SUBMANIFOLD HELICITY

BY

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Abstract

In this thesis we will focus on the topic of helicity. Helicity gives us a way to measure the coiling of flow lines in a vector field, computed by the formula

$$H(V) = \left(\frac{1}{4\pi}\right) \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{(x-y)}{|x-y|^3} d(vol_x) d(vol_y).$$

To start our exploration of helicity we begin with the topic of differential forms and their correspondence with vector fields in $\mathbb{R}^3$. Then we use the correspondence of forms and vector fields to show that Maxwell's Equations can be reduced to two equations of differential forms. We continue our exploration of mathematics in physics with the Biot-Savart operator; this operator calculates the associated magnetic field on a domain Ω given an electric current on Ω. The Biot-Savart operator gives us a way to compute vector field helicity.

We will begin our exploration of helicity with its standard definition above, and where it comes of use in both mathematics and physics. Next we return to differential forms to show how the correspondence to vector fields allows us to define the helicity of forms. Then using helicity of forms we can expand on the three-dimensional idea of helicity to both submanifolds and higher dimension ambient spaces.

Chapter 1: Introduction

The helicity of a vector field finds itself an object of study in both the worlds of physics and mathematics. We can find applications in astrophysics, solar physics, and plasma physics. In these studies helicity can provide a lower bound for energy for a vector field. This is because helicity measures the coiling or wrapping of a plasma flowing through a space or manifold. Helicity also measures the average for the linking number of flow lines. Towards the end of the paper in Examples 7.2, 7.3, we will see that it has connections to the writhe and linking number of curves when we reduce to the right submanifold dimensions.

In this thesis we will look at helicity of vector fields. Similar to how in fluid dynamics helicity measures the crossing of the fluid's flow lines, the helicity of an arbitrary vector field measures the coiling of the flow lines of the vector field. This gives a vector field formula for helicity,

$$H(V) = \left(\frac{1}{4\pi}\right) \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{(x-y)}{|x-y|^3} d(vol_x) d(vol_y).$$

Then using the Biot-Savart operator, $BS(V)$, which we describe in the paper, we will show that helicity is equal to the L_2 inner product of the vector field and $BS(V)$, i.e., $H(V) = \langle V, BS(V)\rangle$. Then we will see a corresponding formula for the helicity of differential forms exists and that it is useful in showing where helicity acts as an invariant. Using forms we are able to extend to cases where $k \neq 0$ for a subdomain $\Omega^{4k+3} \subset \mathbb{R}^{4k+3}$. Also, differential forms allow us to generalize to sub-manifold helicity, an interesting topic which could provide an invariant for a lower-dimension manifold in a higher dimension ambient space.

Helicity was first computed in 1958 during the dutch physicist Woltjer's studies of the Crab Nebula, a supernova. In 1961 Moreau derived helicity through his work in hydrodynamics. Then Moffatt independently derived and named it in 1969 while working in hydrodynamics.

In Chapter 2 we will define differential forms, which will be necessary for looking at submanifold helicity. We provide many important theorems and definitions pertaining to differential forms here, and reference Weintraub's books [2] [3]. Then we provide a proof showing that pullbacks and exterior differentiation commute, necessary for our work on helicity. Then in Chapter 3 we prove that Maxwell's equations have a particularly nice expression when written as differential forms, following an outline by Ruth Gornet [4]. Then in Chapter 4 we introduce the Biot-Savart operator, and present the four main theorems from *The Biot-Savart operator for application to knot theory, fluid dynamics, and plasma physics* by Cantarella, DeTurck, and Gluck [5].

Then in Chapter 5 we introduce helicity. In this chapter we look at the general formula, some applications, and how Biot-Savart is related. In Chapter 6 we continue to study helicity, in particular we show how we can compute helicity with differential forms. We will prove that the helicity of a differential form must equal the helicity of the corresponding vector field. We will also need to cover configuration spaces and Fulton-Macpherson compactification for helicity of forms. Then in Chapter 7 we generalize to helicity of submanifolds– what can we say about the helicity of a manifold of lower dimension than the ambient space.

Chapter 2: Background

2.1 Differential Forms

In this section we will cover some basic ideas and operations of differential forms necessary for future sections. A lot of the information for differential forms is from Steven H. Weintraub's books, *Differential Forms, A Complement to Vector Calculus* [2] and *Differential Forms, Theory and Practice* [3] .

Definition 2.1. We will start by defining differential forms on $\mathbb{R}^n$, for $n \in \mathbb{N}$. Let $(x_1, x_2, ..., x_n)$ be the usual cartesian coordinate system on $\mathbb{R}^n$; then in general we can define k-forms as follows

- A 0-form α is a function A.
- A 1-form $\alpha = A_1dx_1 + A_2dx_2 + ... + A_ndx_n$, where $A_1, A_2, ..., A_n$ are functions in $x_1, x_2, ..., x_n$.
- In general, a k-form is a linear combination of terms like α is $Adx_1dx_2...dx_k$ where A is a function.

Then a **differential form** is a k-form for some k. We denote the class of all smooth k-forms Λ^k.

Some simple operations we will need for differential forms include addition and multiplication, also known as the wedge product. We define addition of differential forms as one would expect, i.e., if $\alpha = dx_1$ and $\beta = dx_2$ are differential forms, then $\alpha + \beta = dx_1 + dx_2$. We also note that addition of differential forms is commutative, $\alpha + \beta = \beta + \alpha$.

Now we will introduce the wedge product of two forms. $dx_1 \wedge dx_2$ will be equal to dx_1dx_2 and takes two one-forms to a two-form, or more generally a k-form and an l-form to a (kl)-form.

Proposition 2.2. *The wedge product is anti-commutative, so for a k-form α and an l-form β,*

$$\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha.$$

For simple forms we notice that $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and hence $dx_i \wedge dx_i = 0$.

Example 2.3. As an example, let $\alpha = dx_1 + dx_2 + dx_3$ and $\beta = dx_2 + dx_3 + dx_4$ be differential forms. Then the product, or wedge product, is as follows,

$$\alpha\beta = \alpha \wedge \beta = dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_1 \wedge dx_4 + dx_2 \wedge dx_2$$

$$+dx_2 \wedge dx_3 + dx_2 \wedge dx_4 + dx_3 \wedge dx_2 + dx_3 \wedge dx_3 + dx_3 \wedge dx_4.$$

We notice that we have a $dx_2 \wedge dx_2$ and $dx_3 \wedge dx_3$ both of which will equal 0. Also we find both $dx_2 \wedge dx_3$ and $dx_3 \wedge dx_2$ which by anti-commutativity will cancel out. Thus,

$$\alpha \wedge \beta = dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_1 \wedge dx_4 + dx_2 \wedge dx_4 + dx_3 \wedge dx_4.$$

For the remainder of this chapter we will denote $\alpha \wedge \beta$ as $\alpha\beta$. The next definition we will need for differential forms is the exterior derivative.

Definition 2.4. Let $I_1, ..., I_m$ be k-subsets of $\{1, 2, ..., n\}$. Then if α is the k-form $A_1dx_{I_1} + ... + A_mdx_{I_m}$, then its **exterior derivative** $d\alpha$ is the $(k+1)$-form acquired from applying the partial derivative operator$\frac{\partial}{\partial x_i}$ to each of the terms in α for all i, $1 \leq i \leq n$. In particular the exterior derivative will be

$$\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}A_1dx_1\right)dx_{I_1} + \cdots + \sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}A_mdx_m\right)dx_{I_m}.$$

We will provide a quick example of an exterior derivative. Let $\alpha = x_2x_3dx_2 + x_3^2dx_4$ then,

$$d\alpha = x_2dx_3dx_2 + 2x_3dx_3dx_4 = -x_2dx_2dx_3 + 2x_3dx_3dx_4$$

As with derivatives of functions we have a product rule or Leibniz Rule for the exterior derivative of forms.

Definition 2.5 (Leibniz Rule of forms)**.** Let α be a k-form and β any form then

$$d(\alpha\beta) = d(\alpha)\beta + (-1)^k\alpha d(\beta).$$

Now we are ready to define closed and exact forms.

Definition 2.6. A k-form α is **closed** if $d\alpha = 0$.

Definition 2.7. k-form β to be **exact** if there exists a $(k-1)$-form α such that $d\alpha = \beta$.

We are now ready for our next proposition.

Proposition 2.8. *Every exact form is also closed.*

Proof. Let β be an exact k-form, then we know there exists a $(k-1)$-form α such that $\beta = d\alpha$. Now consider $d\beta$, we notice that $d\beta = d(d\alpha)$. Then $d(d\alpha) = d(dx_1\alpha_{x_1} + dx_2\alpha_{x_2} + ... + dx_n\alpha_{x_n})$, where dx_i, for $1 \leq i \leq n$, are the simple forms and α_{x_i} is the partial of the entire form with respect to x_i. Then we see that,

$$\begin{aligned} d(dx_1\alpha_{x_1} + dx_2\alpha_{x_2} + ... + dx_n\alpha_{x_n}) =& dx_1dx_1(\alpha_{x_1})_{x_1} + \\ & \cdots + dx_ndx_1(\alpha_{x_n})_{x_1} + dx_1dx_2(\alpha_{x_2})_{x_1} + \\ & \cdots + dx_1dx_n(\alpha_{x_n})_{x_1}dx_ndx_1 + \cdots + dx_ndx_n(\alpha_{x_n})_{x_n}. \end{aligned}$$

Then from above we know that $dx_idx_i = 0$. We also notice that for every dx_idx_j there is a dx_jdx_i, which since $dx_jdx_i = -dx_idx_j$ these will all cancel out since $(\alpha_{x_i})_{x_j} =$

$(\alpha_{x_j})_{x_i}$ by Clairaut's Theorem. Therefore $(\alpha_{x_1})_{x_1}dx_1dx_1 + ... + (\alpha_{x_n})_{x_1}dx_1dx_n + (\alpha_{x_1})_{x_2}dx_2dx_1 + ... + (\alpha_{x_1})_{x_n}dx_ndx_1 + ... + (\alpha_{x_n})_{x_n}dx_ndx_n = 0$. Thus, $d\beta = 0$, telling us that β is closed. $\square$

2.2 The Fundamental Correspondence

With the definitions in the first section, we are ready to talk about the Fundamental Correspondence. To do this we will consider differential forms in $\mathbb{R}^3$. The fundamental correspondence gives the relationship between a vector fields or functions in $\mathbb{R}^3$ and k-forms in $\mathbb{R}^3$. We correspond 0-forms and 3-forms to functions and 1-forms or 2-forms to vector fields. Then we correspond the operator of exterior derivative on a 0-form to a 1-form to taking the gradient. The exterior derivative on a 1-form corresponds to taking the curl of a vector field. Then the operation of an exterior derivative on a 2-form corresponds to finding the divergence of a vector field. In the diagram

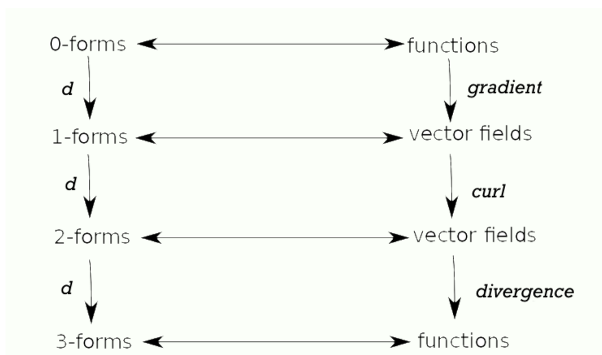

Figure 2.1: A diagram of the Fundamental Correspondence. The diagram is commutative.

d represents the exterior derivative. The diagram shows that the exterior derivative on 0-forms corresponds to the gradient, on 1-forms it corresponds to the curl, and then on 2-forms it will correspond to finding the divergence. To go from 0-forms to a function, we see that a 0-form is a function f and the corresponding function will also be f. For a 1-form $udx + vdy + wdz$ where u, v, w are functions in (x, y, z) the corresponding vector field is $u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Then given the vector field $u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ the corresponding 2-form will be $udy \wedge dz + vdz \wedge dx + wdx \wedge dy$. Finally a 3-form $f(x, y, z)dx \wedge dy \wedge dz$ will correspond to the function $f(x, y, z)$.

We also give names to the maps in the Fundamental Correspondence for $\mathbb{R}^3$. For a vector field F we define ω_F to be the one-form correspondence to a vector field F. Then Φ_F will be the 2-form correspondence to a vector field F. Finally, σ will be the 3-form correspondence to a function A. Then by the commutative nature of the Fundamental Correspondence,

$$d_S\omega_F = \Phi_{curl\,F} = F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy,$$

$$d_S\Phi_F = \sigma_{divF} = Fdx \wedge dy \wedge dz.$$

We also find the following lemma from the definition of the Fundamental Correspondence.

Lemma 2.9. *For a vector field F the following is true,*

$$\omega_F = 0 \text{ if and only if } F = 0,$$
$$\Phi_F = 0 \text{ if and only if } F = 0.$$

Then for a function f,

$$\sigma_f = 0 \text{ if and only if } f = 0.$$

2.3 Push Forwards and Pullbacks

Moving forward we will assume that manifolds are smooth. Here we define push forwards and pullbacks. We define push forwards for points and tangent vectors, while we define pullbacks for functions and differential forms.

Definition 2.10. Let $\phi : M \to N$ be a smooth map from the m-dimensional manifold M to the n-dimensional manifold N. For a point $p \in M$ we define the **push forward** $\phi_*(p) = \phi(p)$, in otherwords we just define the push forward of a point p to be its image under ϕ. Similarly, for a tangent vector $\mathbf{v}$ of M at point p along a curve s, the push forward ϕ_* of $\mathbf{v}$ will be the tangent vector to $\phi_*(p)$ in N along the curve $\phi(s)$, where $\phi(s)$ maps the curve pointwise from M to N. More concisely if $\mathbf{v}$ is the tangent vector to M at p given by $\mathbf{v} = f'(0)$, where $f : \mathbb{R} \to M$ is a smooth curve with $f(0) = p$, then the **tangent vector push-forward** $\mathbf{w} = \phi_*(\mathbf{v})$ is the tangent vector to N at $\phi(p)$ given by $\phi_*(\mathbf{v}) = (\phi f)'(0)$.

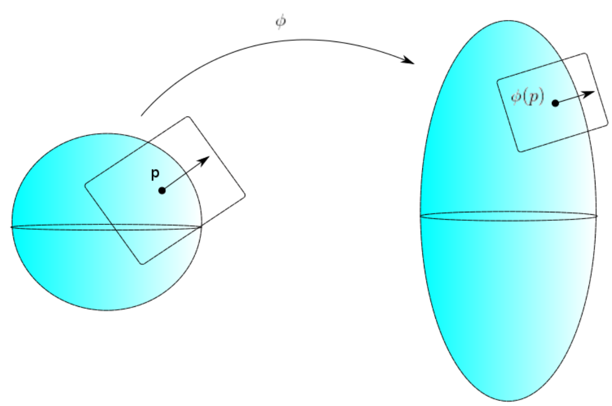

Figure 2.2: An example of a push forward. The point p is mapped directly by ϕ. Where as the tangent vector is mapped to the corresponding vector in the tangent space of $\phi(p)$.

Now we will define pullbacks of functions and forms. First we note that we cannot define push forwards for functions or forms. If we tried to push forward in the analogue way to points and vectors $\phi_*(f)(q) = f(p)$, where $p \in M, q \in N$, and f a function in M, we do not know if q is mapped onto by ϕ, thus $f(p)$ may not be defined. Or we might run into a well-definedness issue if multiple points, p, map to q. These problems will be similar for forms since tangent vectors can run into the same issues as points.

Definition 2.11. Let $f : N \to \mathbb{R}$, and ϕ the map defined in definition (2.10), then we denote the **pullback** of the function, $\phi^*(f)$. Then $\phi^*(f)$ is function on M defined at any point p in M by,

$$\phi^*(f)(p) = f(\phi_*(p)) = f(\phi(p)).$$

We define a form α on N via the projection from N to $\mathbb{R}^n$ on the coordinate system. Similarly, we define the **pull back of forms**, by letting α be a form in N and $\mathbf{v}$ a tangent vector to M then,

$$\phi^*(\alpha)(\mathbf{v}) = \alpha(\phi_*(\mathbf{v})).$$

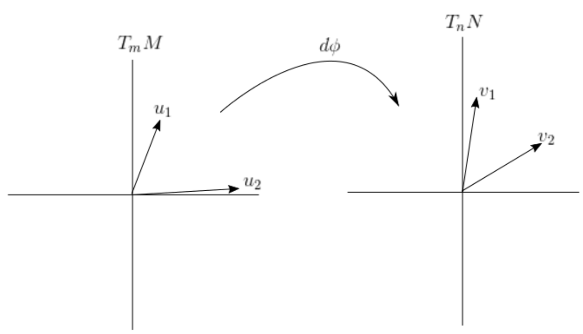

Figure 2.3: This image represents the idea of a pullback of a form. A two form measures the area spanned by two vectors, and the pullback of a two form α is computed by pushing forward the tangent vectors u_1 and u_2 and computing.

We notice that the definition of pullbacks, for functions, work as $p \in M$ has an image under ϕ. Thus, we know that the push-forward of p will give us some point p' in N for which f is defined. As with the pullback of a function, the pullback of a form is really computing the form on the push forward of the tangent vector. This makes sense, because as with points, it is easy to define the push forward of a tangent vector along a map.

Proposition 2.12. *For forms, the pull back commutes with the exterior derivative. More concisely, given a k-form α and a map $\phi : M \to N$, where M and N are manifolds,*

$$d(\phi^*(\alpha)) = \phi^*(d\alpha).$$

Proof. To prove this we first need to show it holds when α is a 0-form, or a function. So we need to prove that the one forms, $d(\phi^*(\alpha))$ and $\phi^*(d\alpha)$ are equal. Since α is a function we know that exterior differentiation will give $d\alpha(\mathbf{v}) = D_{\mathbf{v}}(\alpha)$, where $\mathbf{v}$ is a vector, and where $D_{\mathbf{v}}(\alpha)$ is the directional derivative of α along $\mathbf{v}$. Then given this equality we obtain the following, $d(\phi^*(\alpha))(\mathbf{v}) = D_{\mathbf{v}}(\phi^*(\alpha))$ and similarly $\phi^*(d\alpha)(\mathbf{v}) = d\alpha(\phi_*(\mathbf{v})) = D_{\phi_*(\mathbf{v})}(\alpha)$. So to prove our claim we just need to show that $D_{\mathbf{v}}(\phi^*(\alpha)) = D_{\phi_*(\mathbf{v})}(\alpha)$.

The equality above will be the result of the Chain Rule for multivariable functions. Assume that M has dimension m and N has dimension n. Then,

$$(y_1, ..., y_n) = \phi(x_1, ..., x_m) = (f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m)),$$

where each f_i represents the map from M to y_i in N, under ϕ. Now, let $\mathcal{J}(\phi)$ be the Jacobian matrix of ϕ. Then set $\mathbf{w} = \phi_*(\mathbf{v})$, i.e.

$$\mathbf{w} = \mathcal{J}(\phi)\mathbf{v},$$

or $\mathbf{w}$ is the image of $\mathbf{v}$ under ϕ. The above makes sense because the push forward of a tangent vector $\mathbf{v}$ under ϕ will be the transformation by the Jacobian matrix of ϕ.

We also know that $\mathcal{J}(\alpha) = \left(\frac{\partial x}{\partial y_1}\alpha, .., \frac{\partial x}{\partial y_n}\alpha\right)$, since α is a function, 0-form, from N into $\mathbb{R}$. Then since the Jacobian matrix will give us the derivative of α in matrix form, $D_{\mathbf{w}}(\alpha) = \mathcal{J}(\alpha)\mathbf{w}$. Then by definition of $\mathbf{w}$, we get that,

$$D_{\mathbf{w}}(\alpha) = \mathcal{J}(\alpha)\mathcal{J}(\phi)\mathbf{v}.$$

This is a well-defined product as $\mathbf{v}$ is a $m \times 1$ column vector, $\mathcal{J}(\phi)$ is an $n \times m$ matrix, and $\mathcal{J}(\alpha)$ is a $1 \times n$ row vector.

Now we consider the pullback of α, $\phi^*(\alpha)$. Denote $\phi^*(\alpha)$ by β, then $D_{\mathbf{v}}(\beta) = \mathcal{J}(\beta)\mathbf{v}$. But we know that β is equal to $\alpha \circ \phi$, or the composition of α and ϕ. This is true because computing the tangent vector $\mathbf{v}$ under β is the same as mapping it by ϕ and then computing it under α. But then by the Chain Rule,

$$\mathcal{J}(B) = \mathcal{J}(\alpha \circ \phi) = \mathcal{J}(\alpha)\mathcal{J}(\phi).$$

But then by our equalities we get that $D_{\mathbf{w}}(\alpha) = \mathcal{J}(\alpha)\mathcal{J}(\phi)\mathbf{w} = D_{\mathbf{v}}(B)$, giving us that $d(\phi^*(\alpha))(\mathbf{v}) = \phi^*(d\alpha)(\mathbf{v})$, which was our claim. Hence, we have shown that for 0-forms, or functions, exterior differentiation and pullbacks commute.

Now we want to generalize this to the case where α is a k-form. By linearity, we can assume that α is the k-form $Adx_1 \wedge dx_2 \wedge ... \wedge dx_k$, where A is a function. Then we want to consider $d(\phi^*(Adx_1 \wedge dx_2 \wedge ... \wedge dx_k))$. The map ϕ^* distributes over the wedge product, so we get, $d(\phi^*(Adx_1 \wedge dx_2 \wedge ... \wedge dx_n)) = d(\phi^*(A)\phi^*(dx_1) \wedge ... \wedge \phi^*(dx_n))$. Since we have already shown that pull-backs and exterior differentiation commute for zero forms we get that,

$$d(\phi^*(A)\phi^*(dx_1) \wedge ... \wedge \phi^*(dx_n))(\mathbf{v}) = d((\phi^*(A)d(\phi^*(x_1))...d(\phi^*(x_n)))$$

Then by the Leibniz Rule we can distribute the derivative as follows,

$$d(\phi^*(A))d(\phi^*(x_1))...d(\phi^*(x_n)) + \sum_{i=1}^{n}(-1)^{n-i}\phi^*(A)d(\phi^*(x_1)...d(d(\phi^*(x_i)))...d(\phi^*(x_n))$$

But we notice in the summation at each i, we have $d^2(\phi^*(x_i))$, which we know to be 0. Hence, only the first term, $d(\phi^*(A))d(\phi^*(x_1)...d(\phi^*(x_n))$ remains.

Now we consider taking the derivative and then mapping by ϕ, $\phi^*(d(Adx_1 \wedge dx_2 \wedge ... \wedge dx_n)$. By similarly applying the Leibniz Rule we see that this is $\phi^*(dAdx_1 \wedge dx_2 \wedge$

$... \wedge dx_n)$. But then ϕ^* can be distributed across a wedge product to get,

$$\phi^*(dA)\phi^*(dx_1)...\phi^*(dx_n).$$

We also know that $A, x_1, .., x_n$ are all 0-forms, and having shown the that pull-backs and differentiation commute for 0-forms we get,

$$\phi^*(dA)\phi^*(dx_1)...\phi^*(dx_n)(\mathbf{v}) = d(\phi^*(A))d(\phi^*(x_1))...d(\phi^*(x_n)).$$

Therefore, we have shown that,

$$d(\phi^*(Adx_1 \wedge dx_2 \wedge ... \wedge dx_n)) = d(\phi^*(A))d(\phi^*(x_1)) \wedge ... \wedge d(\phi^*(x_n)) = \phi^*(d(Adx_1 \wedge dx_2 \wedge .. \wedge dx_n)),$$

proving our claim. $\square$

Chapter 3: Maxwell's Equations

In this chapter we will establish Maxwell's equations using differential forms. We will do this by following the outline constructed by Ruth Gornet for the Park City Mathematics Institute [4]. In this chapter we will be working in $\mathbb{R}^4$. We allow for the usual 3-dimensional space components, x, y, z and a time component, t. Then exterior differentiation can be split into $d = d_S + d_T$. Next, we define the Hodge Operator.

Definition 3.1. In n-space the **Hodge Star Operator** takes k forms to $n-k$ forms. If $d\alpha$ is an k-form, then $*d\alpha$ is the Hodge star operator on $d\alpha$ defined as follows. Let $I = \{i_1, ..., i_k\}$ be an ordered subset of $\{1, ..., n\}$. Then let $J = \{j_1, ..., j_{n-k}\}$ be the ordered complement of I, i.e., $j_1 < j_2 < \cdots j_{n-k}$. Then the **Hodge Operator** gives us,

$$*(dx_{i_1}...dx_{i_k}) = \pm dx_{j_1}...dx_{j_{n-k}}.$$

The sign is inherited from the sign of the permutation (I, J) denoted in one-line notation is an odd permutation.

Similarly we see that the Hodge operator distributes over the sum of differential forms. In other words if $\alpha = A_1 dx_{I_1} + ... + A_m dx_{I_m}$ is any k-form then,

$$*\alpha = *A_1 dx_{I_1} + ... + *A_m dx_{I_m}.$$

We also note that for a function A_i the Hodge operator maps A_i to A_i.

Lemma 3.2. *Let α be a k-form on $\mathbb{R}^n$. Then,*

$$*(*\alpha) = (-1)^{k(n-k)}\alpha.$$

In particular if n is odd then $*(*\alpha) = \alpha$ but if n is even and k is odd then $*(*\alpha) = -\alpha$.

Proof. The hodge star operator is linear so it will suffice to prove the lemma for form of the form Adx_I where A is a function. Then by definition of the Hodge operator we know that $*\alpha = \text{sgn}(I,J)Adx_J$. Then $\text{sgn}(I,J) = (-1)^{k(n-k)}sgn(J,I)$ since we need to move k elements past $n-k$ elements. Then

$$\begin{aligned}
**\alpha =&(-1)^{k(n-k)}\text{sgn}(J,I)\text{sgn}(J,I)Adx_I \\
=&(-1)^{k(n-k)}Adx_I \\
=&(-1)^{k(n-k)}\alpha.
\end{aligned}$$

Therefore we have proven our claim. □

Now we will begin to introduce Maxwell's Equations. We will denote the electric field by

$$\mathbf{E} = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k},$$

where each E_i is a function from $\mathbb{R}^4 \to \mathbb{R}$. Next we denote the magnetic field by

$$\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k},$$

with each B_i a function from $\mathbb{R}^4 \to \mathbb{R}$. Also denote the electric current density by

$$\mathbf{J} = J_1\mathbf{i} + J_2\mathbf{j} + J_3\mathbf{k},$$

where each J_i is a function from $\mathbb{R}^4 \to \mathbb{R}$. Finally we let the charge density be denoted by $\rho(x,y,z,t) : \mathbb{R}^4 \to \mathbb{R}$.

Theorem 3.3 (Maxwell's Equations)**.** *In 1861 and 1862, James Clark Maxwell provided four equations governing the vector calculus of electric and magnetic fields.*

These form core tenets of the subdiscipline of electromagnetics and are as follows [4],

$$curl\ E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{3.1}$$

$$curl\ B = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t} \tag{3.2}$$

$$div\ E = 4\pi\rho \tag{3.3}$$

$$div\ B = 0 \tag{3.4}$$

Where we choose the units such that electric permittivity and the speed of light constant are one.

The first equation is also known as Faraday's Law, and gives an electric field which is produced by a changing magnetic field. In other words we can compute an electric field on the left by considering the magnetic field on the right as it changes over time. Meanwhile, the second equation is also know as Ampere's Law. This is similar to Faraday's but gives us a magnetic field produced by an electric field which changes with time. Then the third equation is referred to as Gauss's Law and the last equation, the *Nonexistence of true magnetism.* This last equation tells us that magnetic fields are divergence-free.

Theorem 3.4 (Reduction of Maxwell's Equations)**.** *We define the two-form η to be $\omega_E \wedge dt + \Phi_B$ and the one-form γ to be $\omega_E - \rho dt$. Then Maxwell's Equations in differential forms reduce to the following,*

$$d\eta = 0$$

$$*d*\eta = 4\pi\gamma.$$

To prove this theorem we will prove a sequence of claims, which will show that the equations in Theorem 3.3 are the same as those in Theorem 3.4. We start with the following claim.

Claim 3.5. $*(\omega_E \wedge dt) = \Phi_E$

$*\Phi_B = dt \wedge \omega_B$

Proof. We notice that,

$$*(\omega_E \wedge dt) = *(E_1 dx \wedge dt) + *(E_2 dy \wedge dt) + *(E_3 dz \wedge dt)$$

by the fundamental correspondence and linearity of the Hodge operator. Then taking the Hodge operator of each component we get $E_1 dy \wedge dz + E_2 dz \wedge dx + E_3 dx \wedge dy$, which by the fundamental correspondence we notice to be Φ_E.

Next we show that $*\Phi_B = dt \wedge \omega_B$. Start with Φ_B, which is equal to $B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy$. Then by applying the Hodge operator to this we get, $B_1 dt \wedge dx + B_2 dt \wedge dy + B_3 dt \wedge dz$. Then if we move the dt to the front we notice that this is $dt \wedge \omega_B$. Hence, we have shown

$$*(\omega_E \wedge dt) = \Phi_E \tag{3.5}$$

$$*\Phi_B = dt \wedge \omega_B \tag{3.6}$$

□

Claim 3.6. *Next we show that $d\eta$ is as follows,*

$$d\omega_E \wedge dt + d\Phi_B = d\eta = \sigma_{div\ B} + dt \wedge \Phi_{\frac{\partial B}{\partial t} + curl E}.$$

Proof. First, we show that $d\eta = d\omega_E \wedge dt + d\Phi_B$. We start with $d\eta = d(\omega_E \wedge dt + \Phi_B)$. By the linearity of exterior differentiation we get that this is equal to $d(\omega_E \wedge dt) + d\Phi_B$. Then applying the Leibniz Rule to the first term, we get $d\omega_E \wedge dt - \omega_E \wedge d(dt) + d\Phi_B$. But $d(dt) = 0$ and thus, $d\eta = d\omega_E \wedge dt + d\Phi_B$.

Now we show that $d\omega_E \wedge dt + d\Phi_B$ is equal to $\sigma_{\text{div } B} + dt \wedge \Phi_{\frac{\partial B}{\partial t} + \text{curl } E}$. Consider $d\Phi_B$, first we use that fact that $d = d_S + d_T$ to get $d\Phi_B = d_S \Phi_B + d_T \Phi_B$. Then on the space component we apply the Fundamental Corresponde to find $d_S \Phi_B = \sigma_{\text{div } B}$.

Now we have $d\eta = d\omega_E \wedge dt + d_T\Phi_B + \sigma_{\text{div } B}$. Then we also notice that we can apply the fundamental correspondence to get that $d\omega_E = \Phi_{\text{curl } E}$ and $d_T\Phi_B = \Phi_{\frac{\partial B}{\partial t}} \wedge dt$. Then since Φ is a two form we can move the dt to the front without changing sign giving us the following equality,

$$d\eta = d\omega_E \wedge dt + d\Phi_B = \sigma_{\text{div } B} + dt \wedge \Phi_{\dot{B}+\text{curl } E}. \tag{3.7}$$

□

Lemma 3.7. *The first and fourth equations of Maxwell's equations can be reduced to $d\eta = 0$.*

Proof. We start by setting equation 3.7 equal to zero, i.e.,

$$d\eta = \sigma_{\text{div } B} + dt \wedge \Phi_{\frac{\partial B}{\partial t}+\text{curl } E} = 0.$$

First we notice that we only have one component entirely in space since dt is a form of time, so if $d\eta = 0$ then this component must also be 0. Hence, $\sigma_{\text{div } B} = 0$. But then by Lemma 2.9 above we notice that $\sigma_{\text{div } B} = 0$ tells us div $B = 0$. Next we consider the time component of the equality to find that $dt \wedge \Phi_{\dot{B}+\text{curl } E} = 0$. We know that dt is in the time component, but $\Phi_{\frac{\partial B}{\partial t}+\text{curl } E}$ is a form in space. This tells us that if $dt \wedge \Phi_{\frac{\partial B}{\partial t}+\text{curl } E} = 0$ then $\Phi_{\frac{\partial B}{\partial t}+\text{curl } E}$ must be zero since we cannot get a simple time tensor from a form in space. But then by Lemma 2.9 this tells us $\frac{\partial B}{\partial t} + curlE = 0$. Then, $curlE = -\frac{\partial B}{\partial t}$, since $c = 1$ we can insert this on the right. Hence, we get that,

$$d\eta = 0 \Rightarrow \text{curl } E = -\frac{1}{c}\frac{\partial B}{\partial t}, \quad divB = 0.$$ □

Claim 3.8. *The Hodge star operator on η gives the following,*

$$*\eta = \Phi_E + dt \wedge \omega_B$$

Proof. Consider, $\eta = \omega_E \wedge dt + \Phi_B$. Similar to above we know that $\omega_E \wedge dt = E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$. Applying the Hodge star to this we get $E_1 dy \wedge dz + E_2 dz \wedge dx + E_3 dx \wedge dy$. Similarly,

$$*\Phi_B = *(B_1 dy \wedge dz) + *(B_2 dz \wedge dx) + *(B_3 dx \wedge dy) = -B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt.$$

Then by moving the dt past the spatial differential we lose the negative, and factoring out dt we see that $*\Phi_B = dt \wedge \omega_B$. Therefore,

$$*\eta = \Phi_E + dt \wedge \omega_B.$$ □

Claim 3.9.

$$d(*\eta) = \sigma_{divE} + dt \wedge \Phi_{\frac{\partial E}{\partial t} - curlB}$$

Proof. We start by taking the exterior derivative of $*\eta$. With this we see that, $d(*\eta) = d\Phi_E + d(dt \wedge \omega_B)$. Then applying the Leibniz Rule to the right side of the sum we get,

$$d(*\eta) = d\Phi_E + ddt \wedge \omega_B - dt \wedge \omega_B = d\Phi_E - dt \wedge \omega_B.$$

Now we will find another formula for $d(*\eta)$ starting with the above equation. We will follow a similar technique to the one we used for $d\eta$. We separate $d\Phi_E$ into $d_S\Phi_E + d_T\Phi_E$. Then this will be equal to $\sigma_{divE} + d_T\Phi_E$. Also by the fundamental correspondence we get that $d\omega_B = \Phi_{curl_B}$. Then combining our two-forms involving dt we find

$$d(*\eta) = \sigma_{divE} + dt \wedge \Phi_{\frac{\partial E}{\partial t} - curlB}.$$ □

Definition 3.10. By the definition,

$$\gamma := \omega_J - \rho\ dt.$$

Then γ is also equal to,

$$\gamma = J_1\ dx + J_2\ dy + J_3\ dz - \rho\ dt.$$

Lemma 3.11. *The second (3.2) and third (3.3) Maxwell's Equations reduce to* $*d*\eta = 4\pi\gamma$

Proof. We start by considering $*\gamma$,

$$gamma = *(J_1\ dx) + *(J_2\ dy) + *(J_3\ dz) - *(\rho dt) = -\ J_1\ dy \wedge dz \wedge dt - J_2\ dz \wedge dx \wedge dt \\ -\ J_3\ dx \wedge dy \wedge dt + \rho\ dx \wedge dy \wedge dz.$$

By transposing the dt to the front of the equation we find that this is $\sigma_\rho - dt \wedge \Phi_J$.

Consider the equation $d * \eta = 4\pi * \gamma$. By the work above we see that this tells us that,

$$\sigma_{divE} + dt \wedge \Phi_{\frac{\partial E}{\partial t} - curlB} = 4\pi(\sigma_\rho - dt \wedge \Phi_J).$$

Then on each side the time and space components must be equal. In other words we obtain that,

$$\sigma_{divE} = 4\pi\sigma_\rho$$

and

$$dt \wedge \Phi_{\frac{\partial E}{\partial t} - curlB} = -4\pi dt \wedge \Phi_J.$$

Then since $4\pi\sigma_\rho = \sigma_{4\pi\rho}$, and the preimages of σ on each side must be the same if the image is, we find that $divE = 4\pi\rho$. Similarly, for the latter equation we can cancel the dt on each side and then as Φ is a map the preimages must be the same up to a constant. Hence, we see that $\frac{\partial E}{\partial t} - curlB = -4\pi J$. But taking into account that we allow $c = 1$ and then adding to each side we get that $\frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}J = curlB$. Which we recognize these as Maxwell's second and third equations.

Now we consider $d * \eta = 4\pi * \gamma$. If we take the Hodge operator of each side we notice on the right we are taking just the Hodge operator on the components of $*\gamma$. Since γ is the sum of two one forms, $**\gamma = \gamma$. Therefore, we see that $*d*\eta = 4\pi\gamma$. $\square$

Thus, by Lemma 3.7 and Lemma 3.11 we have shown Theorem 3.1, Maxwell's equations can be reduced into differential forms by the following,

$$d\eta = 0$$

$$*d*\eta = 4\pi\gamma.$$

Chapter 4: Biot-Savart Operator

4.1 The Hodge Decomposition Theorem

In this chapter we will closely follow *The Biot-Savart operator for application to knot theory, fluid dynamics, and plasma physics* by Cantarella, DeTurck, and Gluck [5]. For this chapter we define Ω to be a compact domain with a piece-wise smooth boundary in $\mathbb{R}^3$, so we allow for spaces with corners in the boundary, for example a square since the sides are piecewise smooth.

In this section we state the Hodge Decomposition Theorem for vector fields on a compact domain in $\mathbb{R}^3$, but we will not prove it in this paper. Before we state the theorem we will need to define the components of the decomposition. In the following definitions we let φ represent a function with $\nabla\varphi$ is the gradient of the function. Also we define n to be the outward pointing unit vector field orthogonal to $\partial\Omega$.

Definition 4.1. Consider these five subspaces of $VF(\Omega)$, the smooth vector fields defined on Ω.

FK = **Fluxless Knots** = $\{\nabla \cdot V = 0, V \cdot n = 0,$ all interior fluxes $= 0\}$,

HK = **Harmonic Knots** = $\{\nabla \cdot V = 0, \nabla \times V = 0, V \cdot n = 0\}$,

CG = **Curly Gradients** = $\{V = \nabla\varphi, \nabla \cdot V = 0,$all boundary fluxes $= 0$ $\}$,

HG = **Harmonic Gradients** = $\{V = \nabla\varphi, \nabla \cdot V = 0, \varphi$ locally constant on $\delta\Omega\}$,

GG = **Grounded Gradients** = $\{V = \nabla\varphi, \varphi|_{\delta\Omega} = 0\}$.

We will refer to the direct sum of FK and HK as the *fluid knots*, which are both divergence-free and tangent to the boundary. In the future we will represent fluid knots on the domain Ω by $K(\Omega)$. The word knot arises from the fact that given a knot K in 3-space, if we choose a tubular neighborhood of K and make that Ω we can use a divergence-free vector field on Ω to study the geometry of the knot. We will now provide an example of each of the five subspaces listed above.

Example 4.2 (Fluxless Knots)**.** Consider the ball B of radius one with boundary. Then let $V = -z\mathbf{i} + x\mathbf{k}$. Then we consider $\nabla \cdot V$; which equals $\frac{\partial}{\partial x}(-z) + 0 + \frac{\partial}{\partial z}x = 0$, showing that $\nabla \cdot V = 0$. Next we consider $V \cdot n$. On the sphere n is the radial vectors on the boundary. But then if we extend the radial line through the origin we will find that n and V are perpendicular since the direction of n is represented by (x, y, z). Then the dot product of V and n will be 0. Finally we notice that V rotates around the y-axis. Therefore given any surface Σ in B, and any vector in V which crosses Σ we get a symmetric vector crossing Σ with the same magnitude in the opposite direction.

Example 4.3 (Harmonic Knots)**.** For this consider the solid torus, i.e. $S^2 \times D^1$ where S^2 has radius $\frac{1}{2}$ and D^1 is centered at the origin in the xy-plane with radius two. Then in cylindrical coordinates (r, ϕ, z) let V be the vector field $\frac{1}{r}\hat{\phi}$. V is tangent to the boundary since for any "meridian" slice, S, the vectors in V originating on that slice will be normal to S. Now we compute the divergence, this will be $\frac{1}{r}\frac{\partial}{\partial r}0 + \frac{\partial}{\partial \phi} + 0\frac{\partial}{\partial z}$ which we can see will be zero by partial differentiation. Similarly if we consider the curl we realize that it will be zero as v_r and v_z are both zero and the derivative with respect to ϕ will be zero since the function is a constant with respect to ϕ. Therefore V is a harmonic knot.

Example 4.4 (Curly Gradients)**.** For this example let Ω be a unit sphere. Then we let $\phi = x$. Then $V = \nabla\phi = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. Then the divergence of V will be 0 since v_i, v_j, v_k are all constants. Also the boundary flux over Ω will be 0 since we get an equal flow from the negative x direction and going out the positive x direction.

Example 4.5 (Harmonic Gradients)**.** Let Ω be the region between two concentric spheres of radius 2 and 4, centered at the origin. Then using spherical coordinates, (r, θ, ϕ), consider $\varphi = \frac{1}{r}$. Then its gradient vector field $V = \nabla\varphi = -\frac{1}{r^2}\hat{r}$. Then we consider the divergence of V, in spherical coordinates we will only have to consider the r component, which give $\nabla \cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{1}{r^2}\right)$. Then we notice that this is the partial derivative of a constant which will be zero, therefore $\nabla \cdot V = 0$. Finally the radius on a sphere will be constant so on each boundary component ϕ will be a constant.

Example 4.6 (Grounded Gradients)**.** Let Ω be the torus which is created by rotating a circle around the z-axis in the xz-plane of radius 1 centered at $(2, 0, 0)$. Then let V be the vector field $(\frac{4x}{\sqrt{x^2+y^2}} - 2x)\mathbf{i} + (\frac{4x}{\sqrt{x^2+y^2}} - 2y)\mathbf{j} + 2z\mathbf{k}$. Then V is the gradient of $\phi = z^2 - y^2 - x^2 + 4\sqrt{x^2+y^2} - 3$. If we change this function to cylindrical coordinates we obtain $\phi = 1 - ((2-r)^2 + z^2)$. Then since the boundary of Ω is the circle of radius one centered at the xy-plane circle of radius 2; the function ϕ will be 0 on $\partial\Omega$. Therefore V satisfies the definition of a grounded gradient.

Theorem 4.7. (**Hodge Decomposition Theorem [5]**)

The space $VF(\Omega)$ has a direct sum decomposition into five mutually orthogonal subspaces,

$$VF(\Omega) = FK \oplus HK \oplus CG \oplus HG \oplus GG.$$

Further we get that,

$$\begin{aligned} \ker \operatorname{curl} =& HK \oplus CG \oplus HG \oplus GG, \\ \operatorname{image} \operatorname{grad} =& CG \oplus HG \oplus GG, \\ \operatorname{image} \operatorname{curl} =& FK \oplus HK \oplus CG, \\ \ker \operatorname{div} =& FK \oplus HK \oplus CG \oplus HG, \end{aligned}$$

and furthermore,

$$HK \cong H_1(\Omega; \mathbb{R}) \cong H_2(\Omega, \partial\Omega; \mathbb{R}) \cong \mathbb{R}^{\text{genus of } \delta\Omega},$$

$$HG \cong H_2(\Omega; \mathbb{R}) \cong H_1(\Omega, \partial\Omega) \cong \mathbb{R}^{(\#\text{ components of } \partial\Omega) - (\#\text{ components of } \Omega)}.$$

4.2 The Biot-Savart Operator and its Properties

In this section we will introduce the Biot-Savart operator and some important properties of it. The Biot-Savart operator is an essential part of our story as it is inherent in the computation of helicity. In physics the *Biot-Savart operator* associates each electric current flowing on Ω to the restriction of its induced magnetic field on Ω. In 1820, Jean Baptist Biot and Francis Savart derived the formula through an experiment of electromagnetic physics.

Definition 4.8. In particular if we think of the smooth vector field V, over Ω, in $\mathbb{R}^3$, as the distribution of electric current, then the **Biot-Savart Law**,

$$BS(V)(y) = \left(\frac{1}{4\pi}\right) \int_{\Omega} V(x) \times \frac{(y-x)}{|y-x|^3} d(vol_x)$$

describes the induced magnetic field $BS(V)$ on all of $\mathbb{R}^3$. Then if we restrict the resultant magnetic field $BS(V)$ to the domain Ω, we get the **Biot-Savart operator**, $BS(V) : VF(\Omega) \to VF(\Omega)$.

We will explain the terms in the Biot-Savart operator here. Then for each point $y \in \Omega$, we fix y to get $BS(y)$. Then we allow for the x, in the operator, to traverse through every point in Ω. Then we can think of the resultant vector for each y, $BS(y)$, as the weighted average of the vectors at each x, weighted by the distance from y to x. Since the $BS(y)$ has a cross product, it will result in a vector field. We notice that if $x = y$, then $\frac{x-y}{|x-y|^3}$ will not be defined. Therefore, this integral will be improper, and below (Lemma 4.11) we see that it will be well-defined.

Next we will state some properties of the Biot-Savart operator. Then we will provide a few of the more interesting proofs. Let V, Ω, and $\partial\Omega$ be as above.

Definition 4.9. We define the **vector potential**, $A(V)$, for $BS(V)$ by the formula,

$$A(V)(y) = \left(\frac{1}{4\pi}\right) \int_{\Omega} \frac{V(x)}{|y-x|} d(vol_x).$$

Theorem 4.10 (Properties of Biot-Savart Operator)**.** *Then we get the following list of properties for $BS(V)$ and $A(V)$.*

1. *$BS(V)$ and $A(V)$ are well-defined on all of 3-space, in other words, the improper integrals which define them converge everywhere.*
2. *$BS(V)$ and $A(V)$ are of class C^∞ on Ω, and on the closure Ω' of $\mathbb{R}^3 - \Omega$. $BS(V)$ is continuous on $\mathbb{R}^3$, but its derivatives typically contain jump discontinuities as one crosses $\partial\Omega$. $A(V)$ is of class C^1 on $\mathbb{R}^3$, but its second derivatives tend to contain jump discontinuities as one crosses $\partial\Omega$.*
3. *$\Delta A(V) = -V$ in Ω and $\Delta A(V) = 0$ in Ω', where Δ is the vector Laplacian.*
4. *$\nabla \times A(V) = BS(V)$ on $\mathbb{R}^3$, which justifies its name as the vector potential.*
5. *If $V \in K(\Omega)$, then $A(V)$ is divergence free on $\mathbb{R}^3$.*
6. *$\nabla \cdot BS(V) = 0$ in Ω and in Ω'.*
7. *If $V \in K(\Omega)$, then $\nabla \times BS(V) = V$ in Ω and $\nabla \times BS(V) = 0$ in Ω'.*
8. *If $V \in K(\Omega)$, then $\int_C BS(V) \cdot ds = 0$ for all closed curves C on $\partial\Omega$ which bound in $\mathbb{R}^3 - \Omega$.*
9. *In general, $A(V)$ decays at ∞ like $\frac{1}{r}$ and $BS(V)$ decays at ∞ like $\frac{1}{r^2}$; but, if $V \in K(\Omega)$, then $A(V)$ decays at ∞ like $\frac{1}{r^2}$ and $BS(V)$ decays at ∞ like $\frac{1}{r^3}$.*

Claim 4.11. *Here we will provide a proof for the $BS(V)$ part of (1), showing that $BS(V)$ is well-defined on all of $\mathbb{R}^3$.*

Proof. There are two cases for y. First assume that $y \notin \Omega$, but then we know that for all $x \in \Omega$, that $x \neq y$ and therefore, the integral will be defined.

In the second case we let $y \in \Omega$, then we know that if $y = x$, we will have problems with the integrand. To show that this integral can still be computed we look at the δ neighborhood $B(y, \delta)$ around y. We do this for a fixed y by dividing Ω into $B(y, \delta)$ and $\Omega \setminus B(y, \delta)$. We notice that on $\Omega \setminus B(y, \delta)$ the Biot-Savart integrand is well-defined. Then for any δ, we will show there exists an $\epsilon > 0$ which bounds $BS(V)$ on $B(y, \delta)$. Then if we show that as $\delta \to 0$, the bound $\epsilon \to 0$, then we can properly define a limit for the integral as $x \to y$.

To prove this we will use spherical coordinates, (r, θ, ϕ) centered at y. Then we consider the magnitude of $BS(y)_\delta$, i.e. the Biot-Savart operator over the δ neighborhood around y. This will give us

$$\left| \int_{B(y,\delta)} V(x) \times \frac{(y-x)}{|y-x|^3} d(vol_x) \right|.$$

Then we can split up the inverse-square term

$$\left| \int_{B(y,\delta)} V(x) \times \frac{(y-x)}{|y-x|} \frac{1}{|y-x|^2} d(vol_x) \right|.$$

Given vectors u, v where u is a unit vector we know $|v \times u| \leq |v|$. Since $\frac{y-x}{|y-x|}$ is a unit vector then we conclude,

$$\left| V(x) \times \frac{y-x}{|y-x|} \right| < |V(x)|$$

Then we define, $\max |V(x)|$ to be the max magnitude of a vector in the ball $\overline{B(y, \delta)}$. We will let m_δ be $\max |V(x)|$ on $\overline{B(y, \delta)}$. Then on $B(y, \delta)$

$$\left| V(x) \times \frac{y-x}{|y-x|} \right| < |V(x)| < m_\delta,$$

which is a constant. Then we can see,

$$\left|\int_{B(y,\delta)} V(x) \times \frac{(y-x)}{|y-x|}\frac{1}{|y-x|^2}d(vol_x)\right| \leq m_\delta \left|\int_{B(y,\delta)} \frac{1}{|y-x|^2}d(vol_x)\right|.$$

Then using the fact that for any integrand Φ, $\left|\int_\Omega \Phi\right| \leq \int_\Omega |\Phi|$, and converting to spherical coordinates we find,

$$\left|\int_{B(y,\delta)} V(x) \times \frac{(y-x)}{|y-x|}\frac{1}{|y-x|^2}d(vol_x)\right| \leq m_\delta \int_{B(y,\delta)} \frac{1}{r^2}r^2 sin\phi dr d\theta d\phi.$$

Then since the δ-neighborhood around y has radius δ the integral will be $4\pi\delta$,

$$m_\delta \int_{B(y,\delta)} \frac{1}{r^2}r^2 sin\phi dr d\theta d\phi = 4\pi\delta m_\delta.$$

But since the max magnitude is weakly decreasing as the neighborhood shrinks over all $4\pi\delta m_\delta$ goes to zero. Thus, we have shown that $BS(V)$ is well-defined. □

Next we will expand on Theorem 4.10 part (7). For any given vector field, V, we get that

$$\nabla_y \times BS(V)(y) = \left\{ \begin{array}{cc} V(y) & \text{in } \Omega \\ 0 & \text{in } \Omega' \end{array} \right\} +$$

$$\left(\frac{1}{4\pi}\right)\nabla_y \int_\Omega \frac{(\nabla_x \cdot V(x))}{|y-x|}d(vol_x) - \left(\frac{1}{4\pi}\right)\nabla_y \int_{\delta\Omega} V(x) \cdot \frac{n}{|y-x|}d(area_x).$$

For this expansion we will recall the Hodge Decomposition Theorem. Property (7) states the first component is nonzero only if V has a nonzero fluid knot component. We notice that the second component contains the formula for divergence and thus will be zero if V is divergence-free. Finally the third component contains the normal vector n to $\partial\Omega$, making the dot product with V zero precisely when V is tangent to ∂V. In other words the second term will only be nonzero if $V_{GG} \neq 0$. While the third term will only be nonzero if $V_{CG} = V_{HG}$.

The equation for $\nabla_y \times BS(V)(y)$ can also be viewed in terms of Maxwell's Equations, (Theorem 3.3), as follows

$$\nabla \times B = J + \partial E / \partial t.$$

To start we allow V, our vector field, to represent a current distribution throughout the domain Ω. At time $t = 0$, let the volume charge density ρ throughout Ω and the surface charge density σ along $\partial\Omega$ both be zero. By volume and surface charge density we mean the amount of electric charge per any unit of volume in Ω or unit of area on $\partial\Omega$. Then to construct the formulas for ρ and σ we will use our vector field V. So we set the volume charge to be

$$\rho = -(\nabla \cdot V)t \text{ throughout } \Omega.$$

Then we find the formula for the surface charge density to be what we expect by the Divergence Theorem,

$$\sigma = (V \cdot n)t \text{ along } \delta\Omega.$$

Both of these equations were forced upon us by the continuity equation, $\nabla \cdot V = -\partial\rho/\partial t$. The continuity equation tells us how the flow of particles will impact the density charge in a given region of Ω. In particular we notice that $\nabla \cdot V$ will give us the divergence of the vector field. Then $-\partial\rho/\partial t$ gives us the change, with respect to time, of the charge density at a point in V. These will be equal since positive divergence at a point p, will cause charge to move away from p, lowering the density. Similarly the formula for σ from how V deposits current on the boundary of Ω. In particular, we consider the direction of V in relation to $\partial\Omega$ since the charge is trapped in Ω, and we know a tangent vector field will never move charge to the boundary.

Now we use the volume charge densities to compute time varying electric fields for Ω and $\partial\Omega$. By time varying electric fields we simply mean electric fields which change over a variable of time t. Throughout Ω we will find

$$E_\rho(y,t) = \left[\left(\frac{1}{4\pi}\right)\nabla_y \int_\Omega \frac{\nabla_x \cdot V(x)}{|y-x|} d(vol_x)\right] t.$$

Then similarly, the surface charge along $\partial\Omega$ will give us

$$E_\sigma(y,t) = \left[-\left(\frac{1}{4\pi}\right)\nabla_y \int_{\partial\Omega} \frac{V(x) \cdot n(x)}{|y-x|} d(area_x)\right] t.$$

Both of these fields will extend throughout three space, much like the equation for (7). Then the total electric field,

$$E(y,t) = E_\rho(y,t) + E_\sigma(y,t),$$

has a time rate of change,

$$\partial E/\partial t = \partial E_\rho/\partial t + \partial E_\sigma/\partial t = E'_\rho + E'_\sigma.$$

Then since we want to use the time rate of change to reduce (7) we get the following,

Proposition 4.12.

$$\nabla \times BS(V) = \{V/0\} + E'_\rho + E'_\sigma.$$

where the $\{V/0\}$ is defined the same way as it is in the expansion of (7) above.

To illustrate the above formula for the curl of the Biot-Savart operator we will provide the computations for a few examples from Canterella, DeTurck, and Gluck [5].

Now we will present the main theorems from the literature.

Theorem 4.13. *The equation $\nabla \times BS(V) = V$ holds in Ω if and only if V is divergence-free and tangent to the boundary of Ω, i.e., a fluid knot.*

Proof. We see that the reverse direction is a result of (7) from Theorem 4.10. This is because if V is divergence-free and tangent to the boundary of Ω then V is contained in the set of fluid knots. Therefore we need only show the forward direction.

Assume that $\nabla \times BS(V) = V$ in Ω; we want to show that V is in $K(\Omega)$. We will prove this by contradiction. First, assume that V has a nonzero component in $HG \oplus GG$, then it is impossible for $\nabla \times BS(V)$ to equal V in Ω by the Hodge Decomposition Theorem (4.7), since the image of the curl of a vector field lies in $FK \oplus HK \oplus CG$.

Thus, it remains to show that if V has a component, F in CG, then $\nabla \times BS(V)$ will not equal V in Ω unless $F = 0$.
To prove this we will want to consider Maxwell's Equations (3.3),

$$\nabla_y \times BS(V)(y) = \left\{ \begin{array}{cc} V(y) & \text{in } \Omega \\ 0 & \text{in } \Omega' \end{array} \right\} - \left(\frac{1}{4\pi}\right) \nabla_y \int_{x \in \delta\Omega} V(x) \cdot \frac{n(x)}{|x-y|} d(area_x).$$

The second term on the right-side of this equation is the formula for a time rate of change of an electric static field, E'_σ. Then we can write

$$E'_\sigma(y) = -(\frac{1}{4\pi})\nabla_y \int_{x \in \delta\Omega} V(x) \cdot \frac{n(x)}{|x-y|} d(area_x).$$

Due to the charge density $\sigma(x) = V(x) \cdot n(x)$, along $\partial\Omega$ this will be the same as an electrostatic field. Therefore we can write it as,

$$E'_\sigma(y) = -\nabla_y \psi(y).$$

We notice that $\psi(y) = \left(\frac{1}{4\pi}\right) \int_{x \in \delta\Omega} V(x) \cdot \frac{n(x)}{|x-y|} d(Area_x)$. Although the electrostatic potential function ψ for a surface charge distribution σ is continuous, the electrostatic field E'_σ will in general have a jump discontinuity as we cross the surface. Regardless, we still get that $\nabla \cdot E'_\sigma = 0$ both in Ω and in Ω'.

Now we claim that if F is a nonzero fector field in CG, then E'_σ cannot be identically zero in Ω. This will prove our claim because E'_σ is the second component of the Maxwell Equation 3.2. Since all of F is in the first component on the right this will cause an inequality.

Here we recall that a vector field F defined on Ω is in CG if and only if $F = \nabla\varphi$ where φ is a harmonic function on Ω, and the flux of F through each component of $\partial\Omega$ is zero. Then for each component $\partial\Omega_i$ we have,

$$\int_{\partial\Omega_i} V(x)\cdot n(x) d(area_x) = \int_{\partial\Omega_i} \sigma(x) d(area_x) = 0.$$

We see the equality of integrals from above, and use the fact that the flux is zero through the boundary of each component to get the equality with zero. The right integral will be a measure of the flux through the boundary components. Then the total charge on each component is zero. Now using this, we will show that if $E'_\sigma = 0$ in Ω, then $F = 0$.

Then in our hypothesis we assumed that $E'_\sigma = 0$ in Ω, then we need to show that $E'_\sigma = 0$ in Ω'. The hypothesis will also tell us that ϕ must be constant on each component $\partial\Omega_i$.

Now we consider the field $\psi E'_\sigma$ in Ω and compute its divergence,

$$\nabla\cdot(\psi E'_\sigma) = (\nabla\psi)\cdot E'_\sigma + \psi(\nabla\cdot E'_\sigma) = -E'_\sigma\cdot E'_\sigma = -|E'_\sigma|^2.$$

The equality $(\nabla\psi)\cdot E'_\sigma + \psi(\nabla\cdot E'_\sigma) = -E'_\sigma\cdot E'_\sigma$ stems from the fact that $\nabla\cdot E'_\sigma = 0$. We get this fact from above as $\nabla\cdot(\nabla\psi) = 0$ by properties of differentials. Then using this equality and the Divergence Theorem we get,

$$\int_{\Omega'} |E'_\sigma|^2 d(vol) = -\int_{\Omega'} \nabla\cdot(\psi E'_\sigma) d(vol) = -\int_{\partial\Omega'} \psi E'_\sigma \cdot n' d(area),$$

where n' is the outword pointing normal vector to Ω', so $n' = -n$.

We are able to use the Divergence Theorem on Ω' despite its unbounded components by approximating by using bounded domains, with a boundary near infinity. Then the flux of $\psi E'_\sigma$ through this component goes to zero as the boundary approaches infinity so the integral of its divergence is zero. The boundary of this component will grow like r^2 as it approaches infinity, but from property (9) above we see that the field E'_σ decays like $\frac{1}{r^2}$ and ψ decays like $\frac{1}{r}$. Then $\psi E'_\sigma$ will decay like $\frac{1}{r^3}$, and since the boundary area grows like r^2, the flux will decay like $\frac{1}{r}$.

Now using the fact that ψ_i is constant on each boundary component, $\partial\Omega_i$, we can break the integral into a sum of the integrals on the boundaries of the components to get,

$$\int_{\Omega'} |E'_\sigma|^2 d(vol) = -\int_{\partial\Omega'} \psi E'_\sigma \cdot n' d(area) = -\sum_i \psi_i \int_{\partial\Omega_i} E'_\sigma \cdot n' d(area).$$

Now we use the Divergence Theorem to get

$$\int_{\partial\Omega_i} E'_\sigma \cdot n' d(area) = \text{ total charge "inside" } \partial\Omega_i = \pm \sum_{somej} \int_{\partial\Omega_j} \sigma(x) d(area) = 0$$

The first equality arises from the integral calculating the net charge that enters or leaves the boundary, which gives us the charge inside the boundary. The integral measures this because it looks at the electrostatic field with relation to the inwards pointing normal vectors n'. Then the second equality arises from the fact that the density integral will give us the total charge inside the boundary. Finally the last equality arise from calculations above.

Thus, $\int_{\Omega'} |E'_\sigma|^2 d(vol) = 0$ hence $E'_\sigma = 0$ in Ω'. Then from this and using the fact that $\sigma(x) = F(x) \cdot n(x)$ we get that F is tangent to the boundary of Ω. Then this shows that F is a fluid knot, but the intersection of fluid knots of Ω and CG is empty. Therefore, $F = 0$, proving our claim. $\square$

Theorem 4.14. *The kernel of the Biot-Savart operator is precisely the space of gradient vector fields which are orthogonal to the boundary of* Ω.

Theorem 4.15. *The image of the Biot-Savart operator is a proper subspace of the image of curl, whose orthogonal projection into the subspace of "fluxless knots" is one-to-one.*

Theorem 4.16. *The Biot-Savart operator is a bounded operator, and hence extends to a bounded operator on the L^2 completion of its domain, where it is both compact and self-adjoint.*

Chapter 5: Helicity

5.1 Introduction to Helicity

At this point in the paper we will begin to introduce the main topic, helicity. In an interpretive sense, helicity measures the coiling of a vector field. In physics this can be used to measure the coiling or wrapping of the flow of plasma. Mathematically the formula is similar to that of *"the Gauss integral formula"* for linking number [6],

$$Lk(a,b) = \frac{1}{4\pi}\int a'(\theta)\times b'(\phi)\cdot \frac{a(\theta)-b(\phi)}{|a(\theta)-b(\phi)|^3}d\theta d\phi$$

or Writhing number[7],

$$\frac{1}{4\pi}\oint_{a\times a} a'(x)\times a'(x')\cdot \frac{a(x)-a(x')}{|a(x)-a(x')|^3}dxdx'.$$

Definition 5.1. If we let $H(V)$ be the helicity of the vector field V, which we will also refer to as standard helicity, then,

$$H(V) = \left(\frac{1}{4\pi}\right)\int_{\Omega\times\Omega} V(x)\times V(y)\cdot\frac{(x-y)}{|x-y|^3}d(vol_x)d(vol_y).$$

Gauss linking number is computed by letting a', b' represent our links, and θ, ϕ the position on the link. This makes sense since a', b' are in bijection with a circle. Then the integral computes a value at each position (θ, ϕ), and then weights it based on the distance between $(a'(\theta), b'(\phi))$. Similarly, the writhe integral compares tangent vectors along a curve as we slide points x, x' along a curve. Then we weight contribution based on the distance between x and x'.

We notice the similarities as helicity compares the vectors at each pair of points in Ω. Then similar to the previous two quantities, we will weight the contribution of a pair (x, y) based on their distance apart. This is significant because for all three values, the further apart two points are the less contribution we gain from the triple product. We also remind the reader that the triple product will give the volume of the box defined by the three vectors in the $H(V)$ integrand. But as $y - x$ becomes greater this volume will decrease.

We also notice that the formula for helicity is similar to that of the Biot-Savart operator. We will soon show that the formula helicity can be reduced to an inner product with the Biot-Savart operator. We need to add in a dot product for the helicity integrand because unlike the Biot-Savart operator, applied to a vector field, V, the helicity of V will be a scalar value. This scalar will give us a measure of the coiling of the vector field over Ω. To compute this coiling we will need to consider the vector field at every point $x \in \Omega$ with every point $y \in \Omega$, hence our integral will need to be over $\Omega \times \Omega$.

Now, we will show that $H(V)$ can be reduced to an L^2 inner product of V and $BS(V)$. To begin consider the formula for helicity,

$$H(V) = (\frac{1}{4\pi}) \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{(x-y)}{|x-y|^3} d(vol_x) d(vol_y).$$

Then by the properties of cross products and dot products we can rearrange this to,

$$H(V) = \int_\Omega V(y) \cdot \left[\left(\frac{1}{4\pi} \right) \int_\Omega V(x) \times \left(\frac{(y-x)}{|y-x|^3} \right) d(vol_x) \right] d(vol_y).$$

Then we notice that inside the square brackets we have the formula for $BS(V)$ at each y, so this can be rewritten as,

$$H(V) = \int_\Omega V(y) \cdot BS(V)(y) d(vol_y).$$

Then since everything is with respect to y we can just simplify by writing,

$$H(V) = \int_\Omega V \cdot BS(V) d(vol).$$

We recognize this as the L^2 inner product, and write, $H(V) = \langle V, BS(V)\rangle$.

Similar to the Biot-Savart operator it is important to show that the formula for helicity is well-defined. This is because we notice that when $x = y$ the integrand will be undefined. This happens because of the term $\frac{(x-y)}{|x-y|^3}$, clearly if $x = y$ then the denominator will be zero which is a problem. We know that this will happen because for every y in Ω we consider every x in the same Ω. Then we get a set of points, $\Delta = \{(x, x) : x \in \Omega\} \subset \Omega \times \Omega$, where the integral is undefined which will be equivalent to Ω.

Proposition 5.2. *The helicity of any smooth vector field, $H(V) = \int_\Omega V \cdot BS(V) d(vol) = (\frac{1}{4\pi}) \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \left(\frac{(x-y)}{|x-y|^3}\right) d(vol_x)d(vol_y)$ is well defined.*

Proof. On $\Omega \times \Omega$ we notice that $(\frac{1}{4\pi}) \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \left(\frac{(x-y)}{|x-y|^3}\right) d(vol_x)d(vol_y)$ is well defined everywhere but when $x = y$. We can prove this by using our proof that the Biot-Savart operator is well-defined, lemma 4.11. But by definition, helicity is also $\langle V, BS(V)\rangle$. But, above we proved that $BS(V)$ is zero at $x = y$, so then the integrand of $\langle V, BS(V)\rangle$ will approach 0 at $x = y$. Hence, we see that inner product's contribution to helicity is negligible at $x = y$, and thus $H(V)$ will be well-defined. □

5.2 Applications of Helicity

Next we will begin to introduce some interesting applications of helicity. The first application we will introduce is that helicity can be used as an invariant. This is expected given the similarities to the Gauss linking formula, and that the Gauss linking number is an invariant for links. Given $\Omega \subset \mathbb{R}^3$ we unfortunately do not get an overall diffeomorphism invariant with helicity. But helicity is still an important quantity for applications in fluid dynamics and plasma physics. In mathematics we can still get a somewhat general invariance theorem as follows:

Theorem 5.3. (Invariance of Helicity Theorem) *[8]. The helicity of a divergence-free vector field V on a domain $\Omega \subset \mathbb{R}^3$ is invariant under any volume-preserving diffeomorphism of Ω which is homotopic to the identity. If Ω is simply connected, then helicity is invariant under any volume-preserving diffeomorphism of Ω.*
If V is a null-homologous vector field (meaning that its dual 2-form is exact) on a compact manifold M^3 without boundary, then helicity is invariant under any volume-preserving diffeomorphism of M.

In other words given a vector field V on a domain Ω of $\mathbb{R}^3$ diffeomorphic and with an equivalent volume to the unit ball B^3, we get that helicity will be invariant under diffeomorphisms homotopic to the identity. The last statement tells us that if V is fluxless on Ω then helicity will be an invariant under volume-preserving diffeomorphisms. Otherwise we will get a volume-preserving invariant as long as we do not move any points in a non-homotopic manner, e.g. a Dehn twist or another move involving a cut.

Another useful approach to helicity is that of energy minimization. The problem presented below on energy relaxation was found in Boris Khesin's *Topological Fluid Dynamics*[9]. After presenting the problem we will explain how helicity can be used to solve the problem in a more efficient manner.

To start we consider the topological obstructions to the energy relaxation of a magnetic field in a perfectly conducting medium, for instance plasma. Plasma is an ionized gas, which allows for the free movement of electrical charges. What we want to consider is the magnetic field of a star filled with plasma, carrying a "frozen in" magnetic field. By "frozen in" we mean that the magnetic field is frozen to the plasma; in other words the magnetic field will move if and only if the plasma does. Since the field is bound to the plasma then the topology of the trajectories cannot change, but the trajectories of the magnetic field can change. Now we look at the energy relaxation of the field, the process under which the excess of magnetic energy over its possible minimum is fully dissipated. By Maxwell's equations we see that conducting fluids will continue moving until energy relaxation is realized. In other words, the plasma will continue to flow in a way to minimize energy. But it turns out the linking number of magnetic fields can disrupt this flow and prevent complete dissipation of the excess energy. Then the minimization problem becomes, describing lower bounds for the energy of the magnetic field in terms of the topological characteristics of its trajectories.

In particular if we let V be a divergence-free vector field, tangent to the boundary of a bounded domain $M \subset \mathbb{R}^3$. Then when we refer to the energy of the field V we will mean the square of its L_2 norm, i.e., the integral

$$E(V) = \int_M (V, V) d(vol) = \langle V, V \rangle.$$

Then given a divergence free field V the problem we want to solve is finding the minimum energy (or a lower bound) for the $\inf_h E(h_*)(V)$ of the push forward fields $h_*(V)$ under the action of a volume-preserving diffeomorphism h of M.

An example from the above paper of a topological obstruction to energy relaxation is the magnetic field confined to two linked solid tori. For the example assume that the magnetic field vanishes outside the tori and that the field trajectories are all closed and oriented along the tube axes inside. To minimize the energy we want to shorten the length of most trajectories. But given the volume preserving nature of the diffeomorphisms this leads to a fattening of the solid tori. Therefore eventually we will see that the linking of the two tori will prevent endless fattening, thus limiting the shortening of the trajectories. Hence in the volume preserving relaxation process the magnetic energy of the field on the two tori will be bounded below.

In the above example we notice that the topological obstruction was that of the linking tori. As one torus passes through another, we will see that their respective magnetic fields cross. This crossing will give rise to helicity, since it is the coiling of the fields, creating magnetic energy. Therefore, the amount of helicity arising from a given field or fields will be able to tell us something about the energy minimum. In particular, we notice that if there is more helicity, then we have greater obstruction creating a greater lower bound for the energy minimum. An example of how the helicity could be raised in the above example would be if we started with tori with greater radius, or if we had multiple tori going through the "doughnut hole" of one torus.

More precisely the nonzero helicity of a field V provides a lower bound for the magnetic energy. The following theorem solidifies this idea.

Theorem 5.4. *For a divergence-free vector field V*

$$E(V) \geq C \cdot |H(V)|,$$

where C is a positive constant depending on the shape and size of the compact domain M.

The C term can be taken as the reciprocal of the norm of the compact Biot Savart operator by Theorem 4.16. Therefore, we will usually see that via helicity the lower bound for the energy will be positive.

Normally to determine the minimum energy one would have to use the magnetohydrodynamics equations (MHD). This system contains six different partial differential equations: three equations, governing the fluid flow, which are the Navier Stokes equations and three more representing the electromagnetic effects. But to avoid having to use these tedious equations we can take the presented topological approach to obtain a minimum much more efficiently.

Chapter 6: Helicity of Differential Forms

6.1 Configuration Spaces

Before we begin to discuss the helicity of differential forms, we will define what a configuration space is.

Definition 6.1. To define what we mean by a configuration space, first we allow for X to be a topological space. For example X could be $\mathbb{R}$, I, a torus, etc. Then the configuration space $C_n(x)$ is the set of all ordered n-tuples $(x_1, x_2, ..., x_n)$ of n distinct points in X. Then we define the configuration space,

$$C_n(X) = (X \times X \times ... \times X) - \Delta,$$

where, $\Delta = \{(x_1, x_2, \cdots, x_n) | x_i = x_j \text{ for some } i \neq j\}$, i.e., the set of all subdiagonals.

A more physical interpretation of a configuration space envisions the points, $\{x_1, .., x_n\}$ as robots, letting X be the track on which they can move. Clearly two robots cannot be in the same place, preventing the situation where $x_i = x_j$ if $i \neq j$. Then the configuration space will represent all the possible positions of the robots, where the connected components represent positions in which the robots can move to. An important requirement is that the robots remain attached to the track, in other words we do not allow for the robots to "jump" each other.

Now we will give some examples of configuration spaces.

Example 6.2. Let I be the unit interval; we will find $C_2(I)$. Let A and B be our two points. Then we have two possible options: either A lies to the left of B, or it lies to the right of B. Since I is a closed interval, we cannot switch the orientation of A relative to B, so we will get two connected components in our configuration space. To find these two connected components we first realize that if we did not remove Δ we would get the unit square. This is because both A and B can have values in $[0, 1]$. But, we have to remove the main diagonal as that is where $A = B$. So our configuration space will be two triangles with corners $(0, 0), (0, 1), (1, 1)$ and $(0, 0), (1, 0), (1, 1)$ with closed outside edges and the shared edge an open edge.

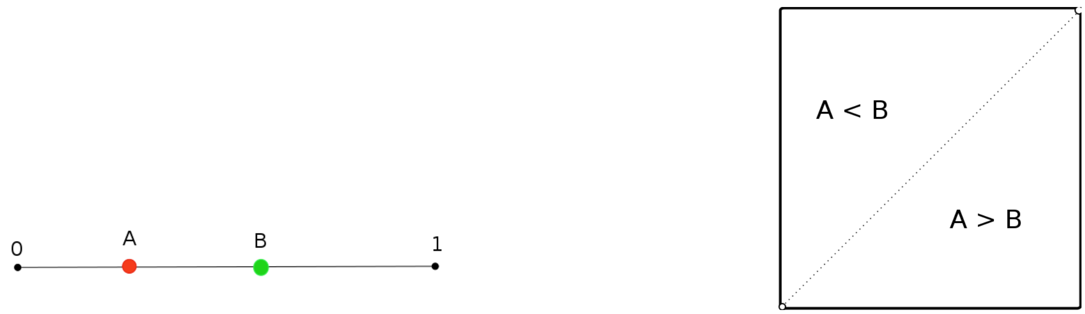

Figure 6.1: The configuration space of $C_2(I)$ has 2 disjoint regions. The union is a square, with the diagonal removed. We see on the interval here that A is to the left of B so we will be in the upper left triangle.

Example 6.3. For the second example we will find $C_2(S^1)$. An easy way to find this configuration space is to notice that S^1 is the unit interval with the endpoints identified. What this means that if a robot B starts to the right of robot A we can move robot B through the right end point to bring it to the left of robot A. Since we still only have two points, along the "unit interval" we will get a square, like above with the diagonal being unobtainable.

The diagram below shows the configuration space, with the left and right edges identified and the top and bottom identified. If the diagonal were still present, this gives us a torus. However, by removing the diagonal we can open up the torus to see a space homeomorphic to the annulus.

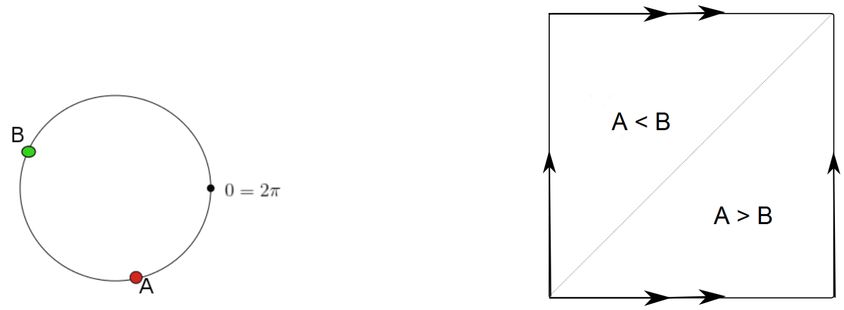

Figure 6.2: This is the configuration space $C_2(S^1)$. Since opposing sides are identified we will get a space homeomorphic to an annulus. On the circle we see that we can switch the relative orientation of A and B.

Example 6.4. One more example we will do is $C_3(S^1)$. This is the configuration space of 3 points (or robots) on a circle. Before we draw a diagram we will think about how many connected components we must have. To do this we proceed as in the previous example and view the circle as the unit interval with 0 and 1 identified. We notice that in the case where the endpoints are not identified there are 6 ways to line up the points, 123, 132, 213, 231, 312, 321. We realize that since these are on S^1 that they are cyclically permutable. In other words we can move from 123 to 312 to 231 to 123, similarly with 132, 213, 321. Another point of interest here is that the two possible orderings up to cyclic permutation are the 3-cycles of S_3. Unsurprisingly the connected components of $C_n(S^1)$, for any n, can be represented by the distinct n-cycles of S_n. More explicitly, an n-cycle and all of its representations will give all of the possible orderings of $\{x_1, ..., x_n\}$.

We should also note that the diagram for the configuration space without removing Δ will be a cube with identified faces. This is because each of the three points can be anywhere in $[0, 1]$ when we think of S^1 as an interval with the endpoints identified. Then we see that there are three dimensions, one for each point or robot. But unlike the previous two examples where Δ is just the main diagonal, we will have to remove multiple planes from this cube. In particular we will have to remove the planes, $x = y$,

$x = z$, and $y = z$, where x, y, z represent the three points. Then when we remove the three planes we will end up with six tetrahedron, as shown in figure 6.4. In the representation of the configuration space below it is important to realize that each of the outer faces (in different shades of purple) are identified with the opposite face. For example the point $(1, \frac{1}{4}, \frac{1}{4})$ is identified with $(0, \frac{1}{4}, \frac{1}{4})$. Then the red faces represent the removed planes, or where Δ lies.

Looking at the diagram above, we realize that each tetrahedron has two faces

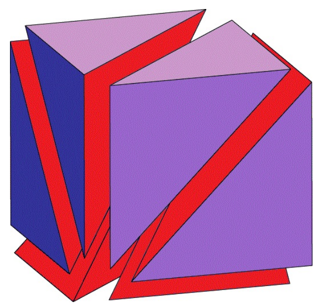

Figure 6.3: Modified from David Eppstein's, *How Many Tetrahedra?* image [1]. $CS^3(S^1)$ will be the following split of the cube into tetrahedra. The red faces represent the diagonal, Δ, and the purple faces are contained in the configuration space.

which are identified with the face of another tetrahedron and two faces which are part of Δ. We also notice that the tetrahedra occur in two sets of three due to cycle orderings. In other words if we label them A, B, C, D, E, F, and if A is identified with B and C then B is identified with A and C, and C is identified with A and B. So then the question becomes what these two spaces are.

To picture our space, imagine taking three of the identified tetrahedra away from the others, while maintaining their orientation. Then choose two identified faces and attach them, let us say the top and bottom. Then attach the third tetrahedron to one of the two open faces. What remains are two identified faces on opposite sides of this object. By connecting the last two identified faces we find an open solid torus. The torus will lack its closure, Δ. We also can confirm that our union of tetrahedra is a torus, because if we draw a loop through all three tetrahedra it will not continuously deform to a point. This tells us we do not have a sphere. We also see that it has genus one, since any single loop gives us a unique generator, making $\pi_1 = \mathbb{Z}$.

6.2 The Fulton-Macpherson Compactification

Next we will need to introduce the Fulton-Macpherson compactification of configuration spaces to talk about helicity with respect to forms. We will want to consider $C_2(\Omega)$ in order to take the helicity of Ω with respect to forms, because as with standard helicity we need to consider two different points in Ω. These two different points will be our forms evaluated at x and y in Ω. Since we have two moving points we will get a configuration space of two points. When we consider $C_2(\Omega)$ we notice that it will not contain its boundary at Δ, which means that $C_2(\Omega)$ will not be compact. But as with standard helicity, we will be taking an integral, which requires a compact region. Hence, we need a way to compactify $C_2(\Omega)$, which is why we will explore the Fulton-Macpherson Compactification.

The general idea of the Fulton-Macpherson compactification is to keep track of the directions and relative rates of approach as configuration points come together. In other words, for any point in Δ where nodes i and j coalesce, we keep track of the limit of the direction vector between i and j as they merge and the "speed" at which they moved together. This way, given a path or higher dimension analog in $C_n(\Omega)$, we can define what happens when we touch Δ based on the trajectory of the path.

We follow the approach of Sinha [10] for defining the Fulton-Macpherson compactification. To start we will introduce some notation; let $\begin{bmatrix} n \\ k \end{bmatrix}$ be the set of ordered k-tuples chosen from a set of n elements. Now we define two maps. Let x_l be a point in $C_n(M)$, where M is an m dimensional domain . The first map is a map from two elements in M to S^{m-1}. Let $(i,j) \in \left[\begin{array}{c} n \\ 2 \end{array} \right]$, then we define $\pi_{ij} : C_n(M) \to S^{m-1}$ to be the map which sends x_l to the unit vector in the direction of $x_i - x_j$. The other map we will define will be for $(i,j,k) \in \left[\begin{array}{c} n \\ 3 \end{array} \right]$, let $s_{ijk} : C_n(M) \to [0,\infty]$ be the map which sends x_l to $\frac{x_i - x_j}{|x_i - x_k|}$.

Let $A_n[M]$ be the product,

$$A_n[M] = (M)^n \times (S^{m-1})^{\#\left[\begin{array}{c} n \\ 2 \end{array} \right]} \times [0,\infty]^{\#\left[\begin{array}{c} n \\ 3 \end{array} \right]}$$

Definition 6.5. The **Fulton-Macpherson compactification** $C_n[M]$ to be the closure of $C_n(M)$ under the image of the map $\gamma_n = \gamma \times (\pi_{ij}|_{C_n(M)}) \times ((s_{ijk})|_{C_n(M)}) : C_n(M) \to A_n[M]$.

We notice that this map, γ_n gives us a product with three terms. The first term of the product will be the point in Ω where are points are coming together. So this will effectively tell us where on the missing boundary we will be. Then the second term of the product gives us vectors telling us the relative direction between the points coming together at γ. So this is the direction component of the Fulton-Macpherson compactification. Then the final term of the product will give us the rate at which points relatively coalesce. It does this by comparing the distance between a point x_i and x_j and x_i and x_k. This tells us that rate at which the points are coming together in comparison to the other points. For the second and third terms we should think of these as limits, so the directional vector and rate will be the limits of these as the points come together. When we lie in $C_n(M)$ we will have a value for the first component, as the other terms are unnecessary. This is because at every point in $C_n(M)$, all of the x_i are distinct.

For our purposes we need only consider $C_2[\Omega]$. This is because when we compute helicity we look at the configuration space of 2 points over our manifold with boundary, Ω. When $n = 2$ the above product $A_n[\Omega]$ will only contain the first and second terms. We find this to be true as we no longer need to worry about how a second and third point relate in distance. The only thing that will remain at the added boundary points is how two paths approach γ, the position in Ω where the points will coalesce.

We briefly define a unit normal bundle. A **unit normal bundle** of a manifold Ω is the pair (x, n), where $x \in \Omega$ and n is a unit vector in the normal space at x. The **normal space** is the space $N \times \Omega$ which is perpendicular to Ω at x. Then our product $A_n[M]$ will be a unit normal bundle. We will provide a few examples for when $n = 2$ to better demonstrate exactly what is happening.

Example 6.6. For the first example we will consider the case where $\Omega = I$. Then we know that our configuration space $C_2(\Omega)$ is going to be $I \times I - \Delta$ where Δ is the main diagonal of the square, i.e., where $x = y$. To compactify this we need to find the unit normal bundle to Δ at a given point $\delta \in \Delta$.

We notice that Δ is homeomorphic to I so the normal space at any $\delta \in \Delta$ will be of dimension 1. But we also notice that a normal vector $\mathbf{v}$ at any point $\gamma \in \Delta$ can either face in a positive or a negative direction. In particular think about the second component of γ_2; it is

$$\frac{y - x}{|y - x|}.$$

Then when we consider starting at a point $(x, y) \in C_2(I)$ we can approach $\delta \in \Delta$ from either the connected component where $y > x$ or the connected component where $x > y$. Then we notice that in the formula when $y > x$ we get a unit normal vector in the direction of the $+$, an example is given by the vector to the $+$ line. Otherwise we get a unit normal vector towards the $-$ line in the diagram below. In both cases we envision the unit vector as lying normal to Δ. Now we see that depending on where our path comes from in $C_2(I)$ into a point δ on Δ, we find the labeling to be $(\delta, \mathbf{v})$ where $\mathbf{v}$ is the normal vector.

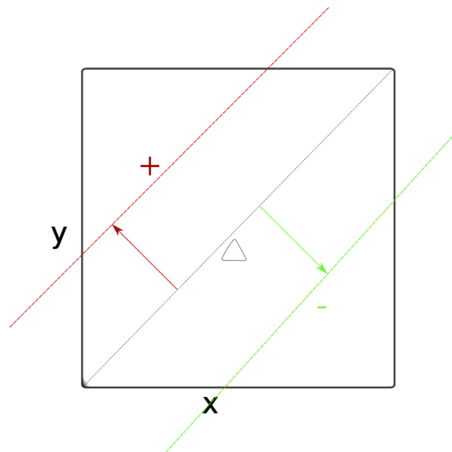

Figure 6.4: This is the Fulton-Macpherson compactification of $C_2(I)$. For each $\delta \in \Delta$, we add points $\delta_\pm$ to form $C_2[I]$, where δ gives the position along Δ, which is the base for the vector which hits either the $+$ or $-$ lines

Example 6.7. Another example to consider is when Ω is a solid torus. In this case Δ of $C_2(\Omega)$ will be isomorphic to a torus. Then the tangent space at any point, $\gamma \in \Omega$, is isomorphic to $\mathbb{R}^3$. Then as our points y and x merge the limit of $\frac{y-x}{|y-x|}$ will approach a normal vector in $\mathbb{R}^3$. Therefore, we will replace Δ with (γ, v), where v is the limit of $\frac{y-x}{|y-x|}$ as x and y converge to γ. Then v contains information on the paths which x and y coalesce along by retaining the limit of their difference vector. Now when x and y merge, instead of finding an undefined value, we get (γ, v) from the unit normal tangent bundle, thus compactifying $C_2(\Omega)$, resulting in $C_2[\Omega]$.

6.3 Bott-Taubes Integration

In this section we will present a way to compute helicity using differential forms. We present this now with standard helicity, as it is the method we base our technique for submanifold helicity later in the paper. It also makes sense that we can convert from the standard vector field formula for the helicity operator into a formula based upon differential forms, given the duality between vector fields and forms. In particular we can use the fundamental corresponce (Figure 2.3) to relate vector fields and differential forms in a convenient manner.

To start talking about helicity with respect to differential forms we will introduce Bott-Taubes integration [11]. Bott-Taubes integration provides us an invariant of knots under isotopy. So to start we will begin by defining $\mathcal{K}$, the configuration space of all possible embeddings of knots in $\mathbb{R}^3$. Equivalently $\mathcal{K}$ is the space of mappings of S^1 into $\mathbb{R}^3$. As we might expect we find that the connected components of $\mathcal{K}$ are the knot classes under isotopy. In other words, as long as we can move between two embeddings of S^1 in $\mathbb{R}^3$ through isotopic moves, they are in the same connected component of $\mathcal{K}$. Given our definition of a configuration space (Definition 6.1) this is the configuration space of S^1 in $\mathbb{R}^3$. Ultimately this tells us that the diagonal for this configuration space will be when our mapping of S^1 contains any singularities.

Figure 6.5 gives an example of how $\mathcal{K}$ separates the different knot classes. On the left we have the unknot and on the right we have a trefoil knot. In between, we show that a singularity arises in $\mathbb{R}^3$ to move between the two knots. Therefore the middle embedding will not be in $\mathcal{K}$, as there exists points x_i, x_j on S^1 such that $x_i = x_j$.

Now we want to describe the Bott-Taubes integral and explain how it will give us an invariant on the knot classes. The Bott-Taubes approach utilizes the following

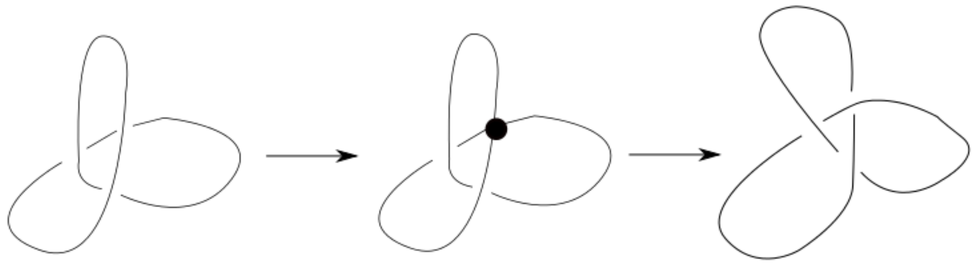

Figure 6.5: This shows that the unknot and trefoil will not be in the same connected component of $\mathcal{K}$. This is because two strands must cross resulting in the singularity shown by the dot.

diagram,

$$\begin{array}{ccc} C_2[S^1] \times \mathcal{K} & \xrightarrow{g} & \frac{x_i - x_j}{|x_i - x_j|} \in S^2 \\ \downarrow & & \\ \mathcal{K} & & \end{array}$$

In the above diagram we introduce the Gauss map as g. The Gauss map here takes the difference in two vectors, x_i, x_j, in the product space of $C_2[S^1] \times \mathcal{K}$, i.e., the compactified configuration space of S^1 and the knot configuration space. Then by transforming this to a unit vector we get a vector on S^2.

6.4 De Rham Cohomology

Before moving into the helicity of forms we will take an interlude into de Rham cohomology. We want to consider de Rham cohomology because when we look at the helicity of forms, it will be in a way looking at the helicity of cohomology classes. To start this exploration we will introduce some basic ideas in linear algebra and homological algebra.

First we will review a few concepts from linear algebra which are useful to know when looking at homological algebra. To start define a vector space V to be the direct sum of subspaces U and W, denoted $V = U \oplus W$ if every $v \in V$ can be written as the sum of a $u \in U$ and $w \in W$, i.e., $u + w = v$, and if U and V are perpendicular as spaces. The direct sum inherits properties, such as basis and dimension from U and W when $V = U \oplus W$, to find a basis for V, the dimension of V and the complement of U in V.

Another important linear algebra idea is that of the *quotient space*. To define the *quotient space* we first define an equivalence relation over V. Let V be a vector space and U be a subspace of V. Then define $\sim$ to be the equivalence relation on V such that $v \sim v'$ if $v - v' \in U$, and let $[v]$ represent the equivalence class of v. Then we define the *quotient* to be the vector space, $V/U := \{[v] | v \in V\}$ with linear vector space operations. In other words the *quotient space* removes the u component of any $v \in V$. In otherwords we reduce every element $v \in V$ to classes based on its respective w, if $V = U \oplus W$.

Now we will present a concept taking ideas from linear algebra to introduce a concept from homological algebra. Define $T : V \to W$ to be a linear transformation. Then we define the kernel and image as one would expect for a linear transformation.

Definition 6.8. If $f : U \to V$ and $g : V \to W$ are linear transformations then the sequence $U \xrightarrow{f} V \xrightarrow{g} W$ is **exact** at V if $\text{Ker}(g) = \text{Im}(f)$. We say a sequence of vector spaces and maps between them is an **exact sequence** if it is exact at every vector space in the sequence. Then a **short exact sequence** is an exact sequence which looks like the following,

$$0 \to U \to V \to W \to 0.$$

Next we define a *cochain complex*, V^*, as a sequence of vector spaces and maps,

$$V^* = \cdots V^{i-1} \to V^i \to V^{i+1} \cdots ,$$

where the map $\delta^i : V^i \to V^{i+1}$. This sequence has the property that $\delta^i \delta^{i-1} = 0$, in other words we find that $V^{i-1} \to V^{i+1}$ is the zero map.

Definition 6.9. Let $Z^i = \text{Ker } \delta^i$ and B^i to be $\text{Im } \delta^{i-1}$. Then we define H^i to be a quotient vector space defined by Z^i/B^i. Then H^i is the i^{th} **cohomology group** of V^* and its elements are called **cohomology classes**.

Now we will begin to look at de Rham cohomology, which means considering cohomology on manifolds. In particular this will give us a way to consider forms over a manifold through an algebra. We will let M be a smooth manifold with boundary, and $\Omega^i(M)$ be the space of smooth i-forms on M. Then from our section on forms earlier in the paper we consider the differentiation operation d, and notice that $d : \Omega^i(M) \to \Omega^{i+1}(M)$ for all i. When $i < 0$ or $i \geq n$, where n is the dimension of M, we find that $\Omega^i(M) = 0$. Also when these values of i hold true the differentiation operator is just the zero operator. This is because nontrivial forms of negative degree and forms of degree greater than n cannot occur on the manifold M.

Then having defined M and $\Omega^i(M)$ as above we get that $\Omega^*(M) = \{\Omega^i(M), d\}$ is a cochain complex. This is a consequence of our work from the section on forms, in knowing that $d^2 = 0$. Now that we have given a definition to a cochain complex for forms over a manifold we are ready to define de Rahm cohomology.

Definition 6.10. Let M be a smooth manifold with boundary. The i^{th} **de Rham cohomology group** $H^i_{dR}(M)$ is the group $H^i_{dR}(M) = H^i(\Omega^*(M))$.

To make sense of this definition we consider what Z^i and B^i will be. In the general definition of cohomology Z^i is the kernel of the i^{th} map. When we think of a co-chain of forms, the mapping is the derivative operator, so Z^i is the collection of forms whose derivative is zero, in other words the closed i-forms. Meanwhile B^i is the image of the $i-1$ map in the normal definition of cohomology. Then for de Rham cohomology this will be the i-forms which are the derivative of an $i-1$ form, i.e. the exact i-forms on M. Then $H^i_{dR}(M)$ is defined by the quotient of the closed i-forms over the exact

i-forms. Further we notice that $H^i_{dR}(M) = 0$ if and only if every closed i-form on M is exact.

Given two de Rham cohomology groups we are able to define the cup product,

Definition 6.11. The **cup product** in the above definition is defined as follows [2]. Let $H^k_{dR}(\Omega)$ and $H^l_{dR}(\Omega)$ be the k^{th} and l^{th} deRahm cohomology groups over Ω. Then the cup product is a map,

$$\cup : H^k_{dR}(\Omega) \times H^l_{dR}(\Omega) \to H^{k+l}_{dR}(\Omega)$$

defined by

$$[\phi] \cup [\psi] = [\phi \wedge \psi],$$

for $\phi \in H^k_{dR}(\Omega)$ and $\psi \in H^l_{dR}(\Omega)$.

Next we will define *relative de Rham cohomology*, but first we will define a smooth pair. A *smooth pair* (M, A) consists of a smooth manifold with boundary M and a smooth submanifold A with boundary of M, where A is closed as a subset of M. Now we need to build up an understanding of the map that creates our cohomology groups. If (M, A) is a smooth pair, then the inclusion $i : A \to M$ is a smooth map. This is a result of A being contained within M. Then our inclusion i will induce the map $i^* : \Omega^*(M) \to \Omega^*(A)$. This map is restriction of the forms on M to the forms on A. Then we define $\Omega^*(M, A)$ to be the kernel of this restriction map i^*. Hence $\Omega^*(M, A)$ gives us the forms which lie on M but are 0 when restricted to A. Then we call $\Omega^*(M, A)$ the complex of differential forms on M relative to A.

Let α be a nontrivial differential form contained in $\Omega^*(M, A)$. This tells us that $\alpha \neq 0$ on some of M but will be equal to zero on A. Then we consider $d\alpha$. By our previous work on differential forms we know that if $\alpha = 0$ on A then $d\alpha = 0$ on A also. Therefore the only place $d\alpha$ could be nonzero is on M outside of A. Hence, we see that $d\alpha \in \Omega^*(M, A)$. Then since $d\alpha$ will also be in $\Omega^*(M, A)$ we get a cochain complex,

$\{\Omega^i(M, A), d\}$. Then using the same definition for de Rham cohomology above we can define relative de Rham cohomology by replacing $\Omega^i(M)$ with $\Omega^i(M, A)$.

Definition 6.12. Letting Z^i and B^i retain their meaning, the i^{th} **relative de Rham cohomology group** of (M, A) will be $H^i_{dR}(M, A) = Z^i(M, A)/B^i(M, A)$.

Now we will provide an example of de Rham cohomology.

Example 6.13. For this example let our manifold be defined as follows $M = \{p\}$ where p is a single point. If we consider the possible forms over p we notice that $\Omega^0(p) \cong \mathbb{R}$ since the zero form can be any constant. But it cannot be anything more since p is a zero-dimensional manifold. Then this also tells us that for all $i \neq 0$, $\Omega^i(p) = 0$. Then $H^0_{dR}(p) = \mathbb{R}$ and 0 for all other i. Then by using a lemma which states that if two manifolds have a smooth homotopy equivalence, then their cohomology groups are isomorphic, we get that any smoothly contractible manifold M will share de Rham cohomology groups with a point. Even more generally since the closed zero forms are always the constant functions, anything else will have a non-zero deriviative, then for any connected, non empty manifold $H^0_{dR}(M) \cong \mathbb{R}$.

6.5 Helicity of Forms

Now we will introduce helicity as a function on differential forms. We are motivated by Bott-Taubes integration to construct helicity in terms of integrating differential forms on a larger space. To do this we will want to find a $(k + 1)$-form, α, using the Fundamental Correspondence (Theorem2.3), for which the integral of α over an appropriate configuration space will give us helicity. To begin we let $\Omega \subset \mathbb{R}^{2k+1}$ be a compact subdomain with piecewise smooth boundary and we let α be a closed Dirichlet $k + 1$-form on Ω. We define a form α to be **Dirichlet**, if $\alpha = 0$ on $\partial\Omega$.

To define helicity of forms we need to establish properties of α and the Gauss map we will be constructing.

Lemma 6.14. *The closed $(k+1)$-form α on Ω pulls back to a pair of closed $(k+1)$-forms α_x and α_y on $C_2[\Omega]$. Then the wedge product, $\alpha_x \wedge \alpha_y$ is a closed $(2k+2)$-form on $C_2[\Omega]$.*

Proof. We construct a surjective map $p : C_2[\Omega] \to \Omega \times \Omega$ as follows. We send points in $C_2(\Omega)$ to themselves, while a point (δ, v) not in $C_2(\Omega)$, we map to $\delta \in \Omega \times \Omega$. Then we can compose it with the projections $\Omega \times \Omega \to \Omega$, to obtain a pair of maps, π_x, π_y from $C_2[\Omega] \to \Omega$. Then if we take (x, y) in $C_2[\Omega]$, which can be seen as a pair of vectors each in Ω, then $\pi_x : (x, y) \mapsto x$and $\pi_y : (x, y) \mapsto y$. Then we define α_x and α_y as the pullback of α over $\pi_x \circ p$ and $\pi_y \circ p$ respectively, i.e., $\alpha_x = p^*(\pi_x^*(\alpha))$ and $\alpha_y = p^*(\pi_y^*(\alpha))$. Then, since we proved earlier that exterior differentiation and pullbacks commute (Proposition 2.12), we get that α_x, α_y, and $\alpha_x \wedge \alpha_y$ are all closed as α is. $\square$

We still need to consider what will happen to $\alpha_x \wedge \alpha_y$ on the boundary of $C_2[\Omega]$. In particular we will find that if α is a Dirichlet form on Ω, then $\alpha_x \wedge \alpha_y$ will be a Dirichlet form on $C_2[\Omega]$. For the boundaries defined by $\partial\Omega \times \Omega - \Delta$ and $\Omega \times \partial\Omega - \Delta$, this arises from the fact that α is 0 on $\partial\Omega$ and therefore the pullback $\alpha_x \wedge \alpha_y$ will also be 0 on a $\partial\Omega$. The argument for Δ is done using a special form of midpoint offset coordinates to show that only $2k + 1$ elementary forms in midpoint variables are in $\alpha_x \wedge \alpha_y$, which is a $(2k + 2)$-form. Therefore there must be a repeated element dx and thus the form will be zero. Hence on $\partial C_2[\Omega]$, $\alpha_x \wedge \alpha_y$ will be 0.

Definition 6.15. The Gauss map $g : C_2(\Omega) \to S^{2k}$ is given by $(x, y) \mapsto \frac{(y-x)}{|y-x|}$.

Lemma 6.16. *The Gauss map is a smooth map defined on all of $C_2[\Omega]$, including the boundary. The pullback of the unit volume form on S^{2k} by g defines a closed $2k$-form g^*dvol on $C_2[\Omega]$.*

Proof. The Gauss map extends naturally to $\Omega \times \partial\Omega$ and $\partial\Omega \times \Omega$. At $(x, y) \in C_2(\Omega)$, wher one lies on $\partial\Omega$, this is because if $x \neq y$ we can still naturally define a vector from x to y. But for $\Delta \subset C_2[\Omega]$ this is not the case. If $(x, v) \in \Delta$ then x is just an expansion of the diagonal of $\Omega \times \Omega$, so we get that the map of π_{12} smoothly extends to the boundary. In other words we can use the map π_{12} from the Fulton-Macpherson compactification (Definition 6.5) to define g by defining $g = v$. This gives rise to the need of $C_2[\Omega]$ instead of $\Omega \times \Omega$. $\square$

We can now use the work above to redefine helicity. Since α is a closed Dirichlet form and hence $\alpha_x \wedge \alpha_y$ is a Dirichlet form on $C_2[\Omega]$, then we know $\alpha_x \wedge \alpha_y$ is represented by a relative cohomology class. In particular, $[\alpha_x \wedge \alpha_y]$ represents a relative deRham cohomology class in $H^{2k+2}(C_2[\Omega], \partial C_2[\Omega]; \mathbb{R})$. Similarly, the pullback of the volume form by g is closed and therefore a de Rham cohomology class $[g^*dvol]$ in $H^{2k}(C_2[\Omega]; \mathbb{R})$. Using these cohomology classes we can define helicity in terms of forms.

Definition 6.17. We let $[dvol_{C_2[\Omega]}] \in H^{4k+2}(C_2[\Omega], \partial C_2[\Omega]) \simeq \mathbb{R}$ be the top-dimensional cohomology class of $C_2[\Omega]$ defined by the standard volume form. Then the cup product $[\alpha_x \wedge \alpha_y] \cup [g^*dvol_{S^{2k}}]$ is in $H^{4k+2}(C_2[\Omega])$ and is hence a multiple of $[dvol_{C_2[\Omega]}]$. Then we can define **helicity** $H(\alpha)$ by,

$$[\alpha_x \wedge \alpha_y] \cup [g^*dvol] = \frac{H(\alpha)}{vol(\Omega)^2}[dvol_{C_2[\Omega]}].$$

The more useful formula for actually computing helicity, and what we will focus on moving forward, is as follows,

$$H(\alpha) = \int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* dvol_{S^{2k}}.$$

Going forward we will denote the integrand above, $\Phi = \alpha_x \wedge \alpha_y \wedge g^* dvol_{S^{2k}}$.

Now we will argue the equality of the above definition of helicity and our original definition, $H = \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} dvol_x dvol_y$.

Lemma 6.18. *Let Ω be a compact subdomain of $\mathbb{R}^3$ with smooth boundary, and let α be a closed Dirichlet 2-form on Ω. Let V be the vector field dual to α. Then using the natural embedding of $\eta : C_2(\Omega) \to \Omega \times \Omega$, the above integrand Φ is equal to the pullback via η of the classical helicity integrand*

$$\left(\frac{1}{4\pi}\right) V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} dvol_x dvol_y.$$

As we said previously we can get a dual vector field for α by the fundamental correspondence 2.3. Also the map η just takes a point in $C_2(\Omega)$ to the equivalent point in $\Omega \times \Omega$, this is because $C_2(\Omega)$ is actually just $\Omega \times \Omega$ with Δ removed. We take the pullback of the classical helicity integrand because Φ is defined on $C_2[\Omega]$ and not $\Omega \times \Omega$. Also we have shown that the regular helicity integrand is zero at Δ and will have negligible contribution to the equality. Then using this lemma we are able to show that our two forms of helicity are equivalent, giving rise to the following theorem.

Theorem 6.19. *For three-dimensional domains, with a vector field V dual to a 2-form α we get that,*

$$\int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* dvol_{S^2} = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(X) \times V(y) \cdot \frac{x-y}{|x-y|^3} dvol_x dvol_y.$$

Proof. For this proof we want to show that for a small neighborhood around around $C_2[\Omega]/C_2(\Omega)$ and Δ on $\Omega \times \Omega$ that both integrals will tend to zero. In other words we will be showing that we can take out small neighborhoods around these boundaries in both integrals. This will ultimately leave us with an integral on $C_2(\Omega)$ and $\Omega \times \Omega / N_\delta(\Delta)$, where $N_\delta(\Delta)$ is a δ-neighborhood of Δ, which then by the lemma above we see that the two integrals will actually be the pullback of each other via η.

Denote $\Psi = \frac{1}{4\pi}\int_{\Omega\times\Omega} V(X) \times V(y) \cdot \frac{x-y}{|x-y|^3} dvol_x dvol_y$. Then let U_ϵ be an ϵ-neighborhood of $\partial C_2[\Omega]$. Then,

$$\int_{C_2[\Omega]} \Phi = \int_{C_2[\Omega]-U_\epsilon} \Phi + \int_{U_\epsilon} \Phi.$$

This follows from properties of integrals, we know that we can split the domain and then sum the integrals to get the integral over the whole space. Then as $\epsilon \to 0$ we see that $\int_{U_\epsilon} \Phi \to 0$. This is because the volume of U_ϵ is going to zero, and that Φ integrated over $C_2[\Omega]$ will be finite. Then by the lemma above we get that $\Phi = \eta^*\Psi$ on $C_2[\Omega] - U_\epsilon$. Then we see that

$$\int_{C_2[\Omega]-U_\epsilon} \Phi = \int_{C_2[\Omega]-U_\epsilon} \eta^*\Psi = \int_{\eta(C_2[\Omega]-U_\epsilon)} \Psi.$$

We are able to change the integrand to Ψ because the domain becomes a subset of $C_2(\Omega)$ with U_ϵ removed. This is because we have taken the diagonal Δ out of the domain. Then the second equality it valid since the integral of the pullback of Ψ is the same as moving the domain along η and then taking the integral over Ψ. Then from above we know that the image of $\eta(C_2[\Omega]-U_\epsilon)$ is $\Omega\times\Omega$ with some neighborhood V_ϵ around Δ removed. Then as $\epsilon \to 0$, the set V_ϵ will approximate Δ. This is because as ϵ approaches zero, the neighborhood will shrink but will always contain Δ.

We know from before that the integral $\int_{\Omega\times\Omega}\Psi$ is improper along Δ. In our proof for the well-definedness of $\int_{\Omega\times\Omega}\Psi$, we showed that as we shrink the neighborhood around

Δ the contribution of these neighborhoods goes to zero. Therefore, $\int_{\eta(C_2(\Omega)-U_\epsilon)} \Psi$ converges to both the classical helicity integral and the integral of forms. Therefore the two expressions for helicity will be equal. $\square$

Next we will prove Lemma 6.18 above, since we used it for the proof of the theorem. In particular we assumed that the integrands were the same in the proof above and showed that the integrals on the relevant domains were the same.

Proof. By choosing coordinates on Ω we will find an induced set of coordinates on $C_2(\Omega)$. We accomplish this by taking the product of the coordinate system. At the point $p = (x, y) \in C_2(\Omega)$, we choose right-handed orthonormal coordinates $\{u_i\}$ such that u_3 points along the vector $y - x$ at p. By our construction of configuration spaces these coordinates will induce coordinates $\{x_i, y_i\}$ on $C_2(\Omega)$. We will now calculate Φ and Ψ in these coordinates at p. We begin by writing

$$V(x) = v_1 \frac{\partial}{\partial u_1} + v_2 \frac{\partial}{\partial u_2} + v_3 \frac{\partial}{\partial u_3}$$

and

$$V(y) = w_1 \frac{\partial}{\partial u_1} + w_2 \frac{\partial}{\partial u_2} + w_3 \frac{\partial}{\partial u_3}.$$

The v_i and w_i are components of these vector fields. This way when we convert these vector fields to forms via the Fundamental Correspondence (2.3) the vector fields will convert to 2-forms. By our construction of helicity of forms it will be necessary to have two-forms. The unit vectors will also cause the two-form for a given v_i to be the one with $du_j \wedge du_k$ where $j \neq i, k \neq i$, $\text{sign}(i, j, k) = 1$. Then we will get that,

$$\alpha_x = v_1 dx_2 \wedge dx_3 + v_2 dx_3 \wedge dx_1 + v_3 dx_1 \wedge dx_2$$

$$\alpha_y = w_1 dy_2 \wedge dy_3 + w_2 dy_3 \wedge dy_1 + w_3 dy_1 \wedge dy_2.$$

Since the v and w are functions in u_i we can naturally map them into $C_2(\Omega)$ to get the separate forms for x and y. Then by the choice of coordinates we get that $\frac{x-y}{|x-y|^3} = -\frac{1}{|x-y|^2}\frac{\partial}{\partial u_3}$. This arises as $\frac{\partial}{\partial u_3}$ will be the unit vector pointing along $y-x$, and therefore can be pulled out of the original fraction by adding a negative. Then we get that the classical helicity integrand Ψ is equal to

$$\frac{1}{4\pi}\frac{1}{|x-y|^2}(v_2w_1 - v_1w_2)dvol_x dvol_y.$$

The $(v_2w_1 - v_1w_2)$ term comes from the cross product and multiplying in the unit vector towards u_3, the other two terms in the cross product will go to zero as they already contain contribution from the vector towards u_3. Hence, the only components remaining are those with a unit vector in all three directions, u_1, u_2, u_3 giving three-dimensional volume.

Now we calculate Φ in these coordinates; we will first consider g^*dvol. We start by noticing that changing the x_3 or y_3 terms will not affect the Gauss map. This is because the u_3 term was defined by the vector $y-x$ and thus adjusting x_3, y_3 really just moves the points x, y closer together or further apart, it does not change the direction on the sphere. Since changing x_3,y_3 has no impact on the Gauss map we know that g^*dvol will have no dx_3, dy_3 terms. That is because these terms represent the impact change has on the Gauss map. But changing the terms, x_1, x_2, y_1, y_2 will move the vector along the sphere, giving g^*dvol as

$$A_1dx_1 \wedge dx_2 + A_2dy_1 \wedge dy_2 + A_3dx_1 \wedge dy_1 + A_4dx_2 \wedge dy_2 + A_5dx_1 \wedge dy_2 + A_6dy_1 \wedge dx_2.$$

Since g^*dvol represents the pullback of the area form on S^2 we need to know which forms span the area of S^2, i.e. which related vectors will give us all of S^2. We notice that $dx_1 \wedge dy_1$ and $dx_2 \wedge dy_2$ will not. This is because change on just the terms in these two forms will only allow a circle on S^2 since x_i and y_i represent movement in the same dimension. The other forms will give us all of the sphere since they allow for the pair of points to move in both the u_1 and u_2 directions. Then since this is a Gauss map we still must normalize the distance between x and y. Also we need to consider the fact that the fact that dvol here represents the unit area form on S^2, so we must divide by area$(S^2) = 1$. Then by considering orientations we get that, $A_1 = A_2 = 1/(4\pi|x-y|^2) = -A_5 = -A_6$. So,

$$g^* dvol = \frac{1}{4\pi|x-y|^2}(dx_1 \wedge dx_2 + dy_1 \wedge dy_2 - dx_1 \wedge dy_2 - dy_1 \wedge dx_2).$$

Now we compute the 6-form Φ under the wedge product to get, $c_5 v_1 w_2 - c_6 v_2 w_1$, which by substituting becomes,

$$\Phi = \frac{1}{4\pi}\frac{1}{|x-y|^2}(v_2 w_1 - v_1 w_2) dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3.$$

The forms at the end come from the wedge product as well, since we are wedging together three, 2-forms each one must contain two distinct dx_i, dy_i. Then we know that $dx_1 \wedge dx_2 \wedge dx_3$ and $dy_1 \wedge dy_2 \wedge dy_3$ are $dvol_x, dvol_y$ respectively on $C_2(\Omega)$; therefore we get that via the pullback on η, $\Phi = \eta^* \Psi$. $\square$

Now we have shown that we can in fact represent helicity using forms as

$$\int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* dvol.$$

When we defined this above we talked about how the form $\alpha_x \wedge \alpha_y$ represent a relative cohomology class and $g^* dvol$ represents an absolute cohomology class. Then using the differential forms definition of helicity we are able to describe some of the properties of helicity via cohomology. Another useful application of the differential forms definition of helicity is that we can use it to show that helicity is invariant under volume-preserving diffeomorphisms, which we will prove (Theorem 7.6) for submanifold helicity in the next chapter, but that same proof is similar to the one need here. Also helicity of forms allows us in the next chapter to define what submanifold helicity is.

Chapter 7: Submanifold Helicity

So far we have been describing "standard" helicity which is calculated on 3-manifolds in 3-space. A lot of what we have introduced can be easily expanded to top dimensional manifolds Ω in $\mathbb{R}^{2k+1}$. As with standard helicity, top-dimensional helicity in $\mathbb{R}^{2k+1}$ is an invariant on volume-preserving diffeomorphisms. For this section though we will be focused on *submanifold helicity*, which is the situation when Ω has smaller dimension than the ambient space.

Definition 7.1. Let α be a $(k+1)$-Dirichlet form on Ω, where Ω is an n-dimensional submanifold of $\mathbb{R}^m$. Then we define **submanifold helicity** $H_{n,m}$ for α on Ω by,

$$H_{n,m}(\phi) = \int_{C_2[\Omega]} \phi_x \wedge \phi_y \wedge g^* dvol_{S^{m-1}},$$

where $C_2[\Omega]$ is the Fulton-Macpherson compactification (6.5) of the configuration space of Ω.

For the integrand $\Phi = \alpha_x \wedge \alpha_y \wedge g^* dvol_{S^{m-1}}$ to be a $2n$-form, we require that $2k+2+m-1 = 2n$. We need Φ to be a $2n$-form because $C_2[\Omega]$ will be $2n$-dimensional. Therefore when we take an integral over $C_2[\Omega]$ the form in the integrand must be of equal dimension. Then we see that the equality $2k+2+m-1 = 2n$ will guarantee α_x, α_y have the appropriate dimension in m-space. This is because α_x and α_y are $(k+1)$-forms. Then the form $g^* dvol_{S^{m-1}}$ is a $(m-1)$-dimensional form, since it comes from the Gauss map onto a S^{m-1} sphere. When this requirement is met $H_{n,m}$ will be defined and is referred to as (n,m)-*helicity*.

Now we will give some examples of submanifold helicity.

Example 7.2. We consider $(1, 3)$ helicity, $H_{1,3}$. Using our equation $2k+2+m-1 = 2n$ we get $2k + 2 + 3 - 1 = 2$, telling us that $k = -1$. Then we will get 0-forms, on a 1-dimensional curve in $\mathbb{R}^3$. Then since we know that helicity measures how a flow wraps around itself, we notice that this will really be the writhing number of the curve. This is because the writhe of a curve measures how it crosses itself. Also this example demonstrates how submanifold helicity is not always an invariant because writhing number is not a diffeomorphism invariant. In particular we can move a curve by a diffeomorphism, while preserving length, but cause the writhe to change. In particular we could imagine twisting the curve.

Example 7.3. Next we consider $(2, 3)$-helicity, $H_{2,3}$. First we calculate for k to see that $2k + 2 + 3 - 1 = 4$, which gives $2k = 0$. So we know that α will be a 1-form. Then α is a 1-form, on a 2-dimensional manifold, i.e. a surface in $\mathbb{R}^3$. A 1-form in $\mathbb{R}^{\nVDash}$ is a vector field, and we are going to be counting how the vector field crosses itself on Ω, a surface. Then this is really going to be the linking number of our 1-form.

Now we will look some of the results of submanifold helicity from Cantarella and Parsley's *Submanifold Helicities* abstract. [12] In particular we consider the questions which naturally arise from our work with top dimensional helicity. The first being for what values of n and m will helicity be an invariant? Also we want to consider when $H_{n,m}$ will be nontrivial.

Example 7.4. One other example is that $H_{4k+3,4k+3}$ generalizes helicity to subdomains of $\mathbb{R}^{4k+3}$. The case when the ambient space is of dimension $4k + 3$ is the only situation where top-dimension helicity is nontrivial.

Lemma 7.5. *First we will prove a case for when helicity is an invariant. To start we need to show that for submanifold helicity, with $k + 1$ odd, $\alpha_x \wedge \alpha_y$ is Dirichlet on $C_2[\Omega]$ if α is Dirichlet on Ω.*

In chapter 6 we showed this for top-dimensional helicity of forms. We also remind the reader that Δ' is the boundary added by the Fulton-Machpherson compactification (Definition 6.5). But for the Δ' part of the boundary of $C_2[\Omega]$, we relied on the fact that $\alpha_x \wedge \alpha_y$ was a $2k+2$ form along Δ' but we only had $n = 2k+1$ elementary forms, dm_i to work with along Δ'. These dm_i represented the change in the $2n+1$ directions on manifold Ω, given by a coordinate system. Therefore in chapter 6 we found we must have $dm_i \wedge dm_i$ resulting in 0. But now we cannot guarantee that $2k+2$ will be greater than n, the dimension of Ω.

Proof. As with above on the boundaries of $C_2[\Omega]$, which are not Δ', we get that only one of α_x,α_y lie on a $\partial\Omega$. Then since α is Dirichlet, this will hold true for both α_x and α_y on the boundary of a copy of Ω. Therefore we are left with the Δ' boundary, that is the boundary added to compactify $C_2(\Omega)$.

Now we consider $\alpha_x \wedge \alpha_y$ on Δ'. On Δ' we notice that $\alpha_x \wedge \alpha_y$ is pulled back to $\alpha \wedge \alpha$. This is because Δ' is diffeomorphic to $\Omega \times S^{m-1}$. Then we are looking at the form on Ω which is just $\alpha \wedge \alpha$, hence why we pull it back. Then by transposing simple forms we find that $\alpha \wedge \alpha = -\alpha \wedge \alpha$. This is because each α is an odd form, so we will get $(k+1)(k+1)$ transpositions, which is odd, when we transpose α and α. Thus, since $\alpha \wedge \alpha = -\alpha \wedge \alpha$ (Lemma 2.2), we know that $\alpha \wedge \alpha = 0$ on Δ'. Therefore if α is Dirichlet on Ω, then $\alpha_x \wedge \alpha_y$ is Dirichlet on $C_2[\Omega]$. □

Proposition 7.6. *If $k+1 = n - \frac{m-1}{2}$ is odd, then $H_{n,m}$ is invariant under any volume preserving diffeomorphism that is homotopic to the identity.*

Proof. This proof relies on the fact that $\alpha_x \wedge \alpha_y$ is a Dirichlet form from the previous lemma, and therefore will also work to show that helicity is invariant for standard helicity.

Now we establish our homotopy. Let $f : \Omega \times I \to \mathbb{R}^{2k+1}$ be a smooth map. Then at each fixed t define $f_t : \Omega \to \Omega_t \subset \mathbb{R}^{2k+1}$, by $f_t(p) = f(p,t)$. Then we assume that each f_t is a diffeomorphism, where f_0 is the identity map. Then α_t will be the pull back of α along f_t^{-1} on each Ω_t, i.e. the image of the diffeomorphism at t. This pullback will give us a way to see where the form in Ω goes when mapped by f_t. In particular we are looking at what happens by the inverse of f_t, since we want to pull α back. This establishes the homotopy required by the hypothesis in the statement of the theorem. Then what we want to show is that $H_{n,m}(\alpha)$ on $\Omega_0 = \Omega$ is equal to $H(\alpha_1)$ on Ω_1, in other words that the helicity remains constant along the homotopy.

Now given the homotopy we constructed above we see that there exists a natural projection map from $\Omega \times I \to \Omega$, namely the map $(p,t) \mapsto p$. This removes the time variable, leaving us with our manifold Ω. Similarly we can get projections π_x, π_y from $C_2[\Omega] \times I \to \Omega \times I$. These maps will project onto the appropriate coordinates, i.e.,

$$\pi_x : (x,y,t) \mapsto (x,t),$$

$$\pi_y : (x,y,t) \mapsto (y,t).$$

Then if we pull back a closed form α in Ω along the composition of these projections we are able to define a form $\alpha_x \wedge \alpha_y$ in $C_2[\Omega] \times I$.

We also need to define an extended Gauss map on $C_2[\Omega] \times I$. We do this by defining

$$G(x,y,t) = \frac{f_t(x) - f_t(y)}{|f_t(x) - f_t(y)|}.$$

Then this Gauss map allows us to define a closed $(m-1)$-form $G^* dvol_{S^{m-1}}$ on $C_2[\Omega] \times I$. This will be an $(m-1)$-form by the dimension of our manifold. The Gauss map represents the unit vector in the direction of the vector between $f_t(x)$ and $f_t(y)$. Then the $(2k+2+m-1)$-form $\alpha_x \wedge \alpha_y \wedge G^* dvol_{S^{m-1}}$ is a closed Dirichlet form on $C_2[\Omega]$. This is a closed form as it is the wedge product of closed forms. It is a Dirichlet form by the proof of Lemma 7.5. Then by Stokes theorem, the integral of

$\alpha_x \wedge \alpha_y \wedge G^* dvol_{S^{m-1}}$ over $\partial(C_2[\Omega] \times I)$ is zero. This is because the exterior derivative of $\alpha_x \wedge \alpha_y \wedge G^* dvol_{S^{m-1}}$ will be 0, since Φ is a closed form.

We notice that the boundary of $C_2[\Omega] \times I$ will be

$$\partial(C_2[\Omega]) \times I \cup C_2[\Omega] \times \{0, 1\}.$$

But since $\alpha_x \wedge \alpha_y \wedge G^* dvol_{S^{m-1}}$ is Dirichlet it will be zero on the boundary of $C_2[\Omega]$ and thus zero over the first component of the union. Then we are left with the boundaries of the interval. Which over an integral will tell us that,

$$\int_{C_2[\Omega] \times \{1\}} \Phi = \int_{C_2[\Omega] \times \{0\}} \Phi$$

So now we want to show that the right-hand side will be the helicity, $H(\alpha_0)$ and the left-hand side $H(\alpha_1)$. In other words we are trying to show that these integrals give the helicity of our two diffeomorphic manifolds, Ω and Ω_1. Since f_0 is the identity map, then $G(x, y, 0) = g(x, y)$ so we get that the right hand side is $H(\alpha_0)$. Now we need to consider the left hand side.

First we observe that,

$$H(\alpha_1) = \int_{C_2[\Omega_1]} (\alpha_1)_x \wedge (\alpha_1)_y \wedge g^* dvol_{S^{m-1}} = \int_{C_2[\Omega_1]} (F^{-1})^* \alpha_x \wedge (F^{-1})^* \alpha_y \wedge g^* dvol_{S^{m-1}},$$

where $F : C_2[\Omega] \to C_2[\Omega_1]$ is the map of configuration spaces induced by f_1. We take the integral measuring helicity for $C_2[\Omega_1]$ and compute it by pulling back the forms α_x,α_y in $C_2[\Omega]$, by the inverse of the diffeomorphism. Now if we pull this integral back to $C_2[\Omega] \times \{1\}$ using F^{-1}, we get the following integral

$$H(\alpha_1) = \int_{F^{-1}(C_2[\Omega_1]) = C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge F^* g^* dvol_{S^{m-1}}$$

Then since F^* is the map $C_2[\Omega] \to C_2[\Omega_1]$ composed with g^*, it will be the pullback by G^* at $t = 1$. Then we see that

$$H(\alpha_1) = \alpha_x \wedge \alpha_y \wedge G^* dvol_{S^{m-1}}.$$

Thus, we have shown that $H(\alpha) = H(\alpha_1)$, meaning that helicity is invariant under a volume preserving diffeomorphism, that is homotopic to the identity map. $\square$

The above proof will also work for top dimensional helicity. This is because the pullbacks over the initial projections will still produce $\alpha_x \wedge \alpha_y$ as a closed Dirichlet form if α is closed and Dirichlet.

Next we will consider the second question above, in what instances will submanifold helicity be trivial. This is important because even if submanifold helicity is invariant in the case of $k+1$ being odd, it will not tell us much if it is always trivial.

Proposition 7.7. *If the ambient space has dimension* $4i+1$*, then* $H_{n,m}$ *is identically zero for any* $(k+1)$*-form* α*.*

Proof. We consider the automorphism ψ of $C_2(\Omega)$ which interchanges x and y. What ψ is doing by flipping x and y, changing which component of $C_2(\Omega)$ we lie in. This automorphism ψ can be extended to $C_2[\Omega]$. We can see this by flipping the direction of the vector in Δ given a point (x, y) in Δ. Then ψ will change the orientation of $C_2[\Omega]$ by a factor of $(-1)^n$.

Now, we take the pullback $\psi^*\Phi = \alpha_y \wedge \alpha_x \wedge \psi^* g^* dvol_{S^{m-1}}$. Since ψ interchanges x and y, we switch α_x and α_y via the pullback. Then ψ induces an antipodal map on S^{m-1}, because by switching x and y we change the orientation of our vector in the Gauss map. Then since m is odd, this map will have degree -1. Then we see that $\psi^* g^* dvol_{S^{m-1}} = -g^* dvol_{S^{m-1}}$. Also via the anti-commutativity of simple forms, $\alpha_y \wedge \alpha_x = (-1)^{(k+1)^2} \alpha_x \wedge \alpha_y$. Then if we combine these results we get that $\psi^*\Phi = (-1)^k \Phi$. Then we know that $\int_{C_2[\Omega]} \psi^*\Phi = \int_{\psi(C_2[\Omega])} \Phi$, since changing x and y still measure helicity, as helicity is symmetric. But by pulling out the constants above we see that this implies,

$$(-1)^k H_{n,m}(\alpha) = (-1)^n H_{n,m}(\alpha).$$

But then using the formula relating k, m, n we see that $k+1 = n - \frac{4i+1-1}{2} = n - 2i$. But then k and n must have opposite parity and thus $H_{n,m}(\alpha) = -H_{n,m}(\alpha)$. So $H_{n,m}(\alpha)$ must be 0. $\square$

Now we know what happens for $m = 4i+1$, but if we try and follow a similar proof for $m = 4i+3$ we will just get that $H_{n,m}(\alpha) = H_{n,m}(\alpha)$. So the question remains as to what happens when $m = 4i+3$. While we know when it will be an invariant, we are not sure of whether it is a useful invariant. In particular does it have nontrivial values for some manifold Ω, and some form α.

The next step would be to try and find some new examples for the $m = 4i+3$ case. While we know what will happen in the $H_{4i+3,4i+3}$ case, it is not as clear for the submanifold case. The hope is that there exists examples of submanifolds for which helicity will be nonzero. This is because a nonzero helicity will give us a useful invariant of diffeomorphic submanifolds in $\mathbb{R}^{4i+3}$.

Bibliography

[1] Eppstein, David. How many tetrahedra?, 2001. [http://www.ics.uci.edu/ eppstein/projects/tetra/; accessed January 20, 2018].

[2] Steven H. Weintraub. *Differential forms.* Academic Press, Inc., San Diego, CA, 1997. A complement to vector calculus.

[3] Steven H. Weintraub. *Differential forms.* Elsevier/Academic Press, Amsterdam, second edition, 2014. Theory and practice.

[4] Ruth Gornet. Maxwell's equations via differential forms. Park City Mathematics Institute, Undergraduate Summer School., 2001.

[5] Jason Cantarella, Dennis DeTurck, and Herman Gluck. The Biot-Savart operator for application to knot theory, fluid dynamics, and plasma physics. *J. Math. Phys.*, 42(2):876–905, 2001.

[6] Jason Cantarella and Jason Parsley. A new cohomological formula for helicity in $\mathbb{R}^{2k+1}$ reveals the effect of a diffeomorphism on helicity. *J. Geom. Phys.*, 60(9):1127–1155, 2010.

[7] Mitchell A. Berger. Topological quantities: calculating winding, writhing, linking, and higher order invariants. In *Lectures on topological fluid mechanics*, volume 1973 of *Lecture Notes in Math.*, pages 75–97. Springer, Berlin, 2009.

[8] V. I. Arnol'd and B. A. Khesin. Topological methods in hydrodynamics. In *Annual review of fluid mechanics, Vol. 24*, pages 145–166. Annual Reviews, Palo Alto, CA, 1992.

[9] Boris Khesin. Topological fluid dynamics. *Notices Amer. Math. Soc.*, 52(1):9–19, 2005.

[10] Dev P. Sinha. Manifold-theoretic compactifications of configuration spaces. *Selecta Math. (N.S.)*, 10(3):391–428, 2004.

[11] Ismar Volić. A survey of Bott-Taubes integration. *J. Knot Theory Ramifications*, 16(1):1–42, 2007.

[12] Jason Cantarella and Jason Parsley. Submanifold helicities. 2016. http://www.matapp.unimib.it/ ricca/IUTAM2016Venice/attachments/ Book-Abstracts.pdf.

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