'$, then $w \in W'$ so that $\bar{w} = 0$.

By passing from V to V/V' and from W to W/W', we may assume that $V' = 0$, $W' = 0$ and the original pairing is nondegenerate. We now prove 3) and 4) .

Let $\phi : V \to W^*$ be given by $(\phi(v))(w) = \langle v, w \rangle$. Note

$$\ker(\phi) = \{v \in V \mid \phi(v) = 0\} = \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\} = \{0\}$$

since $V \otimes W \to \Bbbk$ is nondegenerate. Thus by the Rank-Nullity Theorem

$$\dim V = \operatorname{rank} \phi + \operatorname{nullity}(\phi) \leq \dim W^* + 0 = \dim W.$$

By similar reasoning $\dim W \leq \dim V$. Therefore $\dim V = \dim W$. Further, since $\ker \phi = 0$, ϕ is an isomorphism.

The proof of this portion proceeds similar to the Gram-Schmidt algorithm for finding an orthonormal basis of a vector space with respect to an inner product.

Let w_1 be a nonzero element of W. Then, via non-degeneracy, there exists $\tilde{v}_1 \in V$ such that $\langle \tilde{v}_1, w_1 \rangle = m_1$ where $m_1 \in \Bbbk$ and $m_1 \neq 0$. Let $v_1 = m_1^{-1} \tilde{v}_1$. Thus, since a pairing is a bilinear map, $\langle v_1, w_1 \rangle = 1$.

Definition 4.3.3. Let X be a subspace of V. An element $w \in W$ is in $X^\perp$ relative to $\langle \ , \ \rangle$ if $\langle x, w \rangle = 0$ for all $x \in X$. It is clear that $X^\perp$ is a subspace of W.

Then, take w_2 to be a nonzero element of $(\operatorname{span}\{v_1\})^\perp$ under $\langle \ , \ \rangle$. Such an element exists (as long as $n \geq 2$) from the following claim:

Claim 4.3.4. If $\langle \ , \ \rangle : V \otimes W \to \Bbbk$ is nondegenerate, and X is a submodule of V then $X^\perp$ under $\langle \ , \ \rangle$ has dimension $\dim V - \dim X$.

Proof. Choose some isomorphism $\psi : V \to W$. Define $\{v_1, v_2\} = \langle v_1, \psi(v_2) \rangle$. Then $\{ \ , \ \}$ is nondegenerate. Let $y \in X^\perp$ under $\langle \ , \ \rangle$. This is true if and only if $\langle x, y \rangle = 0 \ \forall x \in X$, which is true if and only if $\{x, \psi^{-1}(y)\} = 0$. This is true if and only if $\psi^{-1}(y) \in X^\perp$ under $\{ \ , \ \}$. Therefore $X^\perp$ under $\{ \ , \ \}$ is just $\psi^{-1}(X)^\perp$ under $\langle \ , \ \rangle$.

Since the dimension of $X^\perp$ is preserved under ψ^{-1}, then via [Axl15]

$$\dim(X^\perp) = \dim(V) - \dim(X).$$

Thus, the result follows. □

Choose $\tilde{v}_2$ such that $\langle \tilde{v}_2, w_2 \rangle = m_2 \neq 0$. Note, by construction $\tilde{v}_2 \notin \text{span}(v_1)$ because otherwise $\langle \tilde{v}_2, w_2 \rangle = 0$. Let $c_1 = \langle \tilde{v}_2, w_1 \rangle$, which may be zero. Set $v_2 = m_2^{-1}\tilde{v}_2 - c_1 m_2^{-1} v_1$. Then:

$$\begin{aligned} \langle v_2, w_2 \rangle &= \langle m_2^{-1}\tilde{v}_2 - c_1 m_2^{-1} v_1, w_2 \rangle \\ &= m_2^{-1}\langle \tilde{v}_2, w_2 \rangle - c_1 m_2^{-1} \langle v_1, w_2 \rangle \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \langle v_2, w_1 \rangle &= \langle m_2^{-1}\tilde{v}_2 - c_1 m_2^{-1} v_1, w_1 \rangle \\ &= m_2^{-1}\langle \tilde{v}_2, w_1 \rangle - c_1 m_2^{-1} \langle v_1, w_1 \rangle \\ &= 0. \end{aligned}$$

Take w_3 to be a nonzero element of $(\text{span}\{v_1, v_2\})^\perp$ under $\langle\ ,\ \rangle$, which we know exists via Claim 4.3.4 as long as $n \geq 3$. Next, choose $\tilde{v}_3$ such that $\langle \tilde{v}_3, w_3 \rangle = m_3 \neq 0$, which we know exists via nondegenracy. Note, by construction $\tilde{v}_2 \notin \text{span}(v_1, v_2)$ because otherwise $\langle \tilde{v}_3, w_3 \rangle = 0$. Let $d_1 = \langle \tilde{v}_3, w_1 \rangle$ and $d_2 = \langle \tilde{v}_3, w_2 \rangle$ which may be zero. Set $v_3 = m_3^{-1}\tilde{v}_3 - d_1 m_3^{-1} v_1 - d_2 m_3^{-1} v_2$. Then:

$$\begin{aligned} \langle v_3, w_3 \rangle &= \langle m_3^{-1}\tilde{v}_3 - d_1 m_3^{-1} v_1 - d_2 m_3^{-1} v_2, w_3 \rangle \\ &= m_3^{-1}\langle \tilde{v}_3, w_3 \rangle - d_1 m_3^{-1} \langle v_1, w_3 \rangle - d_2 m_3^{-1} \langle v_2, w_3 \rangle \\ &= 1, \end{aligned}$$

and

$$\begin{aligned}
\langle v_3, w_1 \rangle &= \langle m_3^{-1}\tilde{v}_3 - d_1 m_3^{-1} v_1 - d_2 m_3^{-1} v_2, w_1 \rangle \\
&= m_3^{-1}\langle \tilde{v}_3, w_1 \rangle - d_1 m_3^{-1}\langle v_1, w_1 \rangle - d_2 m_3^{-1}\langle v_2, w_1 \rangle \\
&= m_3^{-1} d_1 - d_1 m_3^{-1} - 0 \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle v_3, w_2 \rangle &= \langle m_3^{-1}\tilde{v}_3 - d_1 m_3^{-1} v_1 - d_2 m_3^{-1} v_2, w_2 \rangle \\
&= m_3^{-1}\langle \tilde{v}_3, w_2 \rangle - d_1 m_3^{-1}\langle v_1, w_2 \rangle - d_2 m_3^{-1}\langle v_2, w_2 \rangle \\
&= m_3^{-1} d_2 - 0 - d_2 m_3^{-1} \\
&= 0.
\end{aligned}$$

Continue in this way until you complete a basis for V and W. Thus there exists a basis $\{v_1, \ldots, v_n\}$ of V and $\{w_1, \ldots, w_n\}$ of W such that $\langle v_i, w_j \rangle = \delta_{ij}$. □

Proof of correctness of the algorithm: Now, let $A_3' = \mathrm{span}\{g_1^*\}$ be the image of the multiplication map $\mu_{1,2} : A_1 \otimes A_2 \to A_3$. Note that $\dim(A_3') = 1$ by defintion of $G(r)$, which implies that $A_3' \cong \Bbbk$ where $\Bbbk$ is our field. Then $\phi : A_1 \otimes A_2 \to A_3'$ is a pairing as in Lemma 4.4. Choose elements $\{e_1^*, \ldots, e_r^*\}$ of A_1 and $\{f_1^*, \ldots, f_r^*\}$ of A_2 according to the lemma. Let $\{e_{r+1}^*, \ldots, e_m^*\}$ be a basis of $\mathrm{Ann}(f_1^*, \ldots, f_r^*)$ in degree one (which is a basis of V' from the lemma), and let $\{f_{r+1}^*, \ldots, f_{m+n-1}^*\}$ be a basis of $\mathrm{Ann}(e_1^*, \ldots, e_r^*)$ in degree two (which is a basis of W' from the lemma). Finally, complete $\{g_1^*\}$ to a basis of A_3 to obtain a basis meeting the definition of $G(r)$.

4.4 B

Our code uses the following algorithm to build a basis for B :

1. Let $M : A_2 \to A_1^*$ where A_1^* is the dual space and M be the map given by $(M(e_i))(f_j) = e_i f_j$.

2. Choose f_1^* to be the preimage of a nonzero vector in A_1^*

3. Let e_1^* be an element such that $e_1^* f_1^* \neq 0$

4. Let $e_1^* f_1^* = g_1^*$.

5. Let f_2^* be a nonzero vector in $(e_1^*)^\perp \cap \operatorname{preim}(M)$

6. Find $\tilde{e_2}$ such that $\tilde{e_2} f_2^* = m_2 \neq 0$

7. Let $e_2^* = m_2^{-1}\tilde{e_2} - m_2^{-1} c_{21} e_1^*$ where $e_2^* f_1^* = c_{21} g_1^*$.

8. Let $f_3^* = e_1^* e_2^*$

9. Let $\{e_3^*, \ldots, e_m^*\}$ be a basis of $\operatorname{Ann}(f_1^*, f_2^*)$ in degree one.

10. Complete $\{f_3^*\}$ to a basis $\{f_3^*, \ldots, f_{m+n-1}^*\}$ be a basis of $\operatorname{Ann}(e_1^*, e_2^*)$ in degree two.

11. Complete $\{g_1^*\}$ to a basis of A_3.

Proof of correctness of the algorithm: Now, let $A_3' = \operatorname{span}\{g_1^*\}$ be the image of the multiplication map $\mu_{1,2} : A_1 \otimes A_2 \to A_3$. Note that $\dim(A_3') = 1$ which implies that $A_3' \cong \Bbbk$ where $\Bbbk$ is our field. Then $\phi : A_1 \otimes A_2 \to A_3'$ is a pairing as in Lemma . Choose elements $\{e_1^*, e_2^*\}$ of A_1 and $\{f_1^*, f_2^*\}$ of A_2 according to the lemma. Let $f_3^* = e_1^* e_2^*$. We will show that $\{f_1^*, f_2^*, f_3^*\}$ is linearly independent. Suppose

$$a f_1^* + b f_2^* + c f_3^* = 0.$$

Then,

$$\begin{aligned} e_1^*0 &= e_1^*(af_1^* + bf_2^* + cf_3^*) \\ 0 &= ae_1^*f_1^* + be_1^*f_2^* + ce_1^*e_1^*e_2^* \\ 0 &= ag_1^*. \end{aligned}$$

Thus $a = 0$. Further

$$\begin{aligned} e_2^*0 &= e_w^*(af_1^* + bf_2^* + cf_3^*) \\ 0 &= ae_2^*f_1^* + be_2^*f_2^* + ce_2^*e_1^*e_2^* \\ 0 &= bg_1^*. \end{aligned}$$

Thus $b = 0$. Which implies $cf_3^* = 0$. Thus we just need to show $e_1^*e_2^* \neq 0$. We know that since our ideal I is of class B then there must exist a basis $e_1, \ldots, e_m$ for A_1, $f_1, \ldots, f_{m+n-1}$ for A_2, and and $g_1, \ldots, g_n$ for A_3 such that the multiplication structure holds. Thus

$$\begin{aligned} e_1^* &= a_{11}e_1 + a_{12}e_2 + \cdots + a_{1m}e_m \\ e_2^* &= a_{21}e_1 + a_{22}e_2 + \cdots + a_{2m}e_m \\ f_1^* &= b_{11}f_1 + \cdots + b_{1(m+n-1)}f_{m+n-1} \\ f_2^* &= b_{21}f_1 + \cdots + b_{2(m+n-1)}f_{m+n-1} \end{aligned}$$

Assume for the sake of contradiction that $e_1^*e_2^* = 0$, This would imply that $a_{11}a_{22} - a_{21}a_{12} = 0$. Without loss of generality, we may assume there exists some constant d such that $a_{21} = da_{11}$ and $a_{22} = da_{12}$. This would imply that

$$e_2^* = da_{11}e_1 + da_{12}e_2 + a_{23}e_3 + a_{24}e_4 + \cdots + a_{2m}e_m.$$

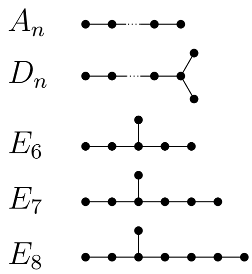
Thus,

$$\begin{aligned}
0 &= e_2^* f_1^* \\
&= (a_{21}b_{11} + a_{22}b_{12})g_1 \\
&= (da_{11}b_{11} + da_{12}b_{12})g_1 \\
&= d(a_{11}b_{11} + a_{12}b_{12})g_1 \\
&= dg_1^*.
\end{aligned}$$

Hence $d = 0$. However, if $d = 0$, then $e_2^* f_2^* = 0$ as well, a contradiction. So $(a_{11}a_{22} - a_{12}a_{21}) \neq 0$, which implies $c = 0$. Thus $\{f_1^*, f_2^*, f_3^*\}$ is linearly independent. Let $\{e_3^*, \ldots, e_m^*\}$ be a basis of $\mathrm{Ann}(f_1^*, f_2^*)$ in degree one (which is a basis of V' from the lemma), and complete $\{f_3^*\}$ to a basis $\{f_3^*, f_4^*, \ldots, f_{m+n-1}^*\}$ of $\mathrm{Ann}(e_1^*, e_2^*)$ in degree two (which is a basis of W' from the lemma). Finally, complete $\{g_1^*\}$ to a basis of A_3 to obtain a basis meeting the definition of the class B.

Chapter 5: Unsolved Problem

We will now discuss a conjecture by Christensen, Veliche, and Weyman in [CVW19], as it was part of the motivation for writing this code. First note, the following graphs are a complete list of simply laced Dynkin Diagrams where n denotes the number of vertices:



These graphs play an important role in representation theory and show up many places in commutative algebra.

To set the stage for the conjecture, we will need several definitions:

Definition 5.0.1. The *height* of a prime ideal $\mathfrak{p}$ is the supremum of all n so that there is a chain $p_0 \subset p_1 \subset \cdots \subset p_n = \mathfrak{p}$ where all p_i are distinct prime ideals. The *Krull dimension of* R is the supremum of all heights of all prime ideals of R.

Definition 5.0.2. A *regular local ring* is a local ring with maximal ideal $\mathfrak{m}$ such that if the Krull dimension of R is d, then $\mathfrak{m}$ can be generated with exactly d elements.

Definition 5.0.3. Let R be a ring and let M be an R-module. A sequence of elements $x_1, \ldots, x_n \in R$ is called a *regular sequence* on M if

1. For all $i = 1, \ldots, n$ x_i is a non-zero-divisor on $M/(x_1, \ldots, x_{i-1})M$
2. $M/(x_1, \ldots, x_n)M \neq 0$

Definition 5.0.4. Let I and J be ideals. We say that I and J are *directly linked* (denoted $I \sim J$) if there is a regular sequence $\langle f_1, \ldots, f_c \rangle$ such that $(\langle f_1, \ldots, f_c \rangle : I) = J$.

Note the following obtains an equivalence relation. We state that two ideals I and J are *linked* if there is a sequence of links

$$I = I_0 \sim I_1 \sim \cdots \sim I_{n-1} \sim I_n = J$$

joining I and J. If this is true we say I and J are in the same *linkage class.*

Definition 5.0.5. A noetherian local ring R is a *complete intersection ring* if $R = S/I$ where S is a regular local ring and I is generated by an S-sequence.

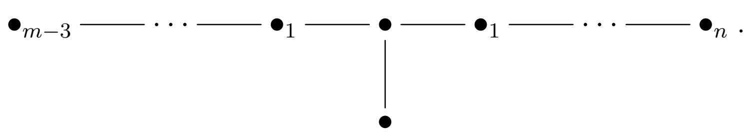
We say an ideal is *licci* if it is in the linkage class of an ideal defining a complete intersection.

Definition 5.0.6. For a free resolution of length 3 $F_\bullet$,

$$F_\bullet : 0 \leftarrow F_0 \overset{\partial_1}{\leftarrow} F_1 \overset{\partial_2}{\leftarrow} F_2 \overset{\partial_3}{\leftarrow} F_3 \leftarrow 0,$$

the *format* is the quadruple (f_0, f_1, f_2, f_3), where $f_i = \text{rank}(F_i)$

Note that for a regular local ring Q with a grade 3 perfect ideal I our format of Q/I over Q is $f = (1, m, m+n-1, n)$ where $m \geq 3$ is the minimal number of generators for I and $n \geq 1$. The diagrams associated with f is as follows:



Note that these diagrams associate with the Dynkin Diagrams from the beginning of this chapter can be seen in Table 5.0.1:

The conjecture is as follows:

Dynkin Diagram	Corresponding Format
A_3	$(1,3,3,1)$
D_m	$(1,m,m,1)$ for odd $m \geq 5$
D_{n+3}	$(1,4,n+3,n)$ for $n \geq 2$
E_6	$(1,5,6,2)$
E_7	$(1,6,7,2)$ and $(1,5,7,3)$
E_8	$(1,7,8,2)$ and $(1,5,8,4)$

Table 5.0.1: Dynkin Diagrams and Resolution Formats

Conjecture 5.0.7. *Let Q be a regular local ring and $\mathfrak{m}$ be its maximal ideal. Let $F = (1, m, m+n-1, n)$ be a resolution format realized by some grade 3 perfect ideal contained in $\mathfrak{m}^2$.*

- *If F is not Dynkin, then there exists a grade 3 perfect ideal $I \subseteq \mathfrak{m}^2$ of format F that is not licci.*
- *If F is Dynkin, then every grade 3 perfect ideal $I \subseteq \mathfrak{m}^2$ of format F is licci.*

During our exploration into Christensen, Veliche, and Weyman's conjecture, there was one ideal that we could not prove nor disprove whether it was linked to a complete intersection. If we are working over the ring $R = \mathbb{Q}[x,y,z]$ then the ideal $I = \langle z^3, y^2z, y^3 - x^2z, x^2y, x^4\rangle$ which is of format $(1,5,6,2)$, and of class $H(0,0)$. We were able to link I using the ideal $J = \langle x^4, y^4, z^3\rangle$, and the result is of class $H(1,0)$. Past this, we were unable to link to an ideal of a different structure.

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[BE77] David A. Buchsbaum and David Eisenbud. Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *Amer. J. Math.*, 99(3):447–485, 1977.

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Code

```
codimThreeAlgStructure = method()

codimThreeAlgStructure(ChainComplex, List) := (F, sym) -> (
   if F.cache#?"Algebra Structure" then return F.cache#"Algebra
Structure";
   if length F != 3 then
     error "Expected a chain complex of length three which is free of
rank one in degree zero.";
   if #sym != 3 or any(sym, s -> (class baseName s =!= Symbol))  then
     error "Expected a list of three symbols.";
   mult := multTables(F);
   Q := ring F;
   m := numcols F.dd_1;
   l := numcols F.dd_2;
   n := numcols F.dd_3;
   degreesP := if isHomogeneous F then
                  flatten apply(3, j -> apply(degrees source
F.dd_(j+1), d -> {j+1} | d))
               else
                  flatten apply(3, j -> apply(degrees source
F.dd_(j+1), d -> {0} | d));
   skewList := toList((0..(m-1)) | ((m+l)..(m+l+n-1)));
   e := baseName (sym#0);
   f := baseName (sym#1);
   g := baseName (sym#2);
   -- use this line if you want to ensure that 'basis' works properly
on the returned ring.
   --P := first flattenRing
(Q[e_1..e_m,f_1..f_l,g_1..g_n,SkewCommutative=>skewList, Degrees =>
degreesP, Join => false]);
   P := Q[e_1..e_m,f_1..f_l,g_1..g_n,SkewCommutative=>skewList,
Degrees => degreesP, Join => false];
   phi := map(P,Q,apply(numgens Q, i -> P_(m+l+n+i)));
   eVector := matrix {apply(m, i -> P_(i))};
   fVector := matrix {apply(l, i -> P_(m+i))};
   gVector := matrix {apply(n, i -> P_(m+l+i))};
   eeGens := apply(pairs mult#0, p -> first flatten entries
(P_(p#0#0-1)*P_(p#0#1-1) - fVector*(phi(p#1))));
   efGens := apply(pairs mult#1, p -> first flatten entries
(P_(p#0#0-1)*P_(m+p#0#1-1) - gVector*(phi(p#1))));
   I := (ideal eeGens) +
        (ideal efGens) +
        (ideal apply(m..(m+l-1), i -> P_i))^2 +
        (ideal apply(0..(m-1), i -> P_i))*(ideal apply((m+l)..
(m+l+n-1), i -> P_i)) +
        (ideal apply(m..(m+l-1), i -> P_i))*(ideal apply((m+l)..
(m+l+n-1), i -> P_i)) +
        (ideal apply((m+l)..(m+l+n-1), i -> P_i))^2;
   A := P/I;
   A.cache#"l" = l;
```

```
   A.cache#"m" = m;
   A.cache#"n" = n;
   F.cache#"Algebra Structure" = A;
   A
)

codimThreeAlgStructure2 = method()

codimThreeAlgStructure2(ChainComplex, List) := (F, sym) -> (
   if F.cache#?"Algebra Structure2" then return F.cache#"Algebra
Structure2";
   if length F != 3 then
     error "Expected a chain complex of length three which is free of
rank one in degree zero.";
   if #sym != 3 or any(sym, s -> (class baseName s =!= Symbol))  then
     error "Expected a list of three symbols.";
   mult := multTables2(F);
   Q := ring F;
   m := numcols F.dd_1;
   l := numcols F.dd_2;
   n := numcols F.dd_3;
   degreesP := if isHomogeneous F then
                  flatten apply(3, j -> apply(degrees source
F.dd_(j+1), d -> {j+1} | d))
               else
                  flatten apply(3, j -> apply(degrees source
F.dd_(j+1), d -> {0} | d));
   skewList := toList((0..(m-1)) | ((m+l)..(m+l+n-1)));
   e := baseName (sym#0);
   f := baseName (sym#1);
   g := baseName (sym#2);
   -- use this line if you want to ensure that 'basis' works properly
on the returned ring.
   --P := first flattenRing
(Q[e_1..e_m,f_1..f_l,g_1..g_n,SkewCommutative=>skewList, Degrees =>
degreesP, Join => false]);
   P := Q[e_1..e_m,f_1..f_l,g_1..g_n,SkewCommutative=>skewList,
Degrees => degreesP, Join => false];
   phi := map(P,Q,apply(numgens Q, i -> P_(m+l+n+i)));
   eVector := matrix {apply(m, i -> P_(i))};
   fVector := matrix {apply(l, i -> P_(m+i))};
   gVector := matrix {apply(n, i -> P_(m+l+i))};
   eeGens := apply(pairs mult#0, p -> first flatten entries
(P_(p#0#0-1)*P_(p#0#1-1) - fVector*(phi(p#1))));
   efGens := apply(pairs mult#1, p -> first flatten entries
(P_(p#0#0-1)*P_(m+p#0#1-1) - gVector*(phi(p#1))));
   I := (ideal eeGens) +
        (ideal efGens) +
         (ideal apply(m..(m+l-1), i -> P_i))^2 +
         (ideal apply(0..(m-1), i -> P_i))*(ideal apply((m+l)..
```

```
(m+l+n-1), i -> P_i)) +
        (ideal apply(m..(m+l-1), i -> P_i))*(ideal apply((m+l)..
(m+l+n-1), i -> P_i)) +
        (ideal apply((m+l)..(m+l+n-1), i -> P_i))^2;
   A := P/I;
   A.cache#"l" = l;
   A.cache#"m" = m;
   A.cache#"n" = n;
   F.cache#"Algebra Structure" = A;
   A
)

codimThreeTorAlgebra = method()

codimThreeTorAlgebra(ChainComplex,List) := (F,sym) -> (
   if F.cache#?"Tor Algebra Structure" then return F.cache#"Tor
Algebra Structure";
   A := codimThreeAlgStructure(F,sym);
   P := ambient A;
   Q := ring F;
   kk := coefficientRing Q;
   PP := kk monoid P;
   I := ideal mingens sub(ideal A, PP);
   B := PP/I;
   B.cache#"l" = A.cache#"l";
   B.cache#"m" = A.cache#"m";
   B.cache#"n" = A.cache#"n";
   F.cache#"Tor Algebra Structure" = B;
   B
)

codimThreeTorAlgebra2 = method()

codimThreeTorAlgebra2(ChainComplex,List) := (F,sym) -> (
   if F.cache#?"Tor Algebra Structure" then return F.cache#"Tor
Algebra Structure";
   A := codimThreeAlgStructure2(F,sym);
   P := ambient A;
   Q := ring F;
   kk := coefficientRing Q;
   PP := kk monoid P;
   I := ideal mingens sub(ideal A, PP);
   B := PP/I;
   B.cache#"l" = A.cache#"l";
   B.cache#"m" = A.cache#"m";
   B.cache#"n" = A.cache#"n";
   F.cache#"Tor Algebra Structure" = B;
   B
)
```

```
eeMultTable = method(Options => {Labels => true, Compact => false})

eeMultTable(Ring) := opts -> A -> (
   if not (A.cache#?"l" and A.cache#?"m" and A.cache#?"n") then
      error "Expected an algebra created with a CodimThree routine.";
   l := A.cache#"l";
   m := A.cache#"m";
   n := A.cache#"n";
   eVector := matrix {apply(m, i -> A_i)};
   if (opts.Compact) then (
       oneTimesOneA := table(m,m, (i,j) -> if i <= j then (A_i)*(A_j)
else 0))
   else (
       oneTimesOneA = matrix table(m,m,(i,j) -> (A_i)*(A_j));
       );
   result := entries ((matrix {{0}} | eVector) || ((transpose eVector)
| oneTimesOneA));
   if (opts.Labels) then result else oneTimesOneA
   )

efMultTable = method(Options => options eeMultTable)

efMultTable(Ring) := opts -> A -> (
   if not (A.cache#?"l" and A.cache#?"m" and A.cache#?"n") then
      error "Expected an algebra created with a CodimThree routine.";
   l := A.cache#"l";
   m := A.cache#"m";
   n := A.cache#"n";
   eVector := matrix {apply(m, i -> A_i)};
   fVector := matrix {apply(l, i -> A_(m+i))};
   oneTimesTwoA := matrix table(m,l,(i,j) -> (A_i)*(A_(m+j)));
   -- put on the row and column labels for fun
   result := matrix entries ((matrix {{0}} | fVector) || ((transpose
eVector) | oneTimesTwoA));
   if (opts.Labels) then entries result else entries oneTimesTwoA
)

multMap = method()
multMap(Ring, ZZ, ZZ) := (A,m,n) -> (
    Abasism := basis(m,A);
    Abasisn := basis(n,A);
    AbasismPlusn := basis(m+n,A);

    AmTimesAn := matrix {flatten entries ((transpose Abasism) *
Abasisn)};
    sub(last coefficients(AmTimesAn, Monomials=>AbasismPlusn),
coefficientRing A)
)

multMap(RingElement,ZZ) := (f,m) -> (
```

```
    -- returns the matrix of left multiplication by f
    A := ring f;
    n := first degree f;
    Abasism := basis(m,A);
    AbasismPlusn := basis(m+n,A);
    fTimesAbasism := f*Abasism;
    sub(last coefficients(fTimesAbasism, Monomials=>AbasismPlusn),
coefficientRing A)
)

rankMultMap = method()
rankMultMap(Ring,ZZ,ZZ) := (A,m,n) -> rank multMap(A,m,n);

homothetyMap = method()
homothetyMap(Ring,ZZ,ZZ) := (A,m,n) -> (
    Abasism := basis(m,A);
    homothetyList := apply(flatten entries Abasism, f -> transpose
matrix {flatten entries multMap(f,n)});
    matrix {homothetyList}
)

multPairingDualMap = method()
multPairingDualMap(Ring,ZZ,ZZ,RingElement) := (A,m,n,w) -> (
   -- this function returns the map A_m --> A_n^*
   -- provided w is a basis of the image of the multiplication
   -- map A_m ** A_n \to A_(m+n)
    Abasism := basis(m,A);
    multMapList := apply(flatten entries Abasism, f -> multMap(f,n));
    matrix {multMapList}
)

rankHomothetyMap = method()
rankHomothetyMap(Ring,ZZ,ZZ) := (A,m,n) -> rank homothetyMap(A,m,n);

-*
A_l \otimes A_m \otimes A_n \to (A_(l+m) \otimes A_n) \oplus (A_l
\otimes A_(m+n))
*-
-- (when we use this function, l=m=n=1
-- imagelm = image multMap(A,l,m)
-- imagemn = image multMap(A,m,n)
-- psi: A_1 \otimes A_1 \otimes A_1 -> (A_2 \otimes A_1) \oplus (A_1
\otimes A_2)
--    (a \otimes b \otimes c) |-> ((ab)\otimes c,a \otimes (bc))
-- image of psi is a submodule of (imagelm \otimes A_1) \oplus (A_1
\otimes imagemn)
-- so tau is the rank of the cokernel of psi after restricting its
codomain as above.
-- A_1 \otimes A_1 \otimes A_1 \to (imagelm \otimes A_1) \oplus (A_1
\otimes imagemn) \to cokernel \to 0
```

```
tauMaps = method()
tauMaps(Ring,ZZ,ZZ,ZZ) := (A,l,m,n) -> (
  kk := coefficientRing A;
  multMaplm := multMap(A,l,m);
  multMapmn := multMap(A,m,n);
  Al := kk^(numcols basis(l,A));
  An := kk^(numcols basis(n,A));
  lTensmn := (id_Al) ** multMapmn;
  lmTensn := multMaplm ** (id_An);
  psi := matrix {{lTensmn},{lmTensn}};
  {rank lTensmn + rank lmTensn - rank psi,lTensmn, lmTensn, psi}
)

torAlgebraClassCodim3 = method()
torAlgebraClassCodim3 QuotientRing := A -> (
  -- check to ensure that A is torAlgebra for codim 3 example?
  p := rank multMap(A,1,1);
  q := rank multMap(A,1,2);
  r := rank homothetyMap(A,2,1);
  -- see the above description for what tau is computing
  tau := first tauMaps(A,1,1,1);
  if (p >= 4 or p == 2) then
      return ("H(" | p | "," | q | ")")
  else if (p == 3) then
  (
      if (q > 1) then return ("H(" | p | "," | q | ")")
      else if (q == 1 and r != 1) then return "C(3)"
      else if (q == 0 and tau == 0) then return ("H(" | p | "," | q |
")")
      else return "T";
  )
  else if (p == 1) then
  (
      if (q != r) then return "B"
      else return ("H(" | p | "," | q | ")");
  )
  else if (p == 0) then
  (
      if (q != r) then return ("G(" | r | ")")
      else return ("H(" | p | "," | q | ")");
  );
)

torAlgebraClassCodim3 ChainComplex := F -> (
   B := codimThreeTorAlgebra(F,{getSymbol "e",getSymbol "f", getSymbol
"g"});
   torAlgebraClassCodim3 B
)

makeMonic = f -> (leadCoefficient(f))^(-1)*f
```

```
performBasisChange = method()
performBasisChange(ChainComplex, List, List, List) := (F, eList,
fList, gList) -> (
   Q := ring F;
   B := ring first eList;

   newF1 := Q^(-apply(eList, x -> drop(degree x,1)));
   newF2 := Q^(-apply(fList, x -> drop(degree x,1)));
   newF3 := Q^(-apply(gList, x -> drop(degree x,1)));

   P1 := matrix entries sub(last coefficients(matrix{eList},
Monomials=>basis(1,B)), Q);
   P2 := matrix entries sub(last coefficients(matrix{fList},
Monomials=>basis(2,B)), Q);
   P3 := matrix entries sub(last coefficients(matrix{gList},
Monomials=>basis(3,B)), Q);

   newdd1 := (F.dd#1)*map(F#1,newF1,P1);
   newdd2 := map(newF1,F#1,P1^(-1))*(F.dd#2)*map(F#2,newF2,P2);
   newdd3 := map(newF2,F#2,P2^(-1))*(F.dd#3)*map(F#3,newF3,P3);

   makeRes(newdd1,newdd2,newdd3)
)


poincareSeriesCodim3 = method()
poincareSeriesCodim3 QuotientRing := R -> (
   I := ideal R;
   F := res I;
   A := codimThreeTorAlgebra(F,{getSymbol "e",getSymbol "f", getSymbol
"g"});
   p := rank multMap(A,1,1);
   q := rank multMap(A,1,2);
   r := rank homothetyMap(A,2,1);
   tau := first tauMaps(A,1,1,1);
   m := numcols basis(1,A);
   n := numcols basis(3,A);
   e := numcols mingens ideal vars R;
   G := ZZ[getSymbol "T",Weights=>{-1},Global=>false];
   num := (1 + G_0)^(e-1);
   den := 1 - G_0 - (m-1)*G_0^2 - (n-p)*G_0^3 + q*G_0^4 - tau*G_0^5;
   (expression num) / (expression den)
)

changeBasisT = method()
changeBasisT(ChainComplex,List) := (F,inputEs) -> (
  --- this function performs the change of basis of F required
  --- so that the multiplication table of the tor algebra computed
from F
```

```
  --- is of the desired form.
  Q := ring F;
  B := ring first inputEs;
  eList := inputEs;
  annEs := ann ideal eList;
  -- tack on the annihilator of inputEs in degree 1 to the eList.
  eList = eList | flatten entries ((gens annEs)*sub(matrix
basis(1,annEs),B));
  fList := flatten entries matrix
{{eList#1*eList#2,eList#2*eList#0,eList#0*eList#1}};
  fList = fList | flatten entries (basis(2,B)*(mingens coker
multMap(B,1,1)));
  gList := flatten entries basis(3,B);
  F.cache#"Good Basis" = {eList,fList,gList};
  G := performBasisChange(F,eList,fList,gList);
  G
)

-- this function tries to find an appropriate choice for e_1,e_2,e_3
-- in the classification theorem of AKM
changeBasisT(ChainComplex) := F -> (
   B := codimThreeTorAlgebra(F,{getSymbol "e",getSymbol "f", getSymbol
"g"});
   kk := coefficientRing B;
   eList := select(gens B, d -> first degree d == 1);

   --- attempt to find the generators we want of the truncated
exterior subalgebra
   m := #eList;
   a := getSymbol "a";
   ee := getSymbol "ee";
   D := kk[ee_1,ee_2,ee_3,SkewCommutative=>true];

   -- for each subset of size three of the degrees set, choose a
random element
   -- until you find a set of size 3 that is isomorphic to a truncated
exterior algebra
   -- for this, it suffices to check that the hom in degree two has
full rank.
   genDegs := unique subsets(eList / degree, 3);
   goodGens := {};
   for genDeg in genDegs list
   (
       potentialGens := apply(genDeg, g -> makeMonic(random(g,B)));
       phi := map(B,D,potentialGens);
       phi2 := sub(last coefficients(phi basis(2,D), Monomials =>
basis(2,B)),kk);
       if (rank phi2 == 3) then
       (
          goodGens = potentialGens;
```

```
          break;
       );
   );
   if goodGens == {} then error "Unable to find an appropriate change
of basis for class T.";
   changeBasisT(F, goodGens)
)

getGoodBasisHpq = method()
getGoodBasisHpq(ChainComplex, RingElement) := (F, ee) -> (
   Q := ring F;
   B := ring ee;

   multMap1 := multMap(ee,1);
   imagMultMap1 := mingens image multMap1;
   eList := flatten entries (basis(1,B)*(imagMultMap1 // multMap1));

   -- now we add in some indeterminants to track the lifts, and choose
   -- the first few es carefully so that they have trivial products
among themselves
   a := getSymbol "aaa";
   p := #eList;
   varRing := (coefficientRing B)[a_1..a_p];
   overC := varRing monoid B;
   newI := sub(ideal B, overC);
   C := overC/newI;
   eeC := sub(ee,C);

   -- eList is a list of preimages of a basis of image of
multiplication by ee
   if eList != {} then (
      -- We purturb the preimages by a multiple of ee, in the hopes of
finding
      -- a set of elements that also have trivial products among
themselves.
      genEList := apply(#eList, i -> sub(eList#i,C) +
sub(varRing_i,C)*eeC);
      products := subsets(genEList,2) / product;
      -- these are the coefficients of the pairwise products.  The
coefficients
      -- give a (consistent!) linear system of equations over
(coefficientRing B)
      prodCoeffs := last coefficients(matrix {products}, Monomials =>
basis(2,C));
      soln := mingens ideal flatten entries prodCoeffs;
      -- reduce modulo the ideal of soln to make the substitution
      eList = apply(genEList, f -> sub(f % sub(soln, C), B));
   );

   -- the rest of the es are the annihilator of ee (modulo ee)
```

```
   eList = eList | {ee};
   dubAnnMod := ann ideal eList;
   newEs := flatten entries ((gens dubAnnMod)*(matrix entries basis(1,
(ann ideal eList)/(ideal {ee}))));
   eList = eList | newEs;

   -- fs are a basis of the image of the multiplication map of ee in
degree 1,
   -- followed by a lift of the image of the multiplication map of ee
in degree 2
   fList := flatten entries (basis(2,B)*imagMultMap1);
   origFList := fList;
   multMap2 := multMap(ee,2);
   imagMultMap2 := mingens image multMap2;
   fList = fList | flatten entries (basis(2,B)*(imagMultMap2 //
multMap2));
   -- complete fList to a basis in degree two now by choosing a basis
   -- of ann(ee) / <origFList>
   newFs := flatten entries ((gens ann ideal {ee})*(matrix entries
basis(2, (ann ideal {ee})/(sub(ideal origFList,B)))));
   fList = fList | newFs;
   -- gs are a basis of the image of the multiplication map of ee in
degree 2
   -- then completed to a basis arbitrarily
   gList := flatten entries (basis(3,B)*imagMultMap2);
   gList = gList | flatten entries (basis(3,B)*(mingens coker
multMap2));

   (eList,fList,gList)
)

-- this method changes basis so that ee is the distinguished element
in the
-- H(p,q) classification
changeBasisHpq = method()
changeBasisHpq(ChainComplex,RingElement) := (F,ee) -> (
   goodBasis := getGoodBasisHpq(F,ee);
   F.cache#"Good Basis" = {goodBasis#0,goodBasis#1,goodBasis#2};
   G := performBasisChange(F,goodBasis#0,goodBasis#1,goodBasis#2);
   G
)

changeBasisHpq(ChainComplex) := F -> (
   --- this function selects an element ee of degree one such that
left mult by ee
   --- has full rank.  This is a generic condition, but we would like
to choose
   --- it homogeneously (if grading exists)

   -- first try the es to see if any of them will work
```

```
   B := codimThreeTorAlgebra(F,{getSymbol "e",getSymbol "f", getSymbol
"g"});
   p := rank multMap(B,1,1);
   eGens := apply(B.cache#"m", i -> B_i);
   goodEGens := select(eGens, ee -> rank multMap(ee,1) == p);
   if goodEGens != {} then return changeBasisHpq(F,first goodEGens);

   -- next try a random (homogeneous) element of each internal degree
in homological degree 1.
   eGenDegs := eGens / degree // unique;
   randElts := apply(eGenDegs, d -> random(d,B));
   goodRands := select(randElts, ee -> rank multMap(ee,1) == p);
   if goodRands != {} then return changeBasisHpq(F,first goodRands);

   << "Unable to determine a distinguished element for multiplication
table." << endl;
)

-- This version finds gTop and then calls the below function to finish
the rest
-- of the computation
changeBasisG = method()
changeBasisG(ChainComplex) := F -> (
   B := codimThreeTorAlgebra(F,{getSymbol "e",getSymbol "f", getSymbol
"g"});
   gTop := first flatten entries (basis(3,B)*(matrix mingens image
multMap(B,1,2)));
   changeBasisG(F,gTop)
)

-- This version finds a basis of es and fs so that 'gTop' is the top
-- class of the Tor algebra
changeBasisG(ChainComplex, RingElement) := (F, gTop) -> (
   B := ring gTop;
   -- complete gTop to a basis
   gList := {gTop} | flatten entries mingens ((ideal basis(3,B))/
(ideal gTop));
   -- compute the matrix of the map corresponding to the dual basis
element of gTop
   -- proj = gTop^* : A_3 --> k
   proj := (last coefficients(matrix{gList},
Monomials=>basis(3,B)))^{0};
   -- for each basis element, restrict codomain of mult map (from
degree 1 to 3)
   -- to the 'top class', and take the dual map.
   -- each map is A_1 --*f--> A_3 --gTop^*--> k
   multMapList := apply(flatten entries basis(2,B), f -> transpose
(proj*multMap(f,1)));
   -- make a matrix out of all these maps.  This gives the map from
   -- M : A_2 --> Hom(gTop^*,A_1^*) \cong A_1^*
```

```
   M := sub(matrix {multMapList}, coefficientRing B);
   -- the 'other fs' are those fs which don't participate in mult
   otherFs := flatten entries (basis(2,B)*sub(gens ker M,B));
   -- the 'other es' are those es which don't participate in mult
   -- these are found using the coker of M above
   otherEs := flatten entries (basis(1,B)*sub(matrix mingens coker
M,B));
   goodEsSoFar := {};
   goodFsSoFar := {};
   r := rank M;
   for i from 1 to r do (
     maybeFs := flatten entries (basis(2,B)*sub(matrix mingens image
M // M,B));
     nextF := first (intersect(ann sub(ideal goodEsSoFar,B),ideal
maybeFs))_*;
     goodFsSoFar = append(goodFsSoFar,nextF);
     nextE := first flatten entries (matrix {{gTop}} // matrix
{{nextF}});
     for j from 0 to i - 2 do (
        fixCoeff := first flatten entries (matrix
{{nextE*goodFsSoFar#j}} // matrix {{gTop}});
        nextE = nextE - fixCoeff*goodEsSoFar#j;
     );
     goodEsSoFar = append(goodEsSoFar,nextE);
   );
   eList := goodEsSoFar | otherEs;
   fList := goodFsSoFar | otherFs;
   -- the good es are the duals of those elements in the image of M
   -- goodEs := flatten entries (basis(1,B)*sub(matrix mingens image
M, B));
   -- the good fs are the lifts of these under M
   -- goodFs := flatten entries (basis(2,B)*sub(matrix mingens image
M // M,B));
   -- eList := goodEs | otherEs;
   -- fList := goodFs | otherFs;
   -- change coordinates
   F.cache#"Good Basis" = {eList,fList,gList};
   G := performBasisChange(F,eList,fList,gList);
   G
)

-- This version finds gTop and then calls the below function to finish
the rest
-- of the computation
changeBasisB = method()
changeBasisB(ChainComplex) := F -> (
   B := codimThreeTorAlgebra(F,{getSymbol "e",getSymbol "f", getSymbol
"g"});
   gTop := first flatten entries (basis(3,B)*(matrix mingens image
multMap(B,1,2)));
```

```
   changeBasisB(F,gTop)
)

-- This version finds a basis of es and fs so that 'gTop' is the top
-- class of the Tor algebra
changeBasisB(ChainComplex, RingElement) := (F, gTop) -> (
   B := ring gTop;
   kk := coefficientRing B;
   -- complete gTop to a basis
   gList := {gTop} | flatten entries mingens ((ideal basis(3,B))/
(ideal gTop));
   -- compute the matrix of the map corresponding to the dual basis
element of gTop
   proj := (last coefficients(matrix{gList},
Monomials=>basis(3,B)))^{0};
   -- for each basis element of degree 2, restrict codomain of mult
map (from degree 1 to 3)
   -- to the 'top class', and take the dual map.
   -- mu_f : B_1 --> B_3 --> k.  this builds a list of \mu_f duals
   -- in the good basis, this takes f_1 to e_1^* and f_2 to e_2^*
   multMapList := apply(flatten entries basis(2,B), f -> transpose
(proj*multMap(f,1)));
   -- make a matrix out of all these maps.  This gives the map from
   -- B_2 --> Hom(span(gTop)^*,B_1^*) \cong B_1^*
   M := sub(matrix {multMapList}, coefficientRing B);
   -- the 'other fs' are those fs which don't participate in mult
   otherFs := flatten entries (basis(2,B)*sub(gens ker M,B));
   -- we choose fstar_1 and fstar_2 to be preimages of the image of
the map Phi : B_2 --> B_1^*
   -- rank M is 2 if we are in case B.
   goodEsSoFar := {};
   goodFsSoFar := {};
   for i from 1 to 2 do (
     maybeFs := flatten entries (basis(2,B)*sub(matrix mingens image
M // M,B));
     nextF := first (intersect(ann sub(ideal goodEsSoFar,B),ideal
maybeFs))_*;
     goodFsSoFar = append(goodFsSoFar,nextF);
     nextE := first flatten entries (matrix {{gTop}} // matrix
{{nextF}});
     for j from 0 to i - 2 do (
        fixCoeff := first flatten entries (matrix
{{nextE*goodFsSoFar#j}} // matrix {{gTop}});
        nextE = nextE - fixCoeff*goodEsSoFar#j;
     );
     goodEsSoFar = append(goodEsSoFar,nextE);
   );
   -- goodFs := flatten entries (basis(2,B)*sub(matrix mingens image
M // M,B));
   -- estar_1 and estar_2 are the partners of the fstar_1 and fstar_2
```

```
just found
   -- fix these two lines according to case B proof
   -- goodEcoords := apply(goodFs, x -> (sub((last
coefficients(gTop,Monomials=>basis(3,B))),kk) // multMap(x,1)));
   -- goodEs := flatten apply(goodEcoords, c -> flatten entries
(basis(1,B)*c));

   -- complete es and fs to a basis
   otherEs := flatten entries (basis(1,B)*sub(matrix mingens coker
M,B));
   f3 := goodEsSoFar#0*goodEsSoFar#1;
   otherFs = flatten entries mingens ((ideal otherFs)/(ideal f3));
   eList := goodEsSoFar | otherEs;
   fList := goodFsSoFar | {goodEsSoFar#0*goodEsSoFar#1} | otherFs;
   -- change coordinates
   F.cache#"Good Basis" = {eList,fList,gList};
   G := performBasisChange(F,eList,fList,gList);
   G
)

-- this function determines the class of the resolution
-- and calls the appropriate change of basis command
changeBasisCodim3 = method()
changeBasisCodim3(ChainComplex) := F -> (
   torClass := torAlgebraClassCodim3 F;
   if torClass#0 == "H" then
      changeBasisHpq F
   else if torClass#0 == "T" then
      changeBasisT F
   else if torClass#0 == "C" then
   (
      B := codimThreeTorAlgebra(F,{getSymbol "e",getSymbol "f",
getSymbol "g"});
      F.cache#"Good Basis" = {flatten entries basis(1,B), flatten
entries basis(2,B), flatten entries basis(3,B)};
      F
   )
   else if torClass#0 == "B" then
      changeBasisB F
   else if torClass#0 == "G" then
      changeBasisG F
)
```

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Education

2019–	Wake Forest University: M.A.in Mathematics, Expected Graduation May 2021, Current GPA: 3.748
Spring 2018	North Carolina State University: Non-Degree Studies, Mathematical Analysis II, GPA 3.667
2015-2018	Campbell University: B.S. December 2018, *Summa cum laude* Major in Mathematics with Minors in Christian Studies and Music, GPA 3.975

Honors and Certificates

Spring 2021	Wake Forest University Mathematics and Statistics Department Doris Steppe Hood Award
2019-2021	Wake Forest University Teaching Assistant
Summer 2018	Campbell University Summer Research Pilot Program Participant
Spring 2018	Campbell University Outstanding Senior in Mathematics
2015-2018	Campbell University President's List
2015-2018	Campbell University Presidential Scholarship
2015-2018	Campbell University Athletic Band Scholarship

Research

Research Interests:
Algebra, commutative algebra, algebraic geometry, algebraic topology, topology
Master's thesis:
On the Classification of Grade Three Perfect Ideals, advisor: Dr. Frank Moore
Campbell University Student Research Fellows Pilot Program:
The Study of Knot Theory and its Applications in DNA, advisor: Dr. Brittany Hansen

Presentations

Spring 2021	AWM Brown Bag Talk, Winston-Salem, NC
Fall 2018	MAA State Dinner, Buies Creek, NC - poster
Summer 2018	Campbell University Student Research Fellows Program, Buies Creek, NC - poster
Spring 2018	8th Annual Wiggins Memorial Library Symposium, Buies Creek, NC

Teaching Experience: Led study sessions, one-on-one tutoring sessions, class coverage, and grading.

Spring 2021	Discrete Mathematics	Wake Forest University
Fall 2020	Calculus and Analytic Geometry I	Wake Forest University
Summer II 2020	Calculus and Analytic Geometry II	Wake Forest University
Summer I 2020	Explorations in Mathematics	Wake Forest University
Spring 2020	Multi-variable Calculus	Wake Forest University
Fall 2019	Calculus and Analytic Geometry II	Wake Forest University
Spring 2018	Calculus and Analytic Geometry I	Campbell University

Service

Spring 2021	AWM Triangle Conference Organizer
2020–	Wake Forest University AWM Chapter Secretary
Spring 2017	Inasmuch EOG Tutoring
2017-2018	Campbell University Math and ITS Club Secretary
2017-2018	Campbell University Kappa Delta Pi Vice President

Memberships

Pi Mu Epsilon Honors Society
Association for Women in Mathematics
American Mathematical Society
Omicron Delta Kappa Honors Society
Kappa Delta Pi Education Honors Society
Phi Kappa Phi Honors Society
Phi Eta Sigma Honors Society

Mathematical Software

LaTeX, Macaulay2