CHOICE FUNCTIONS, DIGRAPHS, AND BALANCED ALLOCATIONS

BY

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Abstract

Brendan P. Lidral-Porter

Choice Functions, Digraphs, and Balanced Allocations

In this thesis, we consider broadly the concept of choice in a variety of settings, focusing on equity in selection. In particular, we introduce the concept of (pair-wise) domination graphs for hypergraphs endowed with a choice function on edges, and are interested, for instance, in minimal numbers of edges for associated domination graphs. Theorems regarding the existence of balanced (zero-edge) domination graphs are presented. In addition, we consider fairness for choice functions, from a sequential perspective, prove existence of fair choice, and consider connections with balanced allocation, path-connected directed graphs and cyclic preference. Several open questions are posed.

Chapter 1: Introduction

Choice plays an integral role in daily life, and manifests in a multitude of ways within various different fields. Just today you probably chose what to wear, where to go to eat, or perhaps which of a few different songs to listen to whilst working out. Indeed, how choices are made, the effects of these choices, and how they relate to the systems upon which they operate is an emerging field. In this thesis, interest lies especially in the representation of choice functions pictorially, with emphasis placed on equity and balance in selections (you don't want to wear the same outfit 10 days in a row).

1.1 Terms and Definitions

In theoretical considerations of choice, it will be helpful to precisely define some terms (whose definitions may differ somewhat from colloquial connotations). What follows should serve as a legend for understanding these terms and how they relate to the fields of study.

1.1.1 Hypertournaments and Hypergraphs

A **hypergraph** is a pair $\mathcal{H} = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ is a set of n vertices (sometimes called nodes) and $E = \{e_1, e_2, ..., e_m\}$ is a set of m nonempty subset of V called hyperedges or edges.

A **hypertournament** is a hypergraph wherein each edge $e \in E$ is endowed with an orientation. That means that if we have a hypergraph with 5 nodes, the edge $(1, 2, 3)$ would be distinct from the edge $(1, 3, 2)$. Though both edges involve the same vertices, the order (referred to as the *orientaion*) makes them distinct. One may think of these orientations as rankings, so $(1, 2, 3)$ would represent an ordering $1 \rightarrow 2 \rightarrow 3$ and $(1, 3, 2)$ would represent $1 \rightarrow 3 \rightarrow 2$. Along the edges, you must travel the vertices in the order of the orientation of that edge.

Similar to hypertournaments, a **k-hypertournament** is a complete k-hypergraph where again $|V| = n$, however complete denotes that E contains all edges of order k, and thus we have $|E| = \binom{n}{k}$. This is a generalization of the standard tournament model. Tournament here does not mean tournaments in the colloquial sense (e.g. March Madness or NFL playoffs), but rather a k-hypertournament with $k = 2$. That is, a **tournament** consists of n vertices where each edge contains 2 elements of V and thus, a tournament consists of $\binom{n}{2}$ many edges. Therefore, in order for March Madness to be a tournament in our framework, every pair of teams would face each other in a total of $\frac{64*63}{2} = 2016$ games (ignoring play-in games). For mathematical work related to tournaments, see for instance [1, 2, 3].

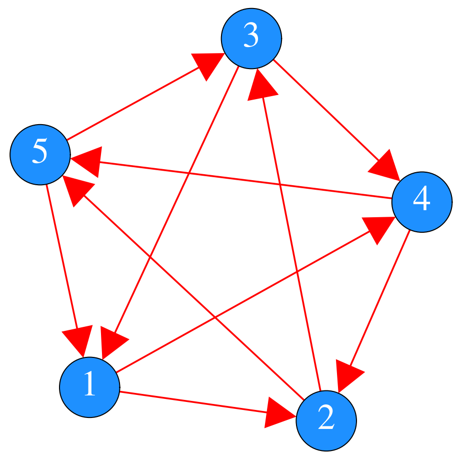
This thesis begins by primarily considering what we term **(n, k)-choice-hypergraphs**. We take a complete k-hypergraph, denoted H, and define a **choice function**, $C : E \rightarrow V$, on the edges of the hypergraph where for each $e \in E$, $C(e)$ "chooses" an element in the edge. Choice functions replace the full orientations found in hypertournaments, and thus gives rise to our term (n, k) choice-hypergraph. Specifically, the pair (H, C) is what we refer to when we use the term (n, k)-choice-hypergraph – recall that H itself is a tuple, $H = (V, E)$.

Within the context of (n, k)-choice-hypergraphs, as with hypertournaments, natural questions arise regarding which vertices (if any) are "dominant", or in some way "beat" other vertices. To do so we will consider various different conceptions of "domination". For example, for two vertices $v, w \in V$ of some (n, k)-choice-hypergraph, we may say that v "dominates" w if w is never chosen in edges where both v and w are present.

If we take the above conception of domination, we may represent an (n, k)-choice-hypergraph with a **directed graph**. A directed graph is simply a pair $G = (V, E)$, where each edge $e \in E$ is of size two and endowed with an orientation. Thus we may represent our (n, k)-choice-hypergraph with a directed edge from vertex v to vertex w if v "dominates" w. This leads to examples such as given in Figure 1.1. The Table in Figure 1.1a gives a $(5, 2)$-choice-hypergraph with Figure 1.1b showing an accompanying directed graph that gives dominance relationships.

	e	orientation	C(e)
1	{1, 2}	(1,2)	1
2	{1, 3}	(3,1)	3
3	{1, 4}	(1,4)	1
4	{1, 5}	(5,1)	5
5	{2, 3}	(2,3)	2
6	{2, 4}	(4,2)	4
7	{2, 5}	(2,5)	2
8	{3, 4}	(3,4)	3
9	{3, 5}	(5,3)	5
10	{4, 5}	(4,5)	4

(a)



(b)

Figure 1.1: An example of a standard tournament with five vertices (a), and an associated graphical display with a directed edge from vertex v to vertex w whenever vertex v is chosen in the presence of w (b).

Note that in the above table, we have given both an orientation and a choice function, C, which selects elements of edges. This is because, for standard tournaments, a 2-hypertournament and an $(n, 2)$-choice-hypergraph are equivalent; orienting an edge of order 2 is the same as selecting a single element. Note that C merely selects the first element of the oriented edge.

In Chapter 2 we consider three possibilities for domination. As made explicit therein, we settle on saying that a vertex v **dominates** another vertex w if the number of times v is chosen in the presence of w is greater than the number of times w is chosen in the presence of v. Taking this formulation of domination, we produce what we term (n, k)**-choice-domination graphs**, or (n, k)**-domination graphs**, which are directed graphs made according to vertex domination relationships.

The above definitions and example lead to some natural questions, such as:

- What constitutes domination?
- What domination graphs arise from various choice functions?
- How many isomorphism classes of resulting domination graphs are possible?

In the context of (n, k)-choice-hypergraphs, these questions amongst others are considered. The examination of (n, k)-choice-hypergraphs is the focus of Chapter 2, and if all the reader desires is an understanding of that, one is encouraged to skip ahead to the beginning of Chapter 2.

1.1.2 Sequential Choice

The other area of focus of this thesis is the concept of sequential choice and balanced allocation considered in Appendix A.

By **sequential choice** we mean any choice made at regular intervals from subsets of the same set of options. For example, from the introduction, consider someone choosing which song from their music library to start their morning commute with every day, or a group of friends who must, once a week, have one of them make a choice for where to eat lunch.

In particular, consider a fixed set $\mathcal{P} = \{p_1, p_2, ..., p_n\}$ of n individuals and suppose selections $x_1, x_2, ...$ are being drawn from $\mathcal{P}$ in successive fashion. We are interesting in notions of "fairness" regarding the distribution of selections in time. To do so we define a **choice function**, $C : \mathscr{P}(\mathcal{P}) \setminus \emptyset \to \mathcal{P}$ where $\mathscr{P}(\mathcal{P})$ is the power set of $\mathcal{P}$, and consider properties of choice functions which will ensure some idea of "fairness".

One naive conception of fairness would be to uniformly randomly select from $\mathcal{P}$, giving each $p \in \mathcal{P}$ a $1/n$ chance of being selected. However, this formulation which seems *a priori* fair could lead to situations wherein an individual selects an unbounded amount of times in a row – something no one would consider properly fair.

We suppose that $x_1, x_2, \ldots, x_Q$ have been selected (via some means) from the set $\mathcal{P}$ (think of these as Q-many initial selections). Following this $x_{Q+1}, x_{Q+2}, \ldots$ are to be drawn under the assumption that for all times $t > Q$, $x_t \in \mathcal{L}_t$, where $\mathcal{L}_t \subseteq \mathcal{P}$ is the set of "least represented" elements in $\mathcal{P}$ among the Q selections directly preceding time t. Moreover suppose $x_t = C(\mathcal{L}_t)$ where C is some fixed choice function on subsets of $\mathcal{P}$.

By way of example, suppose that $\mathcal{P} = \{1, 2, 3, 4, 5\}$ and C is defined on subsets of $\mathcal{P}$ of order 3 as in Table 1.1 below.

$$\begin{array}{llll} C(\{1,2,3\}) = 1, & C(\{1,2,4\}) = 4, & C(\{1,2,5\}) = 5, & C(\{1,3,4\}) = 3, \\ C(\{1,3,5\}) = 5, & C(\{1,4,5\}) = 4, & C(\{2,3,4\}) = 2, & C(\{2,3,5\}) = 5, \\ C(\{2,4,5\}) = 4, & C(\{3,4,5\}) = 3. & & \end{array}$$

Table 1.1: A choice function on the subsets of a five-individual set (restricted to subsets of size three).

For $Q = 7$ and the 7 initial values $(x_1, x_2, \ldots, x_7) = (2, 2, 3, 4, 5, 1, 1)$ this leads to $x_8 = 3$ as, at time $t = 8$, the least represented elements of $\mathcal{P}$ amongst the previous 7 values are $\{3, 4, 5\}$. Continuing in this way, it may be verified that the first 100 values are given by:

```
2 2 3 4 5 1 1 3 4 2 5 2 3 4 1 5 3 4 5 2 1 2 3 4 5 3 4 5 1 2 1 2
3 3 4 5 4 5 1 2 1 3 2 3 4 5 4 5 1 1 2 3 2 3 4 5 4 1 5 1 2 3 2 3
4 4 5 1 5 1 2 3 2 4 3 4 5 1 5 1 2 2 3 4 3 4 5 1 5 2 1 2 3 4 3 4
5 5 1 2.
```

One thing of interest to note, is that this string of selections settles into a period of length 80, given by

```
2 1 2 3 4 5 3 4 5 1 2 1 2 3 3 4 5 4 5 1 2 1 3 2 3 4 5 4 5 1 1 2
3 2 3 4 5 4 1 5 1 2 3 2 3 4 4 5 1 5 1 2 3 2 4 3 4 5 1 5 1 2 2 3
4 3 4 5 1 5 2 1 2 3 4 3 4 5 5 1.
```

Indeed, since both $|\mathcal{P}|$ and Q are finite, the sequence $\boldsymbol{x}$ must be **ultimately periodic** with minimal period length at most n^Q.

We define a choice function C to be **fair** if for <u>all</u> $Q \geq 1$ such that n does not divide Q, and <u>all</u> possible values of $(x_1, x_2, ..., x_Q)$, the resulting eventual cycle is equally distributed over elements of $\mathcal{P}$.

As before, we are able to represent these choice functions and their effects on the system considered via directed graphs. We will do so by creating a directed edge from p_1 to p_2 if p_2 is never selected by C whenever $p_1, p_2 \in \mathcal{L}_t$. For example, in Table 1.1 above, note that 2 is never selected whenever $1, 2 \in \mathcal{L}_t$. This leads to the following directed graph shown below in Figure 1.2.

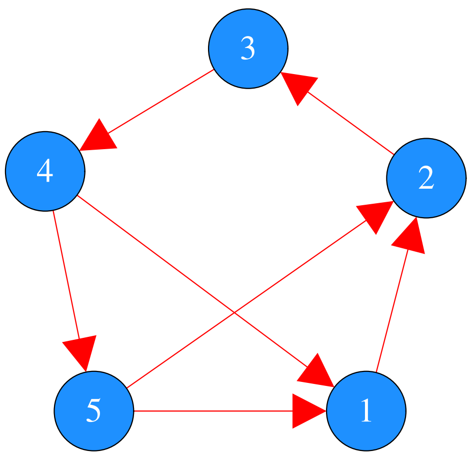


Figure 1.2: A directed graph for $\mathcal{P} = \{1, 2, 3, 4, 5\}$ where an edge is drawn from p_1 to p_2 if p_2 is never chosen when $p_1, p_2 \in \mathcal{L}_t$

From the overview given above, some natural questions arise, such as:

- What conditions are needed to ensure a fair choice function exists?
- How do properties of choice functions and their digraph representations translate into properties of the resulting sequences?
- What is the set of all fair choice functions for a given $\mathcal{P}$?

1.2 Results

For (n, k)-choice-hypergraphs, the concepts of pair-wise domination and domination-hypergraphs are introduced and employed to discern results concerning the existence of "balanced" hypertournaments and zero-edged domination hypergraphs. The concept of pair-wise domination is integral in proving two main results:

Theorem 1. *Suppose $n, k \geq 1$ and $H = (V, E)$ is a complete k-hypergraph on n vertices. If k is odd and $gcd(n, k) = 1$, then there exists a choice function, C, on E resulting in a zero-edge domination graph.*

Theorem 2. *If (H, C) is a choice-hypergraph with a zero-edge domination graph, then for all $v \in V$,*

$$|\{e \in E : C(e) = v\}| = \frac{1}{n} \cdot \binom{n}{k} \tag{1.1}$$

that is, each vertex is chosen for an equal number of edges, in E.

In the context of sequential choice, we prove a variety of results, foremost amongst them in importance is the following:

Theorem 3. *The set of fair choice functions for a given $\mathcal{P}$ is non-empty for all $\mathcal{P}$.*

1.3 Organization of the thesis

The remainder of the thesis proceeds as follows. Chapter 2, contains the paper "Symmetry in Domination for Hypergraphs with Choice" which has appeared in the journal *Symmetry*. Chapter 3 includes discussion of future research. The appendices include the paper "Balanced Allocation, Infinite Words, and Sequential Choice", as well as appendices for Chapter 2. Note that the bibliography for the introduction and conclusion appears after all appendices, this is because Chapter 2 and Appendix A are self-contained papers and as such have their own bibliographies attached.

Chapter 2: Symmetry in Domination for Hypergraphs with Choice

The following paper[1] has been published to the Journal *Symmetry*. Stylistic variations are due to the requirements of the journal. Kenneth S. Berenhaut and Brendan P. Lidral-Porter wrote the paper. Kenneth S. Berenhaut, Brendan P. Lidral-Porter, Theo H. Schoen, and Kyle P. Webb contributed to the results.

[1] K. S. Berenhaut, B. P. Lidral-Porter, T. H Schoen, K. P. Webb. Symmetry in Domination for Hypergraphs with Choice. *S*ymmetry (2017) Mar 22;**9** (3):46.

2.1 Introduction

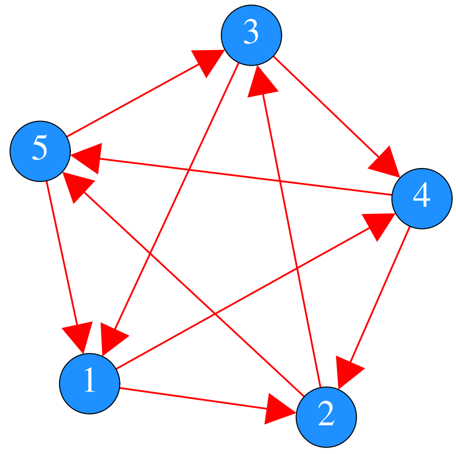
In this paper we introduce the concept of pair-wise domination for hypergraphs endowed with a choice function on edges. A hypergraph is a pair $\mathcal{H} = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of n vertices (or nodes) and $E = \{e_1, e_2, \ldots, e_m\}$ is a set of m non-empty subsets of V called hyperedges or edges (see for instance [2]). A k-hypertournament is a complete k-hypergraph $H = (V, E)$ (i.e. E consists of all the $\binom{n}{k}$ possible k-subsets), with each k-edge endowed with an orientation. Here we are interested in hypergraphs where each edge, e, has a chosen element $C(e) \in e$ (in place of a complete orientation). We will refer to the pair (H, C) as a (complete) k-hypergraph with choice (or an (n, k) choice-hypergraph). For various considerations of choice functions, see for instance [10] and [18]. For some recent work related to choice in the context of Cayley graphs see [1].

In the case $k = 2$, both k-hypertournaments and hypergraphs with choice reduce to standard tournaments. For discussion of tournaments, see for instance [11], [12], and [20]. The following is an example of a standard tournament with $n = 5$ vertices.

Example 1. Consider the complete 2-hypertournament with five nodes and ten edges, depicted via the table in Figure 2.1a. Here, for instance, vertex 1 is chosen in the presence of vertices 2 and 4 (the first and third lines in the table). In fact, in this particular instance, each vertex is chosen for exactly two of the $\binom{5}{2} = 10$ edges. Figure 2.1b gives a graphical display with a directed edge from vertex v to vertex w whenever vertex v is chosen in the presence of w. □

	e	orientation	C(e)
1	{1, 2}	(1,2)	1
2	{1, 3}	(3,1)	3
3	{1, 4}	(1,4)	1
4	{1, 5}	(5,1)	5
5	{2, 3}	(2,3)	2
6	{2, 4}	(4,2)	4
7	{2, 5}	(2,5)	2
8	{3, 4}	(3,4)	3
9	{3, 5}	(5,3)	5
10	{4, 5}	(4,5)	4

(a)



(b)

Figure 2.1: An example of a standard tournament with five vertices (a), and an associated graphical display with a directed edge from vertex v to vertex w whenever vertex v is chosen in the presence of w (b).

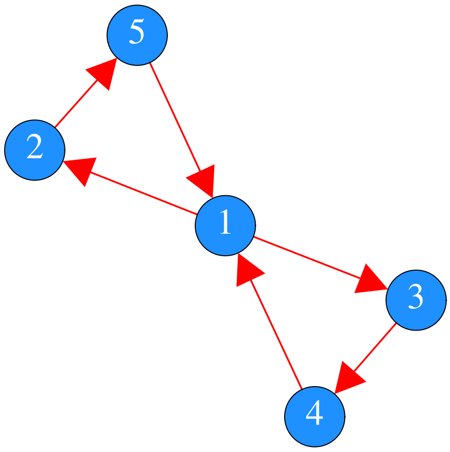
In the next example we consider a complete 3-hypergraph with choice.

Example 2. Consider the complete 3-hypergraph with choice with five vertices and 10 edges, depicted in the table in Figure 2.2a. Here vertex 1 is chosen once in the presence of vertices 2 and 4 (for edges $\{1, 2, 3\}$ and $\{1, 3, 4\}$, respectively), twice in the presence of vertex 3 (again edges $\{1, 2, 3\}$ and $\{1, 3, 4\}$) and never in the presence of vertex 5. Note that again, as in Example 1, each node is chosen in the case of exactly two edges.

In considering possible analogues to Figure 2.1b, summarizing domination, one might include a directed edge from vertex v to vertex w, if and only if for edges that include both v and w, the tally of wins for v exceeds that of w. The resulting graph is depicted in Figure 2.2b. Note that there is a directed edge from vertex 1 to vertex 2, since vertex 1 is chosen for edge $\{1, 2, 3\}$, while there is no edge, e, where vertex 2 is chosen when $1 \in e$. It may be noted that in this case the associated graph is path connected. $\square$

	e	$C_0(e)$
1	$\{1, 2, 3\}$	1
2	$\{1, 2, 4\}$	4
3	$\{1, 2, 5\}$	5
4	$\{1, 3, 4\}$	1
5	$\{1, 3, 5\}$	3
6	$\{1, 4, 5\}$	4
7	$\{2, 3, 4\}$	3
8	$\{2, 3, 5\}$	2
9	$\{2, 4, 5\}$	2
10	$\{3, 4, 5\}$	5

(a)



(b)

Figure 2.2: An example of a possible (5,3) choice-hypergraph (a), with a possible graph summarizing pair-wise domination (b).

Now, define the function $\tau : V \times V \to \mathbb{Z}^+$ via

$$\tau(v, w) = |\{e \in E : v, w \in e \text{ and } C(e) = v\}|, \tag{2.1}$$

i.e. $\tau(v, w)$ is the number of edges for which v is chosen in the presence of w.

Example 2 leads to consideration of potential appropriate graphs on n vertices reflecting domination properties among vertices. Here we mention three possibilities.

(i) There is a directed edge from vertex v to vertex w if

$$\tau(w, v) = 0, \tag{2.2}$$

i.e. if vertex w is never chosen in the presence of vertex v.

(ii) There is a directed edge from vertex v to vertex w if

$$\tau(v, w) > \tau(w, v), \tag{2.3}$$

i.e. among the edges containing both v and w, v is chosen with greater frequency.

(iii) There is a directed edge from vertex v to vertex w if

$$\tau(v,w) > \frac{\binom{n-2}{k-2}}{2}, \tag{2.4}$$

i.e. v is chosen for a majority of the edges containing both v and w.

We will restrict attention henceforth to Option (ii), above, unless stated otherwise. It should be noted that for a standard tournament graph (i.e. $k = 2$) all three formulations are equivalent; furthermore, Option (iii) is a stricter requirement than Option (ii). For discussion of ranking for vertices in hypertournaments, see for instance [17].

We refer to graphs as in Figures 2.2b and 2.3 (below) as (n, k)-*choice-domination graphs* or simply (n, k)-*domination graphs*. When n and k are clear from context, we will at times simply refer to these as domination graphs. As with tournaments, domination graphs could be valuable in considerations of individual dominance in competitive settings, as may arise for instance in biology, game theory or decision analysis. Note that hypergraphs with choice allow for analysis of scenarios wherein selection (but not full orientation) information is available.

Example 3. Table 2.1 gives a 3-hypergraph with $n = 5$ vertices and $m = 10$ edges, along with four possible choice functions, C_0 (from Example 2), C_1, C_2 and C_3 on E, while Figure 2.3 includes the associated domination graphs, for comparison.

	e	$C_0(e)$	$C_1(e)$	$C_2(e)$	$C_3(e)$
1	{1, 2, 3}	1	1	1	1
2	{1, 2, 4}	4	1	2	2
3	{1, 2, 5}	5	5	5	5
4	{1, 3, 4}	1	3	4	4
5	{1, 3, 5}	3	5	3	3
6	{1, 4, 5}	4	4	1	1
7	{2, 3, 4}	3	2	3	4
8	{2, 3, 5}	2	2	2	2
9	{2, 4, 5}	2	4	4	4
10	{3, 4, 5}	5	3	5	3

Table 2.1: Four possible $(5, 3)$ choice-hypergraphs

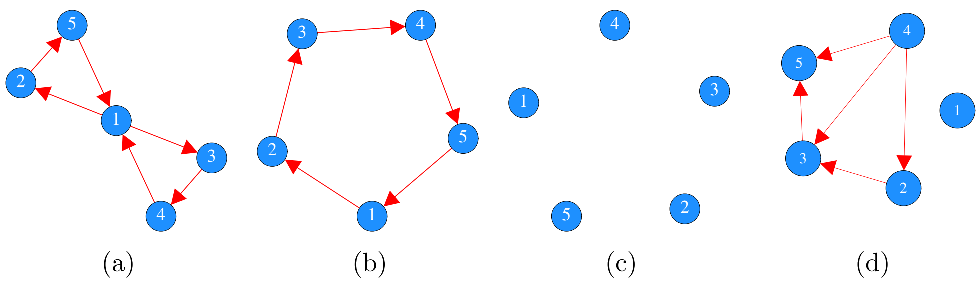


Figure 2.3: Domination graphs arising from choice-hypergraphs with choice functions C_0, C_1, C_2, and C_3 as in Table 2.1

For fixed n and k, many natural questions arise as to properties of resulting domination graphs; for instance:

1. What are the maximal and minimal number of edges possible for an (n, k)-domination graph?

2. What proportion of (n, k)-domination graphs are strongly path connected (for example Figures 2.3a and 2.3b)?

3. What is the distribution of the number of edges in the domination graph for a uniformly selected choice function on the edges of a k-hypergraph on n vertices?

4. What is the number of non-isomorphic (n, k)-domination graphs?

Example 4. ((5, 3) domination graphs.) For $(n, k) = (5, 3)$, we have that $|E| = 10$ and the number of distinct choice functions on E is $3^{10} = 59049$. Tables 2.2 and 2.3 give frequency tables for the number of choice functions leading to domination graphs with a given number of edges, and a given number of strongly connected components, respectively. Note that 3348 choice functions result in strongly connected domination graphs. There are 225 non-isomorphic (5, 3)-domination graphs (of which 21 are strongly connected); plots of these are provided in the Supplementary Materials; Figure 2.4 gives the frequencies for these graphs. The two most frequent domination graphs (each occurring for 1560 distinct choice functions, C), are given in Figure 2.5a and 2.5b respectively; the most frequently occurring strongly connected domination graph is given in Figure 2.5c. $\square$

Table 2.2: Edge distribution for (5, 3)-domination graphs

Edges	0	2	3	4	5	6	7	8	9	10
Frequency	6	60	120	1035	3324	10080	15180	16920	9180	3144

Table 2.3: Component distribution for (5, 3)-domination graphs

Components	1	2	3	5
Frequency	3348	6630	11760	37311

Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.
1	1560	2	510	3	360	4	300	5	840	6	480	7	240
8	480	9	120	10	600	11	840	12	240	13	960	14	120
15	720	16	120	17	480	18	360	19	240	20	120	21	420
22	840	23	240	24	120	25	120	26	960	27	180	28	180
29	180	30	120	31	120	32	240	33	120	34	840	35	240
36	480	37	240	38	120	39	600	40	120	41	240	42	360
43	600	44	180	45	360	46	300	47	15	48	180	49	1200
50	720	51	1560	52	240	53	960	54	480	55	120	56	360
57	600	58	240	59	120	60	240	61	240	62	840	63	360
64	840	65	480	66	840	67	360	68	120	69	240	70	720
71	480	72	360	73	240	74	240	75	480	76	240	77	480
78	360	79	120	80	240	81	120	82	240	83	120	84	120
85	240	86	240	87	120	88	120	89	240	90	360	91	480
92	360	93	480	94	120	95	240	96	240	97	120	98	120
99	120	100	240	101	360	102	360	103	120	104	120	105	720
106	240	107	120	108	210	109	120	110	240	111	120	112	120
113	120	114	600	115	120	116	360	117	360	118	120	119	240
120	360	121	120	122	120	123	120	124	120	125	240	126	120
127	360	128	240	129	240	130	120	131	120	132	120	133	360
134	120	135	240	136	300	137	120	138	120	139	120	140	60
141	360	142	120	143	120	144	120	145	60	146	120	147	300
148	240	149	120	150	120	151	360	152	240	153	240	154	120
155	120	156	120	157	120	158	120	159	120	160	120	161	120
162	120	163	240	164	120	165	480	166	120	167	240	168	240
169	120	170	240	171	120	172	120	173	240	174	120	175	120
176	120	177	240	178	120	179	120	180	120	181	240	182	120
183	120	184	120	185	240	186	120	187	120	188	120	189	120
190	120	191	120	192	120	193	120	194	120	195	120	196	120
197	120	198	120	199	120	200	120	201	120	202	120	203	120
204	120	205	120	206	120	207	120	208	120	209	120	210	60
211	360	212	120	213	120	214	120	215	120	216	24	217	120
218	120	219	120	220	120	221	120	222	240	223	120	224	24
225	6												

Figure 2.4: Frequency distribution for non-isomorphic $(5, 3)$-domination graphs

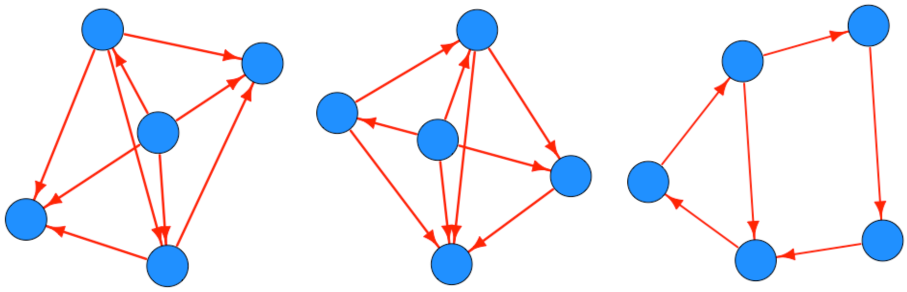


Figure 2.5: Three frequently occurring $(5,3)$-domination graphs. The two most frequent domination graphs are given in (a) and (b). The most frequently occurring strongly connected domination graph is given in (c).

In reference to Question 1 above, in Section 2 below, we will prove the following two results.

Theorem 4. *Suppose $n, k \geq 1$ and $H = (V, E)$ is a complete k-hypergraph on n vertices. If k is odd and $gcd(n, k) = 1$, then there exists a choice function, C, on E resulting in a zero-edge domination graph.*

Theorem 5. *If (H, C) is a choice-hypergraph with a zero-edge domination graph, then for all $v \in V$,*

$$|\{e \in E : C(e) = v\}| = \frac{1}{n} \cdot \binom{n}{k} \tag{2.5}$$

that is, each vertex is chosen for an equal number of edges, in E.

The question of minimal edges in associated domination graphs may be of interest in instances where notions of "fairness" and equitable distribution are of importance, such as in resource allocation, decision theory, data and network processing, and clinical trials. Fairness and choice have been considered in the past, notably in the context of social welfare and information processing. The interested reader might like to consult, for instance [1, 3, 5, 9, 10, 14, 18, 24].

Figure 2.6 provides an example of a $(9,5)$ choice-hypergraph with vertex set $V = \{1, 2, \ldots, 9\}$, possessing a zero edge domination graph (employing the construction in the proof of Theorem 4). Note that $|E| = \binom{9}{5} = 126$, and $\tau(1,2) = 8 = \tau(2,1)$ (as highlighted in red; $C(e)$ for $e \in E$ satisfying $\{1,2\} \in e$ are indicated in bold). It may also be verified that $|\{e \in E : C(e) = 1\}| = 126/9 = 14$, as required by Theorem 5.

1 2 3 4 5	**3**	1 2 4 6 9	**2**	1 3 4 7 8	1	1 4 6 8 9	8	2 3 5 7 9	7	3 4 5 6 7	5
1 2 3 4 6	**3**	1 2 4 7 8	**1**	1 3 4 7 9	1	1 4 7 8 9	4	2 3 5 8 9	2	3 4 5 6 8	5
1 2 3 4 7	**7**	1 2 4 7 9	**1**	1 3 4 8 9	1	1 5 6 7 8	7	2 3 6 7 8	7	3 4 5 6 9	9
1 2 3 4 8	**2**	1 2 4 8 9	**1**	1 3 5 6 7	5	1 5 6 7 9	7	2 3 6 7 9	9	3 4 5 7 8	5
1 2 3 4 9	**2**	1 2 5 6 7	**6**	1 3 5 6 8	1	1 5 6 8 9	8	2 3 6 8 9	9	3 4 5 7 9	5
1 2 3 5 6	**3**	1 2 5 6 8	**8**	1 3 5 6 9	3	1 5 7 8 9	8	2 3 7 8 9	9	3 4 5 8 9	4
1 2 3 5 7	**3**	1 2 5 6 9	**1**	1 3 5 7 8	3	1 6 7 8 9	8	2 4 5 6 7	5	3 4 6 7 8	6
1 2 3 5 8	**2**	1 2 5 7 8	**8**	1 3 5 7 9	5	2 3 4 5 6	4	2 4 5 6 8	5	3 4 6 7 9	6
1 2 3 5 9	**2**	1 2 5 7 9	**9**	1 3 5 8 9	1	2 3 4 5 7	4	2 4 5 6 9	4	3 4 6 8 9	6
1 2 3 6 7	**2**	1 2 5 8 9	**5**	1 3 6 7 8	8	2 3 4 5 8	8	2 4 5 7 8	5	3 4 7 8 9	8
1 2 3 6 8	**1**	1 2 6 7 8	**8**	1 3 6 7 9	9	2 3 4 5 9	3	2 4 5 7 9	9	3 5 6 7 8	6
1 2 3 6 9	**6**	1 2 6 7 9	**9**	1 3 6 8 9	9	2 3 4 6 7	4	2 4 5 8 9	2	3 5 6 7 9	6
1 2 3 7 8	**1**	1 2 6 8 9	**9**	1 3 7 8 9	9	2 3 4 6 8	4	2 4 6 7 8	6	3 5 6 8 9	6
1 2 3 7 9	**1**	1 2 7 8 9	**9**	1 4 5 6 7	1	2 3 4 6 9	3	2 4 6 7 9	2	3 5 7 8 9	7
1 2 3 8 9	**1**	1 3 4 5 6	4	1 4 5 6 8	6	2 3 4 7 8	3	2 4 6 8 9	4	3 6 7 8 9	3
1 2 4 5 6	**4**	1 3 4 5 7	4	1 4 5 6 9	5	2 3 4 7 9	2	2 4 7 8 9	9	4 5 6 7 8	6
1 2 4 5 7	**4**	1 3 4 5 8	3	1 4 5 7 8	7	2 3 4 8 9	2	2 5 6 7 8	2	4 5 6 7 9	6
1 2 4 5 8	**2**	1 3 4 5 9	3	1 4 5 7 9	7	2 3 5 6 7	5	2 5 6 7 9	7	4 5 6 8 9	6
1 2 4 5 9	**2**	1 3 4 6 7	4	1 4 5 8 9	9	2 3 5 6 8	5	2 5 6 8 9	8	4 5 7 8 9	7
1 2 4 6 7	**4**	1 3 4 6 8	8	1 4 6 7 8	7	2 3 5 6 9	3	2 5 7 8 9	8	4 6 7 8 9	7
1 2 4 6 8	**6**	1 3 4 6 9	3	1 4 6 7 9	7	2 3 5 7 8	5	2 6 7 8 9	8	5 6 7 8 9	7

Figure 2.6: A $(9,5)$-choice hypergraph resulting in a zero-edge domination graph. For given $e \in E$, the value of $C(e)$ is listed to the right of the five elements of e in a demarcated column.

Before turning to proofs of Theorems 4 and 5, we will briefly mention some recent related work on hypertournaments, which carry over to choice-hypergraphs. Recall that a k-hypertournament is a complete k-hypergraph $H = (V, E)$, with each edge endowed with an orientation. We will refer to the oriented edges as arcs.

One concept considered extensively in the literature is score sequences (see for instance [8], [20], [19], [13], [21], [12], [15]). In particular, for a given $1 \leq i \leq n$ define the *score*, s_i of a vertex v_i of a k-hypertournament on $H = (V, E)$ as the number of arcs containing v_i in which v_i is not the last element (this is with a complete orientation on the edges, rather than a choice function solely selecting a single element). Similarly, define the *losing score*, r_i as the number of arcs containing v_i in which v_i is the last element. The *total score*, t_i, is then given by $t_i = s_i - r_i$. Finally, we obtain the *score sequences* $(s_1, \ldots, s_n)$, $(r_1, \ldots, r_n)$ and $(t_1, \ldots, t_n)$. Guofei et al. proved the following results regarding existence of score sequences (see also [22] and [13]).

Theorem 6. *(Guofei et al. [8]) Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $R = (r_1, r_2, \ldots, r_n)$ of non-negative integers is a losing score sequence of some k-hypertournament if and only if for each j $(k \leq j \leq n)$,*

$$\sum_{i=1}^{j} r_i \geq \binom{j}{k}, \tag{2.6}$$

with equality when $j = n$.

Theorem 7. *(Guofei et al. [8]) Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers is a score-sequence of some k-hypertournament if and only if for each j $(k \leq j \leq n)$,*

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k}, \tag{2.7}$$

with equality when $j = n$.

A k-hypertournament is said to be *regular* if for each vertex, v, the tally of arcs containing v as the last element is $\binom{n}{k}/n$. Koh and Ree [13] proved the following.

Theorem 8. *(Koh and Ree, [13]) A regular (n, k) hypertournament exists if and only if $n|\binom{n}{k}$.*

For alternative considerations of regularity see [23] and [17].

Compare Theorem 8 with Theorems 4 and 5, above. Note that symmetry in domination (i.e. the existence of choice functions resulting in zero-edge domination graphs) is a stronger requirement than regularity. To see this, simply note that all standard tournaments have $\binom{n}{k}$-edge domination graphs.

For further work on hypertournaments or score sequences, see for instance Pirizda et al. [19], Landau [15], Marshall [17], Khan et al. [12], Guofei et al. [8], Gunderson et al. [6], Li et al. [16], Guo and Surmacs [7], and Chou and Guofei [4].

In the next section we prove Theorems 4 and 5.

2.2 Proof of Theorems 4 and 5

Before moving on to the proofs of Theorems 4 and 5, we introduce some preliminary notation. First, suppose $n \geq k \geq 1$ are fixed and (H, C) is an (n, k)-hypergraph with choice, where $H = (V, E)$ is a complete k-hypergraph on n vertices. Without loss of generality, we assume that $V = \{0, 1, \ldots, n-1\}$. Similar to in [13], define the rotation operator $P : E \to E$, via $P(e) = e + 1 \pmod{n}$, i.e. P acts on k-subsets of V by shifting the elements (cyclically) to the right by one.[2] For $e \in E$ and $j \geq 0$, define P^j as the j-fold iteration of P and the *order* of e, α_e, via

$$\alpha_e \stackrel{def}{=} \min\{\gamma \geq 1 : P^\gamma(e) = e\}. \tag{2.8}$$

We will denote the set of equivalence classes under successive application of P by $\mathcal{R} = \{R_1, \ldots, R_q\}$. Note that for $R \in \mathcal{R}$, $|R|$ is the order of each $e \in R$. We will refer to elements of $\mathcal{R}$ as *rotation classes* of H.

In general, addition of the form $S + c$ for $S \subseteq \mathbb{Z}$ and $c \in \mathbb{Z}$, will be modulo n, unless stated otherwise.

[2]Here, $e + 1$ indicates $\{v + 1 : v \in e\}$.

The set $e \in E$ is said to be *symmetric* if $-e = e \pmod{n}$, and more generally $R \in \mathcal{R}$ is symmetric if, for all $e \in R$, $-e \in R$. Note that if for some $e \in R$, $e = -e$, then for $1 \leq i \leq n$, $-(e+i) = -e - i = e - i \in R$, and hence if $e \in R$ is symmetric then R is symmetric. If R is not symmetric, then there exists an $R' \in \mathcal{R} \setminus R$ such that $e \in R$ implies $-e \in R'$.

Let $\mathcal{H}_{n,k}$ be the set of all (n,k)-choice hypergraphs for fixed $n, k \in \mathbb{Z}^+$ and $\mathcal{G}_n$ be the set of all directed graphs on n vertices. Suppose a domination scheme, D is fixed (see (i)–(iii), above, for examples) and define $G_D : \mathcal{H}_{n,k} \to \mathcal{G}_n$, where $G_D(T)$ is the domination graph for choice-hypergraph $T = (H, C)$ under domination scheme D.

We have the following elementary lemma:

Lemma 1. *Suppose H is a k-hypertournament on n vertices with $gcd(n,k) = 1$. Then, for any $R \in \mathcal{R}$, $|R| = n$.*

Proof. Suppose $gcd(n,k) = 1$, and $\mathcal{R} = \{R_1, R_2, \ldots, R_q\}$ for some $q \geq 1$. For any $R \in \mathcal{R}$ and any $e \in R$, set $\alpha = \alpha_e$ and represent e as a binary vector $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$. We then have, with $a = \sum_{1 \leq i \leq \alpha} \zeta_i$ and $b = n/\alpha$,

$$k = ba, \qquad n = b\alpha. \tag{2.9}$$

Now, suppose $n = \alpha p + r$, with $p \geq 0$ and $0 \leq r \leq \alpha - 1$. Then

$$e = P^{\alpha(p+1)}(e) = P^{n-r+\alpha}(e) = P^{\alpha-r}(e). \tag{2.10}$$

Since α is minimal, we have $r = 0$ and hence $b = n/\alpha \in \mathbb{Z}$. Thus, since $gcd(n,k) = 1$, (2.9) gives that $b = 1$, $\alpha = n$ and finally $|R| = n$. $\square$

For convenience of notation, as in (2.1), define the function $\tau_R : V \times V \to \mathbb{Z}^+$ via

$$\tau_R(v,w) = |\{e \in R : v, w \in e \text{ and } C(e) = v\}|, \tag{2.11}$$

i.e. $\tau_R(v,w)$ is the number of edges in the rotation class $R \in \mathcal{R}$ for which v is chosen in the presence of w. Note that

$$\tau(v,w) = \sum_{R\in\mathcal{R}} \tau_R(v,w). \tag{2.12}$$

Lemma 2. *Suppose $n, k \geq 1$ with k odd, $H = (V, E)$ is a complete k-hypergraph on n vertices, and $R \in \mathcal{R}$. If R is symmetric and $e \in R$, then there exists a $\xi \in V$ and $\mathcal{T} = \{\delta_1, \delta_2, \ldots, \delta_k\}$, such that $e = \xi + \mathcal{T}$, and $\delta \in \mathcal{T}$ implies $-\delta \in \mathcal{T}$.*

Proof. Suppose that R is symmetric and $e = \{x_1, \ldots, x_k\} \in R$. Then, there exists an i such that $e + i = -e$, and hence a permutation $(j_1, j_2, \ldots, j_k)$ of $(1, 2, \ldots, k)$ such that

$$x_{j_q} + i = -x_q, \qquad 1 \leq q \leq k. \tag{2.13}$$

Since k is odd, there exists a Q such that

$$x_Q + i = -x_Q, \tag{2.14}$$

and taking differences, (2.13) and (2.14) implies

$$x_{j_q} - x_Q = x_Q - x_q = -(x_q - x_Q), \qquad 1 \leq q \leq k. \tag{2.15}$$

The result follows upon setting $\xi = x_Q$, and $\mathcal{T} = \{x_q - x_Q : 1 \leq q \leq k\}$. $\square$

We will now prove Theorem 4 regarding the existence of choice functions with symmetry in domination.

Proof of Theorem 1. Suppose $gcd(n, k) = 1$, k is odd, and (H, C) is an (n, k)-choice hypergraph. Consider E, the set of all edges in H, and let $\mathcal{R} = \{R_1, R_2, \ldots, R_q\}$ be the set of all rotation classes of H, where, by Lemma 1, for $R \in \mathcal{R}$, $|R| = n$, and $|\mathcal{R}| = \binom{n}{k}/n$. Fix $v, w \in V$, with $v \neq w$, and choose some $R \in \mathcal{R}$.

Suppose R is symmetric and fix an $e \in R$. Then by Lemma 2, $e = \xi + \mathcal{T}$, where $\xi \in V$ and $\mathcal{T}$ is closed under additive inverses. For $u = e + i \in R$, set $C(u) = \xi + i$. Suppose $v = \xi + j_v$ and $w = \xi + j_w$ for $j_v, j_w \in \{0, 1, ..., n-1\}$. For $f \in R$, $C(f) = v$ implies $f = e + j_v$ and $C(f) = w$ implies $f = e + j_w$. Note that $w \in e + j_v$ if and only if $j_w - j_v \in \mathcal{T}$. Similarly, $v \in e + j_w$ if and only if $j_v - j_w \in \mathcal{T}$. Since $\mathcal{T}$ is closed under inverses, we have

$$\begin{aligned}
\tau_R(v, w) &= |\{e \in R : v, w \in e \text{ and } C(e) = v\}| \\
&= |\{e \in R : v, w \in e \text{ and } C(e) = w\}| \\
&= \tau_R(w, v). \qquad (2.16)
\end{aligned}$$

Suppose R is not symmetric, and consider $R' = \{-e : e \in R\}$, and note that $R \cap R' = \emptyset$. Fix an $e \in R$ and $\xi \in e$, and write $e = \xi + \mathcal{T}$ (note that $\mathcal{T}$ is not closed under inverses). For $f \in R \cup R'$, set

$$C(f) = \begin{cases} \xi + i & \text{if } f = e + i = \xi + i + \mathcal{T} \in R \\ -\xi + i & \text{if } f = -e + i = -\xi + i - \mathcal{T} \in R'. \end{cases} \qquad (2.17)$$

Now, suppose $v = \xi + j_v$ and $w = \xi + j_w$. For $f \in R \cup R'$, $C(f) = v$ implies $f = e + j_v$ or $f = -e + j_v$, and $C(f) = w$ implies $f = e + j_w$ or $f = -e + j_w$. Note that $w \in e + j_v$ if and only if $j_w - j_v \in \mathcal{T}$, and $w \in -e + j_v$ if and only if $j_w - j_v \in -\mathcal{T}$ (i.e. $j_v - j_w \in \mathcal{T}$). Similarly, $v \in e + j_w$ if and only if $j_v - j_w \in \mathcal{T}$, and $v \in -e + j_w$ if and only if $j_v - j_w \in -\mathcal{T}$ (i.e. $j_w - j_v \in -\mathcal{T}$). Thus,

$$\begin{aligned}
\tau_{R \cup R'}(v, w) &= |\{f \in R \cup R' : v, w \in f \text{ and } C(f) = v\}| \\
&= |\{f \in R \cup R' : v, w \in f \text{ and } C(f) = w\}| \\
&= \tau_{R \cup R'}(w, v). \qquad (2.18)
\end{aligned}$$

Employing (2.16), (2.18) and (2.12), the result follows. $\square$

We will now prove Theorem 5.

Proof of Theorem 2. Suppose (H, C) is a choice-hypergraph with a zero-edge domination graph, where $H = (V, E)$ is a complete k-hypergraph on n vertices. For a fixed vertex $v \in V$, define $\sigma_v = \{e \in E : v \in e\}$ and $\omega_v = \{e \in \sigma_v : C(e) = v\}$, i.e. σ_v is the set of edges to which v belongs and ω_v is the set of edges for which v is selected. Note that

$$|\omega_v| \leq |\sigma_v| = \binom{n-1}{k-1}. \tag{2.19}$$

Assume $|\omega_v| < \binom{n}{k}/n$. Since (H, C) has a zero-edge domination graph, $\tau(w, v) = \tau(v, w)$ for all $w \in V$ and hence

$$\begin{aligned} \binom{n-1}{k-1} &= |\sigma_v| = (k-1)|\omega_v| + |\omega_v| && (2.20) \\ &< k\frac{1}{n}\binom{n}{k} = \binom{n-1}{k-1}. && (2.21) \end{aligned}$$

Thus $|\omega_v| \geq \binom{n}{k}/n$ for all $v \in V$. The result follows upon noting that

$$\sum_{v \in V} \omega_v = |E| = \binom{n}{k}. \tag{2.22}$$

$\square$

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Chapter 3: Conclusions and future work

This thesis considers choice functions in relation to balance in selection. In particular, for a given set P of size n, consider a choice function C on the non-empty subsets of $\mathcal{P}$ (i.e. C is a function from $\mathscr{P}(\mathcal{P}) \setminus \emptyset$ to $\mathcal{P}$, where $\mathscr{P}(\mathcal{P})$ is the power set of $\mathcal{P}$)

With respect to hypergraphs and hypertournaments, we employ the concept of pair-wise domination for hypergraphs endowed with a choice function on the edges. A hypergraph is a pair $\mathcal{H} = (V, E)$, where $V = \{v_1, v_2, ..., v_2\}$ is a set of n vertices and $E = \{e_1, e_2, ..., e_m\}$ is a set of m nonempty subset of V called hyperedges or edges. Hypertournaments are a generalization of the standard tournament model, where traditionally n competitors face each other in one-on-one match-ups of size $k = 2$ (for work related to tournaments, see [1, 2, 3]). Hypertournaments extend this concept in considering k-hypertournaments, wherein a complete k-hypergraph $H = (V, E)$ (where $|E| = \binom{n}{k}$) has an orientation for each edge $e \in E$. Here, we consider choosing a single winner via a choice function, C, on the edges of the hypertournament. We call the pair (H, C) an (n, k) choice-hypergraph. Questions which arise include "what constitutes domination?", "what properties of domination graphs arise from various choice functions?", and "How many isomorphism classes of domination graphs are possible".

With respect to balanced allocations and fair representation (considered in Appendix A), related questions are considered. The problem of addressing inequities (dynamically) may arise naturally in instances where notions of "fairness" and equitable distribution are of interest, such as in resource allocation, decision theory, data and network processing, and clinical trials. Of interest in this context are considerations of what constitutes a "fair" choice function, and how to rectify *a priori* conceptions of fairness which may, in practice, lead to unbalanced or inequitable representations of members. Questions of particular interest are "when do fair choice functions exist?", "what sort of balanced representation is possible?", and "what properties of associated directed graphs arise from fair choice functions?", amongst others.

3.1 Future Work

Future avenues of research include natural extensions. In the case of (n, k) choice-hypergraphs, of interest are the following questions (for general (n, k)):

Open Question 1. *What are the maximal and minimal possible number of edges for an (n, k)-domination graph?*

Open Question 2. *What proportion of (n, k)-domination graphs are strongly path connected (for example Figures 2.3a and 2.3b)?*

Open Question 3. *What is the distribution of the number of edges in the domination graph for a uniformly selected choice function on the edges of a k-hypergraph on n vertices?*

Open Question 4. *What is the number of non-isomorphic (n, k)-domination graphs?*

Consideration of these and related questions is work in progress.

Appendix A: Balanced Allocation, Infinite Words, and Sequential Choice

This paper[1] is currently in pre-submission for the journal *Discrete Applied Mathematics*. Stylistic variations are due to the requirements of the journal. K.S.B., K.E.B, K.M.D., B.P.L.-P., and M.H.S. contributed to the research and wrote the paper.

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[1] K. S Berenhaut, K. E. Beeler, K. M. Donadio, B. P. Lidral-Porter, M. H. Stone Balanced allocation, infinite words, and sequential choice. Discrete Applied Mathematics.

Balanced allocation, infinite words and sequential choice

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Abstract

In this paper, we introduce a stringent definition of fairness for choice functions, from a sequential perspective. We prove existence of fair choice, and consider connections with balanced allocation, path-connected directed graphs and cyclic preference. Some open questions are mentioned.

Keywords: balanced allocation, choice function, digraphs, balls-into-bins, fair choice, cyclic preference

1. Introduction

The problem of addressing inequities (dynamically) may arise naturally in instances where notions of "fairness" and equitable distribution are of interest, such as in resource allocation, decision theory, data and network processing, and clinical trials. In this paper, we introduce a stringent definition of fairness for choice functions,

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from a sequential perspective, and prove existence of fair choice, via consideration of cyclic preference.

Although our focus here is on aspects of choice, the work is closely related to the study of balanced allocations (see for instance [1, 2]). Therein one often considers a model consisting of sequential placement of m balls into n bins. Different combinations of random and deterministic placement are considered, with a quantity of interest being the maximal number of balls placed in any one bin. Implications on performance for memory restrictions are considered in [3, 4]. Generally, implied in these methods is that preference in placement is given in some manner to the least loaded of a set of selection of bins, with *ties broken arbitrarily.* The benefits of an asymmetric tie-breaking mechanism are discussed for instance in [5] and in the context of deterministic walks in [6]. Although here we are interested in balanced allocations from a deterministic perspective, and the interactions with choice mechanisms, the work could be applicable in computational resource allocation, wherein the above methods are often applied.

In particular, consider a fixed set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ of n individuals and suppose selections $x_1, x_2, \ldots$ are being drawn from $\mathcal{P}$ in successive fashion. We are interested in notions of *fairness* regarding the distribution of selections in time. We suppose that $x_1, x_2, \ldots, x_Q$ have been selected (via some means) from the set $\mathcal{P}$. Following this $x_{Q+1}, x_{Q+2}, \ldots$ are to be drawn under the assumption that for $t > Q$, $x_t \in \mathcal{L}_t$, where $\mathcal{L}_t \subseteq \mathcal{P}$ is the set of "least represented" elements in $\mathcal{P}$ among the Q selections directly preceding time t, and moreover suppose $x_t = C(\mathcal{L}_t)$ where C is some fixed choice function on subsets of $\mathcal{P}$.

Notationally, for a given k with $0 < k < t$, let $N_t^k(p)$ be the number of selections of individual $p \in \mathcal{P}$ within the k time points directly preceding time t, i.e.

$$N_t^k(p) = |\{1 \leq i \leq k : x_{t-i} = p\}|. \tag{1}$$

For convenience, when considering $k = t - 1$, we will write simply

$$N_t(p) = |\{1 \leq i \leq t - 1 : x_i = p\}|, \tag{2}$$

i.e. $N_t(p)$ represents the number of instances of p prior to time t. In addition define $\mathcal{L}_t$ via

$$\mathcal{L}_t = \mathcal{L}_t(Q) \stackrel{def}{=} \left\{ p \in \mathcal{P} : N_t^Q(p) = \min_{\pi \in \mathcal{P}} (N_t^Q(\pi)) \right\}. \tag{3}$$

Finally, we assume that x_t is selected via a choice function C on the non-empty subsets of $\mathcal{P}$ (i.e. C is a function from $\mathscr{P}(\mathcal{P}) \setminus \emptyset$ to $\mathcal{P}$, where $\mathscr{P}(\mathcal{P})$ is the power set of $\mathcal{P}$), so that

$$x_t = C(\mathcal{L}_t), \qquad t > Q. \tag{4}$$

We begin with a simple five-individual example.

Example 1. Suppose $\mathcal{P} = \{1, 2, 3, 4, 5\}$ and $Q = 7$, with initial selections given by

$$(x_1, x_2, \ldots, x_7) = (2, 2, 3, 4, 5, 1, 1), \tag{5}$$

and the choice function C (restricted to subsets of $\mathcal{P}$ of size three) is as given in Table 1.

$$C(\{1,2,3\}) = 1, \quad C(\{1,2,4\}) = 4, \quad C(\{1,2,5\}) = 5, \quad C(\{1,3,4\}) = 3,$$
$$C(\{1,3,5\}) = 5, \quad C(\{1,4,5\}) = 4, \quad C(\{2,3,4\}) = 2, \quad C(\{2,3,5\}) = 5,$$
$$C(\{2,4,5\}) = 4, \quad C(\{3,4,5\}) = 3.$$

Table 1: A choice function on the subsets of a five-individual set (restricted to subsets of size three).

We then have $x_8 = C(\{3,4,5\}) = 3$, and since

$$(x_2, x_3, \ldots, x_8) = (2,3,4,5,1,1,3), \tag{6}$$

it follows that $x_9 = C(\{2,4,5\}) = 4$. It may be verified that the first 100 values (through x_{100}) are in fact given by

```
2 2 3 4 5 1 1 3 4 2 5 2 3 4 1 5 3 4 5 2 1 2 3 4 5 3 4 5 1 2 1 2
3 3 4 5 4 5 1 2 1 3 2 3 4 5 4 5 1 1 2 3 2 3 4 5 4 1 5 1 2 3 2 3
4 4 5 1 5 1 2 3 2 4 3 4 5 1 5 1 2 2 3 4 3 4 5 1 5 2 1 2 3 4 3 4
5 5 1 2.
```

The sequence is eventually periodic with period length eighty given by

`2 1 2 3 4 5 3 4 5 1 2 1 2 3 3 4 5 4 5 1 2 1 3 2 3 4 5 4 5 1 1 2`

`3 2 3 4 5 4 1 5 1 2 3 2 3 4 4 5 1 5 1 2 3 2 4 3 4 5 1 5 1 2 2 3`

`4 3 4 5 1 5 2 1 2 3 4 3 4 5 5 1.`

Note that, in perhaps surprisingly equal fashion, each of the five elements of $\mathcal{P}$ appears in the resulting period, above, exactly sixteen times. $\square$

Since both $|\mathcal{P}|$ and Q are finite, the sequence $\boldsymbol{x}$ must be ultimately periodic with minimal period length at most n^Q. We will refer to the minimal period length as $P = P_{\boldsymbol{x}}$. Example 1 suggests investigation of dynamic fairness of choice functions in an equalizing sense.

Definition 1. We say that a choice function, C, on an n-set $\mathcal{P}$ is *fair* (or *sequentially fair*) if for <u>all</u> $Q \geq 1$ such that n does not divide Q, and <u>all</u> possible values of $(x_1, x_2, \ldots, x_Q)$, the resulting eventual cycle is equally distributed over elements of $\mathcal{P}$.

The restriction that n does not divide Q in the above definition will be clear later, when we prove (see Corollary 1) that if n divides Q, the resulting eventual cycle is never equally distributed over elements of $\mathcal{P}$.

The next example gives selections on sets of size three, which are not consistent with fair choice in the sense of Definition 1. On the other hand, we will see later that selections as in Table 1 are consistent with fair choice.

Example 2. Suppose $\mathcal{P}$ and $(x_1, x_2, \ldots, x_7)$ are as in Example 1, and C (restricted to sets of size three) is as in Table 1, except that $C(\{1,2,3\}) = 3$ and $C(\{1,3,4\}) = 1$ (note this is simply a swap in chosen values). Then, it may be verified that the resulting minimal period is now of length eight, given by `3 4 5 3 4 5 2 1` and C is not fair; for instance, there are two `5`'s and one `1` in the period. □

Definition 2. Let $\mathcal{C}_{\mathcal{P}}$ be the set of all choice functions on the subsets of $\mathcal{P}$, and define $\mathcal{F}_{\mathcal{P}} \subset \mathcal{C}_{\mathcal{P}}$ to be the set of all fair choice functions (for $\mathcal{P}$).

Among the questions we are interested in is the following.

OPEN PROBLEM 1. Suppose $\mathcal{P}$ is fixed. Determine the set $\mathcal{F}_{\mathcal{P}}$.

□

Among our results, we will prove the following.

Theorem 1. *The set $\mathcal{F}_{P}$ is non-empty for all $\mathcal{P}$.* □

Without loss of generality, we will restrict attention to the set $\mathcal{P} = \{1, 2, \ldots, n\}$, and denote $\mathcal{F}_{\mathcal{P}}$ by $\mathcal{F}_n$.

Fairness and choice have been considered in the past, notably in the context of social welfare and information processing. The interested reader might like to consult, for instance [7, 8, 9, 10, 11, 12, 13, 14]. Balance in sequences has been considered in the context of clinical trials (see for instance [15, 16, 17, 18]), as well as other areas such as word combinatorics and load balancing [19, 20, 5].

The remainder of the paper proceeds as follows. Section 2 contains some preliminaries and notation and Section 3 follows with results related to cyclic preference, domination graphs and the existence of fair choice functions in the sense of Definition 1.

2. Preliminaries and notation

In this section we introduce some preliminary definitions and notation.

The following definitions regarding equitable representation of elements of $\mathcal{P} = \{1, 2, ..., n\}$ in $\boldsymbol{x} = (x_1, x_2, \ldots)$ will be useful in what follows.

Definition 3. If $(\boldsymbol{x}, k, t)$ satisfies $1 < k \leq t - 1$ and

$$|N_t^k(\pi) - N_t^k(\rho)| \leq 1 \text{ for all } \pi, \rho \in \mathcal{P}, \tag{7}$$

we say that $\boldsymbol{x}$ is *k-smooth* at t. If, in addition $k = Q + 1$, we will say simply that $\boldsymbol{x}$ is *smooth* at t.

Definition 4. If the sequence

$$\{|N_{t+1}(\pi) - N_{t+1}(\rho)| : t \geq Q\} \tag{8}$$

is bounded above, for all $\pi, \rho \in \mathcal{P}$, then we say that $\boldsymbol{x}$ is *asymptotically balanced.*

Recall that $P = P_{\boldsymbol{x}}$ is the length of a minimal period of the sequence $\boldsymbol{x}$.

Definition 5. If there exists a $T > 0$ such that for all $t > T$,

$$N_t^{P_{\boldsymbol{x}}}(\pi) = N_t^{P_{\boldsymbol{x}}}(\rho) \text{ for all } \pi, \rho \in \mathcal{P}, \tag{9}$$

then we say that $\boldsymbol{x}$ is *cyclically balanced*.

The following elementary lemma will be helpful.

Lemma 1. If $\boldsymbol{x}$ is asymptotically balanced, then $\lim_{t\to\infty} N_{t+1}(\pi)/t = 1/n$, for all $\pi \in \mathcal{P}$.

Proof. Assume $\boldsymbol{x}$ is asymptotically balanced and $\pi \in \mathcal{P}$. Then, for $t \geq 0$,

$$t = \sum_{\rho\in\mathcal{P}} N_{t+1}(\rho) = N_{t+1}(\pi) + (n-1)N_{t+1}(\pi) + V_t = nN_{t+1}(\pi) + V_t, \tag{10}$$

where $V_t < D$ for some $D > 0$ (independent of t). Thus

$$\frac{N_{t+1}(\pi)}{t} = \frac{t - V_t}{nt} = \frac{1}{n} - \frac{V_t}{nt}, \tag{11}$$

and the lemma follows upon taking the limit in (11). □

Note that since $\boldsymbol{x}$ is ultimately periodic, $\boldsymbol{x}$ is asymptotically balanced if and only if $\boldsymbol{x}$ is cyclically balanced. In addition, Definition 1 can be restated in terms of cyclical balance.

Definition 6. (Definition 1, revisited). We say that a choice function, C, on an n-set $\mathcal{P}$ is *fair* (or *sequentially fair*) if for all $Q \geq 1$ such that n does not divide Q, and all possible values of $(x_1, x_2, \ldots, x_Q)$, $\boldsymbol{x}$ is cyclically balanced .

Now, for convenience define $d = d(Q, n)$ and $r = r(Q, n)$, via

$$Q + 1 = nd + r, \qquad 0 \le r \le n - 1. \tag{12}$$

We have the following lemma.

Lemma 2. If $\boldsymbol{x}$ is smooth at some $T > Q + 1$, then $\boldsymbol{x}$ is smooth at all $t \ge T$. In addition, for all $t \ge T$

$$|\mathcal{L}_t| \in \{1, n - r + 1\}. \tag{13}$$

Proof. For $t > Q + 1$, set

$$A_t^+ = \{\pi : N_t^{Q+1}(\pi) = d + 1\} \text{ and } A_t^- = \{\pi : N_t^{Q+1}(\pi) = d\}, \tag{14}$$

and note that by (12) and (7), $|A_T^+| = r$ and $|A_T^-| = n - r$. Suppose $x_{T-(Q+1)} = p$. If $p \in A_T^-$, then $\mathcal{L}_T = \{p\}$, $|\mathcal{L}_t| = 1$, $x_T = p$, and $(A_{T+1}^+, A_{T+1}^-) = (A_T^+, A_T^-)$. Otherwise $\mathcal{L}_T = A_T^- \cup \{p\}$, $|\mathcal{L}_t| = n - r + 1$ and $(|A_{T+1}^+|, |A_{T+1}^-|) = (|A_T^+|, |A_T^-|)$. Note that in both cases, $A_{T+1}^+ \cup A_{T+1}^- = A_T^+ \cup A_T^- = \mathcal{P}$ and $|\mathcal{L}_T| \in \{1, n - r + 1\}$. Inductively, we have $(|A_t^+|, |A_t^-|) = (r, n - r)$, and $|\mathcal{L}_t| \in \{1, n - r + 1\}$ for all $t \ge T$, and the result follows. $\square$

Theorem 2. *The sequence* $\boldsymbol{x}$ *is smooth for all* $t > 2Q+1$, *i.e.*

$$N_t^{Q+1}(\pi) \in \{d, d+1\} \text{ for all } \pi \in \mathcal{P}, \tag{15}$$

and in addition (13) holds for all $t > 2Q+1$.

Proof. Suppose $N_T^{Q+1}(\pi) \geq d+2$, for some $\pi \in \mathcal{P}$ and $T > 2Q+1$, and set

$$J = \max\{T-(Q+1) \leq i \leq T-1 : x_i = \pi\}. \tag{16}$$

Then, $N_J^Q(\pi) \geq d+1$, and $x_J = \pi$, and by (4), $N_J^Q(\rho) \geq d+1$, for all $\rho \in \mathcal{P}$. Hence, by (12),

$$nd + r - 1 = Q \geq (d+1)n = nd + n, \tag{17}$$

and hence $r \geq n+1$ (which contradicts (12)). Thus, $N_t^{Q+1}(\pi) \leq d+1$ for all $t > 2Q+1$ and $\pi \in \mathcal{P}$. If $N_t^{Q+1}(\pi) = d$ for all π, then (15) holds. Hence, assume $N_{T_0}^{Q+1}(\pi) = d+1$ for some π and $T_0 > 2Q+1$, and similar to (16)

$$J_0 = \max\{T_0-(Q+1) \leq i \leq T_0-1 : x_i = \pi\}. \tag{18}$$

Then, $N_{J_0}^Q(\pi) \geq d$, and $x_{J_0} = \pi$, and thus by (4) and (12), $N_{J_0}^Q(\rho) \geq d$ (and hence $N_{J_0}^{Q+1}(\rho) \geq d$) for all $\rho \in \mathcal{P}$, and $\boldsymbol{x}$ is smooth at $t = J_0$. Employing Lemma 2, $\boldsymbol{x}$ is smooth for all $t \geq J_0$ (and in particular for $t > 2Q+1$), and (13) holds. $\square$

Note that Theorem 2 implies that when examining long term behavior of $\boldsymbol{x}$, it is sufficient to consider the behavior of C on subsets $\mathcal{L}_t \subset \mathcal{P}$ of size $n-r+1$.

The following theorem provides conditions under which $\boldsymbol{x}$ is not asymptotically balanced.

Theorem 3. *Suppose $r \geq 1$ and there exists a nonempty subset satisfying $\mathcal{A} \subset \mathcal{P}$, such that $C(\mathcal{L}) \notin \mathcal{A}$ whenever $\mathcal{A} \subset \mathcal{L} \subseteq \mathcal{P}$ and $|\mathcal{L}| = n-r+1$. Then, there exist initial values such that $\boldsymbol{x}$ is not asymptotically balanced.*

Proof. Consider initial values such that $N_{Q+1}(\pi) = d$ for $\pi \in \mathcal{A}$ and $N_{Q+1}(\pi) \leq d+1$ for $\pi \notin \mathcal{A}$. This is possible via (12), since $|A| \leq n-r$, $(n-r+1)+(r-1) = n$ and

$$(n-r+1)d + (r-1)(d+1) = nd + r - 1 = Q. \tag{19}$$

Now, note that $\mathcal{A} \subset \mathcal{L}_{Q+1}$ and $|\mathcal{L}_{Q+1}| = n-r+1$, and hence $x_{Q+1} = C(\mathcal{L}_{Q+1}) \notin \mathcal{A}$, and $N^{Q+1}_{Q+2}(\pi) = N_{Q+2}(\pi) = d$, for all $\pi \in \mathcal{A}$. Suppose $N^{Q+1}_t(\pi) = d$ for some $t > Q+1$ and all $\pi \in \mathcal{A}$. If $x_{t-(Q+1)} \in \mathcal{A}$, then $\mathcal{L}_t = \{x_{t-(Q+1)}\}$, thus $x_t = x_{t-(Q+1)}$ and $N^{Q+1}_{t+1}(\pi) = d$ for all $\pi \in \mathcal{A}$. Otherwise $|\mathcal{L}_t| = n-r+1$, $\mathcal{A} \subset \mathcal{L}_t$, $x_t = C(\mathcal{L}_t) \notin \mathcal{A}$, and again $N^{Q+1}_{t+1}(\pi) = d$. Thus $N^{Q+1}_t(\pi) = d$, for all $t > Q+1$ and $\pi \in \mathcal{A}$. If $d = 0$, then $N_{t+1}(\pi) = 0$ for all $t > Q+1$ and $\pi \in \mathcal{A}$. Otherwise, for $j \geq 1$

$$\frac{N_{j(Q+1)+1}(\pi)}{j(Q+1)} = \frac{dj}{j(Q+1)} = \frac{d}{nd+r} = \frac{1}{n+r/d}. \tag{20}$$

In either case, $\liminf_{t\to\infty} N_{t+1}(\pi)/t < 1/n$, and $\boldsymbol{x}$ is not asymptotically balanced. $\square$

Corollary 1. If n divides Q (i.e. $r = 1$) then there exist initial values such that $\boldsymbol{x}$ is not asymptotically balanced.

Proof. Set $\pi = C(\mathcal{P})$ and $\mathcal{A} = \mathcal{P} \setminus \{\pi\}$. We have $|\mathcal{A}| = n - 1 < n = n - r + 1$, and the hypotheses of Theorem 3 are satisfied. $\square$

Complimentary to Corollary 1, we have the following.

Theorem 4. *If n divides $Q + 1$ (i.e. $r = 0$), then $\boldsymbol{x}$ is asymptotically balanced.*

Proof. Theorem 2 implies that $N_t^{Q+1}(\pi) = d$ for all $t > 2Q + 1$ and $\pi \in \mathcal{P}$. Thus, as a crude bound, $|N_{t+1}(\pi) - N_{t+1}(\rho)| \leq 2Q + 1$ for all $t > 2Q + 1$ and $\pi, \rho \in \mathcal{P}$, and the theorem holds. $\square$

In the next section we prove the existence of fair choice functions, via consideration of cyclic preference properties.

3. Graph representations, cyclic preference and the existence of fair choice.

In this section, we consider cyclic preference for C as it relates to cyclic balance of $\boldsymbol{x}$.

Akin to (2), for convenience, when considering only values determined via (4) (i.e. $\{x_t : t \geq Q + 1\}$), we will write simply

$$\widetilde{N}_t(p) = |\{Q + 1 \leq i \leq t - 1 : x_i = p\}|, \tag{21}$$

i.e. $\widetilde{N}_t(p)$ represents the number of instances of p between time $Q + 1$ and time $t - 1$, inclusively.

We begin with the following key lemma regarding pair-wise domination for elements of $\mathcal{P}$.

Lemma 3. Suppose $\pi, \rho \in \mathcal{P}$ and for $t \geq Q+1$, $\rho \in \mathcal{L}_t$ implies $C(\mathcal{L}_t) \neq \pi$. Then, for $t > 0$,

$$\Delta_{2Q+1+t}(\pi, \rho) \stackrel{def}{=} \widetilde{N}_{2Q+1+t}(\pi) - \widetilde{N}_{2Q+1+t}(\rho) \leq d+1. \tag{22}$$

Proof. Suppose that $\pi, \rho \in \mathcal{P}$ satisfy the requirements in the statement of the lemma. Theorem 2 implies that (22) holds for $0 \leq t \leq Q$. Assume that (22) does not hold for some $t > 0$, and let

$$T_0 = \min\{t : \Delta_{2Q+1+t}(\pi, \rho) = d+2\}. \tag{23}$$

Then $x_{(2Q+1)+(T_0-1)} = x_{2Q+T_0} = \pi$, which, by the assumptions in the lemma, implies

$$N^Q_{2Q+T_0}(\pi) < N^Q_{2Q+T_0}(\rho). \tag{24}$$

Thus

$$\Delta_{2Q+T_0+1}(\pi, \rho) = \Delta_{Q+T_0}(\pi, \rho) + N^Q_{2Q+T_0}(\pi) - N^Q_{2Q+T_0}(\rho) + 1, \tag{25}$$

(see Figure 1). Therefore,

$$\Delta_{Q+T_0}(\pi, \rho) = \Delta_{Q+1+T_0}(\pi, \rho) + N^Q_{Q+T_0}(\rho) - N^Q_{Q+T_0}(\pi) - 1 \geq \Delta_{Q+1+T_0}(\pi, \rho) = d+2,$$

which contradicts the definition of T_0 in (23). The lemma follows. $\square$

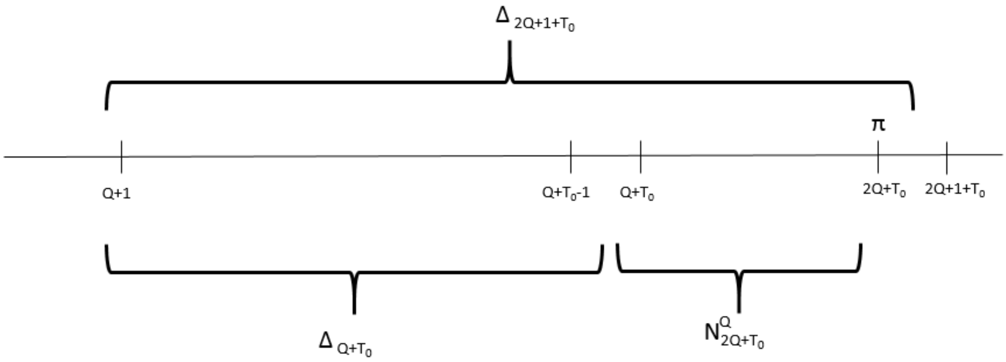

Figure 1: Schematic for Equation (25) in the proof of Theorem 3, indicating sequence values under consideration.

Theorem 5. *Suppose the set $\mathcal{P}$, a choice function C (on $\mathscr{P}(\mathcal{P})$) and $Q \geq 1$ are fixed, and r is as in (12). Now, for $2 \leq c \leq n-1$, consider graphs $G = G_c(C)$ with vertex set $\mathcal{P}$ and directed edge set $E = \{(\pi_1, \pi_2)\}$, determined via $(\pi_1, \pi_2) \in E$, if and only if, $\pi_1 \in \mathcal{L}$ with $|\mathcal{L}| = c$ implies $C(\mathcal{L}) \neq \pi_2$. If the graph G_{n-r+1} is path connected, i.e., there exists a path of the form*

$$(p_{i_0}, p_{i_1})(p_{i_1}, p_{i_2}) \cdots (p_{i_w}, p_{i_0}) \tag{26}$$

such that $\{p_{i_j} : 0 \leq j \leq w\} = \mathcal{P}$, then $\boldsymbol{x}$ is asymptotically balanced.

Before proving Theorem 5, we give some some resulting graphs as described in the statement of Theorem 5, for $\mathcal{P} = \{1, 2, 3, 4, 5, 6\}$.

Example 2. The choice function given in Table 2, below, leads to the digraphs in Figure 2, for $2 \leq |\mathcal{L}| \leq 5$. Note that each of the graphs is path connected.

$\mathcal{L}$	$C(\mathcal{L})$	$\mathcal{L}$	$C(\mathcal{L})$	$\mathcal{L}$	$C(\mathcal{L})$	$\mathcal{L}$	$C(\mathcal{L})$
$\{1\}$	1	$\{3,5\}$	5	$\{2,3,5\}$	5	$\{1,3,4,6\}$	6
$\{2\}$	2	$\{3,6\}$	6	$\{2,3,6\}$	6	$\{1,3,5,6\}$	5
$\{3\}$	3	$\{4,5\}$	4	$\{2,4,5\}$	4	$\{1,4,5,6\}$	4
$\{4\}$	4	$\{4,6\}$	6	$\{2,4,6\}$	6	$\{2,3,4,5\}$	2
$\{5\}$	5	$\{5,6\}$	5	$\{2,5,6\}$	5	$\{2,3,4,6\}$	6
$\{6\}$	6	$\{1,2,3\}$	1	$\{3,4,5\}$	3	$\{2,3,5,6\}$	5
$\{1,2\}$	1	$\{1,2,4\}$	4	$\{3,4,6\}$	6	$\{2,4,5,6\}$	4
$\{1,3\}$	3	$\{1,2,5\}$	5	$\{3,5,6\}$	5	$\{3,4,5,6\}$	3
$\{1,4\}$	4	$\{1,2,6\}$	6	$\{4,5,6\}$	4	$\{1,2,3,4,5\}$	1
$\{1,5\}$	5	$\{1,3,4\}$	3	$\{1,2,3,4\}$	1	$\{1,2,3,4,6\}$	6
$\{1,6\}$	6	$\{1,3,5\}$	5	$\{1,2,3,5\}$	5	$\{1,2,3,5,6\}$	5
$\{2,3\}$	2	$\{1,3,6\}$	6	$\{1,2,3,6\}$	6	$\{1,2,4,5,6\}$	4
$\{2,4\}$	4	$\{1,4,5\}$	4	$\{1,2,4,5\}$	4	$\{1,3,4,5,6\}$	3
$\{2,5\}$	5	$\{1,4,6\}$	6	$\{1,2,4,6\}$	6	$\{2,3,4,5,6\}$	2
$\{2,6\}$	6	$\{1,5,6\}$	5	$\{1,2,5,6\}$	5	$\{1,2,3,4,5,6\}$	1
$\{3,4\}$	3	$\{2,3,4\}$	2	$\{1,3,4,5\}$	3		

Table 2: Choice function for Example 2

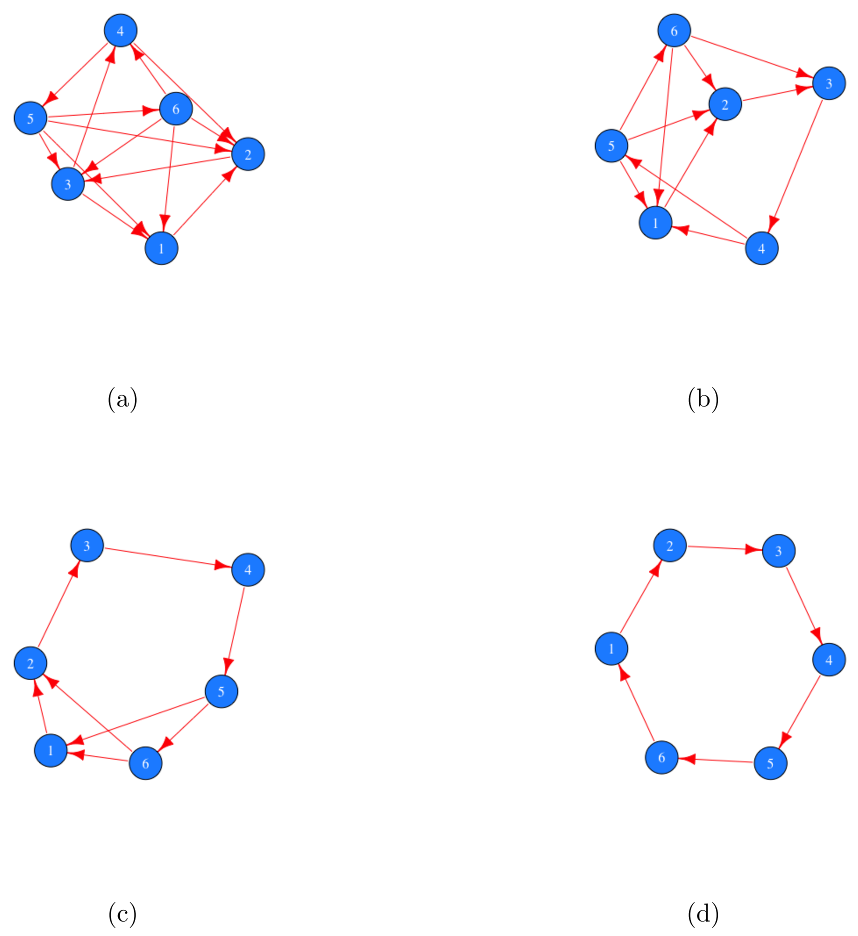

Figure 2: Plots of associated digraphs, $G_c(C)$ for the choice function, C, in Table 2 for $c = 2, 3, 4, 5$ respectively.

Proof of Theorem 5. Suppose G is path connected, with a path as in (26), and $t \geq 0$. Then, employing Lemma 3, for $v_1, v_2 \in \mathcal{P}$, there exist $(\gamma_i)_{1\leq i\leq w_1}$ and $(\sigma_i)_{1\leq i\leq w_2}$

with $w_1, w_2 \le w$ such that,

$$\widetilde{N}_{\xi_t}(v_1) \le \widetilde{N}_{\xi_t}(p_{\gamma_1}) + \delta \le \cdots \le \widetilde{N}_{\xi_t}(p_{\gamma_{w_1}}) + w_1\delta \le \widetilde{N}_{\xi_t}(v_2) + (w_1+1)\delta$$

and

$$\widetilde{N}_{\xi_t}(v_2) \le \widetilde{N}_{\xi_t}(p_{\sigma_1}) + \delta \le \cdots \le \widetilde{N}_{\xi_t}(p_{\sigma_{w_2}}) + w_2\delta \le \widetilde{N}_{\xi_t}(v_1) + (w_2+1)\delta,$$

where $\xi_t = 2Q + 1 + t$ and $\delta = d + 1$. Hence

$$|\widetilde{N}_{2Q+1+t}(v_1) - \widetilde{N}_{2Q+1+t}(v_2)| \le (w+1)(d+1), \tag{27}$$

and $\boldsymbol{x}$ is asymptotically balanced. $\square$

Note that the argument employed in Theorem 5 carries over to connected components. This fact is provided in the following theorem.

Theorem 6. *For the construction of G_{n-r+1} given in Theorem 5, if $\mathcal{P}_1 \subset \mathcal{P}$ are the vertices in a path-connected component of G_{n-r+1} then, the sequence*

$$\{|\widetilde{N}_{2Q+1+t}(v_1) - \widetilde{N}_{2Q+1+t}(v_2)| : t \ge 0\} \tag{28}$$

is bounded above, for all $\rho_1, \rho_2 \in \mathcal{P}_1$.

For $\pi, \rho \in \mathcal{P}$, we will say $\pi \succ_c \rho$ if and only if $|\mathcal{L}| = c$ and $\pi \in \mathcal{L}$ implies $C(\mathcal{L}) \ne \rho$.

The following theorem follows directly from Theorem 5.

Theorem 7. *Suppose $c = n-r+1$ and $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a permutation of $(1, 2, \ldots, n)$. If $\pi_{\sigma_i} \succ_c \pi_{\sigma_{i+1}}$, for $1 \leq i \leq n-1$ and $\pi_{\sigma_n} \succ_c \pi_{\sigma_1}$, i.e. there is a cyclic preference ordering on $\mathcal{P}$, then $\boldsymbol{x}$ is asymptotically balanced.*

The next result considers existence of asymptotically balanced sequences, when $r > 1$.

Theorem 8. *Suppose $r > 1$. Then there exists a choice function C such that $\boldsymbol{x}$ is asymptotically balanced for all initial values.*

Proof. Suppose $t \geq 2Q+1$, and note that, by (13), $|\mathcal{L}_t| > 1$ implies $|\mathcal{L}_t| = n-r+1 < n$. Now, set $c = n-r+1$, and for $\mathcal{L} \in \mathscr{P}(\mathcal{P})$, with $|\mathcal{L}| = c$, define

$$\mathcal{U}_{\mathcal{L}} = \{2 \leq i \leq n : p_{i-1} \notin \mathcal{L} \text{ and } p_i \in \mathcal{L}\} \tag{29}$$

and

$$u_{\mathcal{L}} = \begin{cases} \max\{u : u \in \mathcal{U}_{\mathcal{L}}\}, & \text{If } \mathcal{U}_{\mathcal{L}} \neq \emptyset, \\ 1, & \text{otherwise} \end{cases}, \tag{30}$$

and set $C(\mathcal{L}) = p_{u_{\mathcal{L}}}$. Suppose $i \geq 2$ and $p_{i-1} \in \mathcal{L}$. Then $u_t \neq i$ and hence $p_{i-1} \succ_c p_i$. On the other hand, if $u_{\mathcal{L}} = 1$, then $\mathcal{U}_{\mathcal{L}} = \emptyset$, $\mathcal{L} = \{p_1, p_2, \ldots, p_c\} \neq \mathcal{P}$, and $p_n \notin \mathcal{L}$. Thus $p_n \succ_c p_1$, and by Theorem 7, $\boldsymbol{x}$ is asymptotically balanced. $\square$

Defining a choice function on subsets of $\mathcal{P}$ as described in the proof of Theorem

8, and combining with Theorem 4 leads directly to Theorem 1.

Example 3. Table 3 provides some additional preference restrictions in the case $n = 5$ and $c = 3$. The associated digraphs are given in Figure 3. The final row in Table 3 provides the minimal cycle length of $\boldsymbol{x}$, for the initial values $(x_1, x_2, \ldots, x_7) = (1, 5, 2, 3, 4, 2, 3)$.

$\mathcal{L}$	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
$\{1,2,3\}$	2	2	3	3	2	1	2	1	1
$\{1,2,4\}$	1	1	2	2	2	1	2	2	4
$\{1,2,5\}$	1	1	1	1	5	5	1	5	5
$\{1,3,4\}$	3	4	3	3	1	4	4	4	3
$\{1,3,5\}$	1	3	5	3	5	1	1	1	5
$\{1,4,5\}$	4	1	1	1	1	4	4	1	5
$\{2,3,4\}$	4	3	4	3	2	2	4	4	2
$\{2,3,5\}$	5	3	5	5	3	3	5	3	5
$\{2,4,5\}$	4	5	4	4	4	5	4	4	2
$\{3,4,5\}$	5	4	4	4	5	5	3	3	5
period length	64	8	80	80	8	17	17	25	32

Table 3: Choices on subsets of size three for Example 3.

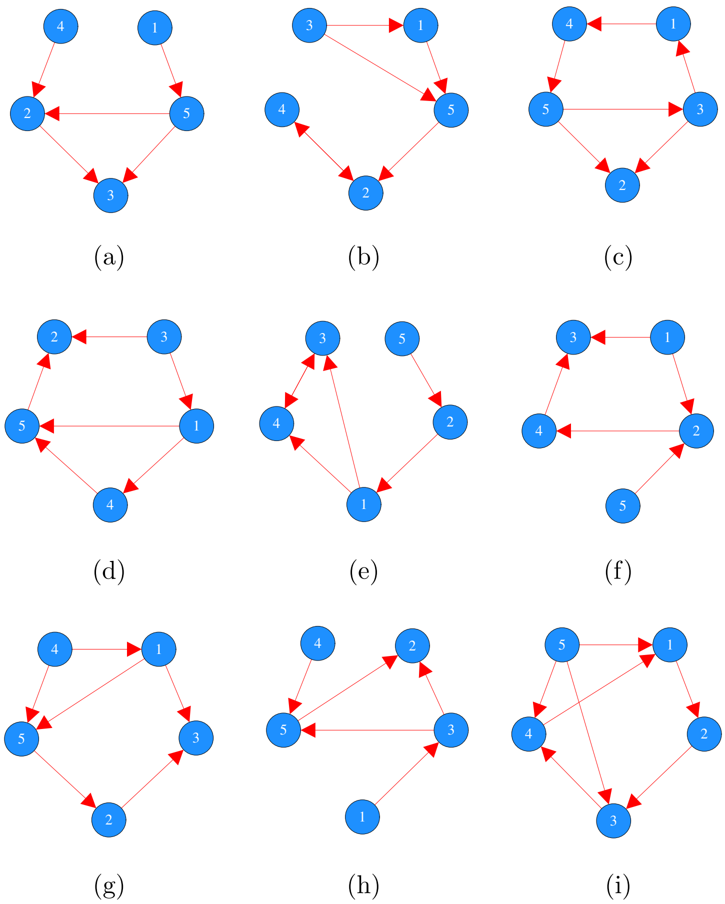

Figure 3: Plots of the digraphs for the choice functions in Table 3. (a) corresponds to C_1, (b) to C_2, etc.

We close with some final related open questions.

Open Question 1. Suppose that $x_1, x_2, \ldots, x_Q$ are drawn uniformly (randomly) from the set $\mathcal{P}$, and x_t is drawn uniformly from the set $\mathcal{L}_t$. What is the distribution of $N_t(\pi)$, for $\pi \in \mathcal{P}$?

Open Question 2. Fix $\mathcal{P}$ and $2 \leq c \leq n-1$, and consider the set $\mathcal{G}_c = \{G_c(C) :$

$C \in \mathcal{C}_{\mathcal{P}}\}$ (where $G_c(C) = G_c$ is as defined in the statement of Theorem 5). What is $|\mathcal{G}_c|$? For what proportion of $C \in \mathcal{C}_{\mathcal{P}}$ is $G_c(C)$ path connected?

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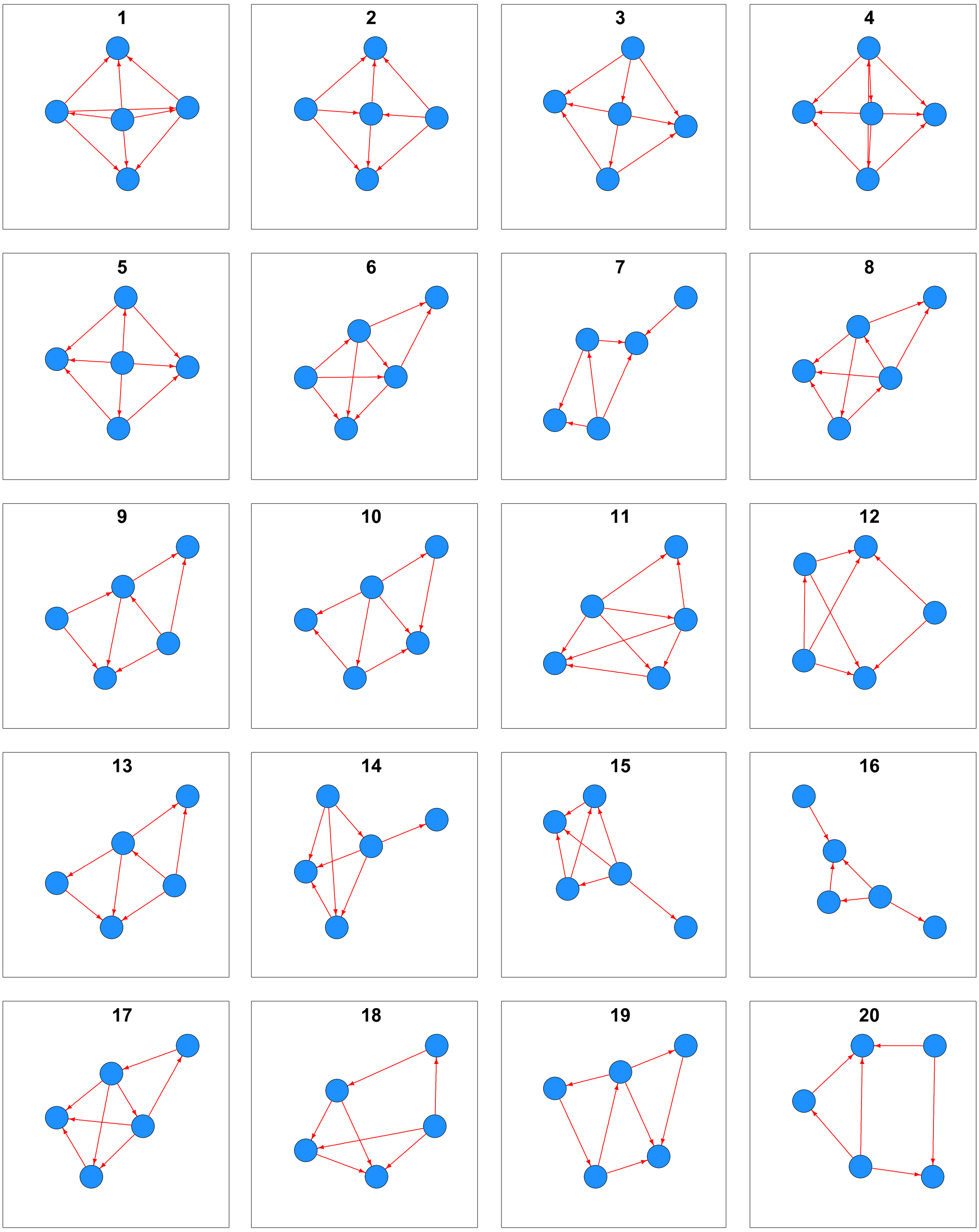
Appendix B: S1: Plots of the 225 distinct non-isomorphic $(5, 3)$ domination graphs.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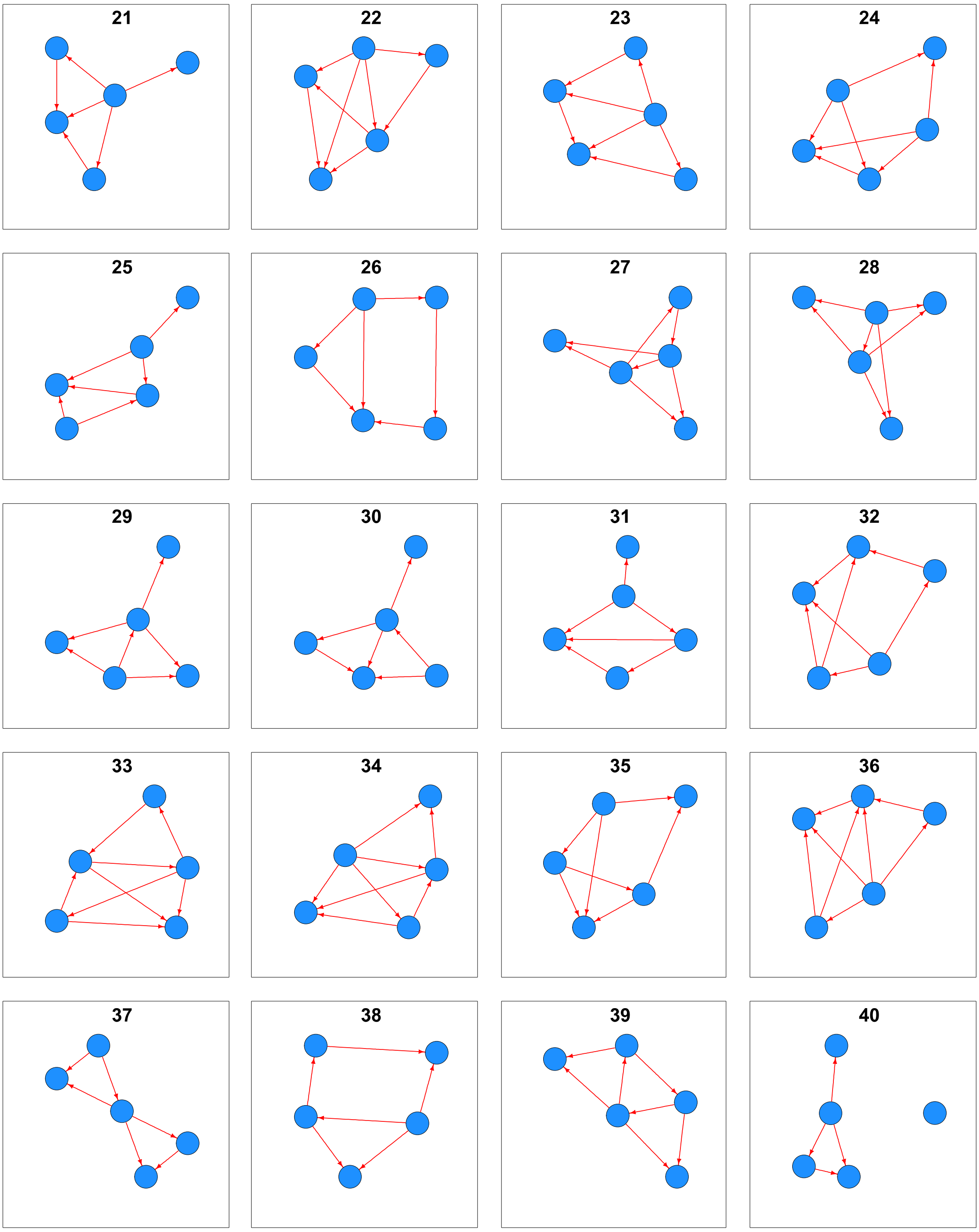
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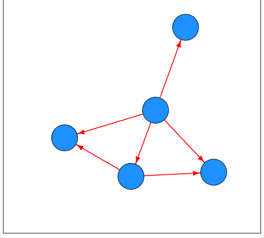

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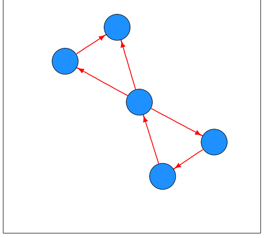

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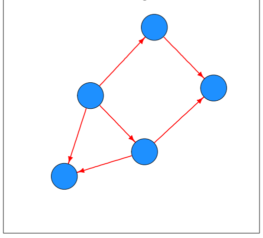

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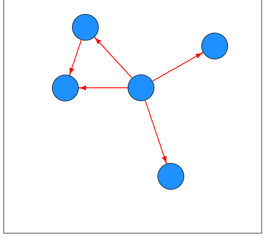

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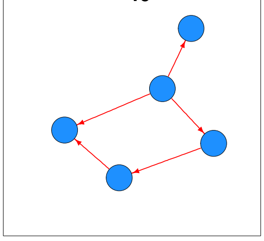

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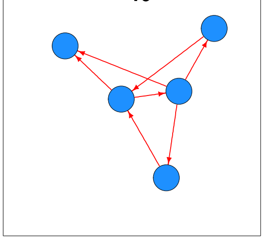

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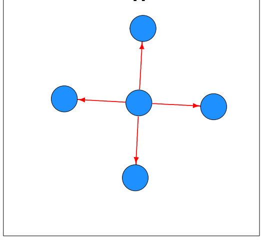

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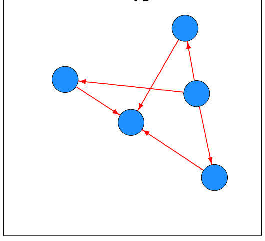

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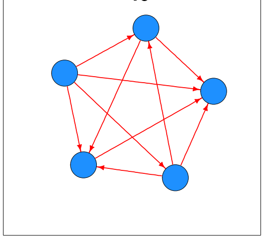

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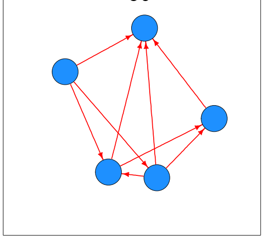

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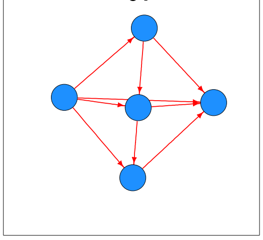

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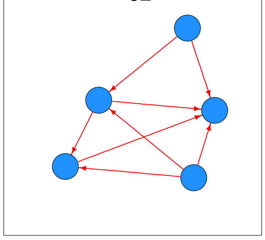

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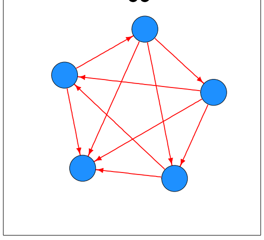

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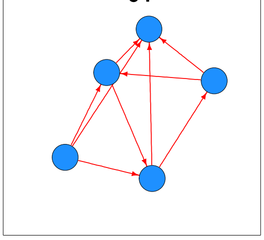

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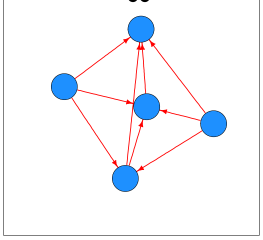

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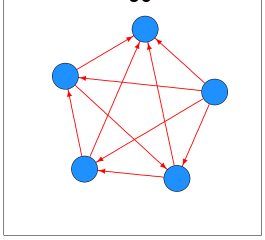

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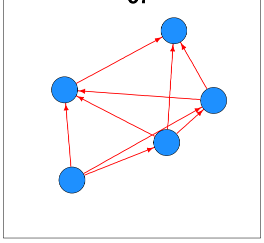

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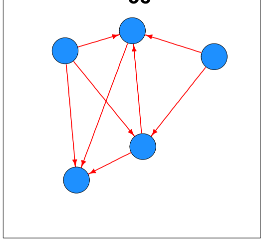

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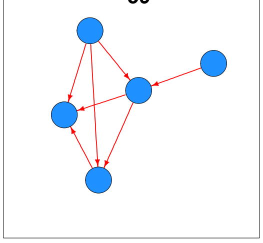

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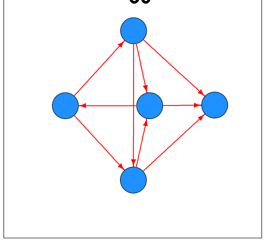

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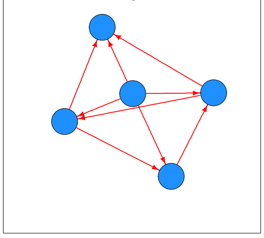

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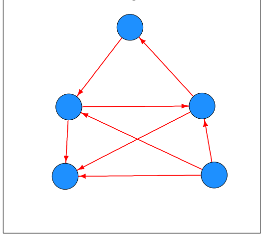

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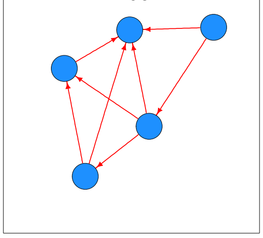

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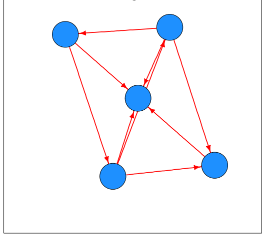

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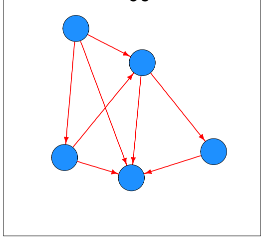

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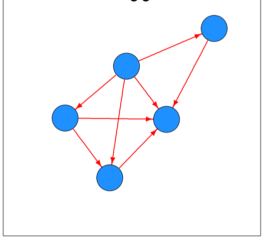

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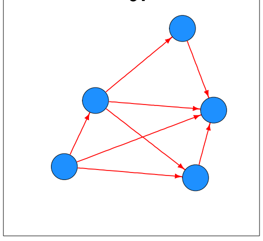

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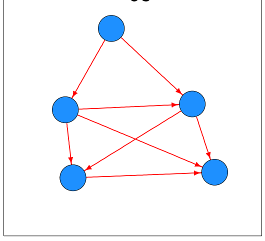

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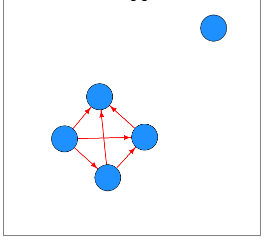

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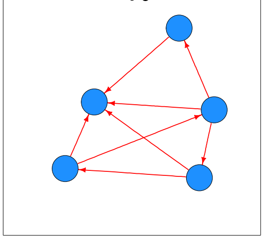

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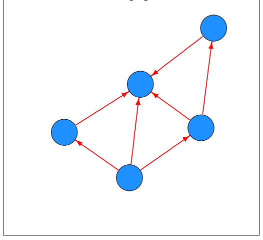

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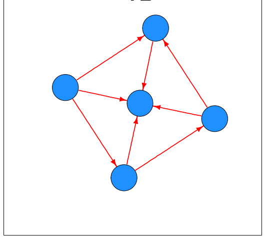

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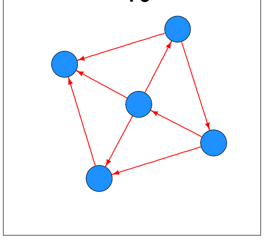

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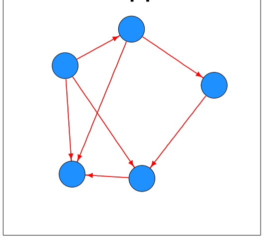

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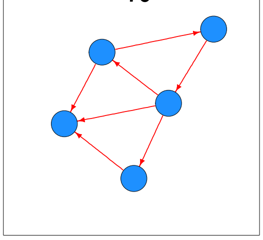

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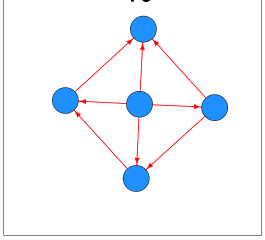

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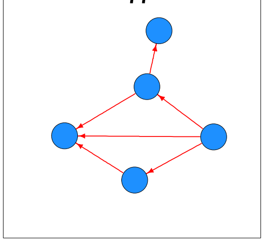

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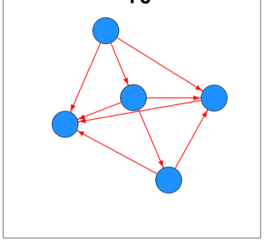

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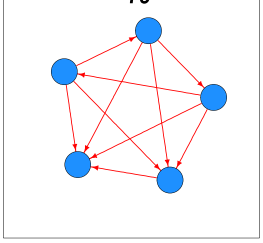

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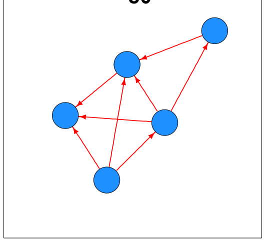

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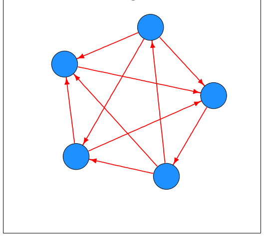

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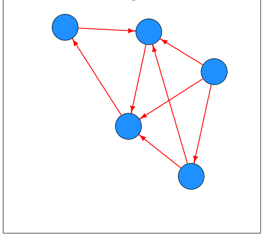

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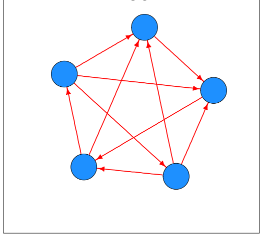

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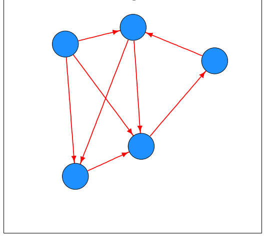

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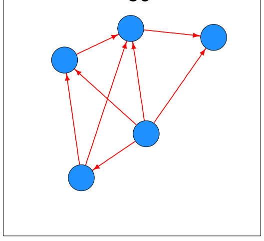

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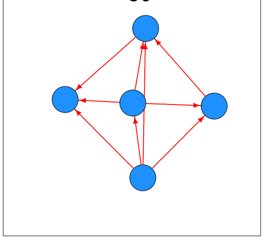

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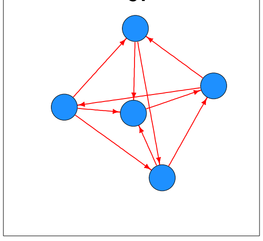

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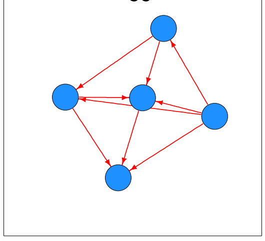

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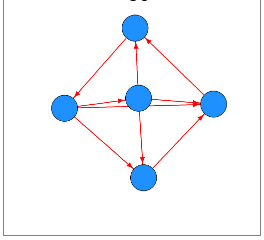

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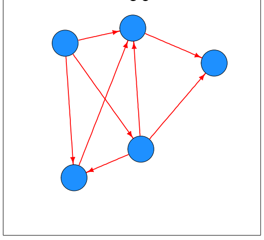

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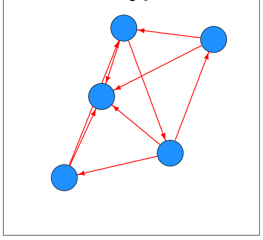

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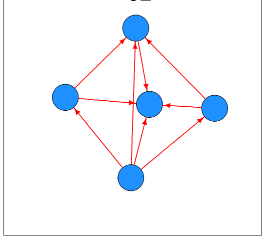

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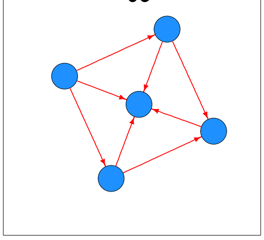

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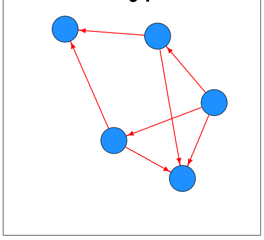

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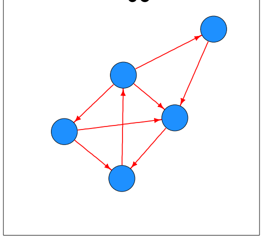

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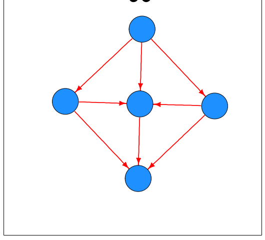

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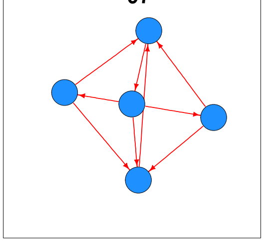

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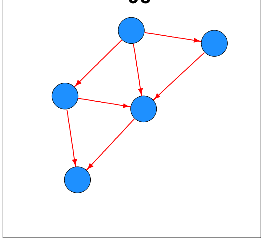

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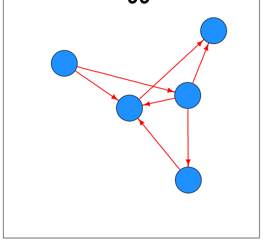

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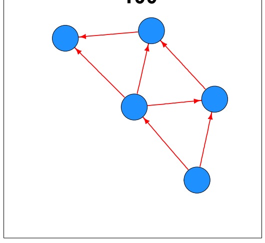

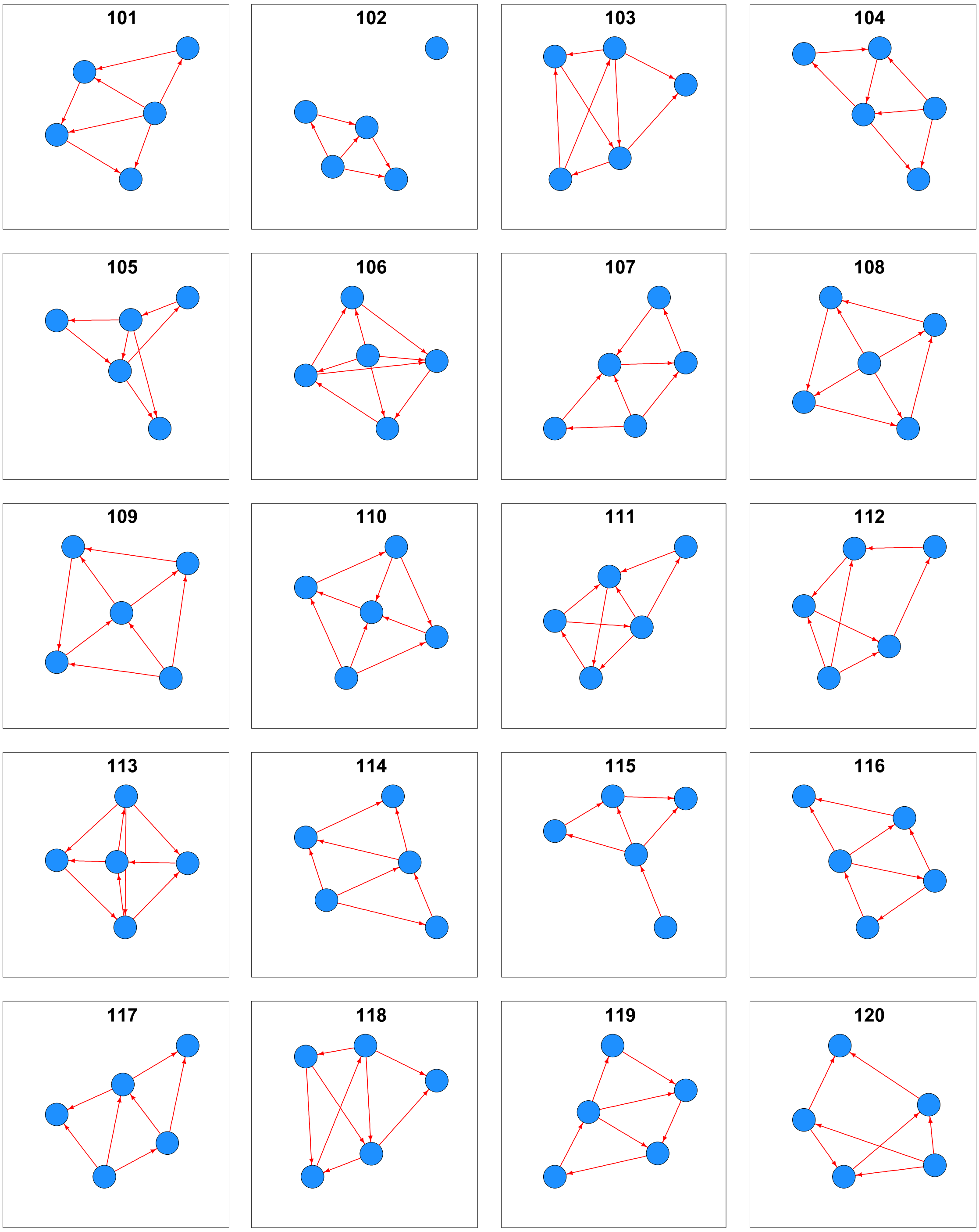
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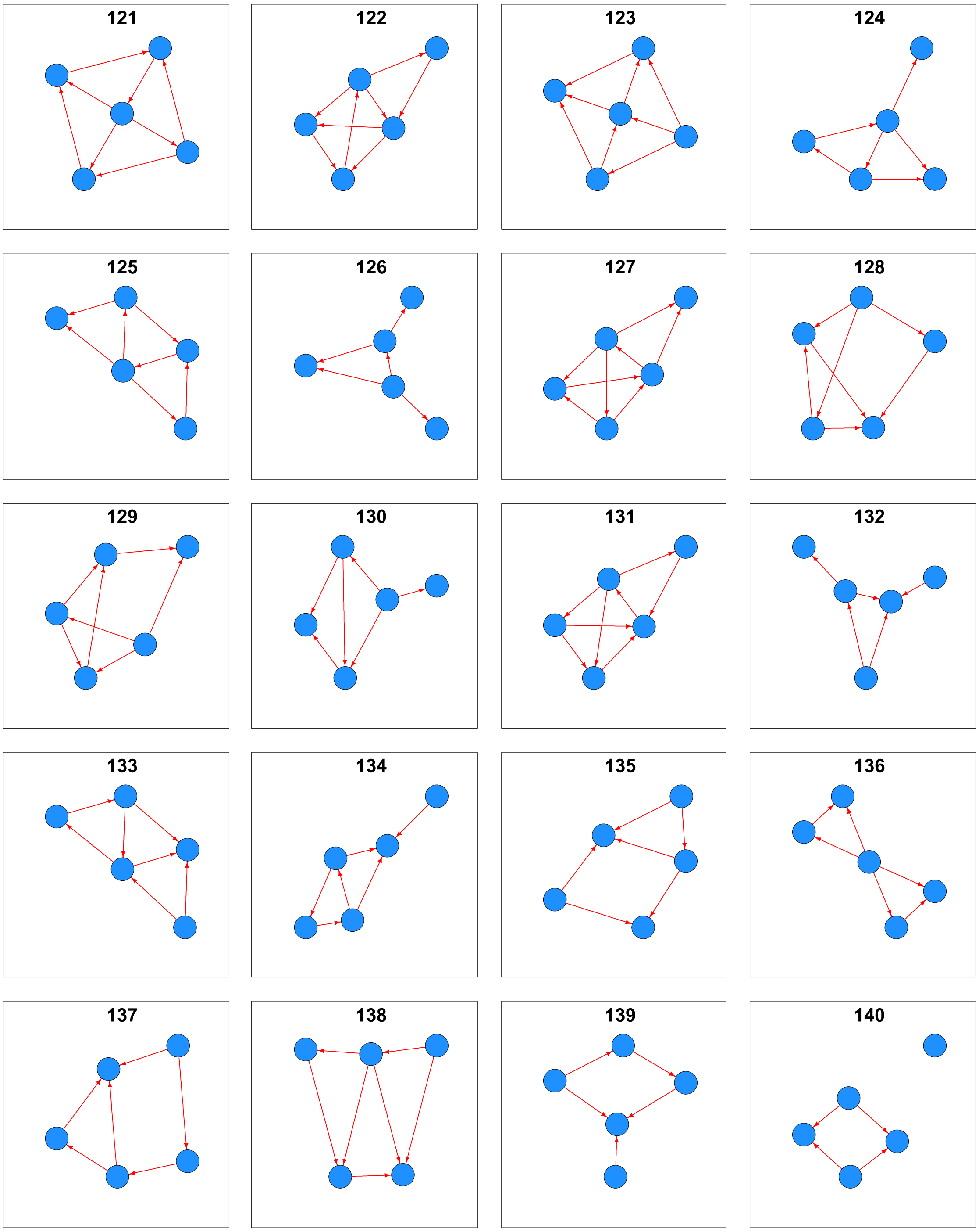
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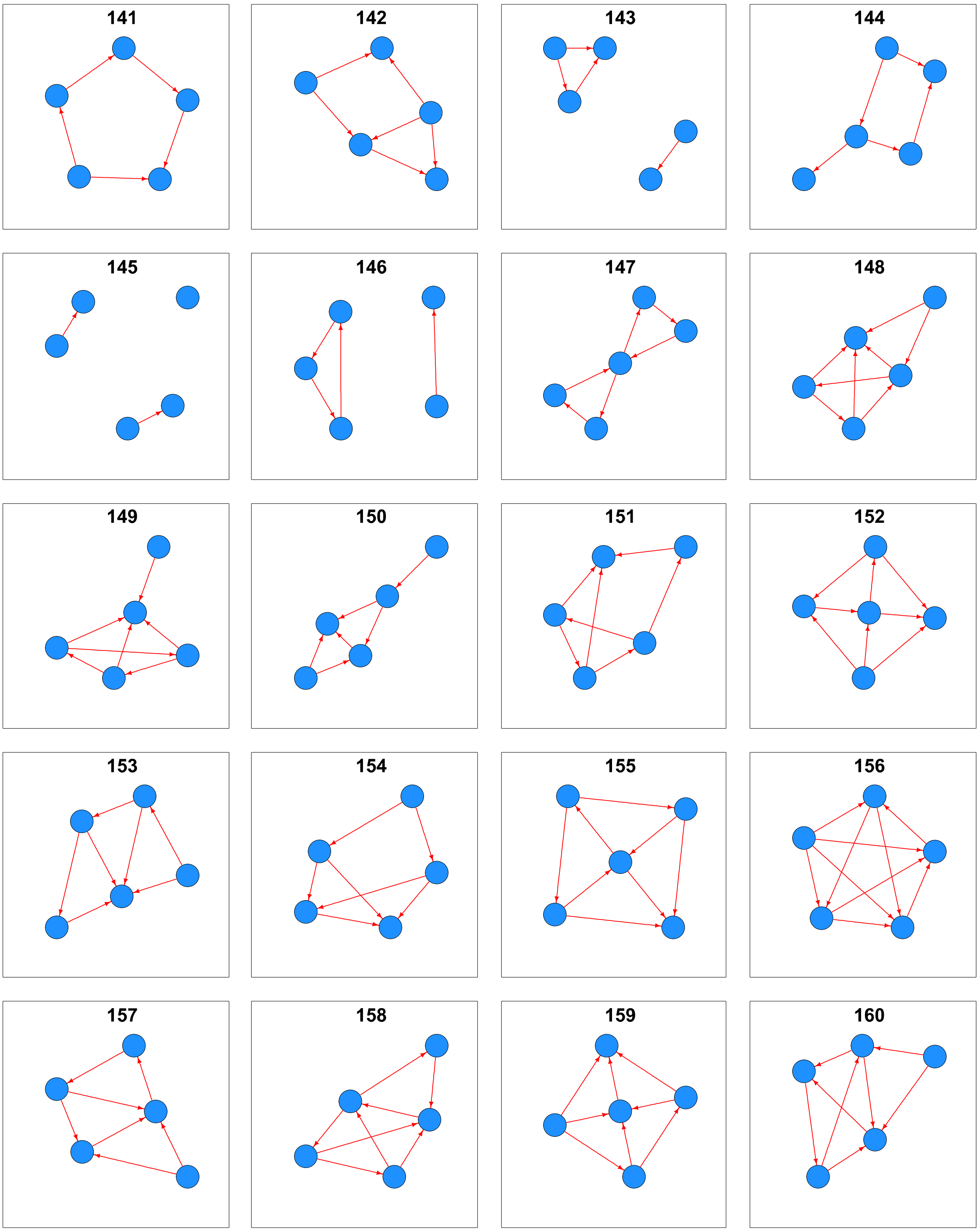
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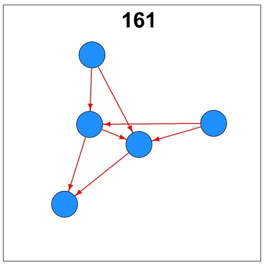
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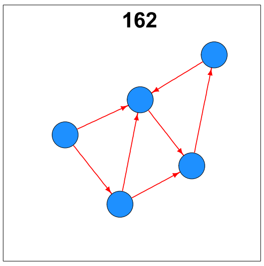
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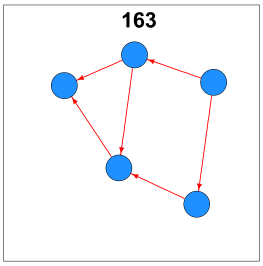
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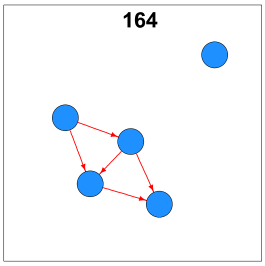
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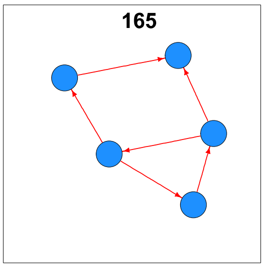
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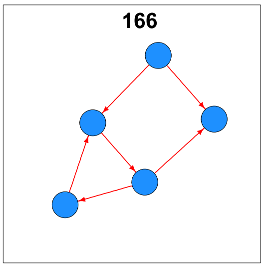
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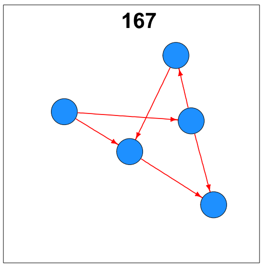
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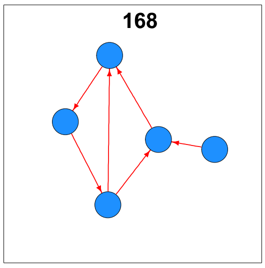
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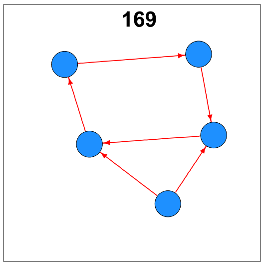
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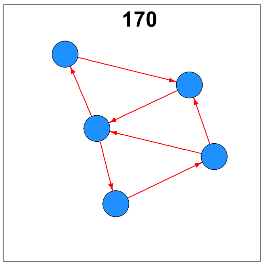
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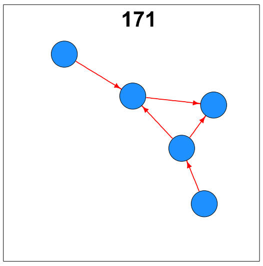
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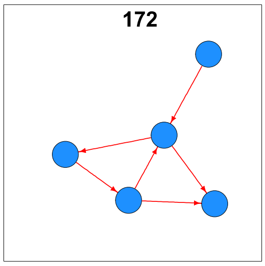
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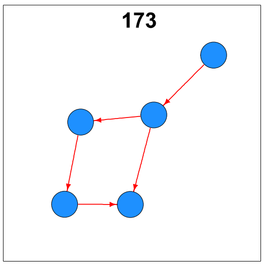
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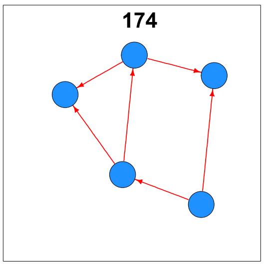
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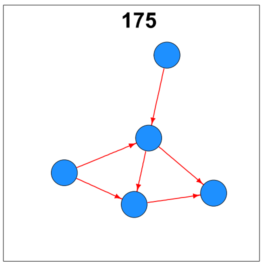
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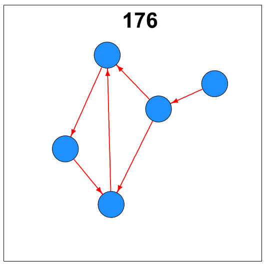
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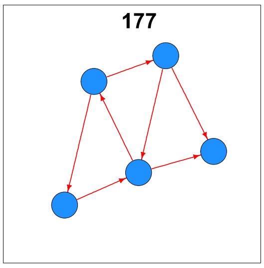
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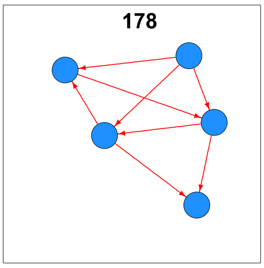
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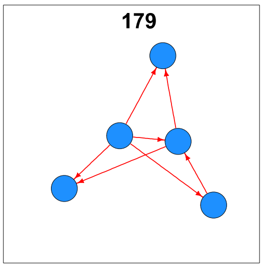
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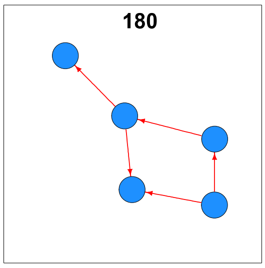
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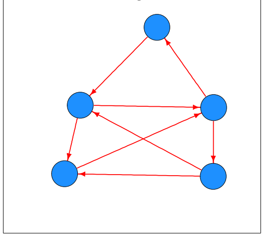

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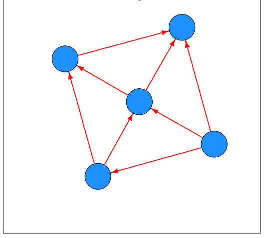

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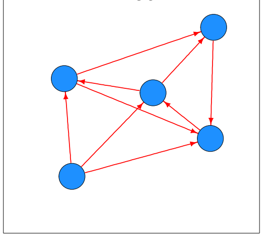

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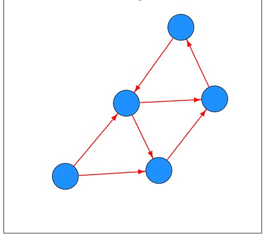

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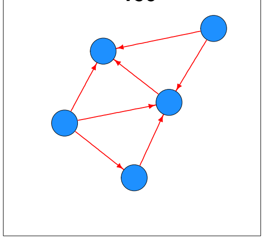

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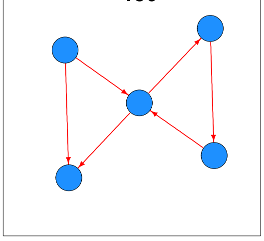

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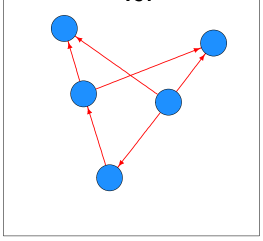

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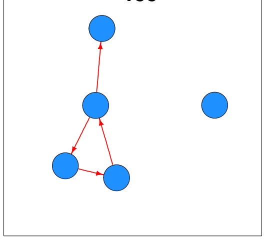

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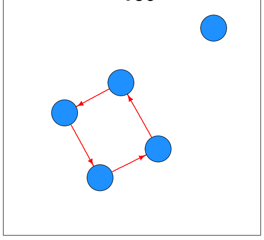

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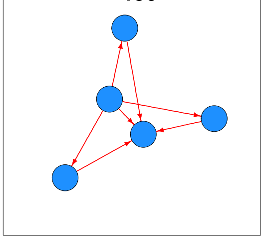

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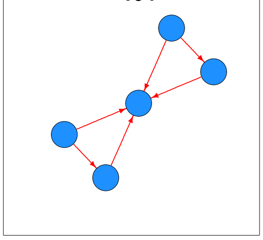

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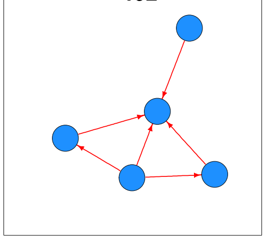

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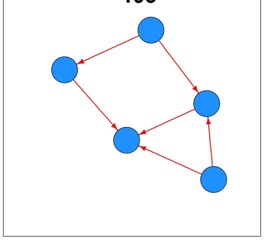

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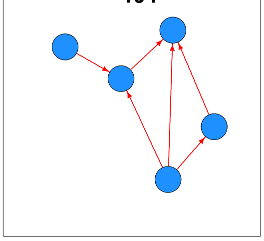

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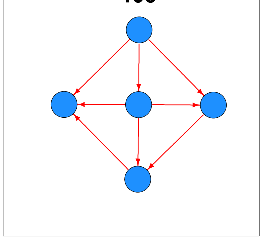

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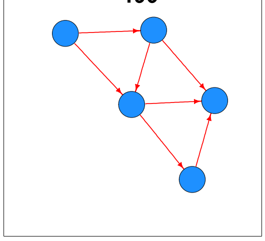

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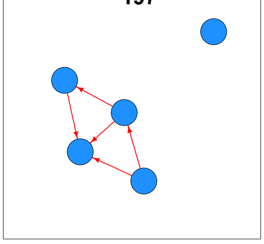

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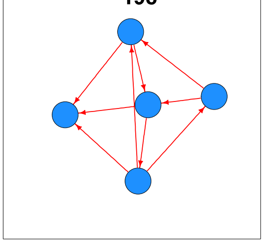

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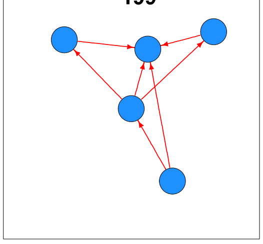

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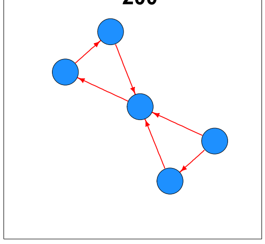

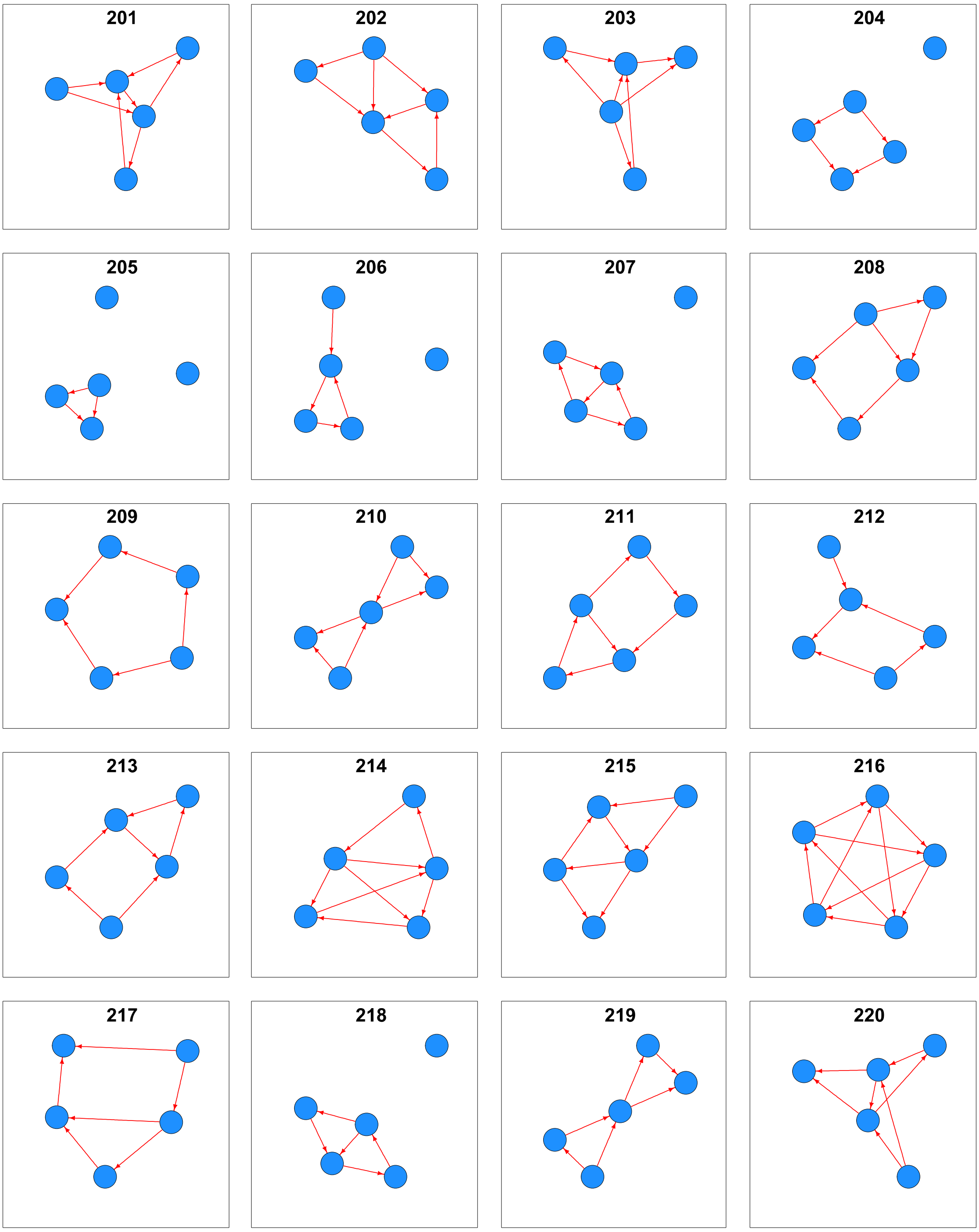
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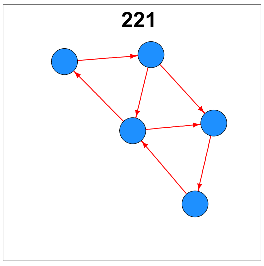
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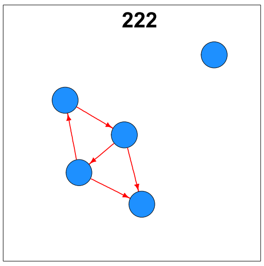
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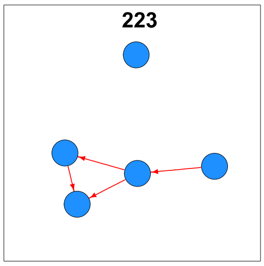
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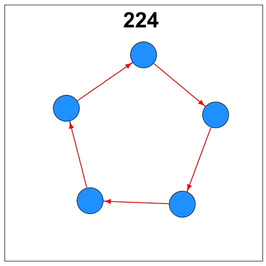
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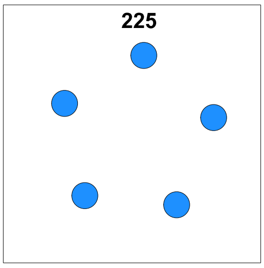
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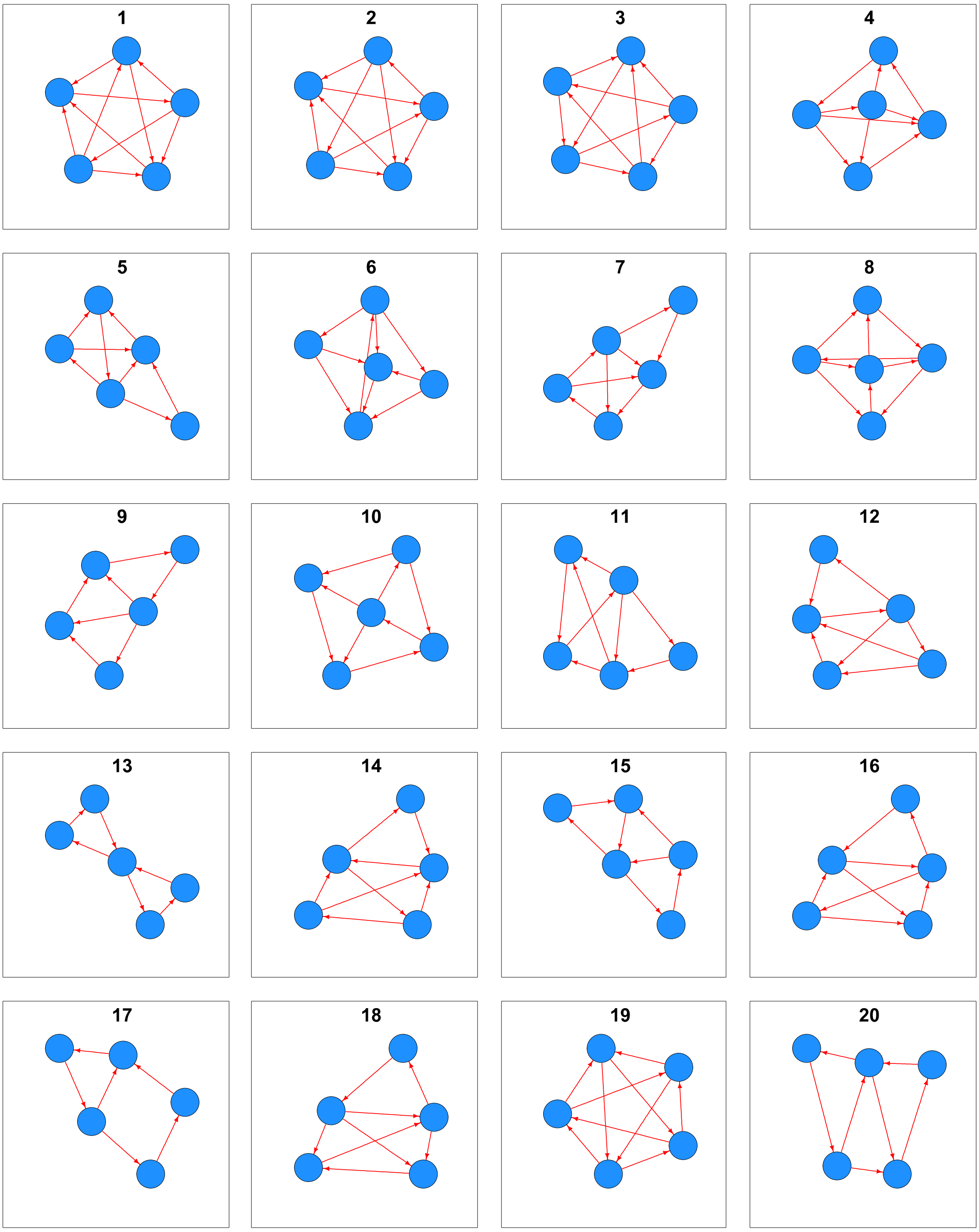
Appendix C: S2: Plots of the 21 distinct strongly connected, non-isomorphic $(5, 3)$ domination graphs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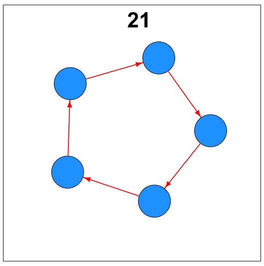
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[2] S. Pirzada, T. Naikoo, On score sets in tournaments, Vietnam J. of Mathematics 34 (2) (2006) 157–161.

[3] M. A. Khan, S. Pirzada, K. K. Kayibi, Scores, inequalities and regular hypertournaments, J. Math. Ineq. Appl.

Curriculum Vitae

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Journal Articles

[1] K. S. Berenhaut, B. P. Lidral-Porter, T. H Schoen, K. P. Webb. Symmetry in Domination for Hypergraphs with Choice. *S*ymmetry (2017) Mar 22;**9** (3):46.

[2] K. S Berenhaut, K. E. Beeler, K. M. Donadio, B. P. Lidral-Porter, M. H. Stone Balanced allocation, infinite words, and sequential choice. To be submitted to *Discrete Applied Mathematics.*

Presentation

"Digraphs and choice functions", CanadAM, Toronto, Canada, June 2017.